Last time: the Divergence Theorem

Assume $\mathbf{E}$ is a vector field with continuous first derivatives on an open set $D \subset \mathbb{R}^3$. Assume that $B \subset D$ is a "nice" solid. Then

Theorem (Divergence Theorem)

$\iiint_B div\ \mathbf{E}\ dV = \iint_{\partial B} \mathbf{E} \cdot d\mathbf{S}.$

Example. Assume $D = \mathbb{R}^3 \setminus \{(0,0,0)\}$ is everything except the origin. Suppose that div$\mathbf{E} = 0$ on D. Let $S_r = \{x^2 + y^2 + z^2 = r^2\}$ and let $S'_r = \{(x-3)^2 + (y-1)^2 + z^2 = r^2\}$.
Find $I_1 = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ and $I_2 = \iint_{S'_1} \mathbf{E} \cdot d\mathbf{S}$.

(a) Not enough information to find either.

(b) $I_1 = I_2 = 0$.

(c) Not enough information to find I_1; but $I_2 = 0$.

(d) $I_1 = 0$; not enough information to find I_2.

Solution

Let B be any solid that doesn't contain $(0,0,0)$, so that $B \subset D$. (e.g. the solid inside the sphere S'_r.)

By the Divergence theorem,

$$\iint_{\partial B} \mathbf{E} \cdot d\mathbf{S} = \iiint_B \text{div}\mathbf{E}\ dV = \iiint_B 0\ dV = 0.$$

Note that this argument doesn't work for S_1, because the solid inside of S_1 contains $(0,0,0)$, so $\mathbf{E}$ is not defined over the whole solid, and the Divergence Theorem doesn't apply.

In fact, we will see later how to calculate $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ explicitly for a certain example of $\mathbf{E}$, and we will see that it's not 0.

Announcements

- Final exam is next Friday. Register for conflict by Monday.
- Office hours/review session next week:
    - Ordinary office hours Tuesday 11–11:50am.
    - Extra office hours Wednesday evening (probably 6–7pm, maybe 7–8pm—it's fine with me if you bring your dinner). **AH 341**
    - Extra office hours Thursday 12–1pm. **AH 341**
    - Possibly office hours also on Friday, but I can't confirm yet.
    - Come with questions (or you can listen to other people's questions).
- Fact: today is our ante-penultimate lecture. Wednesday was our **pre-ante-penultimate** lecture, but I forgot to say so.

Electric field and electric flux

Given a particle of charge Q at $(0,0,0)$, its electric field is

$$\mathbf{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}}\langle x,y,z\rangle$$

or equivalently

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}|^3}\mathbf{r}.$$

Inverse square law

This means that the force experienced by a particle of charge q at position $\mathbf{r}$ is $q\mathbf{E}(\mathbf{r})$.

Where is the vector field $\mathbf{E}$ defined?

(a) all of $\mathbb{R}^3$

(b) everywhere except $(0,0,0)$

(c) everywhere except the z-axis, $\{x=y=0\}$

(d) I don't know

Practice with Gauss' Law

Suppose we have particles of charge Q_i at points P_i, with $Q_i = i$ for $i = 1, 2, 3, 4, 5$. Suppose that B is a solid region containing P_1, P_3, and P_4, but not P_2 or P_5. What is

$$\iint_{\partial B} \mathbf{E} \cdot d\mathbf{S}?$$

(a) 0

(b) $\frac{1}{\epsilon_0}$

(c) $\frac{2}{\epsilon_0}$

(d) $\frac{4}{\epsilon_0}$

(e) $\frac{8}{\epsilon_0}$

Solution

The enclosed charge is $Q_1 + Q_3 + Q_4 = 1 + 3 + 4 = 8$.

So by Gauss' Law,

$$\iint_{\partial B} \mathbf{E} \cdot d\mathbf{S} = \frac{8}{\epsilon_0}$$