Math 241 - Lecture 19 - Fundamental Theorem of Line Integrals.

Friday 1 March 2019

__Last time__ - vector fields; integrating them along curves.

[1] Let C be the line segment from $(0,0)$ to $(1,2)$.

Let $\vec{F}(x,y) = \langle 1, 2y \rangle$.

Find $\int_C \vec{F} \cdot d\vec{r}$.

Does the integral depend on the parametrization?

__Answer:__ No, only up to the direction.

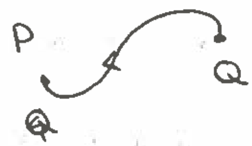
(e.g. the integral from $(1,2)$ to $(0,0)$ has the opposite sign.)

__Definition:__ Any curve C has a choice of two **orientations**; once we make the choice, we call C an **oriented curve**;

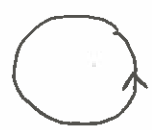
$-C$ denotes the same curve with the opposite orientation

from P to Q

OR

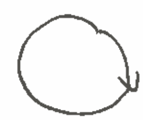


from Q to P



counterclockwise

OR



clockwise.

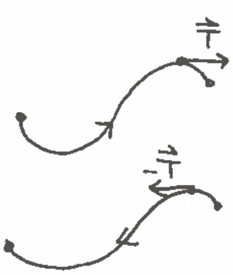
Recall that given a parametrization of C, $\vec{r} : [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$)

we defined the **unit tangent vector** $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

(at the point $P = \vec{r}(t) \in C$)

If we choose a different parametrization with the same orientation, $\vec{T}$ doesn't change at each point P.

But if we choose a parametrization corresponding to the opposite orientation $-C$, the unit tangent vector is $-\vec{T}$.



Rewrite $\int_C \vec{F}\cdot d\vec{r}$ in terms of $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

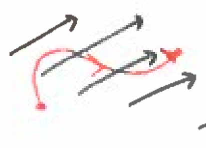
$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\,|\vec{r}'(t)|\,dt$$

$$= \int_C \vec{F}\cdot\vec{T}\,ds$$

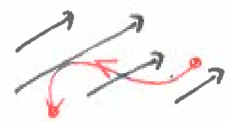
↑ depends only on the orientation of C, not the parametrization.

$$\therefore \int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}.$$

Example:



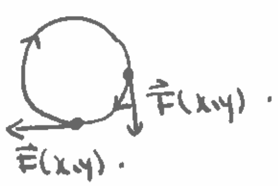
work is positive



work is negative

Example: Use $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$ to find $\int_C \langle y, -x\rangle\, d\vec{r}$,

where $C = \{x^2+y^2=1\}$ oriented clockwise



- $\vec{F}(x,y) = \langle y, -x\rangle$
- $\vec{T}$ points in the same direction, and for $(x,y)\in C$ both have length 1

$\Rightarrow \vec{F} = \vec{T}$, and $\vec{F}\cdot\vec{T} = |\vec{F}|^2 = 1.$

$\boxed{1}$ $\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (1)\,ds = L$

$= 2\pi.$

Note that we didn't need to choose a parametrization here (we could have taken $\vec{r}(t) = \langle \cos(-t), \sin(-t)\rangle$, $t\in[0,2\pi]$ and calculated from the definition $\int_C \vec{F}\cdot d\vec{r}$)

§16.3. FUNDAMENTAL THEOREM OF LINE INTEGRALS.

Recall: Fundamental theorem of calculus:

Given $f: [a,b] \to \mathbb{R}$ with $f': [a,b] \to \mathbb{R}$ continuous,

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

- We think of the gradient ∇f ~~as~~ the analogue of "f'" for $f: C \to \mathbb{R}^2$ or $\mathbb{R}^3$.

Let's use FTC to understand $\int_C \nabla f \cdot d\vec{r}$.

- Assume C is a smooth curve parametrized by

$$\vec{r}: [a,b] \to \mathbb{R}^3$$

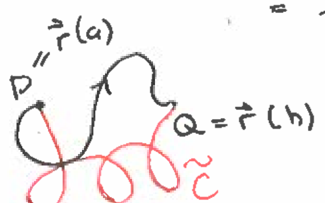
and f is a differentiable function $f: C \to \mathbb{R}$ with continuous gradient ∇f.

$$\leadsto \int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\left(f(\vec{r}(t))\right) dt \quad \text{by the Chain Rule.}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)).$$



$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)}$$

Note: if $\tilde{C}$ is another path from P to Q, then

$$\int_{\tilde{C}} \nabla f \cdot d\vec{r} = f(Q) - f(P) = \int_C \nabla f \cdot d\vec{r}.$$

Definition We say that the line integral of ∇f is independent of path: it only depends on the starting point and the ending point.

[1] Let C be a circle. Find $\int_C \nabla f \cdot d\vec{r}$.] do next example first.

Example: Let C be the line segment from $(0,0)$ to $(1,2)$

Let $g(x,y) = x + y^2$; Find $\int_C \nabla g \cdot d\vec{r}$

$\Rightarrow \int_C \nabla g \cdot d\vec{r} = g(1,2) - g(0,0) = 5.$

(compare to your work on the first i-clicker slide)

1. Look at the vector field describing wind velocity.
Is it conservative?