Last time: integrating vector fields

Let $C_1 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$.
Let $C_2 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y \leq 0\}$
Orient both from $(-1, 0)$ to $(1, 0)$.
Let $\mathbf{F}(x, y) = \langle y, -x \rangle$. (Note: I had the opposite sign on the slide in class, but that was wrong.)

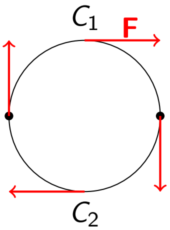
Use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \; ds$ to calculate

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

(a) 0
(b) 2π
(c) -2π
(d) $-\pi$
(e) I don't know what to do.

Correct answer: (b)

Solution



For C_1**:**
$\mathbf{F} = \mathbf{T}$, so

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T} ds = \int_{C_1} |\mathbf{T}|^2 ds$$
$$= \int_{C_1} 1 ds = \pi.$$

For C_2**:**
$\mathbf{F} = -\mathbf{T}$, so

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T} ds = \int_{C_2} (-1) ds = -\pi.$$

So $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \pi - (-\pi) = 2\pi.$

Example 1

Let $\mathbf{F} = \nabla f$ be a conservative vector field on $\mathbf{R}^2$ or $\mathbf{R}^3$, and let C be a curve with initial point P and terminal point Q. Assume that ∇f is continuous.

The Fundamental Theorem of Line Integrals tells us that

$$\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P).$$

This implies that $\mathbf{F}$ is independent of path.

Example 2

Let $\mathbf{F}(x, y) = \langle -y, x \rangle$.

At the beginning of class, we found two curves C_1 and C_2 with the same initial point $(-1, 0)$ and the same terminal point $(1, 0)$, but we showed that the integrals of $\mathbf{F}$ over C_1 and C_2 were not equal.

So $\mathbf{F}$ is not path independent.

Remark: Combining this observation with the previous slide, we can conclude that $\mathbf{F}$ is not conservative.

Is the vector field conservative?

We're going to look at the vector field describing wind velocity.
Discuss with your neighbour: is this vector field conservative?
https://earth.nullschool.net/
(Remember the options below:)

(a) Yes, we think it is.
(b) No, we think it's not.
(c) We don't agree/we don't know.

Answer: the vector field is not conservative. You can find circles around which the integral is not zero.

Comments on the proof

Theorem: For D open and connected,
the integral of $\mathbf{F}$ is path independent $\Leftrightarrow$ $\mathbf{F}$ is conservative.

We have to prove two things.

- The integral of $\mathbf{F}$ is path independent $\Rightarrow$ $\mathbf{F}$ is conservative.
- The vector field $\mathbf{F}$ is conservative $\Rightarrow$ the integral is path independent.

We already showed the second line, using the Fundamental Theorem of Line Integrals.

The integral of $\mathbf{F}$ is path independent $\Rightarrow$ $\mathbf{F}$ is conservative.

We're mostly going to skip the proof, but here is the main idea.

Choose any point P in D.

Define $f : D \to \mathbb{R}$ as follows.

Given any point Q in D, choose a path C from P to Q.
We can do this because D is connected!

Now let $f(Q) = \int_C \mathbf{F} \cdot d\mathbf{r} \in \mathbb{R}$.
It doesn't matter what path C we chose, because the integral is path independent!

We claim that $\nabla f = F$, which shows that F is conservative.