Monday, 4 March 2019

Last time: Let $C_1 = \{x^2+y^2=1\}, y \geq 0\}$; let $C_2 = \{x^2+y^2=1, y \leq 0\}$

Orient both from $(-1,0)$ to $(0,1)$ and let $\vec{F}(x,y) = \langle -y, x \rangle$.

$\boxed{1}$ Use $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, ds$ to find $\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$.

§16.3. CONSERVATIVE VECTOR FIELDS.

Let $\vec{F}$ be a vector field on $D \subset \mathbb{R}^n$.

Def $\int_C \vec{F}\cdot d\vec{r}$ is **path independent** if $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ whenever C_1 and C_2 have the same initial points and endpoints.

Examples (on slide)

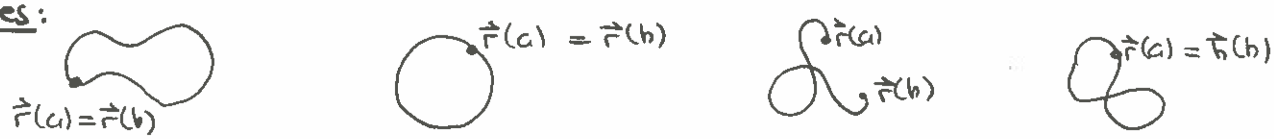
- [Last time]. The fundamental theorem of line integrals implies that a conservative vector field is independent of path
- $\vec{F} = \langle -y, x \rangle$ is not independent of path.

Let C be a curve parametrized by $\vec{r}(t)$, $a \leq t \leq b$.

We say that C is **closed** if it begins and ends at the same point

i.e. if $\vec{r}(a) = \vec{r}(b)$.

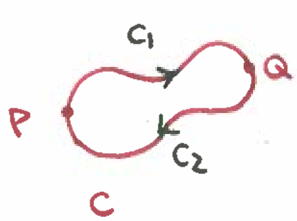
Examples:



closed   closed   not closed   closed.

Fact 1: If $\vec{F}$ is independent of path and C is a closed curve in D, $\int_C \vec{F}\cdot d\vec{r} = 0$.

Why?



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

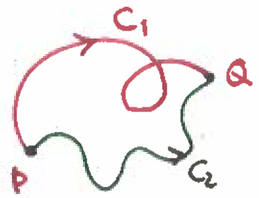
$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$$

$C_1, -C_2$ are paths from P to Q.

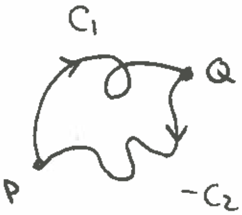
Fact 2: Suppose $\int_C \vec{F}\cdot d\vec{r} = 0$ $\forall$ (for all) closed curves C in D.

Then $\vec{F}$ is independent of path.

Why?



define C



$$0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}.$$

Theorem: $\int_C \vec{F}\, d\vec{r}$ is independent of path

$\iff \int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves C.

$\boxed{1}$ is wind-~~speed~~ velocity a conservative vector field?

Recall: $D \subset \mathbb{R}^n$ is **closed** if it contains all of its boundary points.

Definition $D \subset \mathbb{R}^n$ is **open if it doesn't contain any of its boundary points.**

Examples

$x^2+y^2 \le 1$

closed

contains all boundary points

$x^2+y^2 < 1$

open

contains no boundary points

$1 \le x^2+y^2 < 4$

neither open nor closed

contains some boundary points

$\mathbb{R}^2$

both open and closed.

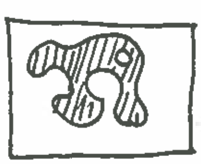
there aren't any boundary points.

Equivalently: For any $P \in D$, there is a small disk around P contained in D.

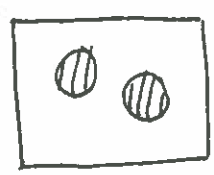
In practice: Defined by $<, >$ or $\neq$

(not $\le, \ge$ or $=$).

Definition: We say $D \subset \mathbb{R}^n$ is **connected** if any two points in D can be joined by a path in D.



connected



Not connected.

In practice - it's only one piece.

Theorem: If D is open and connected, then

$\int_C \vec{F}\cdot d\vec{r}$ is path independent $\Longleftrightarrow$ $\vec{F}$ is conservative.

(See slides for sketch of proof.)

Note that ∇f has the same path integrals as $\vec{F}$:

(evidence to believe $\nabla f = \vec{F}$).

$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$ (FTLI)

$= f(Q)$ ($f(P) = 0$ by definition)

$= \int_C \vec{F}\cdot d\vec{r}$ (by definition of f).

In practice, given $\vec{F} = \langle P, Q\rangle$ conservative, we can explicitly write ~~down~~ a formula for f as follows.

Want $\nabla f = \vec{F} = \langle P, Q\rangle$.

Step 1: $f_x(x,y) = P$.

↳ integrate w.r.t. x to find a function $h(x,y)$ with $h_x(x,y) = P$.

Step 2. $f(x,y) - h(x,y) = c(y)$ depends only on (y).

$\underbrace{f_y(x,y)}_{Q} - h_y(x,y) = c'(y)$

So we can integrate w.r.t. y to find $c(y)$.

Step 3: $f(x,y) = h(x,y) + c(y)$

Example: $\vec{F} = \langle \sin y, x\cos y + 2y\rangle$

$f_x(x,y) = \sin y$

$\Rightarrow f(x,y) = \underbrace{x\,\overset{\sin}{\cancel{\cos}}\,y}_{h(x,y)} + c(y)$. $(*)$

- $f_y(x,y) = x\cos y + 2y = Q$
- $f_y(x,y) = x\cos y + c'(y)$ $(*)$

$\Rightarrow c'(y) = 2y$

$\Rightarrow c(y) = y^2 + \underbrace{k}_{\text{some constant}}$.

- $f(x,y) = x\sin y + y^2 + k$.

(can take $k = 0$)

$f(x,y) = x\sin y + y^2$.