Friday, 8 March, 2019

· Begin with an example problem from 2018 Midterm.
[1] (See slides for announcements about Midterm 2)

<u>Definition:</u> Let f be a function on a set D.
f is **bounded** if there exists a constant K so that
$|f(P)| \le K$ for all P in D

Review of integration of a function $g: [a,b] \to \mathbb{R}$.
[see slides]

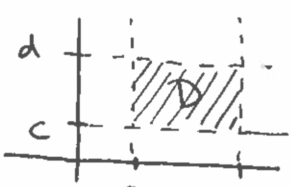
<u>Theorem:</u> If g is bounded on $[a,b]$ and continuous except at a finite number of points, then $\int_a^b g(x)dx$ is well-defined.

Note: · g is continuous, except for a finite number (maybe 0) of jump discontinuities.

· if $g \geq 0$, $\int_a^b g(x)dx$ = area under the graph of g.

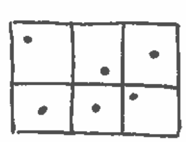
Now fix a function f on the rectangle D.

$$D = [a,b] \times [c,d] = \{(x,y) \mid a \le x \le b, \quad c \le y \le d\}.$$



· Divide $[a,b]$ into m pieces $[x_{i-1}, x_i]$ of width $\Delta x = \frac{b-a}{m}$.

Divide $[c,d]$ into n pieces $[y_{j-1}, y_j]$ of width $\Delta y = \frac{d-c}{n}$.



Pick $(x_{ij}, y_{ij})^* \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for each i, j

Define $\iint_D f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$.

where $\Delta A = \Delta x \Delta y$ = area of a subrectangle.

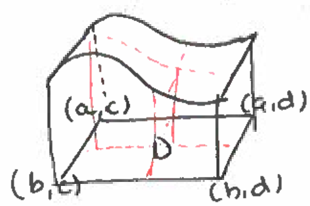
(if the limit exists & doesn't depend on choices of (x_{ij}^*, y_{ij}^*))

Def. in this case we say f is **integrable**

Theorem: If f is bounded, and is continuous except at a finite number of smooth curves, then f is integrable.

Geometric interpretations:

- $\iint_D f(x,y)\,dA = (\text{average value of } f)\cdot(\text{area of } D)$
- if $f \geq 0$:



$\iint_D f(x,y)\,dA =$ volume of the solid under the graph of f, over D.

- approximated by dividing it into columns of height $f(x_{ij}^*, y_{ij}^*)$.

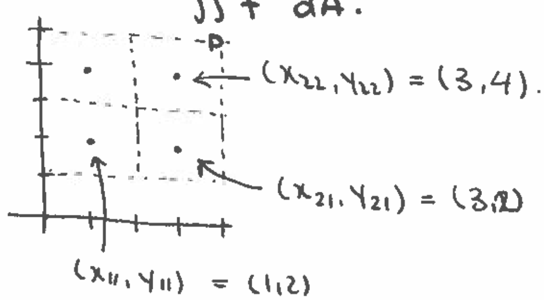
Estimating integrals - "Midpoint rule"

- Fix values of m & n that you (or your computer) can handle.
- Pick $x_{ij}^* = \dfrac{x_{i-1} + x_i}{2}$, $y_{ij}^* = \dfrac{y_{j-1} + y_j}{2}$ ← midpoint of rectangle

$$\iint_D f\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

Example: let $D = [0,4] \times [1,5]$ and $f(x,y) = x+y$.

Use the midpoint rule ~~to~~ with $m=n=2$ to estimate $\iint f\,dA$.



$\boxed{1}$. (See slides for solution)

Theorem: **[Fubini's theorem]**

Let f be continuous on $D = [a,b] \times [c,d]$. Then

$$\iint_D f(x,y)\,dA = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

- In practice:
  - Do inner integral first. (treat the other variable like a constant).
  - Then do the outer integral.
- Sometimes one order is easier than the other, so if you get stuck, try the other way.

Example: Let D = [0,4] × [1,5]. Find $\iint_D (x+y)\,dA$.

$$\iint_D (x+y)\,dA = \int_0^4 \left(\int_1^5 (x+y)\,dy \right) dx$$
$$= \int_0^4 \left[xy + \tfrac{1}{2}y^2 \right]_{y=1}^5 dx$$
$$= \int_0^4 \left[\left(5x + \tfrac{25}{2}\right) - \left(x + \tfrac{1}{2}\right) \right] dx$$
$$= \int_0^4 (4x + 12)\,dx$$
$$= \left[2x^2 + 12x\right]_0^4 = 32 + 48 = 80$$

OR $$\iint_D (x+y)\,dA = \int_1^5 \left(\int_0^4 x+y\,dx \right) dy$$
$$= \int_1^5 \left(\tfrac{1}{2}x^2 + xy \right)_0^4 dy$$
$$= \int_1^5 \left[(8 + 4y) \right] dy = \left[8y + 2y^2\right]_1^5$$
$$= (40 + 50) - 10 = 80$$

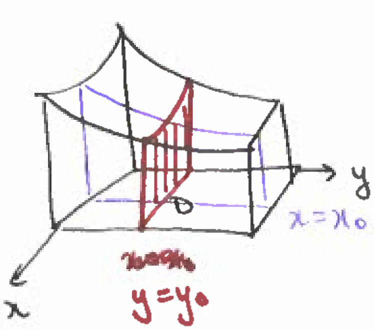
Example: Let D = [0,2] × [−3,1]. Find $\iint_D (3x^2 + 3y^2)\,dA$.

[2] (See slides for solution)

Why is Fubini's theorem true? [Optional]

- Assume $f(x,y) \geq 0$, so we're computing the area of a solid.

$$V = \iint_D f\,dA.$$



- We can compute/approximate V by slicing.

At $\boxed{y = y_0}$ slice has area $\int_a^b f(x, y_0)\,dx$.

"adding up" all the slices ⟺ integrate over y

$$\Rightarrow V = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

But we could also have taken slices in the other direction:

area of slice at $x = x_0$ is $\int_c^d f(x_0, y)\,dy$

$\Rightarrow$ Volume $= \int_a^b \int_c^d f(x,y)\,dy\,dx.$