Tuesday, January 15 $*$ **Solutions** $*$ *A review of some important calculus topics*

1. Chain Rule:

   (a) Let $h(t) = \sin\big(\cos(\tan t)\big)$. Find the derivative with respect to t.

   Solution.

$$\begin{aligned}\frac{d}{dt}(h(t)) &= \frac{d}{dt}(\sin(\cos(\tan t)))\\ &= \cos(\cos(\tan t)) \cdot \frac{d}{dt}(\cos(\tan t))\\ &= \cos(\cos(\tan t)) \cdot (-\sin(\tan t)) \cdot \frac{d}{dt}(\tan t)\\ &= \cos(\cos(\tan t)) \cdot (-\sin(\tan t)) \cdot \sec^2 t\end{aligned}$$

   (b) Let $s(x) = \sqrt[4]{x}$ where $x(t) = \ln\big(f(t)\big)$ and $f(t)$ is a differentiable function. Find $\dfrac{ds}{dt}$.

   Solution. From $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt}$, we get

$$\frac{ds}{dt} = \frac{1}{4x^{3/4}} \cdot \frac{f'(t)}{f(t)}.$$

   But we need to make sure that $\frac{ds}{dt}$ is a single variable function of f, so

$$\frac{ds}{dt} = \frac{1}{4\left[\ln(f(t))\right]^{3/4}} \cdot \frac{f'(t)}{f(t)}.$$

2. Parameterized curves:

   (a) Describe and sketch the curve given parametrically by

$$\begin{cases} x = 5\sin(3t) \\ y = 3\cos(3t) \end{cases} \quad \text{for} \quad 0 \le t < \frac{2\pi}{3}.$$

   What happens if we instead allow t to vary between 0 and 2π?

   Solution. Note that

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2(3t) + \cos^2(3t) = 1.$$

   So this parameterizes (at least part of) the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$.

By examining differing values of t in $0 \le t \le \frac{2\pi}{3}$, we see that this parametrization travels the ellipse in a clockwise fashion exactly once.

$$\begin{aligned}
&t = 0 : (x(0), y(0)) = (0, 3)\\
&t = \pi/6 : (x(\pi/6), y(\pi/6)) = (5, 0)\\
&t = \pi/3 : (x(\pi/3), y(\pi/3)) = (0, -3)\\
&t = \pi/2 : (x(\pi/2), y(\pi/2)) = (-5, 0)
\end{aligned}$$

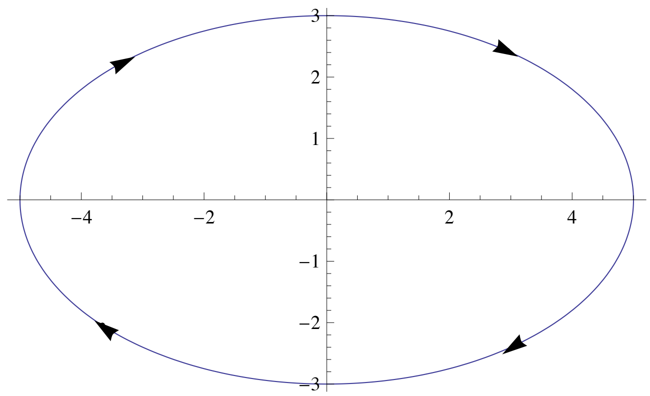

Figure 1: Ellipse.

If we let t vary between 0 and 2π, we will traverse the ellipse 3 times.

(b) Set up, but **do not evaluate** an integral that calculates the arc length of the curve described in part (a).

Solution. Arc length

$$\begin{aligned}
s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\\
&= \int_0^{\frac{2\pi}{3}} \sqrt{(15\cos(3t))^2 + (-9\sin(3t))^2}\, dt.
\end{aligned}$$

(c) Consider the equation $x^2 + y^2 = 16$. Graph the set of solutions of this equation in $\mathbb{R}^2$ and find a parametrization that traverses the curve once counterclockwise.

Solution. If we let $x = 4\cos t$ and $y = 4\sin t$, then $x^2 + y^2 = (4\cos t)^2 + (4\sin t)^2 = 16$. Moreover, as t increases, this parametrization traverses the circle in a counterclockwise fashion:

$$\begin{aligned} &t = 0 : (x(0), y(0)) = (4, 0) \\ &t = \pi/2 : (x(\pi/2), y(\pi/2)) = (0, 4) \\ &t = \pi : (x(\pi), y(\pi)) = (-4, 0) \\ &t = 3\pi/2 : (x(3\pi/2), y(3\pi/2)) = (0, -4) \\ &t = 2\pi : (x(2\pi), y(2\pi)) = (4, 0) \end{aligned}$$

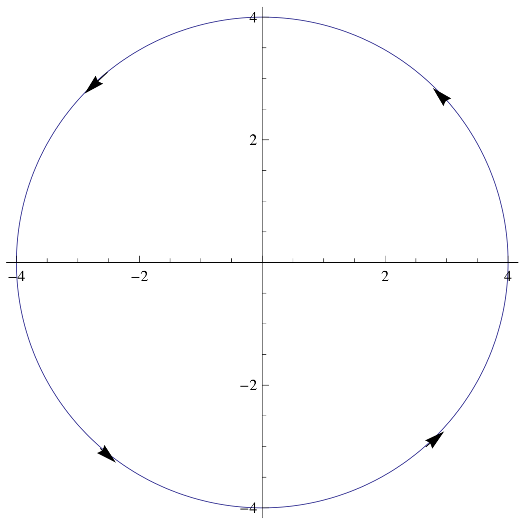

Figure 2: Circle.

To ensure that we travel the curve only once, we restrict t to the interval $[0, 2\pi)$. So the parametrization is

$$\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases} \qquad \text{when} \quad 0 \le t \le 2\pi.$$

3. 1st and 2nd Derivative Tests:

   (a) Use the 2nd Derivative Test to classify the critical numbers of the function $f(x) = x^4 - 8x^2 + 10$.

   Solution. First, we find the critical points of $f(x)$.

   $$f'(x) = 4x^3 - 16x.$$

   $f'(x) = 0$ when $4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2) = 0$. Hence $f'(x) = 0$ when $x = 0$, $x = 2$ or $x = -2$.

Now apply the 2nd Derivative Test to the three critical points. From $f''(x) = 12x^2 - 16$, we get:
$f''(0) = -16 < 0$, so $y = f(x)$ is concave down at the point $(0, f(0))$. So a local max occurs at $(0, 10)$.
$f''(-2) = 32 > 0$, so $y = f(x)$ is concave up at the point $(-2, f(-2))$. So a local min occurs at $(-2, -6)$.
$f''(2) = 32 > 0$, so $y = f(x)$ is concave up at the point $(2, f(2))$. So a local min occurs at $(2, -6)$.

(b) Use the 1st Derivative Test and find the extrema of $h(s) = s^4 + 4s^3 - 1$.

Solution. First, find the critical points of $h(s)$.

$$h'(s) = 4s^3 + 12s^2.$$

Then $h'(s) = 0$ when $4s^3 + 12s^2 = 4s^2(s+3) = 0$. So $h'(s) = 0$ when $s = 0$ and $s = -3$.
For the 1st Derivative Test, we need to determine if h is increasing or decreasing on the intervals $(-\infty, -3)$, $(-3, 0)$ and $(0, \infty)$.
On $(-\infty, -3)$ choose any test point (for example, choose $s = -1000$). The sign of $h'(s) = 4s^3 + 12s^2 < 0$ on this interval. Hence $h(s)$ is decreasing on $(-\infty, -3)$.
On $(-3, 0)$ choose any test point (for example, choose $s = -1$). The sign of $h'(s) = 4s^3 + 12s^2 > 0$ on this interval. Hence $h(s)$ is increasing on $(-3, 0)$.
On $(0, \infty)$ choose any test point (for example, choose $s = 1000$). The sign of $h'(s) = 4s^3 + 12s^2 > 0$ on this interval. Hence $h(s)$ is increasing on $(0, \infty)$.

Since at $s = -3$ the function changes from decreasing to increasing, the function must have obtained a local min at $s = -3$.
At $s = 0$, neither a max or a min occurs in the value of h.

(c) Explain why the 2nd Derivative test is unable to classify all the critical numbers of $h(s) = s^4 + 4s^3 - 1$.

Solution. When $s = -3$, $h''(-3) = 36 > 0$. A local min occurs when $s = -3$ by the 2nd Derivative Test.
When $s = 0$, $h''(0) = 0$. The 2nd Derivative Test is inconclusive. The graph of $y = h(s)$ has no concavity at $(0, h(0))$. Without more information (the 1st Derivative Test), we are unable to identify $(0, h(0))$ as a local max, min or a point of inflection.

4. Consider the function $f(x) = x^2 e^{-x}$.

(a) Find the best linear approximation to f at $x=0$.

Solution. Recall that in Calc I and II, the "best linear approximation" is synonymous with the equation of the tangent line or the 1st order Taylor polynomial. Hence, $f'(x) = 2xe^{-x} + x^2(-e^{-x})$.

Since $f'(0)=0$, the tangent line has no slope at $(0, f(0)) = (0,0)$. The equation of the tangent line is $y=0$.

(b) Compute the second-order Taylor polynomial at $x=0$.

Solution. By definition, the second-order Taylor polynomial at $x=0$ is

$$T_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2.$$

Since $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$, we compute that $f''(0) = 2$. Hence

$$T_2(x) = 0 + \frac{0}{1!}(x-0) + \frac{2}{2!}(x-0)^2 = x^2.$$

(c) Explain how the second-order Taylor polynomial at $x=0$ demonstrates that f must have a local minimum at $x=0$.

Solution. The second-order Taylor polynomial is the best quadratic approximation to the curve $y=f(x)$ at the point $(0, f(0))$. Since $T_2(x) = x^2$ clearly has a local minimum at $(0,0)$, and $(0,0)$ is the location of a critical point of f, then f must also have a local minimum at $(0,0)$.

5. Consider the integral $\displaystyle\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx$.

(a) Sketch the area in the xy-plane that is implicitly defined by this integral.

Solution. The shadow area in the following picture is the area defined by the integral.

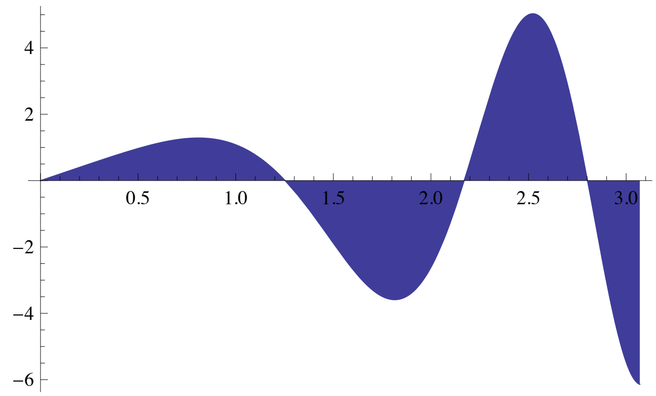

Figure 3: 5(a).

(b) To evaluate, you will need to perform a substitution. Choose a proper $u = f(x)$ and rewrite the integral in terms of u. Sketch the area in the uv-plane that is implicitly defined by this integral.

Solution. Let $u = x^2$. Then $du = 2x\,dx$, so the integral becomes

$$\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx = \int_0^{3\pi} \cos u\,du.$$

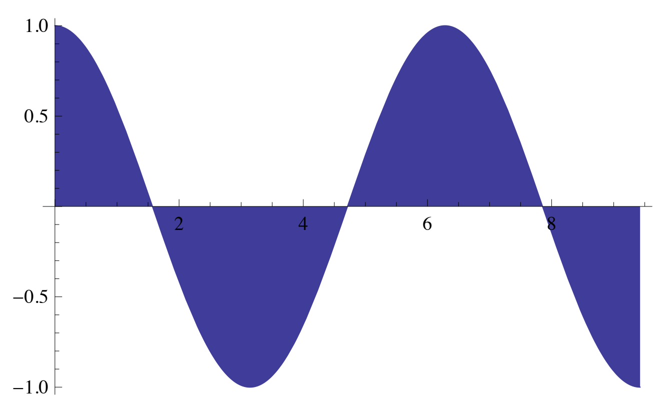

Figure 4: 5(b).

(c) Evaluate the integral $\displaystyle\int_0^{\sqrt{3\pi}} 2x\cos\left(x^2\right)dx$.

Solution.

$$\int_0^{\sqrt{3\pi}} 2x\cos(x^2)\,dx = \int_0^{3\pi} \cos u\,du = \Big[\sin u\Big]_{u=0}^{u=3\pi} = \sin(3\pi) - \sin 0 = 0.$$