Segal–Sugawara vectors for orthosymplectic Lie superalgebras

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Abstract

We consider the centre of the affine vertex algebra at the critical level associated with the orthosymplectic Lie superalgebra. It is well-known that the centre is a commutative superalgebra, and we construct a family of its elements in an explicit form. In particular, this gives a new proof of the formulas for the central elements for the orthogonal and symplectic Lie algebras. Our arguments rely on the properties of a new extended Brauer-type algebra.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$ that is equipped with an invariant supersymmetric bilinear form. Consider the corresponding affine Kac–Moody superalgebra $\widehat{\mathfrak{g}}$ defined as a central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

of the Lie superalgebra of Laurent polynomials $\mathfrak{g}[t, t^{-1}]$. The vacuum module $V_\kappa(\mathfrak{g})$ at the level $\kappa \in \mathbb{C}$ is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K - \kappa$. The vacuum module has a vertex algebra structure and is known as the (*universal*) *affine vertex algebra*; see e.g. [8] and [10] for definitions. The *centre* of this vertex algebra is a commutative associative superalgebra which can be regarded as a subalgebra of $\mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$.

In the case of a simple Lie algebra $\mathfrak{g}$, the centre is trivial, except at the critical level $\kappa = -h^\vee$, where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$. The vertex algebra $V_{-h^\vee}(\mathfrak{g})$ has a large centre $\mathfrak{z}(\widehat{\mathfrak{g}})$ which can be described by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{S \in V_{-h^\vee}(\mathfrak{g}) \mid \mathfrak{g}[t]\, S = 0\}. \tag{1.1}$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a *Segal–Sugawara vector*. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is equipped with the derivation $\tau = -d/dt$ arising from the vertex algebra structure. By the celebrated theorem of Feigin and Frenkel [5], the differential algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ possesses generators $S_1, \dots, S_n$ so that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of polynomials

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[\tau^r S_l \mid l = 1, \dots, n,\ \ r \geqslant 0],$$

where $n = \operatorname{rank} \mathfrak{g}$; see also [8]. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is known as the *Feigin–Frenkel centre*, and the elements $S_1, \dots, S_n$ form a *complete set of Segal–Sugawara vectors*.

Explicit formulas for complete sets of Segal–Sugawara vectors were given in [3] and [4] for the Lie algebras $\mathfrak{g}$ of type A, and in [13] for types B, C and D with the use of the Brauer algebra. A detailed exposition of these results, together with applications to commutative subalgebras in enveloping algebras and to higher order Hamiltonians in the Gaudin models, can be found in [14]. A complete set of Segal–Sugawara vectors for the Lie algebra of type G_2 was produced in [18] using computer-assisted calculations. A different method to construct generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ was developed in [22] which led to new explicit formulas in the case of the Lie algebras of types B, C, D and G_2. It was shown in [15] that they coincide with those in [13] in the classical types. Note also the recent work [20] where interpolating families of Segal–Sugawara operators were constructed in the context of categorical vertex algebra theory.

Although no general analogue of the Feigin–Frenkel theorem is known in the super case, a few families of Segal–Sugawara vectors for the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ were constructed in [17], and it was conjectured therein that these vectors generate the centre $\mathfrak{z}(\widehat{\mathfrak{gl}}_{m|n})$ of the associated affine vertex algebra at the critical level. The conjecture was proved for $\mathfrak{gl}_{1|1}$ in [16], and for $\mathfrak{gl}_{2|1}$ in [2]. The affine vertex algebra associated with the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1|2}$ was investigated in [1]. A conjectural description of its centre at the critical level was pointed out in [1, Remark 10].

Our goal in this paper is to construct a family of Segal–Sugawara vectors for the Lie superalgebra $\mathfrak{osp}_{M|2n}$. As with the case of the orthogonal and symplectic Lie algebras considered in [13] (see also [14, Ch. 8]), we rely on the properties of a distinguished element $s^{(m)}$ (the *symmetriser*) of the Brauer algebra $\mathcal{B}_m(\omega)$. According to the super version of the Schur–Weyl duality, the actions of $\mathfrak{osp}_{M|2n}$ and $\mathcal{B}_m(\omega)$ (with $\omega = M - 2n$) on the tensor product space $(\mathbb{C}^{M|2n})^{\otimes m}$ commute, thus allowing us to apply a similar approach to the orthosymplectic Lie superalgebra. However, the arguments used for the Lie algebras do not readily extend to the super case. The main obstacle is the singularity of the symmetriser $s^{(m)}$; it is a rational function of ω which may have a pole at $\omega = M - 2n$.

We solve the problem by using a new Brauer-type algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$ containing $\mathcal{B}_{2m+1}(\omega)$ as a subalgebra. We construct *abstract Segal–Sugawara vectors* as elements of $\hat{\mathcal{B}}_{2m+1}(\omega)$, keeping ω as an indeterminate. Then we show that the abstract Segal–Sugawara vectors admit an equivalent 'integral form' where the symmetriser $s^{(m)}$ is replaced by the symmetriser in the symmetric group algebra, thus allowing the required evaluation of ω.

Our main result is Theorem 2.1; it provides explicit formulas for Segal–Sugawara vectors Φ_m for $\mathfrak{osp}_{M|2n}$. In particular, these elements are even, and they generate a commutative subalgebra of the universal enveloping algebra of the Lie superalgebra $t^{-1}\mathfrak{osp}_{M|2n}[t^{-1}]$; see Corollary 2.2. We believe that if M is odd, then the vectors Φ_{2k} generate the centre of the affine vertex algebra, as stated in Conjecture 2.3.

With the significance of the Feigin–Frenkel centre in applications to Gaudin models and Vinberg's quantisation problem [6], [7], [21], we expect that the explicit Segal–Sugawara vectors Φ_m will play a due role in understanding the higher Hamiltonians in the Gaudin models with the $\mathfrak{osp}$-symmetry and the quantum shift of argument subalgebras in $\mathrm{U}(\mathfrak{osp}_{M|2n})$.

2 Segal–Sugawara vectors

We use the involution $i \mapsto i' = M + 2n - i + 1$ on the set $\{1, 2, \dots, M + 2n\}$. Set

$$\bar{\imath} = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n, n', \dots, 1', \\ 0 & \text{for} \quad i = n+1, \dots, (n+1)' \end{cases}$$

and

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, M+n, \\ -1 & \text{for} \quad i = M+n+1, \dots, M+2n. \end{cases}$$

2.1 Lie superalgebras

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{M|2n}$ is formed by elements E_{ij} of the parity $\bar{\imath} + \bar{\jmath} \mod 2$ for $1 \leqslant i, j \leqslant M + 2n$, with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}\, E_{il} - \delta_{il} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}\, E_{kj}.$$

We will regard the orthosymplectic Lie superalgebra $\mathfrak{osp}_{M|2n}$ as the subalgebra of $\mathfrak{gl}_{M|2n}$ spanned by the elements

$$F_{ij} = E_{ij} - (-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}\, \varepsilon_i \varepsilon_j E_{j'i'}.$$

The corresponding affine Kac–Moody superalgebra $\widehat{\mathfrak{osp}}_{M|2n} \coloneqq \mathfrak{osp}_{M|2n}[t, t^{-1}] \oplus \mathbb{C}K$ has Lie superbracket

$$\begin{aligned} \Big[F_{ij}[r], F_{kl}[s]\Big] &= \delta_{jk} F_{il}[r+s] - \delta_{il} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})} F_{kj}[r+s] - \delta_{ik'} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \varepsilon_i \varepsilon_j F_{j'l}[r+s] \\ &\quad + \delta_{jl'} (-1)^{\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}} \varepsilon_i \varepsilon_j F_{ki'}[r+s] + r\delta_{r,-s} \Big((-1)^{\bar{\imath}} \delta_{il} \delta_{jk} - (-1)^{\bar{\imath}\bar{\jmath}} \varepsilon_i \varepsilon_j \delta_{ik'} \delta_{jl'} \Big) K, \end{aligned}$$

where $F_{ij}[r] \coloneqq F_{ij} t^r$ for each $r \in \mathbb{Z}$ and $1 \leqslant i, j \leqslant M + 2n$, and K is central.

2.2 Vacuum module

The vacuum module $V_{-h^\vee}(\mathfrak{osp}_{M|2n})$ over $\widehat{\mathfrak{osp}}_{M|2n}$ at the critical level is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{osp}}_{M|2n})$ by the left ideal generated by $K + h^\vee$ and $\mathfrak{osp}_{M|2n}[t]$, where $h^\vee = M - 2n - 2$ is the dual Coxeter number for $\mathfrak{osp}_{M|2n}$.

Equipping the vacuum module $V_{-h^\vee}(\mathfrak{osp}_{M|2n})$ with the derivation $\tau \coloneqq -d/dt$ yields a vertex algebra called the affine vertex algebra; see e.g. [10] for details. As vector spaces, we have the isomorphism

$$V_{-h^\vee}(\mathfrak{osp}_{M|2n}) \cong \mathrm{U}\Big(t^{-1}\mathfrak{osp}_{M|2n}[t^{-1}]\Big). \tag{2.1}$$

The centre $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$ of the affine vertex algebra defined as in (1.1) is called the Feigin–Frenkel centre, and its elements are called Segal–Sugawara vectors. Due to (2.1), the Feigin–Frenkel centre can be regarded as a commutative subalgebra of $\mathrm{U}\Big(t^{-1}\mathfrak{osp}_{M|2n}[t^{-1}]\Big)$; see e.g. [8, Sec. 3.3] and [14, Sec. 6.2]. Moreover, this subalgebra is invariant under the derivation τ so that we can regard $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$ as a differential superalgebra.

2.3 Main results

Consider the Lie superalgebra $\widehat{\mathfrak{osp}}_{M|2n} \oplus \mathbb{C}\tau$, where τ is even and satisfies

$$[\tau, K] = 0 \qquad \text{and} \qquad \Big[\tau, F_{ij}[r]\Big] = -rF_{ij}[r-1].$$

We let U denote the universal enveloping algebra $\mathrm{U}\Big(\widehat{\mathfrak{osp}}_{M|2n} \oplus \mathbb{C}\tau\Big)$. Consider the tensor product superalgebra

$$\Big(\operatorname{End}\mathbb{C}^{M|2n}\Big)^{\otimes m} \otimes \mathrm{U}, \tag{2.2}$$

and for $1 \leqslant a \leqslant m$ and $r \in \mathbb{Z}$ define the elements

$$F[r]_a \coloneqq \sum_{i,j=1}^{M+2n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ij}[r](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}+\bar{\jmath}}, \tag{2.3}$$

where the $e_{ij} \in \operatorname{End}\mathbb{C}^{M|2n}$ are the standard matrix units.

The symmetric group $\mathfrak{S}_m$ acts on the space $(\mathbb{C}^{M|2n})^{\otimes m}$ by permuting tensor factors. Denote by $H^{(m)}$ the element of the algebra (2.2) (with identity component in U) which is the image of the symmetriser $h^{(m)} \in \mathbb{C}\mathfrak{S}_m$ defined by

$$h^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} s \tag{2.4}$$

under the action of $\mathfrak{S}_m$. Furthermore, let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of m of length $\ell = \ell(\lambda)$, so that $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$ and $\lambda_1 + \dots + \lambda_\ell = m$. We denote by c_λ the number of permutations in the symmetric group $\mathfrak{S}_m$ of cycle type λ. Set

$$F[-\lambda] = F[-\lambda_1]_1 \dots F[-\lambda_\ell]_\ell, \tag{2.5}$$

and for any $m \geqslant 2$, introduce elements $\Phi_m \in \mathrm{U}\Big(t^{-1}\mathfrak{osp}_{M|2n}[t^{-1}]\Big)$ by

$$\Phi_m = \sum_{\lambda \vdash m,\ \ell(\lambda)\ \text{even}} \mathcal{Y}_{m,\ell}(M-2n-1)\, c_\lambda \operatorname{str}_{1,\dots,\ell} H^{(\ell)} F[-\lambda]. \tag{2.6}$$

Here we use the polynomials $\mathcal{Y}_{m,\ell}(T)$ in a variable T defined by

$$\mathcal{Y}_{m,\ell}(T) = \frac{\ell!}{m!} \prod_{k=\ell}^{m-1} (T+k);$$

cf. [20], while the supertrace

$$\operatorname{str} : \operatorname{End}\mathbb{C}^{M|2n} \to \mathbb{C}, \qquad e_{ij} \mapsto \delta_{ij}(-1)^{\bar{\imath}}$$

is taken over the first ℓ copies of $\operatorname{End}\mathbb{C}^{M|2n}$. The following is our main result.

Theorem 2.1. *All elements Φ_m belong to the Feigin–Frenkel centre $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$.*

The proof of the theorem will be given in Sec. 4, with some preliminary results discussed in Sec. 3. In the following corollary, we use the isomorphism (2.1).

Corollary 2.2. *The elements Φ_m with $m = 2, 3, \dots$ generate a commutative subalgebra of* $\mathrm{U}\Big(t^{-1}\mathfrak{osp}_{M|2n}[t^{-1}]\Big)$.

Conjecture 2.3. *If M is odd, then the elements $\Phi_2, \Phi_4, \Phi_6, \dots$ generate $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$ as a differential superalgebra.*

The conjecture holds for $n = 0$ by [15]. It is likely that for even values of M some additional elements of $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$ arising from the super Pfaffian are necessary to generate this superalgebra; cf. [11] and [12].

Remark 2.4. By taking $M = 0$ or $n = 0$ in Theorem 2.1, we get elements of the respective Feigin–Frenkel centres $\mathfrak{z}(\widehat{\mathfrak{sp}}_{2n})$ or $\mathfrak{z}(\widehat{\mathfrak{o}}_M)$. We thus obtain a new proof of the formulas for the Segal–Sugawara vectors for the orthogonal and symplectic Lie algebras given in [15]. □

3 Extended Brauer-type algebra

To prove Theorem 2.1, we will consider its abstract version first. Namely, we will introduce a new algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$, and construct abstract analogues of the Segal–Sugawara vectors ϕ_m as elements of $\hat{\mathcal{B}}_{2m+1}(\omega)$, regarding ω as a variable. We show that the vectors satisfy the desired properties and that they are well-defined at $\omega = M - 2n$. We then use a homomorphism from $\hat{\mathcal{B}}_{2m+1}(M-2n)$ to the tensor product superalgebra to get the actual Segal–Sugawara vectors Φ_m in the vacuum module over $\widehat{\mathfrak{osp}}_{M|2n}$. The details will be given in Sec. 4, while this section is devoted to some preliminary results on the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$.

3.1 Brauer algebra

Let ω be an indeterminate. Recall that the Brauer algebra $\mathcal{B}_m(\omega)$ is the associative unital algebra with identity 1, generated by s_a and ϵ_a with $1 \leqslant a \leqslant m-1$, subject only to the relations

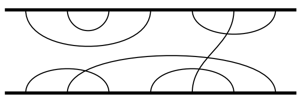
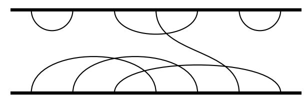
$$s_a^2 = 1, \qquad \epsilon_a^2 = \omega\, \epsilon_a, \qquad s_a\epsilon_a = \epsilon_a s_a = \epsilon_a,$$
$$s_a s_b = s_b s_a, \qquad \epsilon_a\epsilon_b = \epsilon_b\epsilon_a, \qquad s_a\epsilon_b = \epsilon_b s_a, \qquad |a-b| > 1,$$
$$s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}, \qquad \epsilon_a\epsilon_{a+1}\epsilon_a = \epsilon_a, \qquad \epsilon_{a+1}\epsilon_a\epsilon_{a+1} = \epsilon_{a+1},$$
$$s_a\epsilon_{a+1}\epsilon_a = s_{a+1}\epsilon_a, \qquad \epsilon_{a+1}\epsilon_a s_{a+1} = \epsilon_{a+1} s_a.$$

For $1 \leqslant a < b \leqslant m$ we can also define

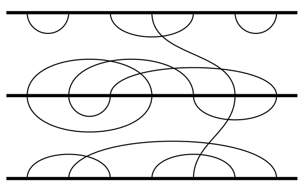
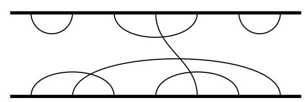
$$s_{ab} = s_a s_{a+1} \dots s_{b-2} s_{b-1} s_{b-2} \dots s_{a+1} s_a,$$
$$\epsilon_{ab} = s_a s_{a+1} \dots s_{b-2} \epsilon_{b-1} s_{b-2} \dots s_{a+1} s_a,$$

and set $s_{ba} = s_{ab}$ and $\epsilon_{ba} = \epsilon_{ab}$.

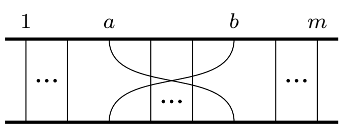
The Brauer algebra $\mathcal{B}_m(\omega)$ also has a diagrammatic presentation as follows. The algebra has a basis of diagrams, where each diagram consists of two horizontal lines, each with m nodes, and m strings connecting the nodes pairwise. The product xy of two diagrams x and y is computed by concatenation; we draw y directly above x, connect the strings at the nodes in the middle, remove the middle line, and replace each loop formed by a factor of ω. For example, in $\mathcal{B}_7(\omega)$, if

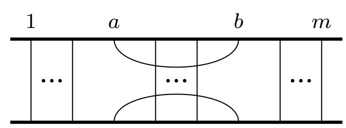
$x =$  , $\qquad y =$  ,

then

$xy =$  $= \omega^2$  .

This can be identified with the non-diagrammatic presentation of $\mathcal{B}_m(\omega)$ by taking

$s_{ab} =$ 

, $\qquad \epsilon_{ab} =$ 

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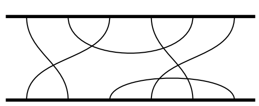
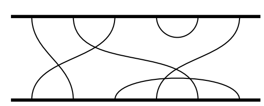
For each $1 \leqslant a \leqslant m$, we define the *partial transposition* $t_a : \mathcal{B}_m(\omega) \to \mathcal{B}_m(\omega)$ as the linear map taking each basis diagram d to the diagram obtained by swapping the string endpoints connected to the nodes numbered a on the top and bottom of d. For example, in $\mathcal{B}_6(\omega)$, we have

$d =$  , $\qquad d^{t_5} =$  .

The *Brauer symmetriser* $s^{(m)} \in \mathcal{B}_m(\omega)$ is the unique nonzero element satisfying

$$s^{(m)}s^{(m)} = s^{(m)}, \qquad s_a s^{(m)} = s^{(m)} s_a = s^{(m)}, \qquad \epsilon_a s^{(m)} = s^{(m)} \epsilon_a = 0$$

for all $1 \leqslant a \leqslant m-1$. It follows that

$$s_{ab} s^{(m)} = s^{(m)} s_{ab} = s^{(m)}, \qquad \epsilon_{ab} s^{(m)} = s^{(m)} \epsilon_{ab} = 0$$

for all $1 \leqslant a < b \leqslant m$. A few explicit expressions for $s^{(m)}$ are collected in [14, Ch. 1]. In particular,

$$s^{(m)} = \frac{1}{m!} \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \binom{\omega/2 + m - 2}{r}^{-1} \sum_{d \in \mathcal{D}^{(r)}} d, \tag{3.1}$$

where $\mathcal{D}^{(r)} \subset \mathcal{B}_m(\omega)$ denotes the set of diagrams which have exactly r horizontal strings in the top row; see [9].

The Brauer algebra $\mathcal{B}_m(\omega)$ has a subalgebra isomorphic to the symmetric group algebra $\mathbb{C}\mathfrak{S}_m$, generated by s_a with $1 \leqslant a \leqslant m-1$. We thus have elements $h^{(m)} \in \mathcal{B}_m(\omega)$ defined as in (2.4).

Let J_m be the vector subspace of $\mathcal{B}_m(\omega)$ spanned by sums of the form $d + d^{t_a}$, for any basis diagram d and $1 \leqslant a \leqslant m$. From Lemma 2.4 of [15], the Brauer and symmetric group symmetrisers in $\mathcal{B}_m(\omega)$ satisfy

$$\gamma_m(\omega) s^{(m)} \equiv h^{(m)} \mod J_m, \tag{3.2}$$

where

$$\gamma_m(\omega) := \frac{\omega + m - 2}{\omega + 2m - 2}. \tag{3.3}$$

This result is readily generalised using the embeddings $\mathcal{B}_k(\omega) \hookrightarrow \mathcal{B}_m(\omega)$ for $1 \leqslant k \leqslant m$ obtained by adding $m-k$ pairs of nodes on the right of each diagram, connected by vertical strings. That is, in $\mathcal{B}_m(\omega)$, we have

$$\gamma_k(\omega) s^{(k)} \equiv h^{(k)} \mod J_m^{(k)}, \tag{3.4}$$

where $J_m^{(k)}$ is the vector subspace of $\mathcal{B}_m(\omega)$ spanned by sums of the form $d + d^{t_a}$, for any basis diagram d whose rightmost $m-k$ pairs of nodes are connected by vertical strings, and $1 \leqslant a \leqslant k$.

The Brauer algebra $\mathcal{B}_m(\omega)$ with parameter $\omega = M - 2n$ acts on the tensor product space $(\mathbb{C}^{M|2n})^{\otimes m}$ so that there is a homomorphism

$$\mathcal{B}_m(M-2n) \to \left(\operatorname{End} \mathbb{C}^{M|2n}\right)^{\otimes m} \tag{3.5}$$

which is defined by $s_{ab} \mapsto P_{ab}$ and $\epsilon_{ab} \mapsto Q_{ab}$, where

$$P_{ab} := \sum_{i,j=1}^{M+2n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)} (-1)^{\bar{\jmath}},$$
$$Q_{ab} := \sum_{i,j=1}^{M+2n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}+\bar{\jmath}} \varepsilon_i \varepsilon_j$$

for $1 \leqslant a < b \leqslant m$. As we will work with the extended tensor product superalgebra, we will usually identify these elements with $P_{ab} \otimes 1$ and $Q_{ab} \otimes 1$ in (2.2), respectively. For $b < a$, define $P_{ab} := P_{ba}$ and $Q_{ab} := Q_{ba}$.

The following easily verified property of the operators Q_{ab} will be essential in the proof of Theorem 2.1. Suppose that X is an element of the superalgebra (2.2). We will also identify X with the element $X \otimes 1$ of this superalgebra but with the parameter m replaced by $m+1$. Then for any $1 \leqslant a \leqslant m$ we have

$$Q_{a\,m+1} X Q_{a\,m+1} = (\operatorname{str}_a X)\, Q_{a\,m+1}, \tag{3.6}$$

where str_a denotes the partial supertrace taken over the a-th copy of the endomorphism superalgebra in (2.2). We will use a version of relation (3.6) in the extended Brauer algebra to define the 'supertrace' on the algebra.

3.2 Affine extension of the Brauer algebra

Our definition of the extended Brauer-type algebra is motivated by the matrix form of the defining relations for the affine Kac–Moody superalgebra $\widehat{\mathfrak{osp}}_{M|2n}$; see Sec. 2.1. Namely, we can derive from the defining relations of $\widehat{\mathfrak{osp}}_{M|2n}$ described in Sec. 2.1 that the elements $F[r]_a$ defined in (2.3) satisfy the relations

$$F[r]_a F[s]_b - F[s]_b F[r]_a \\ = (P_{ab} - Q_{ab}) F[r+s]_b - F[r+s]_b (P_{ab} - Q_{ab}) + r\delta_{r,-s}(P_{ab} - Q_{ab})K$$

for all $1 \leqslant a < b \leqslant m$. Moreover, we also have $Q_{ab} F[r]_a Q_{ab} = 0$,

$$P_{ab} F[r]_a = F[r]_b P_{ab} \qquad \text{and} \qquad Q_{ab}\Big(F[r]_a + F[r]_b\Big) = \Big(F[r]_a + F[r]_b\Big) Q_{ab} = 0,$$

whereas P_{ab} and Q_{ab} commute with $F[r]_c$ for $c \neq a, b$.

Following the approach of [19], where the degenerate affine Wenzl algebra was introduced (it is also known as the Nazarov–Wenzl algebra), we use these matrix relations to give the following definition.

Definition 3.1. *Let ω be an indeterminate. Define $\mathcal{B}_m^{\mathrm{aff}}(\omega)$ as the associative unital algebra generated by the Brauer algebra $\mathcal{B}_m(\omega)$ together with additional elements $f[r]_a$ with $1 \leqslant a \leqslant m$ and r running over $\mathbb{Z}$, and τ, K, subject to the following relations. The element K is central, τ commutes with any element of $\mathcal{B}_m(\omega)$, and we also have*

$$f[r]_a f[s]_b - f[s]_b f[r]_a \\ = (s_{ab} - \epsilon_{ab})\, f[r+s]_b - f[r+s]_b\, (s_{ab} - \epsilon_{ab}) + r\delta_{r,-s}\, (s_{ab} - \epsilon_{ab})\, K, \tag{3.7}$$

$$s_{ab} f[r]_a = f[r]_b s_{ab} \qquad \textit{and} \qquad \epsilon_{ab}\Big(f[r]_a + f[r]_b\Big) = \Big(f[r]_a + f[r]_b\Big)\epsilon_{ab} = 0,$$

$$s_{ab} f[r]_c = f[r]_c s_{ab} \qquad \textit{and} \qquad \epsilon_{ab} f[r]_c = f[r]_c \epsilon_{ab} \qquad \textit{for } c \neq a, b,$$

$$\epsilon_{ab} f[r]_a \epsilon_{ab} = 0, \qquad f[r]_a \tau - \tau f[r]_a = r f[r-1]_a. \tag{3.8}$$

We have the epimorphism $\mathcal{B}_m^{\mathrm{aff}}(\omega) \to \mathcal{B}_m(\omega)$ identical on elements of $\mathcal{B}_m(\omega)$ and sending $f[r]_a, \tau$ and K to zero. Hence, we have a natural embedding $\mathcal{B}_m(\omega) \hookrightarrow \mathcal{B}_m^{\mathrm{aff}}(\omega)$ so that $\mathcal{B}_m(\omega)$ can be regarded as a subalgebra of $\mathcal{B}_m^{\mathrm{aff}}(\omega)$.

The defining relations of $\mathcal{B}_m^{\mathrm{aff}}(\omega)$ show that this algebra is defined over the algebra of polynomials $\mathbb{C}[\omega]$. However, we will also need to consider it over the field of rational functions $\mathbb{C}(\omega)$, as occurs already in (3.1). We will not introduce new notation for the extended algebra as this will be specified in the context.

Remark 3.2. As with the Nazarov–Wenzl algebra, it should be natural to impose some additional relations in $\mathcal{B}_m^{\mathrm{aff}}(\omega)$ to develop its reasonable structure theory; cf. [19, Sec. 4]. We will leave this topic outside the current paper, as this algebra will play only an auxiliary role here. □

We will need the following key property of $\mathcal{B}_m^{\mathrm{aff}}(\omega)$. Recall the tensor product superalgebra defined in (2.2).

Proposition 3.3. *The map that sends τ and K to the elements with the same names, and sends*

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad f[r]_a \mapsto F[r]_a,$$

defines a homomorphism

$$\mathcal{B}_m^{\mathrm{aff}}(M-2n) \to \left(\operatorname{End}\mathbb{C}^{M|2n}\right)^{\otimes m} \otimes \mathrm{U}.$$

Proof. The verification of the relations is straightforward; they follow from the matrix form of the defining relations in U pointed out above. □

3.3 Cyclic properties in the extended Brauer algebra

We will work with a slightly modified version of the algebra $\mathcal{B}_m^{\mathrm{aff}}(\omega)$. Define $\hat{\mathcal{B}}_{2m+1}(\omega)$ as the associative unital algebra, generated by the Brauer algebra $\mathcal{B}_{2m+1}(\omega)$ with the elements s_{ab} and ϵ_{ab} labelled by $a, b \in \{0, 1, \dots, 2m\}$, $a \neq b$, together with the additional elements $f[r]_a$ with $0 \leqslant a \leqslant m$ and r running over $\mathbb{Z}$, and τ, K, subject to the same relations as in Definition 3.1. The formulas as in Proposition 3.3 define a homomorphism

$$\rho : \hat{\mathcal{B}}_{2m+1}(M-2n) \to \left(\operatorname{End}\mathbb{C}^{M|2n}\right)^{\otimes(2m+1)} \otimes \mathrm{U}, \tag{3.9}$$

where the copies of the endomorphism algebra are labelled by $0, 1, \dots, 2m$. In $\hat{\mathcal{B}}_{2m+1}(\omega)$, for $1 \leqslant k \leqslant m$, set

$$q^{(k)} = \epsilon_{1\,m+1} \dots \epsilon_{k\,m+k}. \tag{3.10}$$

We wish to use left- and right-multiplication by $q^{(k)}$ as an analogue of the partial trace operator $\operatorname{str}_{1,\dots,k}$; cf. (3.6). We will need two cyclic properties in $\hat{\mathcal{B}}_{2m+1}(\omega)$ described in the following propositions. In the notation below, the angle brackets indicate the subalgebras generated by the listed elements.

Proposition 3.4 (Cyclic property 1)**.** *For $1 \leqslant k \leqslant m$, suppose*

$$x \in \langle s_a, \epsilon_a \mid 1 \leqslant a \leqslant k-1 \rangle = \mathcal{B}_k(\omega) \subseteq \hat{\mathcal{B}}_{2m+1}(\omega),$$

$$y \in \langle s_a, \epsilon_a, f[r]_b, \tau, K \mid 0 \leqslant a \leqslant k-1,\ 0 \leqslant b \leqslant k,\ r \in \mathbb{Z} \rangle \subseteq \hat{\mathcal{B}}_{2m+1}(\omega).$$

Then

$$q^{(k)} x y q^{(k)} = q^{(k)} y x q^{(k)}.$$

Proof. We first note that, if we wish to verify a relation in $\hat{\mathcal{B}}_{2m+1}(\omega)$ that involves only the Brauer generators, it suffices to verify the corresponding relation in $\mathcal{B}_{2m+1}(\omega)$, because of the embedding $\mathcal{B}_{2m+1}(\omega) \hookrightarrow \hat{\mathcal{B}}_{2m+1}(\omega)$; see Sec. 3.2. In particular, we can make use of the diagrammatic presentation of $\mathcal{B}_{2m+1}(\omega)$.

For $1 \leqslant a < b \leqslant m$, then, it is easy to verify that

$$\epsilon_{a\,m+a}\epsilon_{b\,m+b}s_{ab} = \epsilon_{a\,m+a}\epsilon_{b\,m+b}s_{m+a\,m+b}, \tag{3.11}$$

$$\epsilon_{a\,m+a}\epsilon_{b\,m+b}\epsilon_{ab} = \epsilon_{a\,m+a}\epsilon_{b\,m+b}\epsilon_{m+a\,m+b}, \tag{3.12}$$

$$s_{m+a\,m+b}\epsilon_{a\,m+a}\epsilon_{b\,m+b} = s_{ab}\epsilon_{a\,m+a}\epsilon_{b\,m+b}, \tag{3.13}$$

$$\epsilon_{m+a\,m+b}\epsilon_{a\,m+a}\epsilon_{b\,m+b} = \epsilon_{ab}\epsilon_{a\,m+a}\epsilon_{b\,m+b}. \tag{3.14}$$

Now, note that all of the indices of the ϵ generators involved in $q^{(k)}$ are distinct, so those generators commute. It follows that the first generator of x, either s_{ab} or ϵ_{ab}, can be converted to $s_{m+a\,m+b}$ or $\epsilon_{m+a\,m+b}$, using (3.11) or (3.12), respectively. The resulting generator commutes with the rest of x and all of y, and can be converted back to the generator we started with using (3.13) or (3.14), respectively. Thus the first generator of x can be moved to the right of y. This process can be repeated for all generators of x, because each s_{ab} or ϵ_{ab} becomes $s_{m+a\,m+b}$ or $\epsilon_{m+a\,m+b}$ respectively, and then commutes with all other generators of x, and with y. The generators of x can thus be moved to the right of y, ending up in the same order as they were in x, which proves the result. □

Proposition 3.5 (Cyclic property 2)**.** *For $1 \leqslant k \leqslant m$, suppose*

$$x \in \langle s_a, \epsilon_a, f[r]_b, \tau, K \mid 1 \leqslant a \leqslant k-1,\ 1 \leqslant b \leqslant k,\ r \in \mathbb{Z}\rangle \subseteq \hat{\mathcal{B}}_{2m+1}(\omega).$$

Then for any $1 \leqslant a \leqslant k$, and $y \in \{s_{0a}, \epsilon_{0a}\}$, we have

$$q^{(k)}xyq^{(k)} = q^{(k)}yxq^{(k)}.$$

Proof. One can check diagrammatically that, for $1 \leqslant a \leqslant k$,

$$\epsilon_{a\,m+a}s_{0a} = \epsilon_{a\,m+a}\epsilon_{0\,m+a}, \qquad \epsilon_{0\,m+a}\epsilon_{a\,m+a} = s_{0a}\epsilon_{a\,m+a},$$

$$\epsilon_{a\,m+a}\epsilon_{0a} = \epsilon_{a\,m+a}s_{0\,m+a}, \qquad s_{0\,m+a}\epsilon_{a\,m+a} = \epsilon_{0a}\epsilon_{a\,m+a}.$$

It follows that

$$q^{(k)}s_{0a}xq^{(k)} = q^{(k)}\epsilon_{0\,m+a}xq^{(k)} = q^{(k)}x\epsilon_{0\,m+a}q^{(k)} = q^{(k)}xs_{0a}q^{(k)},$$

since each generator of x commutes with $\epsilon_{0\,m+a}$. The relation for $y = \epsilon_{0a}$ is proven analogously. □

4 Abstract Segal–Sugawara vectors

Now we explain our strategy to prove Theorem 2.1 in more detail. The theorem will follow by verifying that the elements $\Phi_m \in V_{-h^\vee}(\mathfrak{osp}_{M|2n})$ are annihilated by the action of $\mathfrak{osp}_{M|2n}[t]$; see (1.1). Since the orthosymplectic Lie superalgebra is simple, it suffices to show that

$$F_{ij}[0]\Phi_m = F_{ij}[1]\Phi_m = 0 \tag{4.1}$$

for all i, j. In the case $n = 0$ (i.e., for the orthogonal Lie algebra $\mathfrak{o}_M$), the verification of (4.1) was carried out with the use of the matrix techniques, where Φ_m is first represented as a weighted trace of the symmetriser in the Brauer algebra [13], and then brought to the form (2.6); see [15]. In the super case for arbitrary n, the corresponding initial expression would take the form

$$\Phi_m = \gamma_m(M-2n)\ \mathrm{str}_{1,\dots,m}\ S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big)\,1, \tag{4.2}$$

where $\gamma_m(\omega)$ is defined in (3.3), while $S^{(m)}$ denotes the image of the symmetriser $s^{(m)}$ introduced in (3.1), under the homomorphism (3.5), and we assume $\tau 1 = 0$. However, since $\gamma_m(\omega)$ and $s^{(m)}$ are rational functions in ω, the expression in (4.2) is not defined for some values of M and n. The same problem already occurs for the symplectic Lie algebra $\mathfrak{sp}_{2n}$ (i.e., for $M = 0$); a way around this was found in [13] (see also [14, Sec. 8.3]) with the use of 'analytic continuation' over n. Its extension to arbitrary M appears to be problematic as a super version of the argument should rely on orthosymplectic invariant theory, which is much less well-understood; cf. [11].

Our way to settle the singularity issue of the expression in (4.2) is to 'lift' it to the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$ while keeping ω as an indeterminate, thus working over the field of rational functions $\mathbb{C}(\omega)$. Namely, we find some 'abstract' counterparts $\phi_m \in \hat{\mathcal{B}}_{2m+1}(\omega)$ of Φ_m satisfying the desired annihilation properties. This requires consistent definitions of the supertraces in $\hat{\mathcal{B}}_{2m+1}(\omega)$ and in the tensor product superalgebra appearing in (3.9). For the latter, we use the observation (3.6), which implies a counterpart of (4.2) without an explicit use of the supertrace:

$$\Phi_m Q^{(m)} = Q^{(m)}\Big(\gamma_m(M-2n)\, S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big)\,1\Big)Q^{(m)},$$

assuming Φ_m is defined, where for each $1 \leqslant k \leqslant m$ we set

$$Q^{(k)} := Q_{1\,m+1}Q_{2\,m+2}\dots Q_{k\,m+k} \in \Big(\mathrm{End}\,\mathbb{C}^{M|2n}\Big)^{\otimes 2m}.$$

As a final step, we use (3.2) to find an equivalent 'integral form' of the abstract Segal–Sugawara vectors ϕ_m, implying that the singularities in (4.2) are removable, by showing that the vectors ϕ_m allow for a well-defined evaluation at $\omega = M - 2n$, yielding formula (2.6).

4.1 Annihilation properties

To implement the program, begin by setting

$$f_a := \tau + f[-1]_a \in \hat{\mathcal{B}}_{2m+1}(\omega), \qquad a = 1, \dots, m.$$

Use notation (3.10) and introduce the element

$$q^{(m)}s^{(m)}f_1\dots f_m q^{(m)} \in \hat{\mathcal{B}}_{2m+1}(\omega). \tag{4.3}$$

We can regard it as a polynomial in τ by moving the powers of τ to their right-most positions by using the second relation in (3.8). Our goal is to show that all coefficients of this polynomial are *abstract Segal–Sugawara vectors* in the sense that both expressions

$$f[0]_0\, q^{(m)}s^{(m)}f_1\dots f_m q^{(m)} \qquad \text{and} \qquad f[1]_0\, q^{(m)}s^{(m)}f_1\dots f_m q^{(m)} \tag{4.4}$$

are zero modulo the left ideal in $\hat{\mathcal{B}}_{2m+1}(\omega)$ generated by the subspaces $f[r]_a\mathcal{B}_{2m+1}(\omega)$ for $r \geqslant 0$ and $a = 0, \dots, m$, assuming that the 'abstract level' is critical: $K = -\omega + 2$.

Proposition 4.1. *In $\hat{\mathcal{B}}_{2m+1}(\omega)$, we have*

$$f[0]_0\, q^{(m)}s^{(m)}f_1\dots f_m q^{(m)} = q^{(m)}s^{(m)}f_1\dots f_m q^{(m)}f[0]_0.$$

Proof. Let $\varphi_{xy} := s_{xy} - \epsilon_{xy}$. For $1 \leqslant a \leqslant m$, we have from the defining relations in $\hat{\mathcal{B}}_{2m+1}(\omega)$ and (3.7) that

$$f[0]_0 f_a - f_a f[0]_0 = \varphi_{0a}f_a - f_a\varphi_{0a}.$$

Then

$$f[0]_0 q^{(m)}s^{(m)}f_1\dots f_m q^{(m)} = \sum_{a=1}^{m} q^{(m)}s^{(m)}f_1\dots f_{a-1}\big(f[0]_0 f_a - f_a f[0]_0\big)f_{a+1}\dots f_m q^{(m)} + q^{(m)}s^{(m)}f_1\dots f_m f[0]_0 q^{(m)}.$$

The sum equals

$$\sum_{a=1}^{m} q^{(m)}s^{(m)}f_1\dots f_{a-1}\big(\varphi_{0a}f_a - f_a\varphi_{0a}\big)f_{a+1}\dots f_m q^{(m)} = \sum_{a=1}^{m} q^{(m)}s^{(m)}\varphi_{0a}f_1\dots f_m q^{(m)} - \sum_{a=1}^{m} q^{(m)}s^{(m)}f_1\dots f_m\varphi_{0a}q^{(m)}.$$

Applying cyclic property 2, and noting that $s^{(m)}$ commutes with the sum of φ_{0a}, this becomes

$$\sum_{a=1}^{M} q^{(m)}\varphi_{0a}s^{(m)}f_1\dots f_m q^{(m)} - \sum_{a=1}^{M} q^{(m)}\varphi_{0a}s^{(m)}f_1\dots f_m q^{(m)} = 0,$$

as desired. □

The second expression in (4.4) is more difficult to handle, requiring a few lemmas.

Lemma 4.2. *In $\hat{\mathcal{B}}_{2m+1}(\omega)$, for $1 \leqslant k \leqslant m$, we have*

$$\epsilon_{k\,m+k}s^{(k)}\varphi_{0k}\epsilon_{k\,m+k} = \frac{\omega+2k-2}{k(\omega+2k-4)}s^{(k-1)}\left(\sum_{a=1}^{k-1}\varphi_{0a}\right)\epsilon_{k\,m+k}. \tag{4.5}$$

Proof. In the Brauer algebra $\mathcal{B}_k(\omega)$, it is known that

$$s^{(k)} = \frac{1}{k(\omega+2k-4)}\left(1+\sum_{a=1}^{k-1}(s_{ak}-\epsilon_{ak})\right)\left(\omega+k-3+\sum_{a=1}^{k-1}(s_{ak}-\epsilon_{ak})\right)s^{(k-1)};$$

see [14, proof of Lemma 1.3.2]. Expanding this and applying Brauer relations, we find

$$s^{(k)} = \frac{1}{k}\left(1+\sum_{a=1}^{k-1}s_{ak} - \frac{2}{\omega+2k-4}\left(\sum_{a=1}^{k-1}\epsilon_{ak} + \sum_{1\leqslant a<b\leqslant k-1}s_{ak}\epsilon_{bk}\right)\right)s^{(k-1)}. \tag{4.6}$$

Due to the embeddings $\mathcal{B}_k(\omega) \hookrightarrow \mathcal{B}_m(\omega) \hookrightarrow \hat{\mathcal{B}}_{2m+1}(\omega)$, this result holds in $\hat{\mathcal{B}}_{2m+1}(\omega)$ also.

Substituting the above expression for $s^{(k)}$ into the left-hand side of (4.5), we can commute the φ_{0k} left past the $\epsilon_{k\,m+k}$ and $s^{(k-1)}$, expand, and simplify the resulting expression by noting that

$$\epsilon_{k\,m+k}\,\varphi_{0k}\,\epsilon_{k\,m+k} = 0, \qquad \epsilon_{k\,m+k}\,s_{ak}\,\varphi_{0k}\,\epsilon_{k\,m+k} = \varphi_{0a}\,\epsilon_{k\,m+k},$$
$$\epsilon_{k\,m+k}\,\epsilon_{ak}\,\varphi_{0k}\,\epsilon_{k\,m+k} = -\varphi_{0a}\,\epsilon_{k\,m+k}, \qquad \epsilon_{k\,m+k}\,s_{ak}\,\epsilon_{bk}\,\varphi_{0k}\,\epsilon_{k\,m+k} = \epsilon_{ab}\;\varphi_{0a}\,\epsilon_{k\,m+k}.$$

This gives

$$\epsilon_{k\,m+k}\,s^{(k)}\,\varphi_{0k}\,\epsilon_{k\,m+k} = \frac{1}{k}\left(\frac{\omega+2k-2}{\omega+2k-4}\sum_{a=1}^{k-1}\varphi_{0a} - \frac{2}{\omega+2k-4}\sum_{1\leqslant a<b\leqslant k-1}\epsilon_{ab}\varphi_{0a}\right)\epsilon_{k\,m+k}s^{(k-1)}$$

which equals

$$\frac{\omega+2k-2}{k(\omega+2k-4)}\left(\sum_{a=1}^{k-1}\varphi_{0a}\right)\epsilon_{k\,m+k}s^{(k-1)},$$

since $s^{(k-1)}$ commutes with $\epsilon_{k\,m+k}$, and

$$\epsilon_{ab}\,\varphi_{0a}\,\epsilon_{k\,m+k}\,s^{(k-1)} = \epsilon_{ab}\,\varphi_{0a}\,s^{(k-1)}\,\epsilon_{k\,m+k} = 0.$$

The latter holds by noting that

$$\epsilon_{ab}\,\varphi_{0a}\,s^{(k-1)} = \epsilon_{ab}\,\varphi_{0a}\,s_{ab}s^{(k-1)} = \epsilon_{ab}\,s_{ab}\,\varphi_{0b}\,s^{(k-1)} = \epsilon_{ab}\,\varphi_{0b}\,s^{(k-1)} = -\epsilon_{ab}\,\varphi_{0a}\,s^{(k-1)},$$

completing the proof. □

Lemma 4.3. *In $\hat{\mathcal{B}}_{2m+1}(\omega)$, for $1 \leqslant k \leqslant m$ we have*

$$\epsilon_{k\,m+k}\,s^{(k)}\epsilon_{k\,m+k} = \frac{(\omega+k-3)(\omega+2k-2)}{k(\omega+2k-4)}s^{(k-1)}\epsilon_{k\,m+k}.$$

Proof. We substitute the expression (4.6) for $s^{(k)}$ into the left-hand side, and then use that $s^{(k-1)}$ and $\epsilon_{k\,m+k}$ commute to pull both copies of $\epsilon_{k\,m+k}$ inside the brackets. We then simplify each term using the Brauer relations, and pull a remaining copy of $\epsilon_{k\,m+k}$ back out of the brackets, to the right. This gives

$$\begin{aligned}&\epsilon_{k\,m+k}\,s^{(k)}\epsilon_{k\,m+k}\\ &= \frac{1}{k}\left(\omega+\sum_{a=1}^{k-1}1 - \frac{2}{\omega+2k-4}\sum_{a=1}^{k-1}1 - \frac{2}{\omega+2k-4}\sum_{1\leqslant a<b\leqslant k-1}\epsilon_{ab}\right)s^{(k-1)}\epsilon_{k\,m+k},\end{aligned}$$

which may be simplified to give the result by noting that $\epsilon_{ab}s^{(k-1)} = 0$. □

Lemma 4.4. *Let $1 \leqslant a < b \leqslant k \leqslant m$. Then, in $\hat{\mathcal{B}}_{2m+1}(\omega)$, we have*

$$s_{ab}\,f_1\ldots f_k s^{(k)} = f_1\ldots f_k s^{(k)}.$$

Proof. It suffices to show this for $b = a + 1$, since s_{ab} is equal to a product of generators s_c with $a \leqslant c < b$. Using $f_a = \tau + f[-1]_a$ and the defining relations of $\hat{\mathcal{B}}_{2m+1}(\omega)$, it is tedious but straightforward to show that

$$f_a f_{a+1} \left(\frac{1 + s_a}{2} - \frac{\epsilon_a}{\omega} \right) = f_{a+1} f_a \left(\frac{1 + s_a}{2} - \frac{\epsilon_a}{\omega} \right).$$

We note also that the expression in the brackets is absorbed by $s^{(k)}$, and commutes with f_b for $b \geqslant a + 2$, while s_a commutes with f_b for $b \leqslant a - 1$. The result follows readily from these observations. □

Corollary 4.5. *Let* $1 \leqslant k \leqslant m$*. Then for any* $1 \leqslant a < b \leqslant k$*, we have*

$$q^{(k)} s^{(k)} \varphi_{0a} f_1 \dots f_k q^{(k)} = q^{(k)} s^{(k)} \varphi_{0b} f_1 \dots f_k q^{(k)}$$

in $\hat{\mathcal{B}}_{2m+1}(\omega)$.

Proof. This follows from $s^{(k)} s_{ab} = s_{ab} s^{(k)} = s^{(k)}$, Lemma 4.4, and cyclic property 1. □

We are now in a position to establish the desired property of the second expression in (4.4), as given in the next proposition.

Proposition 4.6. *In* $\hat{\mathcal{B}}_{2m+1}(\omega)$*, we have*

$$\begin{aligned} &f[1]_0\, q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)} \\ &= (\omega + K - 2) \frac{\omega + 2m - 2}{\omega + 2m - 4} \left(\sum_{a=1}^{m-1} q^{(m-1)} s^{(m-1)} \varphi_{0a} f_1 \dots f_{m-1} q^{(m-1)} \right) \epsilon_{m\,2m} \\ &\qquad - m q^{(m)} \varphi_{0m} s^{(m)} f_1 \dots f_{m-1} f[0]_m q^{(m)} + m q^{(m)} s^{(m)} f_1 \dots f_{m-1} q^{(m)} f[0]_0 \\ &\qquad + m q^{(m)} s^{(m)} \varphi_{0m} f_1 \dots f_{m-1} f[0]_m q^{(m)} + q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)} f[1]_0. \end{aligned}$$

Proof. We first note that

$$f[1]_0 f_a - f_a f[1]_0 = f[0]_0 + \varphi_{0a} f[0]_a - f[0]_a \varphi_{0a} + \varphi_{0a} K, \tag{4.7}$$

and

$$f[0]_a f_b - f_b f[0]_a = \varphi_{ab} f_b - f_b \varphi_{ab}. \tag{4.8}$$

Similar to the proof of Proposition 4.1, we begin by rewriting our expression as a telescoping sum. We then apply (4.7), finding

$$\begin{aligned} &f[1]_0\, q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)} \\ &= \sum_{a=1}^{m} q^{(m)} s^{(m)} f_1 \dots f_{a-1} \Big(f[0]_0 + \varphi_{0a} f[0]_a - f[0]_a \varphi_{0a} + \varphi_{0a} K \Big) f_{a+1} \dots f_m q^{(m)} \\ &\qquad + q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)} f[1]_0. \end{aligned}$$

Expanding the brackets gives four sums; we rewrite the first three as telescoping sums in an additional index b, and apply (4.8) to represent the sum over a in the form

$$\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}f_1\dots f_{a-1}f_{a+1}\dots f_{b-1}\Big(\varphi_{0b}f_b - f_b\varphi_{0b}\Big)f_{b+1}\dots f_m q^{(m)}$$
$$+\sum_{a=1}^{m} q^{(m)}s^{(m)}f_1\dots f_{a-1}f_{a+1}\dots f_m q^{(m)} f[0]_0$$
$$+\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}\varphi_{0a}f_1\dots f_{a-1}f_{a+1}\dots f_{b-1}\Big(\varphi_{ab}f_b - f_b\varphi_{ab}\Big)f_{b+1}\dots f_m q^{(m)}$$
$$+\sum_{a=1}^{m} q^{(m)}s^{(m)}\varphi_{0a}f_1\dots f_{a-1}f_{a+1}\dots f_m f[0]_a q^{(m)}$$
$$-\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}f_1\dots f_{a-1}f_{a+1}\dots f_{b-1}\Big(\varphi_{ab}f_b - f_b\varphi_{ab}\Big)f_{b+1}\dots f_m\varphi_{0a} q^{(m)}$$
$$-\sum_{a=1}^{m} q^{(m)}s^{(m)}f_1\dots f_{a-1}f_{a+1}\dots f_m f[0]_a\varphi_{0a} q^{(m)}$$
$$+K\sum_{a=1}^{m} q^{(m)}s^{(m)}\varphi_{0a}f_1\dots f_{a-1}f_{a+1}\dots f_m q^{(m)}.$$

We now manipulate the three sums over a and b, and the final sum. Since $s^{(m)}s_{xy} = s^{(m)}$ for $1 \leqslant x < y \leqslant m$, we can insert $s_{m-1\,m}\dots s_{a\,a+1}$ after the $s^{(m)}$ in each sum. We move this product to the right in each word, using

$$s_{wx}\varphi_{yz} = \varphi_{yz}s_{wx}, \qquad s_{wx}\varphi_{xy} = \varphi_{wy}s_{wx}, \qquad s_{wx}\varphi_{wx} = \varphi_{wx}s_{wx},$$

and

$$s_{wx}f_y = f_y s_{wx}, \qquad s_{wx}f_x = f_w s_{wx}$$

for w, x, y, z distinct, and then apply cyclic property 1 to bring it back to the left of $s^{(m)}$, where it is absorbed. We also move φ_{xy} throughout the word, using $\varphi_{xy}f_z = f_z\varphi_{xy}$ for distinct x, y, z, and cyclic property 1 or 2 as appropriate; that is, for $x, y \neq 0$, and $x = 0$, respectively. Additionally, for $1 \leqslant x < y \leqslant m$, we use $\varphi_{xy}s^{(m)} = s^{(m)}\varphi_{xy} = s^{(m)}$. The resulting summands are then independent of one or more of the summation indices, and we simplify the sums accordingly.

In the first sum, for example, inserting the product $s_{m-1\,m}\dots s_{a\,a+1}$ gives

$$\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}s_{m-1\,m}\dots s_{a\,a+1}f_1\dots f_{a-1}f_{a+1}\dots f_{b-1}\,\varphi_{0b}\,f_bf_{b+1}\dots f_m q^{(m)}$$
$$-\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}s_{m-1\,m}\dots s_{a\,a+1}f_1\dots f_{a-1}f_{a+1}\dots f_{b-1}f_b\,\varphi_{0b}\,f_{b+1}\dots f_m q^{(m)},$$

which equals

$$\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}f_1\dots f_{b-2}\,\varphi_{0\,b-1}\,f_{b-1}\dots f_{m-1}s_{m-1\,m}\dots s_{a\,a+1}q^{(m)}$$
$$-\sum_{1\leqslant a<b\leqslant m} q^{(m)}s^{(m)}f_1\dots f_{b-2}f_{b-1}\,\varphi_{0\,b-1}\,f_b\dots f_{m-1}s_{m-1\,m}\dots s_{a\,a+1}q^{(m)}.$$

We then apply cyclic property 1 and the absorption property of the symmetriser, followed by $\varphi_{xy} f_z = f_z \varphi_{xy}$ for distinct x, y, z, to write the first sum as

$$\sum_{1 \leqslant a < b \leqslant m} \Big(q^{(m)} s^{(m)} f_1 \dots f_{b-2}\, \varphi_{0\,b-1}\, f_{b-1} \dots f_{m-1} q^{(m)} - q^{(m)} s^{(m)} f_1 \dots f_{b-1}\, \varphi_{0\,b-1}\, f_b \dots f_{m-1} q^{(m)} \Big),$$

$$= \sum_{1 \leqslant a < b \leqslant m} \Big(q^{(m)} s^{(m)}\, \varphi_{0\,b-1}\, f_1 \dots f_{m-1} q^{(m)} - q^{(m)} s^{(m)} f_1 \dots f_{m-1}\, \varphi_{0\,b-1}\, q^{(m)} \Big).$$

The summands do not depend on a, so may be simplified to

$$\sum_{a=1}^{m-1} a q^{(m)} s^{(m)}\, \varphi_{0a}\, f_1 \dots f_{m-1} q^{(m)} - \sum_{a=1}^{m-1} a q^{(m)} s^{(m)} f_1 \dots f_{m-1}\, \varphi_{0a}\, q^{(m)}$$
$$= \sum_{a=1}^{m-1} a q^{(m)} s^{(m)}\, \varphi_{0a}\, f_1 \dots f_{m-1} q^{(m)} - \sum_{a=1}^{m-1} a q^{(m)}\, \varphi_{0a}\, s^{(m)} f_1 \dots f_{m-1} q^{(m)},$$

where the last line uses cyclic property 2.

Simplifying the other sums similarly, and grouping some like terms, we represent the initial expression $f[1]_0\, q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)}$ as

$$\sum_{a=1}^{m-1} a q^{(m)} s^{(m)} \varphi_{0a} f_1 \dots f_{m-1} q^{(m)} - \sum_{a=1}^{m-1} a q^{(m)}\, \varphi_{0a}\, s^{(m)} f_1 \dots f_{m-1} q^{(m)}$$
$$+ \sum_{a=1}^{m-1} a q^{(m)} s^{(m)} \varphi_{0m} \varphi_{a\,m} f_1 \dots f_{m-1} q^{(m)} + \sum_{a=1}^{m-1} a q^{(m)} \varphi_{a\,m} \varphi_{0m} s^{(m)} f_1 \dots f_{m-1} q^{(m)}$$

plus the sum of the terms

$$m(K - m + 1) q^{(m)} s^{(m)} \varphi_{0m} f_1 \dots f_{m-1} q^{(m)} - m q^{(m)} \varphi_{0m} s^{(m)} f_1 \dots f_{m-1} f[0]_m q^{(m)}$$
$$+ m q^{(m)} s^{(m)} f_1 \dots f_{m-1} q^{(m)} f[0]_0 + m q^{(m)} s^{(m)} \varphi_{0m} f_1 \dots f_{m-1} f[0]_m q^{(m)}$$
$$+ q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)} f[1]_0.$$

Now, using the Brauer relations and the properties of the symmetriser, we have

$$s^{(m)} \varphi_{0m} s_{am} = s^{(m)} s_{am}\, \varphi_{0a}\, = s^{(m)} \varphi_{0a}$$

and

$$s^{(m)} \varphi_{0m} \epsilon_{am} = s^{(m)} s_{0m} \epsilon_{am} - s^{(m)} \epsilon_{0m} \epsilon_{am} = s^{(m)} s_{0m} \epsilon_{am} - s^{(m)} s_{0m} \epsilon_{am} = 0,$$

so that

$$s^{(m)} \varphi_{0m} \varphi_{am} = s^{(m)} \varphi_{0a},$$

and analogously,

$$\varphi_{am} \varphi_{0m} s^{(m)} = \varphi_{0a} s^{(m)}.$$

The four sums over a may thus be simplified to

$$2\sum_{a=1}^{m-1} aq^{(m)}s^{(m)}\varphi_{0a}f_1\dots f_{m-1}q^{(m)} = 2\sum_{a=1}^{m-1} aq^{(m-1)}\epsilon_{m\,2m}s^{(m)}\epsilon_{m\,2m}\varphi_{0a}f_1\dots f_{m-1}q^{(m-1)}$$

$$= \frac{2(\omega+m-3)(\omega+2m-2)}{m(\omega+2m-4)}\sum_{a=1}^{m-1} aq^{(m-1)}s^{(m-1)}\varphi_{0a}f_1\dots f_{m-1}q^{(m-1)}\epsilon_{m\,2m},$$

using $q^{(m)} = q^{(m-1)}\epsilon_{m\,2m} = \epsilon_{m\,2m}q^{(m-1)}$ and Lemma 4.3 with $k = m$. From Corollary 4.5 with $k = m-1$, however, $q^{(m-1)}s^{(m-1)}\varphi_{0a}f_1\dots f_{m-1}q^{(m-1)}$ has the same value for each a, so this can be further simplified to

$$\frac{(\omega+m-3)(\omega+2m-2)}{\omega+2m-4}\sum_{a=1}^{m-1} q^{(m-1)}s^{(m-1)}\varphi_{0a}f_1\dots f_{m-1}q^{(m-1)}\epsilon_{m\,2m}.$$

Similarly, using Lemma 4.2 with $k = m$, we also have

$$m(K-m+1)q^{(m)}s^{(m)}\varphi_{0m}f_1\dots f_{m-1}q^{(m)}$$

$$= \frac{(K-m+1)(\omega+2m-2)}{\omega+2m-4}\sum_{a=1}^{m-1} q^{(m-1)}s^{(m-1)}\varphi_{0a}f_1\dots f_{m-1}q^{(m-1)}\epsilon_{m\,2m}.$$

Substituting these results back into the expansion of $f[1]_0\, q^{(m)}s^{(m)}f_1\dots f_m q^{(m)}$ gives the desired expression. $\square$

4.2 Integral form of the abstract Segal–Sugawara vectors

Propositions 4.1 and 4.6 show that all coefficients of the polynomial in τ defined in (4.3) are abstract Segal–Sugawara vectors. However, the coefficients are defined in the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$ over the field $\mathbb{C}(\omega)$. Since we aim to evaluate ω at $M-2n$, we would like to bring the coefficients to an 'integral form' to be regarded as elements of $\hat{\mathcal{B}}_{2m+1}(\omega)$ over $\mathbb{C}[\omega]$. As we point out below (see Remark 4.9(i)), all coefficients are easily expressible in terms of the constant terms of the polynomials in τ. Therefore, we will only be concerned with the constant terms modified by the scalar defined in (3.3):

$$\phi_m := \gamma_m(\omega)q^{(m)}s^{(m)}f_1\dots f_m q^{(m)}\,1, \tag{4.9}$$

assuming $\tau\, 1 = 0$. We will keep the notation introduced in Sec. 2.3, and for a partition of $\lambda \vdash m$ of length $\ell = \ell(\lambda)$, set

$$f[-\lambda] = f[-\lambda_1]_1\dots f[-\lambda_\ell]_\ell;$$

cf. (2.5).

Proposition 4.7. *The element* (4.9) *is given by the formula*

$$\phi_m = \sum_{\lambda\vdash m,\ \ell(\lambda)\ \text{even}} \mathcal{Y}_{m,\ell}(\omega-1)\, c_\lambda\, q^{(\ell)}h^{(\ell)}f[-\lambda]\,q^{(m)}. \tag{4.10}$$

Proof. Expand each f_a as $\tau + f[-1]_a$ and use the second relation in (3.8) to move all copies of τ to the right. It follows that ϕ_m is a linear combination of terms of the form

$$\gamma_m(\omega)q^{(m)}s^{(m)}f[-r_1]_{a_1}\dots f[-r_\ell]_{a_\ell}q^{(m)}, \tag{4.11}$$

where $1 \leqslant a_1 < \cdots < a_\ell \leqslant m$ and the r_i are positive integers with $r_1 + \cdots + r_\ell = m$. Now, by (3.7) for $r, s > 0$, we have

$$f[-r]_a f[-s]_b = f[-s]_b f[-r]_a + \varphi_{ab} f[-r-s]_b - f[-r-s]_b \varphi_{ab}.$$

Applying this to each term, we can swap adjacent $f[-r_i]_{a_i}$ until the r_i are weakly decreasing. Since $s^{(m)}\varphi_{ab} = \varphi_{ab}s^{(m)} = s^{(m)}$, the additional terms with φ_{ab} cancel out by cyclic property 1. Then, using $s^{(m)}s_{ab} = s_{ab}s^{(m)} = s^{(m)}$ together with $s_{ab}f[-r]_a = f[-r]_b s_{ab}$ and cyclic property 1, we can insert appropriate permutations s_{ab} after $s^{(m)}$ to change the indices of the $f[-r]_a$ factors to 1 up to ℓ, in order. It follows that

$$\phi_m = \sum_{\lambda \vdash m} c_\lambda \, \gamma_m(\omega)q^{(m)}s^{(m)}f[-\lambda]\, q^{(m)}.$$

for some nonnegative integers c_λ. By the same argument as in [15, Theorem 2.1], we find that c_λ is the number of permutations in $\mathfrak{S}_m$ of cycle type λ.

Furthermore, by repeated application of Lemma 4.3, we have

$$\phi_m = \sum_{\lambda \vdash m} \mathcal{Y}_{m,\ell}(\omega - 1)\, c_\lambda \, \gamma_\ell(\omega)q^{(\ell)}s^{(\ell)}f[-\lambda]\, q^{(m)}.$$

Now we need a lemma which relates these expressions to similar expressions involving $h^{(\ell)}$.

Lemma 4.8. *Let $\lambda \vdash m$ be a partition of length ℓ. Then*

$$\gamma_\ell(\omega)q^{(\ell)}s^{(\ell)}f[-\lambda]q^{(m)} = q^{(\ell)}h^{(\ell)}f[-\lambda]q^{(m)}.$$

Proof. From (3.4), using the embedding $\mathcal{B}_\ell(\omega) \hookrightarrow \hat{\mathcal{B}}_{2m+1}(\omega)$, we have

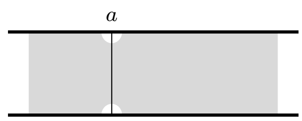
$$h^{(\ell)} - \gamma_\ell(\omega)s^{(\ell)} \in J_m^{(\ell)} \subseteq \hat{\mathcal{B}}_{2m+1},$$

where $J_m^{(\ell)}$ is spanned by elements $d + d^{t_a}$, where d is a Brauer diagram in $\mathcal{B}_\ell(\omega) \subseteq \hat{\mathcal{B}}_{2m+1}(\omega)$ and $1 \leqslant a \leqslant \ell$. It thus suffices to show that

$$q^{(\ell)}\left(d + d^{t_a}\right) f[-\lambda]q^{(\ell)} = 0 \tag{4.12}$$

for all d. We now consider four cases, depending on the diagrams d and d^{t_a}. Diagrams with a shaded area are used to represent Brauer diagrams where the nodes within the shaded area are connected by strings that lie within the shaded area; within each case, each diagram with a shaded area has the same arrangement of strings within the shaded area.

Case (a): In this case,

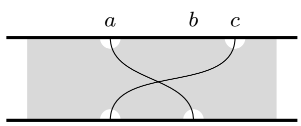
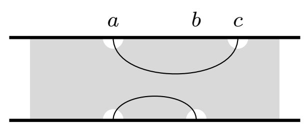
$d = d^{t_a} =$  $\in \mathcal{B}_\ell(\omega).$

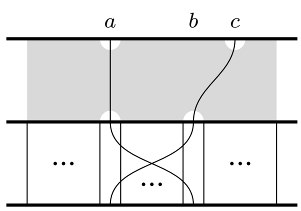
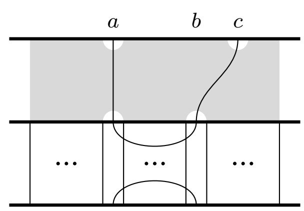
Noting that d commutes with s_{ab} and ϵ_{ab} for any $b > \ell$, we have

$$\epsilon_{a\,m+a}\left(d + d^{t_a}\right) f[-\lambda]\epsilon_{a\,m+a} = 2\epsilon_{a\,m+a} d\, f[-\lambda_1]_1 \dots f[-\lambda_\ell]_\ell \epsilon_{a\,m+a}$$
$$= 2d\, f[-\lambda_1]_1 \dots f[-\lambda_{a-1}]_{a-1}\epsilon_{a\,m+a} f[-\lambda_a]_a \epsilon_{a\,m+a} f[-\lambda_{a+1}]_{a+1} \dots f[-\lambda_\ell]_\ell$$

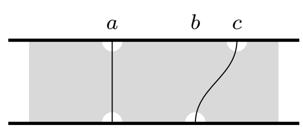
which is zero due to the relation $\epsilon_{xy} f[r]_x \epsilon_{xy} = 0$.

Case (b): In this case,

$d + d^{t_a} =$  $+$ 

$=$  $+$  $= (s_{ab} + \epsilon_{ab})\,\tilde{d},$

where

$\tilde{d} =$  .

Note that we do not specify which diagram is which, so this case includes the cases where d is either of these two diagrams. Observe that $\tilde{d}$ commutes with $s_{a\,m+a}$. From $s_{xy}\epsilon_{xy} = \epsilon_{xy}s_{xy} = \epsilon_{xy}$ and $\epsilon_{xy}s_{ay} = \epsilon_{xy}\epsilon_{ax}$ we have

$$q^{(\ell)} = q^{(\ell)} s_{b\,m+b} = s_{b\,m+b} q^{(\ell)} \qquad \text{and} \qquad q^{(\ell)}(s_{ab} + \epsilon_{ab}) = q^{(\ell)}(s_{a\,m+b} + \epsilon_{a\,m+b}). \tag{4.13}$$

Also, from $f[r]_x\epsilon_{xy} = -f[r]_y\epsilon_{xy}$, it follows that

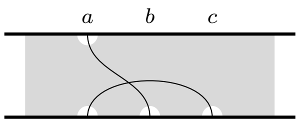
$$f[-\lambda]\, q^{(\ell)} = -f[-\lambda_1]_1 \dots f[-\lambda_b]_{m+b} \dots f[-\lambda_\ell]_\ell\, q^{(\ell)}.$$

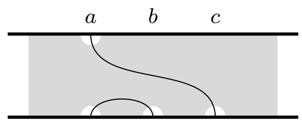
Using these properties, we find

$$q^{(\ell)}\left(d + d^{t_a}\right) f[-\lambda] q^{(\ell)} = q^{(\ell)}\left(s_{ab} + \epsilon_{ab}\right) \tilde{d}\, f[-\lambda] q^{(\ell)}$$
$$= -q^{(\ell)}\left(s_{a\,m+b} + \epsilon_{a\,m+b}\right) \tilde{d}\, f[-\lambda_1]_1 \dots f[-\lambda_b]_{m+b} \dots f[-\lambda_\ell]_\ell q^{(\ell)}.$$

Using the first relation in (4.13), we find that this coincides with $-q^{(\ell)}\,(d+d^{t_a})\,f[-\lambda]q^{(\ell)}$ and hence is equal to zero. Note that this proof does not rely on the ordering of a, b and c, and holds even if $b=c$.

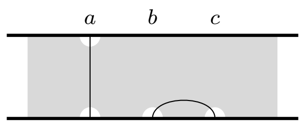
Case (c): In this case,

$d+d^{t_a}=$ 

$+$ 

$=(s_{ab}+\epsilon_{ab})\,\tilde{d},$

where

$\tilde{d}=$ 

.

Then, since this $\tilde{d}$ also commutes with $s_{a\,m+a}$, the computation from the previous case holds here, so $q^{(\ell)}\,(d+d^{t_a})\,f[-\lambda]q^{(\ell)}=0$.

Case (d): In this case, both d and $\tilde{d}$ are obtained from the respective diagrams in Case (c) by reflection about a horizontal line, so that $d+d^{t_a}=\tilde{d}(s_{ab}+\epsilon_{ab})$. Again, $s_{a\,m+a}$ commutes with $\tilde{d}$. The computation to show $q^{(\ell)}\,(d+d^{t_a})\,f[-\lambda]q^{(\ell)}=0$ in this case is similar to the one in the previous two cases: simply replace $(s_{ab}+\epsilon_{ab})\,\tilde{d}$ with $\tilde{d}\,(s_{ab}+\epsilon_{ab})$, and $(s_{a\,m+b}+\epsilon_{a\,m+b})\,\tilde{d}$ with $\tilde{d}\,(s_{a\,m+b}+\epsilon_{a\,m+b})$, throughout. □

Note that if ℓ is odd, then $h^{(\ell)}$ belongs to the subspace $J_m^{(\ell)}$. This follows by writing

$$2\,h^{(\ell)}=\frac{1}{\ell!}\sum_{s\in\mathfrak{S}_\ell}(s+s^{-1})$$

and using the telescoping sum

$$s+s^{-1}=(s+s^{t_1})-(s^{t_1}+s^{t_1t_2})+(s^{t_1t_2}+s^{t_1t_2t_3})-\cdots+(s^{t_1\dots t_{\ell-1}}+s^{t_1\dots t_\ell}).$$

Therefore, $q^{(\ell)}h^{(\ell)}f[-\lambda]q^{(m)}=0$ by (4.12) (in particular, $\phi_1=0$). This observation and Lemma 4.8 complete the proof of the proposition. □

We can now use Propositions 4.1, 4.6 and 4.7 to complete the proof of Theorem 2.1. The element ϕ_m given in (4.10) belongs to the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$ defined over $\mathbb{C}[\omega]$. Consider this algebra at the critical level by taking its quotient $\hat{\mathcal{B}}_{2m+1}(\omega)_{\text{cri}}$ over the ideal generated by the central element $\omega+K-2$. We conclude that both $f[0]_0\,\phi_m$ and $f[1]_0\,\phi_m$ belong to the left ideal of the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)_{\text{cri}}$ over $\mathbb{C}(\omega)$ generated by the subspaces $f[r]_a\mathcal{B}_{2m+1}(\omega)$ for $r\geqslant 0$ and $a=0,\dots,m$. To be able to evaluate ω at $M-2n$ we need to verify that the elements of the left ideal are defined over $\mathbb{C}[\omega]$. This is clear for $f[0]_0\,\phi_m$ from Propositions 4.1 and 4.7, whereas the formula of Proposition 4.6 implies the relation in the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)_{\mathrm{cri}}$:

$$\begin{aligned} f[1]_0\, \phi_m = & -m\, \gamma_m(\omega)\, q^{(m)} \varphi_{0m} s^{(m)} f_1 \dots f_{m-1} f[0]_m q^{(m)}\, 1 \\ & + m\, \gamma_m(\omega)\, q^{(m)} s^{(m)} f_1 \dots f_{m-1} q^{(m)} f[0]_0\, 1 \\ & + m\, \gamma_m(\omega)\, q^{(m)} s^{(m)} \varphi_{0m} f_1 \dots f_{m-1} f[0]_m q^{(m)}\, 1 \\ & + \gamma_m(\omega)\, q^{(m)}\, s^{(m)} f_1 \dots f_m q^{(m)} f[1]_0\, 1. \end{aligned}$$

By cyclic property 2 and relations

$$\varphi_{0m}\, f[0]_m = f[0]_0\, s_{0m} + \epsilon_{0m}\, f[0]_0 \qquad \text{and} \qquad f[0]_m\, \varphi_{0m} = s_{0m}\, f[0]_0 + f[0]_0\, \epsilon_{0m},$$

the sum of the first and third terms on the right hand side equals

$$m\, \gamma_m(\omega)\, q^{(m)} s^{(m)} f_1 \dots f_{m-1} f[0]_0 \varphi_{0m} q^{(m)}\, 1 - m\, \gamma_m(\omega)\, q^{(m)} s^{(m)} f_1 \dots f_{m-1} \varphi_{0m}\, f[0]_0\, q^{(m)}\, 1.$$

With this replacement, the argument used in the proof of Proposition 4.7 applies to all four terms in the expansion of $f[1]_0\, \phi_m$ with obvious modifications to show that all of them are defined over $\mathbb{C}[\omega]$. For example, expanding the second of the new terms, we can write it as a $\mathbb{C}$-linear combination of the expressions analogous to (4.11):

$$\gamma_m(\omega) q^{(m)} s^{(m)} f[-r_1]_{a_1} \dots f[-r_{\ell-1}]_{a_{\ell-1}} \varphi_{0m}\, f[0]_0\, q^{(m)},$$

where $1 \leqslant a_1 < \dots < a_{\ell-1} \leqslant m-1$ and the r_i are positive integers with $r_1+\dots+r_{\ell-1} = m-1$. We can insert appropriate permutations s_{ab} after $s^{(m)}$ to change the indices to write such an expression as

$$\gamma_m(\omega) q^{(m)} s^{(m)} f[-r_1]_1 \dots f[-r_{\ell-1}]_{\ell-1} \varphi_{0\,\ell}\, f[0]_0\, q^{(m)}.$$

Now apply Lemma 4.3 repeatedly to find that it equals

$$\gamma_\ell(\omega) q^{(\ell)} s^{(\ell)} f[-r_1]_1 \dots f[-r_{\ell-1}]_{\ell-1} \varphi_{0\,\ell}\, f[0]_0\, q^{(m)}$$

times an element of $\mathbb{C}[\omega]$. Finally, Lemma 4.8 applies in the same form (the role of $f[-\lambda_\ell]_\ell$ is played by $\varphi_{0\,\ell}$) allowing us to replace $\gamma_\ell(\omega) s^{(\ell)}$ with $h^{(\ell)}$ to produce an element of the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)_{\mathrm{cri}}$ over $\mathbb{C}[\omega]$.

By applying the homomorphism ρ defined in (3.9), we find that the image $\rho(\phi_m)$ coincides with $\Phi_m\, Q^{(m)}$, where Φ_m is defined in (2.6). Under the evaluation $\omega = M - 2n$, the central element $\omega + K - 2$ becomes $K + h^\vee$ so that by the annihilation properties of ϕ_m in $\hat{\mathcal{B}}_{2m+1}(\omega)_{\mathrm{cri}}$, Φ_m belongs to the Feigin–Frenkel centre $\mathfrak{z}(\widehat{\mathfrak{osp}}_{M|2n})$, thus completing the proof of Theorem 2.1.

Remark 4.9. (i) It was pointed out in [15, Remark 2.5] that if $M = 0$ or $n = 0$, then all coefficients of the polynomial in τ appearing in (4.2) coincide with the Segal–Sugawara vectors Φ_k for certain k, up to a constant factor. A similar property is shared by the polynomial $\gamma_m(\omega) q^{(m)} s^{(m)} f_1 \dots f_m q^{(m)}$ or, more generally, by

$$\gamma_k(\omega) q^{(k)} s^{(k)} f_1 \dots f_k\, q^{(m)} = \psi_{k0}\, \tau^k + \dots + \psi_{kk}, \tag{4.14}$$

where we keep m fixed, while $1 \leqslant k \leqslant m$. Namely, we have the relations

$$\psi_{ka} = \binom{\omega + k - 2}{k - a} \psi_{aa}.$$

They are easily verified by noting that for any $u \in \mathbb{C}$ the map $\tau \mapsto u + \tau$ extends to an automorphism of the algebra $\hat{\mathcal{B}}_{2m+1}(\omega)$. We apply it to both sides of (4.14) and compare the coefficients of $u^{k-a}\tau^0$.

(ii) We can apply the evaluation homomorphism $\mathrm{ev}_z : \mathfrak{osp}_{M|2n}[t^{-1}] \to \mathfrak{osp}_{M|2n}$ which takes t to a nonzero complex number z, to the Segal–Sugawara vectors Φ_m. As with the non-super case considered in [14, Sec. 6.5], the images $\mathrm{ev}_z(\Phi_m)$ belong to the centre of the universal enveloping algebra $\mathrm{U}(\mathfrak{osp}_{M|2n})$; cf. [12, Thm 3.16].

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