DEGENERATE LYAPUNOV FUNCTIONALS AND A PREY-PREDATOR MODEL WITH DISCRETE DELAYS

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ABSTRACT. The dynamics of the classical Lotka-Volterra prey-predator equation with discrete delays $\dot{x}(t) = x(t)[r_1 - x(t-\tau_1) - ay(t-\tau_2)]$, $\dot{y}(t) = y(t)[-r_2 + bx(t-\tau_3)]$ is concerned in this paper. We first show that, in some circumstances, the positive equilibrium of the model is locally asymptotically stable for small delays and the Hopf bifurcation occures for large delays. To study the stability of the positive equilibrium of the general system, we then introduce the concept of degenerate Lyapunov functionals. By constructing suitable degenerate Lyapunov functionals, we obtain some sufficient conditions on both local and global stabilities of the positive equilibrium. As a corollary, we show that small delays do not change the stability of the system. Furthermore, some explicit estimates on the delays are given.

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1. Introduction

It is well known that, for the classical autonomous Lotka-Volterra prey-pedrator system

$$\begin{aligned}\dot{N}_1 &= N_1[b_1 - a_{11}N_1 - a_{12}N_2]\\ \dot{N}_2 &= N_2[-b_2 + a_{21}N_1]\end{aligned} \tag{1.1}$$

with $b_i, a_{ij} > 0 (i,j = 1,2)$, if it has a positive equilibrium $N^* = (N_1^*, N_2^*)$, then it must be globally asymptotically stable. In order to account for time lag effects in this model, Cushing [3, 4] considered the following prey-predator integrodifferential system with infinitely distributed delays

$$\begin{aligned}\dot{N}_1(t) &= N_1(t)\Big[b_1 - a_{11}\int_{-\infty}^t k_{11}(t-s)N_1(s)ds - a_{12}\int_{-\infty}^t k_{12}(t-s)N_2(s)ds\Big]\\ \dot{N}_2(t) &= N_2(t)\Big[-b_2 + a_{21}\int_{-\infty}^t k_{21}(t-s)N_1(s)ds\Big],\end{aligned} \tag{1.2}$$

where $k_{ij}(t) \in C^o(R_+, R_+), k_{ij} \geq 0, \int_0^\infty k_{ij}(t)dt = 1\,(i,j = 1,2)$. As we known (see, for example, Cushing [3, 4, 5], Gopalsamy [10, 9], Huang et al. [21], Kuang [25], Leung [27, 28] and Stepan [32]), in general, that delays can complicated dynamics; equilibrium can be distabilized and periodic solutions may arise. For the system (1.2), it was shown (see Cushing [4]) that for *'small delays'* the positive equilibrium N^* is locally stable while for *'large delays'* the solutions tend to have divergent oscillations. However, the question of if *'small delays'* can change the global asymptotical stability of the positive equilibrium N^* of the system is still open. Indeed, this is a highly formidable problem to work on.

Recently, the stability problem of Lotka-Volterra systems with various types of delays has been studied extensively (see, for example, Gopalsamy [10] and Kuang [25]). Among the systems considered, they are basically of two types. One type of models contains delay-independent terms (see, for example Kuang [25], Leung [28], Leung and Zhou [29], Shukla [31] and Wörz-Busekros [33]), which dominate other intraspecific and interspecific interaction effects (with and/or without delays); while the other type of models contains no such delay-independent dominated terms and we refer this type of systems as pure-delay-type ones (see, for example, Gopalsamy [10, 9], He and Gopalsamy [20], He [18, 19], Kolmanovskii et.al. [24] and Kuang and Smith [26]). In both cases, negative feedback terms (delay dependent or independent) are required and the standard Lyapunov function(al)s method and/or Razumikhin-type arguments can be used to study the global stability (see [9, 10, 18, 19, 20, 24, 25, 26, 29, 28, 31]). In particular, it was shown (see He [18]) that for the general n-species autonomous Lotka-Volterra pure-delay-type systems with both finite and infinite delays, if the coefficient matrix is VL-stable (see Goh [8] and Hafbauer and Sigmund [14]) and if the delays are small, then the positive equilibrium (if any) remains globally attractive. However, as pointed in [18], the assumption on the VL-stability of the coefficient matrix may not necessary and this is indeed the case for the ODE system (1.1). Note that, compared with the perivous two types of systems, the system (1.2) has no such negative feedback (even with delay) for the predator and this renders the traditional Lyapunov function(al)s and Lyapunov-Razumikhin methods useless.

In this paper, as a special case of (1.2), we consider the following prey-predator system with different discrete delays

$$\begin{aligned} \dot{N}_1(t) &= N_1(t)[b_1 - a_{11}N_1(t-\tau_{11}) - a_{12}N_2(t-\tau_{12})] \\ \dot{N}_2(t) &= N_2(t)[-b_2 + a_{21}N_1(t-\tau_{21})]. \end{aligned} \tag{1.3}$$

System (1.3) is one of the simplest prey-perdator model with delays. When the system involves only one delay in the predator equation; that is the system (1.3) with $\tau_{11} = \tau_{12} = 0$, $\tau_{21} > 0$, its stability and Hopf bifurcations, both local and global, have been widely investigated. For details, we refer to He [19, 18] on the global stability and Farkas [6], Leung [27], Huang et al.[21] and Stepan [32] on the existence, both local and global, and stability of Hopf bifurcations. However, for the more general system (1.3), its dynamics, including stability, both local and global, and Hopf bifurcation, become more complicated and difficult and this paper is devoted to such effort. We first consider the local stability of the positive equilibrium and the existence of Hopf bifurcation of the system. Following the stardand argument (see, for example, Cooke and Grossman [1], Cooke and van den Driessche [2] and Kuang [25]), when the delays in (1.3) satisfy either (i) $\tau_{11} > 0, \tau_{12} = \tau_{21} = 0$, or (ii) $\tau_{11} = 0$, $\tau_{12} + \tau_{21} > 0$, or (iii) $\tau_{11} = 0, \tau_{12} + \tau_{21} > 0$ or (iv) $\tau_{11} = \tau_{12} = \tau_{21} > 0$, we show that, for small delays, the positive equilibrium of (1.3) is locally asymptotically stable and for large delays, Hopf bifurcations occur. As we know, such analysis can be very complicated and difficult when the systems involve more than two different delays (see Freedman and Rao [7] and Kuang [25]). To deal with the stability problem of (1.3), we then introduce the concepts of degenerate Lyapunov functionals. By constructing suitable degenerate Lyapunov functionals, we are able to obtain explicit conditions on the both local and global asymptotical stability of the positive equilibrium N^* of (1.3). As a corollary, we show that for *'small delays'* the positive equilibium of (1.3) remains globally asymptotically stable, which gives an affirmative answer to the problem on the system (1.3) mentioned early. Moreover, our conditions lead to some explicit estimate for the delays.

This paper is organised as follows. We first present a result on the local stability of the positive equilibrium and the Hopf bifurcation in section 2 and include the detailed proof of this result in the appendix. In section 3, we introduce the concepts of degenerate Lyapunov functionals and state our main results of this paper on the stability, both local and global, and we then give the detailed proofs in the following sections. The paper is ended with some remarks.

2. Local stability and Hopf bifurcation

For convenience, we resclar the variables in (1.3) and consider the following system

$$\begin{aligned}\dot{x}(t) &= x(t)[r_1 - x(t-\tau_1) - ay(t-\tau_2)]\\ \dot{y}(t) &= y(t)[-r_2 + bx(t-\tau_3)].\end{aligned} \tag{2.1}$$

Throughout this paper we assume that r_1, r_2, a and b are positive constants and $\tau_i\,(i = 1,2,3)$ are nonnegative satisfying

$$r_1 b > r_2. \tag{2.2}$$

It is easy to see that, under (2.2), the system (2.1) has a unique positive equilibrium (x^*, y^*) with

$$x^* = \frac{r_2}{b}, \qquad y^* = \frac{r_1 b - r_2}{ab}. \tag{2.3}$$

We shall assume that the system (2.1) is supplemented by initial conditions of the form

$$x(s) = \phi(s) \geq 0, \quad y(s) = \psi(s) \geq 0 \quad \text{for } s \in [-\tau, 0], \quad \phi(0) > 0, \quad \psi(0) > 0 \tag{2.4}$$

with $\tau = \max\{\tau_1, \tau_2, \tau_3\}$ and $\phi, \psi \in C([-\tau, 0], R_+)$. It is then found that the solutions of (2.1) and (2.4) are positive for all $t > 0$.

Let $(x(t), y(t))$ be any positive solution of (2.1) and denote

$$u(t) = ln\frac{x(t)}{x^*}, \qquad v(t) = ln\frac{y(t)}{y^*} \tag{2.5}$$

and thus

$$x(t) = x^* \exp[u(t)], \qquad y(t) = y^* \exp[v(t)]. \tag{2.6}$$

It then follows from (2.1) and (2.5) that

$$\begin{aligned}\dot{u}(t) &= -x^*[e^{u(t-\tau_1)} - 1] - ay^*[e^{v(t-\tau_2)} - 1]\\ \dot{v}(t) &= bx^*[e^{u(t-\tau_3)} - 1].\end{aligned} \tag{2.7}$$

Therefore, the linearised system of (2.1) with respect to (x^*, y^*) is governed by

$$\begin{aligned}\dot{u}(t) &= -x^* u(t-\tau_1) - ay^* v(t-\tau_2)\\ \dot{v}(t) &= bx^* u(t-\tau_3).\end{aligned} \tag{2.8}$$

Correspondingly, the characteristic equation is

$$D(\lambda) = \lambda^2 + A\lambda e^{-\lambda\tau_1} + Be^{-(\tau_2+\tau_3)\lambda} = 0, \tag{2.9}$$

where $A = x^*, B = x^* y^* ab$.

Theorem 2.1. *Let τ_o be defined in the following way:*

1. *If* $\tau_1 = \tau > 0$ *and* $\tau_2 = \tau_3 = 0$, *then*

$$\tau_o = \frac{\pi}{\sqrt{2}\sqrt{A^2 + 2B + A\sqrt{A^2+4B}}};$$

2. *If* $\tau_1 = 0$ *and* $\tau_2 + \tau_3 = \tau > 0$, *then*

$$\tau_o = \frac{\theta}{\omega}$$

with $\theta \in (0, \pi/2]$ *and* $\omega > 0$ *satisfying* $\sin\theta = A\omega/B$ *and* $\omega^2 = \frac{1}{2}[-A^2 + \sqrt{A^4 + 4B^2}]$;

3. *If* $\tau_1 = \tau > 0$, $\tau_2 + \tau_3 = \tau > 0$, *then*

$$\tau_o = \frac{\theta}{\omega}$$

with $\theta \in (0, \pi/2]$ *and* $\omega > 0$ *satisfying* $\tan\theta = A\omega/B$ *and* $\omega^2 = \frac{1}{2}[A^2 + \sqrt{A^4 + 4B^2}]$;

4. *If* $\tau_1 = \tau_2 = \tau_3 = \tau > 0$, *then*

$$\tau_o = \frac{1}{\sqrt{B}} \sin^{-1} \frac{A}{2\sqrt{B}}$$

for $A^2 < 4B$ *and*

$$\tau_o = \frac{\pi}{2\omega}$$

for $A^2 > 4B$ *with* $\omega = \frac{1}{2}[A - \sqrt{A^2 - 4B}]$.

Then, correspondingly, the positive equilibrium (x^*, y^*) *of (2.1) is locally asymptotically stable when* $\tau < \tau_0$ *and is unstable when* $\tau > \tau_0$. *Moreover, the system (2.1) has a Hopf bifurcation of periodic solution from the positive equilibrium* (x^*, y^*) *near* τ_0.

By Hopf's bifurcation theorem, a family of periodic solutions exists for values of τ near τ_o. Using the algorithm in Hassard, Kazarinoff and Wan [17], one can determine the stability of the bifurcating periodic solutions. For the system (2.1) with $\tau_1 = \tau_2 = 0$ and $\tau_3 = \tau > 0$, the global existence and stability of the Hopf bifurcation had been discussed by Leung [27] and Huang et al. [21]. Under the conditions $B > A^2$ and $\tau > \frac{\pi}{2}\frac{A}{B}$, Leung [27] proved that the system has a nontrivial positive periodic solution with least period greater than 2τ. In Huang et al. [21], they treated b, instead of the delay τ, as a parameter and proved that the system has a Hopf bifurcation of periodic solution from the positive equilibrium near some $b_o > 0$ and the direction of the stability of the Hopf bifurcation was also established. For other related discussion, we refer to Farkas [6] and Stepan [32]. The detailed proof of Theorem 2.1 can be found in the appendix.

3. Degenerate Lyapunov functionals and main results

It is well known that Lyapunov function(al)s method is often one of the most powerful and effective tool to establish both local (when the systems involve more than two delays, see, for example, Gopalsamy [10]) and global stability of delay differential equation

$$\dot{x}(t) = f(t, x_t), \qquad x_{t_o} = \phi(s), \tag{3.1}$$

where $f : R \times C \to R^n$ is completely continuous and $f(t, 0) = 0$. Following the notations and definition in Hale and Verduyn Lunel [16], a functional $V : R \times C \to R$ is called a Lyapunov functional for (3.1) if V satisfies

(i) $u(|\phi(0)|) \le V(t, \phi) \le v(\|\phi\|)$;

(ii) $\dot{V}(t, \phi)|_{(3.1)} \le -w(|\phi(0)|)$,

where $u(s), v(s), w(s) : R_+ \to R_+$ are continuous and nondecreasing, $u(s) > 0, v(s) > 0$ for $s > 0$ and $u(0) = v(0) = w(0) = 0$. We know (see Theorem 5.2.1, Hale and Verduyn Lunel [16]) that the existence of such Lyapunov functional implies the uniform stability of $x = 0$. If, in addition, $u(s) \to \infty$ as $s \to \infty$, then solutions of (3.1) are uniformly bounded. On the other hand, if, in addition to (i) and (ii), $w(s) > 0$ for $s > 0$, then $x = 0$ is uniformly asymptotically stable.

The concept of degenerate Lyapunov functionals was first introduced by Gopalsamy [9] when he considered the asymptotical stability of the trivial solution of the following nonautonomous linear delay system

$$\dot{x}_i(t) + \sum_{j=1}^{2} a_{ij}(t) x_j(t - \tau_{ij}) = 0, \qquad i = 1, 2. \tag{3.2}$$

As pointed in [9], it is usually difficult to find or construct Lyapunov function(al)s with negative definite rates of change along the solutions of the systems considered. Also, the approach of a combination of an invariance principle and Razumikhin-type arguments (see, Yoneyama [34, 35], Haddock and Terjeki [13], Yorke [36], Grossman and Yorke [12], Grossman [11], Halanay and Yorke [15], Kato [22, 23] and Razumikhin [30]) appears quite difficult in applications to equations such as Lotka-Volterra system. To overcome these difficult, Gopalsamy [9] used a functional $V(x_1, x_2) = (V_1 + V_2)(x_1, x_2)$ with

$$V_1(x_1, x_2) = \sum_{i=1}^{2} \left[x_i(t) + \sum_{j=1}^{2} \int_{t-\tau_{ij}}^{t} a_{ij}(s + \tau_{ij}) x_j(s) ds \right]^2 \tag{3.3}$$

and some $V_2(x_1, x_2) \ge 0$. With the help of such functional and Barbalat's lemma (see Gopalsamy [10]), he was able to derive the asymptotic stability of (3.2). As an application, he obtained some sufficient conditions for the global asymptotic stability of the positive equilibrium of the competition delay system

$$\dot{N}_i(t) = N_1(t) \left[r_i - \sum_{j=1}^{2} b_{ij} N_j(t - \tau_{ij}) \right], \qquad i = 1, 2. \tag{3.4}$$

(We refer to He [18] and He and Gopalsamy [20] for the further generalization on the system (3.4)). One can see that the functional defined by (3.3) does not satisfy the condition (i) of the Lyapunov functional defined above and this type of functionals was called a "degenerate" Lyapunov functional in Gopalsamy [9].

On the other hand, as a special case of (1.3), the global stability of the delay system

$$\begin{aligned} \dot{N}_1(t) &= N_1(t)[r_1 - aN_1(t) - bN_2(t-\tau)] \\ \dot{N}_2(t) &= N_2(t)[-r_2 + cN_1(t-\sigma)] \end{aligned} \tag{3.5}$$

was considered recently (see He [18]). By assuming (3.5) has a positive equilibrium $N^* = (N_1^*, N_2^*)$ and constructing suitable Lyapunov functional V satisfying

$$\dot{V}(N_1, N_2) \leq -\alpha[N_1(t) - N_1^*]^2 \tag{3.6}$$

for all large t, small delays τ and σ and some positive constant α, we were able to derive the global attractivity of N^*. Noting that there is no negative term related to $(N_2 - N_2^*)^2$ in (3.6) and hence this functional does not fit the condition (ii) of the Lyapunov functional defined above.

Motivated by the above observation, and also for convenience, we introduce the following definition on the degenerate Lyapunov functionals.

Definition 3.1. *A functional $V : R \times C \to R$ is called a weakly degenerate Lyapunov function for (3.1) if for all large t (say $t \geq T$ for some $T > 0$)*

(a) there exists a continuous and nodecreasing function $v(s) : R_+ \to R_+$ with $v(0) = 0$ and $v(s) > 0$ for $s > 0$ such that $0 \leq V(t, \phi) \leq v(\|\phi\|)$;

(b) there exists a continuous and nodecreasing function $w(s) : R_+ \to R_+$ with $w(0) = 0$ and $w(s) > 0$ for $s > 0$ such that $\dot{V}(t, \phi) \leq -w(|\phi(0)|)$.

If, instead of (b),

(b1) there exists a continuous and nodecreasing function $w(s) : R_+^m \to R_+$ with $m < n$, $w(0) = 0$ and $w(s_1, \cdots, s_m) > 0$ for $(s_1, \cdots, s_m) \neq 0$ such that $\dot{V}(t, \phi) \leq -w(\phi_1(0), \cdots, \phi_m(0))$,

then such functional V is called a strongly degenerate Lyapunov functional for (3.1).

Existence of degenerate Lyapunov functionals implies that $w(|\phi(0)|) \in L_1[T, \infty)$ in the first case and $w(\phi_1(0), \cdots, \phi_m(0)) \in L_1[T, \infty)$ in the second case. Then, with the help of the functionals constructed and the equations themselves, it is possible to examinate the boundedness and uniform continuity of the solutions. Once this is confirmed, one can apply Barbalat's lemmas (Lemmas 1.2.2, 1.2.3, Gopalsamy [10]) to conclude the desired asymptotic behaviour. However, as we may have realized, the difficulty is how to construct the required degenerate Lyapunov functionals. Quite often, the Lyapunov functions of the corresponding ODEs are very useful (as we will see in our proofs later). In this paper, to study the stability of the positive equilibrium of (2.1), we propose an effective technique to construct the required degenerate Lyapunov functionals. The construction will be carried on step by step so that one can see the basic idea and method to follow.

We now state the main results of this paper. Our first result is on the local stability of the positive equilibrium of (2.1).

Theorem 3.1. *Assume that $\tau_i \geq 0\,(i = 1, 2, 3)$. Then the positive equilibrium (x^*, y^*) of (2.1) is locally asymptotically stable provided $\Delta < 1$ with*

$$\Delta = x^*\tau_1 + aby^*[\tau_2 + \tau_3 + x^*(\tau_1/2 + \tau_2 + \tau_3)\tau_1]. \tag{3.7}$$

The condition $\Delta < 1$ of Theorem 3.1 gives an estimation for the smallness of the delays. Numerical calculations show that, in some cases, this estimation is very close to τ_o defined by Theorem 2.1. For example, consider the case $\tau_1 = \tau > 0$ and $\tau_2 = \tau_3 = 0$. Let $A = 1, B = 1$, then $\tau_o = 0.97$, while $\Delta < 1$ is satisfied when $\tau < \tilde{\tau} = 0.73$. Let $A = 1, B = 2$, then $\tau_o = 0.705$ and $\tilde{\tau} = 0.618$. If $A = 1, B = 100$, then $\tau_o = 0.15$ and $\tilde{\tau} = 0.13$, while if $A = 8, B = 1$, then $\tau_o = 0.19$ and $\tilde{\tau} = 0.12$.

To derive the global stability, we need an explicit estimate for the upper bound of the positiove solutions of (2.1) and this is given by the following lemma.

Lemma 3.1. *Let $(x(t), y(t))$ be any positive solution of (2.1). Then*

$$\limsup_{t\to\infty}(x(t), y(t)) \leq (M_1, M_2), \tag{3.8}$$

where

$$M_1 = r_1 e^{r_1\tau_1}, \qquad M_2 = \frac{r_1 b - r_2 e^{-r_1\tau_1}}{ab} e^{\delta(\tau_2+\tau_3)} \tag{3.9}$$

with $\delta = bM_1 - r_2 (> 0)$.

Theorem 3.2. *Assume that $\tau_i \geq 0\,(i = 1, 2, 3)$ and*

$$\bar{\Delta} := (M_1 + aM_2)\tau_1 + abM_2\big[\tau_2 + (1 + x^*\tau_1)\tau_3 + M_1\tau_1(\tau_1/2 + \tau_2)\big] < 1$$

with M_1 and M_2 defined by (3.9). Then the positive equilibrium (x^*, y^*) of (2.1) is globally asymptotically stable.

Remark. Noting that the conditions in Theorems 3.1 and 3.2 are delay-dependent and this characterizes the stability of such pure-delay-type systems, while for the no-pure-delay-type systems, such conditions are often delay-independent (see He and Gopalsamy [20], Kuang [25] and Kuang and Smith [26]). Also, Theorems 3.1 and 3.2 indicate that the *'small delays'* do not change both the local and global asymptotical stabilities of the positive equilibrium (x^*, y^*) of (2.1). This gives an affermative answer to the open problem mentioned early. The conditions also give some explicit estimate on the smallness of the delays, which make them easy to be used in practice.

4. Proof of Theorem 3.1

To prove Theorem 3.1, it is sufficient to show that the zero solution of (2.8) is asymptotically stable. Although our construction of required degenerate Lyapunov functionals and subsequent calculations are lengthy, we have some underline ideas to follow. To see this, we first look at some simple facts which come from the case when $\tau_i = 0\,(i = 1, 2, 3)$ in (2.8), that is, the ODEs

$$\dot{u} = -x^*u - ay^*v, \qquad \dot{v} = bx^*u. \tag{4.1}$$

If we define $U_1(u) = u^2/2$ and $U_2(v) = v^2/2$, then along the solutions of (4.1),

$$\dot{U}_1 = -x^*u^2 - ay^*uv, \qquad \dot{U}_2 = bx^*uv. \tag{4.2}$$

Let $U(u, v) = U_1 + ay^*U_2/(bx^*)$. Then

$$\dot{U} = -x^*u^2, \tag{4.3}$$

which is negatively semi-definite. One can use (4.3) and (4.1) to show that the trivial solution of (4.1) is asymptotically stable. In the following proof, we try to construct functionals such that, for the system (2.8), they have the same properties as U_1, U_2 and U in (4.2) and (4.3).

Proof of Theorem 3.1. Let $w(t) = (u(t), v(t))$ and define

$$V_{11}(w(t)) = \frac{1}{2}u^2(t). \tag{4.4}$$

Then, along the solution of (2.8) and for $t \geq T_1 \geq \tau$, one has

$$\begin{aligned}
\dot{V}_{11}(w(t)) &= u(t)[-x^*u(t-\tau_1) - ay^*v(t-\tau_2)] \\
&= -u(t)[x^*u(t) + ay^*v(t)] \\
&\quad + u(t)\left(x^* \int_{t-\tau_1}^{t} \dot{u}(s)\,ds + ay^* \int_{t-\tau_2}^{t} \dot{v}(s)\,ds\right) \\
&= -x^*u^2(t) - ay^*u(t)v(t) \\
&\quad + x^*u(t) \int_{t-\tau_1}^{t} \big[-x^*u(s-\tau_1) - ay^*v(s-\tau_2)\big]\,ds \\
&\quad + ay^*u(t) \int_{t-\tau_2}^{t} bx^*u(s-\tau_3)\,ds \\
&= -x^*u^2(t) - ay^*u(t)v(t) + L_1(w(t)) + L_2(w(t)),
\end{aligned} \tag{4.5}$$

where

$$L_1(w(t)) = -(x^*)^2 u(t) \int_{t-\tau_1}^{t} u(s-\tau_1)\,ds + abx^*y^*u(t) \int_{t-\tau_2}^{t} u(s-\tau_3)\,ds$$

and

$$L_2(w(t)) = -ax^*y^*u(t) \int_{t-\tau_1}^{t} v(s-\tau_2)\,ds.$$

The existence of a functional which has the property of U_1 in (4.2) follows from two steps:

Step 1. There exists a nonnegative functional L_{11} such that $L_1 + \dot{L}_{11} \le A_1 u^2$ for some constant $A_1 > 0$ and all large t.

Using the inequality $ab \le (a^2 + b^2)/2$, we have the following estimate on L_1:

$$\begin{aligned} L_1(w(t)) \quad &\le \frac{1}{2}(x^*)^2 \int_{t-\tau_1}^{t} [u^2(t) + u^2(s-\tau_1)]\,ds + \frac{1}{2}abx^*y^* \int_{t-\tau_2}^{t} [u^2(t) + u^2(s-\tau_3)]\,ds \\ &= \frac{1}{2}(x^*)^2 \left[\tau_1 u^2(t) + \int_{t-\tau_1}^{t} u^2(s-\tau_1)\,ds\right] \\ &\qquad + \frac{1}{2}abx^*y^* \left[\tau_2 u^2(t) + \int_{t-\tau_2}^{t} u^2(s-\tau_3)\,ds\right]. \end{aligned} \tag{4.6}$$

We now define

$$\begin{aligned} L_{11}(w(t)) \quad &= \frac{1}{2}(x^*)^2 \left[\int_{t-\tau_1}^{t}\int_{p}^{t} u^2(s-\tau_1)\,ds\,dp + \tau_1 \int_{t-\tau_1}^{t} u^2(s)\,ds\right] \\ &\qquad + \frac{1}{2}abx^*y^* \left[\int_{t-\tau_2}^{t}\int_{p}^{t} u^2(s-\tau_3)\,ds\,dp + \tau_2 \int_{t-\tau_3}^{t} u^2(s)\,ds\right]. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), it is easy to varify that

$$L_1(w(t)) + \dot{L}_{11}(w(t)) \le x^*[x^*\tau_1 + aby^*\tau_2]u^2(t) \qquad \text{for} \qquad t \ge T_1 \tag{4.8}$$

and the proof of step 1 is completed.

Step 2. There exists a functional $\bar{L}_2$ such that $L_2 + \dot{\bar{L}}_2 \le -A_2 u(t)v(t) + A_3 u^2(t)$ for some positive constants A_2 and A_3 and all large t.

From the second equation of (2.8) and for $t \geq T_2 \geq T_1 + \tau$,

$$\begin{aligned} L_2(w(t)) &= ax^*y^*u(t)\int_{t-\tau_1}^{t}[-v(t)+(v(t)-v(s-\tau_2))]\,ds \\ &= -ax^*y^*\tau_1 u(t)v(t) + ax^*y^*u(t)\int_{t-\tau_1}^{t}\int_{s-\tau_2}^{t}\dot{v}(p)\,dp\,ds \\ &= -ax^*y^*\tau_1 u(t)v(t) + ab(x^*)^2y^*u(t)\int_{t-\tau_1}^{t}\int_{s-\tau_2}^{t}u(p-\tau_3)\,dp\,ds \\ &\leq -ax^*y^*\tau_1 u(t)v(t) + \frac{1}{2}ab(x^*)^2y^*\int_{t-\tau_1}^{t}\int_{s-\tau_2}^{t}[u^2(t)+u^2(p-\tau_3)]\,dp\,ds \\ &= -ax^*y^*\tau_1 u(t)v(t) + \frac{1}{2}ab(x^*)^2y^*\Big[\tau_1(\tau_1/2+\tau_2)u^2(t) \\ &\qquad + \int_{t-\tau_1}^{t}\int_{s-\tau_2}^{t}u^2(p-\tau_3)\,dp\,ds\Big]. \end{aligned} \tag{4.9}$$

Let

$$L_{21}(w(t)) = \frac{1}{2}ab(x^*)^2y^*\int_{t-\tau_1}^{t}\int_{q}^{t}\int_{s-\tau_2}^{t}u^2(p-\tau_3)\,dp\,ds\,dq. \tag{4.10}$$

It turns out from (4.9) and (4.10) that, for $t \geq T_2$,

$$\begin{aligned} L_2(w(t)) + \dot{L}_{21}(w(t)) &\leq -ax^*y^*\tau_1 u(t)v(t) + \frac{1}{2}ab(x^*)^2y^*\Big[\tau_1(\tau_1/2+\tau_2)u^2(t) \\ &\qquad + \int_{t-\tau_1}^{t}\int_{t-\tau_2}^{t}u^2(p-\tau_3)\,dp\,ds + \int_{t-\tau_1}^{t}\int_{q}^{t}ds\,dq u^2(t-\tau_3)\Big] \\ &= -ax^*y^*\tau_1 u(t)v(t) + \frac{1}{2}ab(x^*)^2y^*\Big[\tau_1(\tau_1/2+\tau_2)u^2(t) \\ &\qquad + \tau_1\int_{t-\tau_2}^{t}u^2(p-\tau_3)\,dp + \frac{1}{2}\tau_1^2u^2(t-\tau_3)\Big]. \end{aligned} \tag{4.11}$$

Denote

$$\begin{aligned} L_{22}(w(t)) = {} & \frac{1}{2}ab(x^*)^2y^*\Big[\tau_1\int_{t-\tau_2}^{t}\int_{s}^{t}u^2(p-\tau_3)\,dp\,ds \\ & + \tau_1\tau_2\int_{t-\tau_3}^{t}u^2(s)\,ds + \frac{1}{2}\tau_1^2\int_{t-\tau_3}^{t}u^2(s)\,ds\Big]. \end{aligned} \tag{4.12}$$

Then one can see from (4.9), (4.11) and (4.12) that, for $t \geq T_2$,

$$(L_2 + \dot{L}_{21} + \dot{L}_{22})(w(t)) \leq -ax^*y^*\tau_1 u(t)v(t) + ab(x^*)^2y^*\tau_1[\tau_1/2+\tau_2]u^2(t) \tag{4.13}$$

and this leads to the proof of step 2.

Based on the above steps, for $t \geq T_2$, we define

$$V_1(w(t)) = (V_{11} + L_{11} + L_{21} + L_{22})(w(t)). \tag{4.14}$$

Then (4.8) and (4.13) imply that, for $t \geq T_2$,

$$\begin{aligned} \dot{V}_1(w(t)) &\leq -x^* u^2(t) - ay^* u(t)v(t) + x^*[x^*\tau_1 + aby^*\tau_2]u^2(t) \\ &\quad -ax^*y^*\tau_1 u(t)v(t) + ab(x^*)^2 y^*\tau_1[\tau_1/2 + \tau_2]u^2(t) \\ &= -x^* u^2(t) - ay^*[1 + x^*\tau_1]u(t)v(t) \\ &\quad +x^*[x^*\tau_1 + aby^*\tau_2 + abx^*y^*\tau_1(\tau_1/2 + \tau_2)]u^2(t). \end{aligned} \tag{4.15}$$

On the other hand, to construct a functional having the samilar property as U_2 in (4.2), we rewrite the second equation of (2.8) as

$$\frac{d}{dt}\left[v(t) + bx^* \int_{t-\tau_3}^{t} u(s)\,ds\right] = bx^* u(t). \tag{4.16}$$

Define

$$V_{21}(w(t)) = \frac{1}{2}\left[v(t) + bx^* \int_{t-\tau_3}^{t} u(s)\,ds\right]^2. \tag{4.17}$$

Then, from (4.16) and (4.17),

$$\begin{aligned} \dot{V}_{21}(w(t)) &= bx^* u(t)\left[v(t) + bx^* \int_{t-\tau_3}^{t} u(s)\,ds\right] \\ &= bx^* u(t)v(t) + (bx^*)^2 u(t) \int_{t-\tau_3}^{t} u(s)\,ds \\ &\leq bx^* u(t)v(t) + \frac{1}{2}(bx^*)^2 \int_{t-\tau_3}^{t} [u^2(t) + u^2(s)]\,ds. \end{aligned} \tag{4.18}$$

Let

$$V_{22}(w(t)) = \frac{1}{2}(bx^*)^2 \int_{t-\tau_3}^{t} \int_{p}^{t} u^2(s)\,ds\,dp \tag{4.19}$$

and

$$V_2(w(t)) = (V_{21} + V_{22})(w(t)). \tag{4.20}$$

We then have from (4.18)–(4.20) that

$$\dot{V}_2(w(t)) \leq bx^* u(t)v(t) + (bx^*)^2 \tau_3 u^2(t). \tag{4.21}$$

Now, for $t \geq T_2$, we define a functional

$$V(w(t)) = (V_1 + \alpha V_2)(w(t))$$

with some positive constant α (to be determined). One can see that $V(w(t)) \geq 0$ and, by (4.15) and (4.21), it satisfies

$$\begin{aligned}\dot{V}(w(t)) \ &\leq -x^*u^2(t) - ay^*[1 + x^*\tau_1]u(t)v(t) \\ &+x^*[x^*\tau_1 + aby^*\tau_2 + abx^*y^*\tau_1(\tau_1/2 + \tau_2)]u^2(t) \\ &+\alpha bx^*u(t)v(t) + \alpha(bx^*)^2\tau_3u^2(t). \end{aligned} \tag{4.22}$$

Choose $\alpha = ay^*(1 + x^*\tau_1)/(bx^*)$, then (4.22) leads to

$$\dot{V} \leq -x^*[1 - \Delta]u^2(t) \tag{4.23}$$

with Δ defined by (3.7). Obviousely, under the assuimption that $\Delta < 1$, V is a strongly degenerate Lyapunov functional. Integrating (4.23) from T_2 to $t \geq T_2$, we have

$$V(w(t)) + x^*[1 - \Delta] \int_{T_2}^{t} u^2(s)ds \leq V(w(T_2)), \tag{4.24}$$

which implies $u^2(t) \in L_1[T_2, \infty)$. Also, by (4.24), (4.4) and (4.17), $u(t)$ and hence $v(t)$ are bounded. This, together with (2.8), implies that both $u(t)$, $v(t)$ and their derivatives are uniformly continuous. Applying Barbalat's lemmas (lemmas 1.2.2 and 1.2.3, Gopalsamy [10]), we conclude that $(u(t), \dot{u}(t)) \to 0$ as $t \to \infty$. Then from the first equation of (2.8), this leads to $v(t) \to 0$ as $t \to \infty$. Therefore, $(u(t), v(t)) \to 0$ as $t \to \infty$.

Note that the characteristic equation of (2.8) is governed by

$$f(\lambda) = \lambda^2 + x^*e^{-\lambda\tau_1}\lambda + abx^*y^*e^{-(\tau_2+\tau_3)\lambda} = 0.$$

Since every solution of (2.8) approaches zero as $t \to \infty$, $f(\lambda)$ has no roots with $Re(\lambda) \geq 0$. Therefore, all roots of $f(\lambda)$ have negative real parts and (x^*, y^*) is thus locally asymptotically stable. This completes the proof.

5. Proof of Lemma

Proof. By the positivity of the solution $(x(t), y(t))$ and (2.1), we have $\dot{x}(t) \leq r_1x(t)$. Integrating this inequality from $t - \tau_1$ to t, one obtain that $ln[x(t)/x(t - \tau_1)] \leq r_1\tau_1$ and hence

$$x(t - \tau_1) \geq e^{-r_1\tau_1}x(t). \tag{5.1}$$

This, together with (2.1), implies

$$\dot{x}(t) \leq x(t)[r_1 - e^{-r_1\tau_1}x(t)], \tag{5.2}$$

from which, one can conclude that

$$\limsup_{t\to\infty} x(t) \leq M_1 = r_1e^{r_1\tau_1}. \tag{5.3}$$

We now try to get the upper bound for $y(t)$. Since (5.3), for $\epsilon_1 > 0$ (sufficiently small), there exists a $T_1 > 0$ such that

$$x(t) \leq M_1 + \epsilon_1 \qquad \text{for} \qquad t \geq T_1. \tag{5.4}$$

Then, for $t \geq T_2 = T_1 + \tau_3$, we have from (2.1) and (5.4) that

$$\dot{y}(t) \leq y(t)[-r_2 + b(M_1 + \epsilon_1)].$$

Let $\delta(\epsilon_1) = -r_2 + b(M_1 + \epsilon_1)$, then it follows from $r_1 b > r_2$ that $\delta(\epsilon_1) = -r_2 + b[r_1 e^{r_1\tau_1} + \epsilon_1] > 0$. So,

$$y(t_2) \leq y(t_1) e^{\delta(\epsilon_1)(t_2 - t_1)} \qquad \text{for} \qquad t_2 \geq t_1 \geq T_2 \tag{5.5}$$

and thus

$$t_2 - t_1 \geq \frac{1}{\delta(\epsilon_1)} \ell n \frac{y(t_2)}{y(t_1)} \qquad \text{for} \qquad t_2 \geq t_1 \geq T_2. \tag{5.6}$$

Now, for $t \geq T_3 = T_2 + \tau$, it follows from (2.1), (5.1) and (5.5) that

$$\dot{x}(t) \leq x(t)\Big[r_1 - e^{-r_1\tau_1} x(t) - a e^{-\delta(\epsilon_1)\tau_2} y(t)\Big]. \tag{5.7}$$

Let $\epsilon_2 > 0$ be sufficiently small such that $y_o = r_2/b - \epsilon_2 > 0$. Denote

$$M = \frac{1}{ab}[r_1 b e^{r_1\tau_1} - (r_2 - \epsilon_2)] e^{\delta(\epsilon_1)\tau_2 - r_1\tau_1}.$$

Then $M > 0$. We claim that

$$\limsup_{t\to\infty} y(t) \leq M_2(\epsilon_1, \epsilon_2) = M e^{\delta(\epsilon_1)\tau_3}. \tag{5.8}$$

To prove this, there are two cases to be considered:

Case (i). $y(t)$ does not oscillate about M eventually; that is, there exists $T_4 \geq T_3$ such that either $y(t) \leq M$ or $y(t) > M$ for $t \geq T_4$. If $y(t) \leq M$ eventually, then (5.8) follows obviousely. If $y(t) > M$ for $t \geq T_4$, then from (5.7),

$$\dot{x}(t) \leq x(t)[r_1 - a e^{-\delta(\epsilon_1)\tau_2} M - e^{-r_1\tau_1} x(t)]$$

and which implies

$$x(t) \leq [r_1 - a e^{-\delta(\epsilon_1)\tau_2} M] e^{r_1\tau_1} = y_o$$

Thus, for $t \geq T_4 + \tau$,

$$\dot{y}(t) \leq y(t)[-r_2 + b y_o] = -b\epsilon_2 y(t),$$

which implies $y(t) \to 0$ as $t \to \infty$ and leads to a contradiction. This prove that if $y(t)$ is not oscillatory about M eventually then $y(t)$ is bounded above by M eventually.

Case (ii). $y(t)$ oscillates about M eventually. Note that $M < M_2(\epsilon) = M_2(\epsilon_1, \epsilon_2)$. In this case, $y(t)$ must be bounded above eventually by $M_2(\epsilon)$. Otherwise, for $t \geq T_3$, there must exist s_n and t_n satisfying $T_3 < s_n < t_n$ such that

$$y(s_n) = M, \quad y(t_n) = M_2(\epsilon), \quad \dot{y}(t_n) \geq 0 \quad \text{and} \qquad y(t) \geq M \quad \text{for} \quad t \in [s_n, t_n].$$

From (5.6),

$$t_n - s_n \geq \frac{1}{\delta(\epsilon_1)} \ell n \frac{y(t_n)}{y(s_n)} = \frac{1}{\delta(\epsilon_1)} \ell n \frac{M e^{\delta_1(\epsilon_1)\tau_3}}{M} = \tau_3,$$

that is, $t_n - \tau_3 > s_n$. Note that, for $t \in [s_n, t_n]$, $y(t) \geq M$. Hence, as shown in case (i), $x(t) \leq y_o$ for $t \in [s_n, t_n]$. This, together with (2.1), implies

$$\dot{y}(t_n) \leq y(t_n)[-r_2 + bx(t_n - \tau_3)] \leq -\epsilon_2 y(t_n) < 0$$

This contradicts $\dot{y}(t_n) \geq 0$. Thus $y(t)$ is bounded above eventually by $M_2(\epsilon)$. The conclusion thus follows by letting $\epsilon_i \to 0 (i = 1, 2)$. This completes the proof.

6. Proof of Theorem 3.2

From (2.6), we know that it is sufficient to show that the trivial solution of (2.7) is globally asymptotically stable. As in section 4, to guid the following construction of the required degenerate Lyapunov functionals, we look at some simple facts corresponding to the system (2.7) with no delays, that is, the ODEs

$$\begin{aligned} \dot{u} &= -x^*[e^u - 1] - ay^*[e^v - 1] \\ \dot{v} &= bx^*[e^u - 1]. \end{aligned} \tag{6.1}$$

Let $P_1 = \int_0^u [e^s - 1] ds$ and $P_2 = \int_0^v [e^s - 1] ds$. Then $P_i(s) > 0$ for $s > 0$ and $P_i(0) = 0$ for $i = 1, 2$. Along the solutions of (6.1), we have

$$\dot{P}_1(u, v) = -x^*[e^u - 1]^2 - ay^*[e^u - 1][e^v - 1], \qquad \dot{P}_2(u, v) = bx^*[e^u - 1][e^v - 1]. \tag{6.2}$$

Similarly, if we take $P(u, v) = P_1 + bx^* P_2/(ay^*)$, then

$$\dot{P}(u, v) = -x^*[e^u - 1]^2, \tag{6.3}$$

from which, one can derive the global asymptotical stability of the trivial solution of (6.1). With those facts in mind, we now give the detailed proof of Theorem 3.2.

Proof of Theorem 3.2. Consider the system (2.7) and let

$$V_{11}(u(t)) = \int_0^{u(t)} [e^s - 1]\, ds. \tag{6.4}$$

Then it follows from (6.4) and (2.7) that, for $t \geq T_1 \geq \tau$,

$$\begin{aligned} \dot{V}_{11}(u(t)) &= [e^{u(t)} - 1]\dot{u}(t) \\ &= -x^*[e^{u(t)} - 1]^2 - ay^*[e^{u(t)} - 1][e^{v(t)} - 1] \\ &\quad + [e^{u(t)} - 1]\left\{ x^* \int_{t-\tau_1}^{t} e^{u(s)}\dot{u}(s)\, ds + ay^* \int_{t-\tau_2}^{t} e^{v(s)}\dot{v}(s)\, ds \right\} \\ &= -x^*[e^{u(t)} - 1]^2 - ay^*[e^{u(t)} - 1][e^{v(t)} - 1] \\ &\quad + W_1(w(t)) + W_2(w(t)), \end{aligned} \tag{6.5}$$

where

$$W_1(w(t)) = x^*[e^{u(t)} - 1] \int_{t-\tau_1}^{t} e^{u(s)} \dot{u}(s)\, ds$$

and

$$W_2(w(t)) = ay^*[e^{u(t)} - 1] \int_{t-\tau_2}^{t} e^{v(s)} \dot{v}(s)\, ds.$$

We divide two steps on the estimation for W_1 and W_2.

Step 1. Estimation for W_2. By (2.7), for $t \geq T_1$,

$$\begin{aligned} W_2(w(t)) &= abx^*y^*[e^{u(t)} - 1] \int_{t-\tau_2}^{t} e^{v(s)} [e^{u(s-\tau_3)} - 1]\, ds \\ &\leq \frac{1}{2} abx^*y^* \int_{t-\tau_2}^{t} e^{v(s)} \Big([e^{u(t)} - 1]^2 + [e^{u(s-\tau_3)} - 1]^2 \Big)\, ds \\ &= \frac{1}{2} abx^*y^* \Big\{ \Big(\int_{t-\tau_2}^{t} e^{v(s)}\, ds \Big) [e^{u(t)} - 1]^2 \\ &\qquad + \int_{t-\tau_2}^{t} e^{v(s)} [e^{u(s-\tau_3)} - 1]^2\, ds \Big\}. \end{aligned} \tag{6.6}$$

For $t \geq T_1$, let

$$\begin{aligned} W_{21}(w(t)) = \; & \frac{1}{2} abx^*y^* \Big\{ \int_{t-\tau_2}^{t} \int_{p}^{t} e^{v(s)} [e^{u(s-\tau_3)} - 1]^2\, ds\, dp \\ & \qquad + \tau_2 \int_{t-\tau_3}^{t} e^{v(s+\tau_3)} [e^{u(s)} - 1]^2 \Big\}. \end{aligned} \tag{6.7}$$

Then, from (6.6) and (6.7),

$$(W_2 + \dot{W}_{21})(w(t)) \leq \frac{1}{2} abx^*y^* \Big\{ \Big(\int_{t-\tau_2}^{t} e^{v(s)}\, ds \Big) + \tau_2 e^{v(t+\tau_3)} \Big\} [e^{u(t)} - 1]^2 \tag{6.8}$$

for $t \geq T_1$.

Step 2. Estimation for W_1. For $t \geq T_1$, using (2.7), we have

$$\begin{aligned} W_1(w(t)) &= x^*[e^{u(t)} - 1] \int_{t-\tau_1}^{t} e^{u(s)} \Big(-x^*[e^{u(s-\tau_1)} - 1] - ay^*[e^{v(s-\tau_2)} - 1] \Big)\, ds \\ &= W_{11}(w(t)) + W_{12}(w(t)) \end{aligned} \tag{6.9}$$

with

$$W_{11}(w(t)) = -(x^*)^2 [e^{u(t)} - 1] \int_{t-\tau_1}^{t} e^{u(s)} [e^{u(s-\tau_1)} - 1]\, ds$$

and

$$W_{12}(w(t)) = -ax^*y^*[e^{u(t)} - 1] \int_{t-\tau_1}^{t} e^{u(s)} [e^{v(s-\tau_2)} - 1]\, ds.$$

We first examine W_{11}. For $t \geq T_1$,

$$
\begin{aligned}
W_{11}(w(t)) \ &\leq \frac{1}{2}(x^*)^2 \int_{t-\tau_1}^{t} e^{u(s)}\Big([e^{u(t)}-1]^2 + [e^{u(s-\tau_1)}-1]^2\Big)\, ds \\
&= \frac{1}{2}(x^*)^2 \bigg\{ \Big(\int_{t-\tau_1}^{t} e^{u(s)}\, ds \Big) [e^{u(t)}-1]^2 \\
&\qquad\qquad + \int_{t-\tau_1}^{t} e^{u(s)}[e^{u(s-\tau_1)}-1]^2\, ds \bigg\}
\end{aligned}
\tag{6.10}
$$

To get ride of the delay term in (6.10), we take

$$
\begin{aligned}
Z_1(w(t)) = \ & \frac{1}{2}(x^*)^2 \Big(\int_{t-\tau_1}^{t} \int_{p}^{t} e^{u(s)}[e^{u(s-\tau_1)}-1]^2\, ds\, dp \\
&\qquad + \tau_1 \int_{t-\tau_1}^{t} e^{u(s+\tau_1)}[e^{u(s)}-1]^2\, ds \Big).
\end{aligned}
\tag{6.11}
$$

Then we have from (6.10) and (6.11) that, for $t \geq T_1$,

$$
(W_{11} + \dot{Z}_1)(w(t) \leq \frac{1}{2}(x^*)^2 \bigg\{ \Big(\int_{t-\tau_1}^{t} e^{u(s)}\, ds \Big) + \tau_1 e^{u(t+\tau_1)} \bigg\} [e^{u(t)}-1]^2. \tag{6.12}
$$

Secondly, we examine the term W_{12}. We rewrite it in the form

$$
W_{12}(w(t)) = -ax^*y^*[e^{u(t)}-1] \int_{t-\tau_1}^{t} e^{u(s)} \Big([e^{v(t)}-1] - [e^{v(t)} - e^{v(s-\tau_2)}] \Big)\, ds.
$$

Let

$$
Z_2(w(t)) = -ax^*y^*[e^{u(t)}-1][e^{v(t)}-1] \int_{t-\tau_1}^{t} e^{u(s)}\, ds.
$$

and

$$
Z_3(w(t)) = ax^*y^*[e^{u(t)}-1] \int_{t-\tau_1}^{t} e^{u(s)}[e^{v(t)} - e^{v(s-\tau_2)}]\, ds.
$$

Then

$$
W_{12}(w(t)) = (Z_2 + Z_3)(w(t)) \qquad \text{for} \qquad t \geq T_1.
$$

For $t \geq T_1$, we have the following estimation on Z_2.

$$\begin{aligned}
Z_2(w(t)) &= -ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]\int_{t-\tau_1}^{t}\Big([e^{u(s)}-1]+1\Big)\,ds \\
&= -ax^*y^*\tau_1[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad -ax^*y^*[e^{u(t)}-1][e^{v(t)}-1]\int_{t-\tau_1}^{t}[e^{u(s)}-1]\,ds \\
&\leq -ax^*y^*\tau_1[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +\frac{1}{2}ax^*y^*|e^{v(t)}-1|\int_{t-\tau_1}^{t}\Big([e^{u(t)}-1]^2+[e^{u(s)}-1]^2\Big)\,ds \\
&= -ax^*y^*\tau_1[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +\frac{1}{2}ax^*y^*\Big\{\tau_1|e^{v(t)}-1|[e^{u(t)}-1]^2 \\
&\qquad\qquad +\Big(\int_{t-\tau_1}^{t}[e^{u(s)}-1]^2\,ds\Big)|e^{v(t)}-1|\Big\}. \qquad (6.13)
\end{aligned}$$

Similarly, to convert the delay term in (6.13) into no-delay term, we choose

$$Z_{21}(w(t)) = \frac{1}{2}ax^*y^*\int_{t-\tau_1}^{t}|e^{v(s+\tau_1)}-1|\int_{p}^{t}[e^{u(s)}-1]^2\,ds\,dp.$$

This, together with (6.13), implies that, for $t \geq T_1$,

$$\begin{aligned}
(Z_2+\dot{Z}_{21})(w(t)) &\leq -ax^*y^*\tau_1[e^{u(t)}-1][e^{v(t)}-1]+\frac{1}{2}ax^*y^*\Big(\tau_1|e^{v(t)}-1| \\
&\qquad +\int_{t-\tau_1}^{t}|e^{v(s+\tau_1)}-1|\,ds\Big)[e^{u(t)}-1]^2. \qquad (6.14)
\end{aligned}$$

To estimate Z_3, we choose $T_2 \geq T_1+\tau$. Then, for $t \geq T_2$,

$$\begin{aligned}
Z_3(w(t)) &= ax^*y^*[e^{u(t)}-1]\int_{t-\tau_1}^{t}e^{u(s)}\int_{s-\tau_2}^{t}e^{v(p)}\dot{v}(p)\,dp\,ds \\
&= ab(x^*)^2y^*[e^{u(t)}-1]\int_{t-\tau_1}^{t}e^{u(s)}\int_{s-\tau_2}^{t}e^{v(p)}[e^{u(p-\tau_3)}-1]\,dp\,ds \\
&\leq \frac{1}{2}ab(x^*)^2y^*\int_{t-\tau_1}^{t}e^{u(s)}\int_{s-\tau_2}^{t}e^{v(p)}\Big([e^{u(t)}-1]^2+[e^{u(p-\tau_3)}-1]^2\Big)\,dp\,ds \\
&= \frac{1}{2}ab(x^*)^2y^*\Big\{\Big(\int_{t-\tau_1}^{t}e^{u(s)}\int_{s-\tau_2}^{t}e^{v(p)}\,dp\,ds\Big)[e^{u(t)}-1]^2 \\
&\qquad +\int_{t-\tau_1}^{t}e^{u(s)}\int_{s-\tau_2}^{t}e^{v(p)}[e^{u(p-\tau_3)}-1]^2\Big)\,dp\,ds\Big\}. \qquad (6.15)
\end{aligned}$$

Denote

$$Z_4(w(t)) = \frac{1}{2}ab(x^*)^2y^* \int_{t-\tau_1}^{t} e^{u(s)} \int_{s-\tau_2}^{t} e^{v(p)}[e^{u(p-\tau_3)} - 1]^2\, dp\, ds$$

and define

$$Z_{41}(w(t)) = \frac{1}{2}ab(x^*)^2y^* \int_{t-\tau_1}^{t} \int_{q}^{t} e^{u(s)} \int_{s-\tau_2}^{t} e^{v(p)}[e^{u(p-\tau_3)} - 1]^2\, dp\, ds\, dq.$$

Then for $t \geq T_2$,

$$\begin{aligned}(Z_4 + \dot{Z}_{41})(w(t)) \;\; &\leq \frac{1}{2}ab(x^*)^2y^* \biggl(\int_{t-\tau_1}^{t} e^{u(t)} \int_{t-\tau_2}^{t} e^{v(p)}[e^{u(p-\tau_3)} - 1]^2\, dp\, ds \\ &\qquad + \int_{t-\tau_1}^{t} \int_{q}^{t} e^{u(s)} e^{v(t)}[e^{u(t-\tau_3)} - 1]^2\, ds\, dq \biggr) \\ &= \frac{1}{2}ab(x^*)^2y^* \biggl(A(t)[e^{u(t-\tau_3)} - 1]^2 \\ &\qquad + \tau_1 e^{u(t)} \int_{t-\tau_2}^{t} e^{v(p)}[e^{u(p-\tau_3)} - 1]^2\, dp \biggr) \end{aligned} \tag{6.16}$$

with

$$A(t) = e^{v(t)} \int_{t-\tau_1}^{t} \int_{q}^{t} e^{u(s)}\, ds\, dq.$$

To remove the delays in (6.16), we take

$$\begin{aligned} Z_{42}(w(t)) \;\; &= \frac{1}{2}ab(x^*)^2y^* \biggl\{ \int_{t-\tau_3}^{t} A(s+\tau_3)[e^{u(s)} - 1]^2\, ds \\ &\qquad + \tau_1 \int_{t-\tau_2}^{t} e^{u(s+\tau_2)} \int_{s}^{t} e^{v(p)}[e^{u(p-\tau_3)} - 1]^2\, dp\, ds \biggr\}. \end{aligned} \tag{6.17}$$

It turns out from (6.16) and (6.17) that, for $t \geq T_2$,

$$(Z_4 + \dot{Z}_{41} + \dot{Z}_{42})(w(t)) \leq \frac{1}{2}ab(x^*)^2y^* \Bigl(A(t+\tau_3)[e^{u(t} - 1]^2 + B(t)[e^{u(t-\tau_3)} - 1]^2 \Bigr),$$

where

$$B(t) = \tau_1 e^{v(t)} \int_{t-\tau_2}^{t} e^{u(s+\tau_2)}\, ds.$$

Similarly, defining

$$Z_{43}(w(t)) = \frac{1}{2}ab(x^*)^2y^* \int_{t-\tau_3}^{t} B(s+\tau_3)[e^{u(s)} - 1]^2\, ds,$$

we have

$$(Z_4 + \dot{Z}_{41} + \dot{Z}_{42} + \dot{Z}_{43})(w(t)) \leq \frac{1}{2}ab(x^*)^2y^*[A(t+\tau_3) + B(t+\tau_3)][e^{u(t)} - 1]^2$$

for $t \geq T_2$. Let

$$C(t) = \int_{t-\tau_1}^{t} e^{u(s)} \int_{s-\tau_2}^{t} e^{v(p)}\, dp\, ds.$$

Then, using (6.15),

$$(Z_3 + \dot{Z}_{41} + \dot{Z}_{42} + \dot{Z}_{43})(w(t)) \leq \frac{1}{2}ab(x^*)^2 y^* [A(t+\tau_3) + B(t+\tau_3) + C(t)][e^{u(t} - 1]^2$$

for $t \geq T_2$.

Based on the above estimation for Z_2 and Z_3, we choose

$$Z_5(w(t)) = (Z_{21} + Z_{41} + Z_{42} + Z_{43})(w(t)).$$

It follows that

$$\begin{aligned}(W_{12} + \dot{Z}_5)(w(t)) \leq\ & -ax^*y^*\tau_1[e^{u(t)} - 1][e^{v(t)} - 1] \\ & + \frac{1}{2}ax^*y^*\Big\{ |e^{v(t)} - 1|\tau_1 + \int_{t-\tau_1}^{t} |e^{v(s+\tau_1)} - 1|\, ds \\ & + bx^*[C(t) + A(t+\tau_3) + B(t+\tau_3)] \Big\} [e^{u(t)} - 1]^2 \qquad (6.18)\end{aligned}$$

for $t \geq T_2$ and this completes the estimate on W_{12}.

Combining the above analysis in steps 1 and 2, we define a nonnegative functional

$$V_1(w(t)) = (V_{11} + W_{21} + Z_1 + Z_5)(w(t)).$$

It turns out that, for $t \geq T_2$,

$$\dot{V}_1(w(t)) \leq -x^*[e^{u(t)} - 1]^2 - ay^*(1 + x^*\tau_1)[e^{u(t)} - 1][e^{v(t)} - 1] + D(w(t))[e^{u(t)} - 1]^2,$$

where

$$\begin{aligned} D(t) =\ & \frac{1}{2}x^* \Big\{ aby^* \Big[\int_{t-\tau_2}^{t} e^{(v(s)}\, ds + \tau_2 e^{v(t+\tau_3)} \Big] \\ & + x^* \Big[\int_{t-\tau_1}^{t} e^{u(s)} ds + \tau_1 e^{u(t+\tau_1)} \Big] \\ & + ay^* \Big[\tau_1 |e^{v(t)} - 1| + \int_{t-\tau_1}^{t} |e^{v(s+\tau_1)} - 1|\, ds \Big] \\ & + abx^*y^*[C(t) + A(t+\tau_3) + B(t+\tau_3)] \Big\}. \qquad (6.19)\end{aligned}$$

Note that from Lemma and (2.6),

$$x^* e^{u(t)} = x(t) \leq M_1, \qquad y^* e^{v(t)} = y(t) \leq M_2$$

and

$$y^* |e^{v(t)} - 1| \leq \max\{y(t), y^*\} \leq M_2$$

for all large t. (More preciesly, the above M_i should be replaced by $M_i+\epsilon$ for sufficiently small $\epsilon>0$ and $i=1,2$. With these replacements, corresponding to $\bar{\Delta}$ defined by Theorem 2, we denote it as $\bar{\Delta}(\epsilon)$. Obviousely, $\bar{\Delta}<1$ implies $\bar{\Delta}(\epsilon)<1$ for some small ϵ. Therefore, for simple, we keep M_i, instead of $M_i+\epsilon\,(i=1,2)$, in our analysis.) Thus, for all large t, we have the following estimations for A,B,C and D:

$$x^*y^*C(t)\le M_1M_2\int_{t-\tau_1}^{t}\int_{s-\tau_2}^{t}dp\,ds=M_1M_2\tau_1(\tau_1/2+\tau_2),$$

$$x^*y^*A(t)\le M_1M_2\int_{t-\tau_1}^{t}\int_{q}^{t}ds\,dq=M_1M_2\tau_1^2/2,$$

$$x^*y^*B(t)\le M_1M_2\tau_1\tau_2$$

and

$$D(t)\le E:=x^*\{(M_1+aM_2)\tau_1+abM_2[\tau_2+M_1\tau_1(\tau_1/2+\tau_2)]\}.$$

Therefore, for all large t, say $t\ge T\ge T_2$,

$$\dot{V}_1(w(t))\le -x^*[e^{u(t)}-1]^2-ay^*(1+x^*\tau_1)[e^{u(t)}-1][e^{v(t)}-1]+E[e^{u(t)}-1]^2.$$

On the other hand, from the second equation of (2.7),

$$\frac{d}{dt}\left[v(t)+bx^*\int_{t-\tau_3}^{t}[e^{u(s)}-1]\,ds\right]=bx^*[e^{u(t)}-1].$$

Let

$$z(w(t))=v(t)+bx^*\int_{t-\tau_3}^{t}[e^{u(s)}-1]\,ds$$

and

$$V_{21}(z(w(t)))=\int_{0}^{z(t)}[e^s-1]\,ds.$$

Then, for $t \geq T$,

$$\begin{aligned}
\dot{V}_{21}(z(w(t))) &= [e^{z(w(t))}-1]\dot{z}(w(t)) \\
&= bx^*[e^{u(t)}-1]\left[\exp\left(v(t)+bx^*\int_{t-\tau_3}^{t}[e^{u(s)}-1]\,ds\right)-1\right] \\
&= bx^*[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +bx^*[e^{u(t)}-1]\left[\exp\left(v(t)+bx^*\int_{t-\tau_3}^{t}[e^{u(s)}-1]\,ds\right)-\exp(v(t))\right] \\
&= bx^*[e^{u(t)}-1][e^{v(t)}-1]+bx^*[e^{u(t)}-1][e^{v(t+\tau_3)}-e^{v(t)}] \\
&= bx^*[e^{u(t)}-1][e^{v(t)}-1]+bx^*[e^{u(t)}-1]\int_{t}^{t+\tau_3}e^{v(s)}\dot{v}(s)\,ds \\
&= bx^*[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +(bx^*)^2[e^{u(t)}-1]\int_{t}^{t+\tau_3}e^{v(s)}[e^{u(s-\tau_3)}-1]\,ds \\
&\leq bx^*[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +\frac{1}{2}(bx^*)^2\left\{\int_{t}^{t+\tau_3}e^{v(s)}\,ds[e^{u(t)}-1]^2+\int_{t}^{t+\tau_3}e^{v(s)}[e^{u(s-\tau_3)}-1]^2\,ds\right\} \\
&\leq bx^*[e^{u(t)}-1][e^{v(t)}-1] \\
&\quad +\frac{1}{2}(bx^*)^2(M_2/y^*)\left\{\tau_3[e^{u(t)}-1]^2+\int_{t-\tau_3}^{t}[e^{u(s)}-1]^2\,ds\right\}. \qquad (6.20)
\end{aligned}$$

Let

$$V_2(w(t)) = (V_{21}+V_{22})(w(t)),$$

where

$$V_{22}(w(t)) = \frac{1}{2}(bx^*)^2(M_2/y^*)\int_{t-\tau_3}^{t}\int_{p}^{t}[e^{u(s)}-1]^2\,ds\,dp.$$

This, together with (6.20), leads to

$$\dot{V}_2(w(t)) \leq bx^*[e^{u(t)}-1][e^{v(t)}-1]+(bx^*)^2(M_2/y^*)\tau_3[e^{u(t)}-1]^2$$

for $t \geq T$.

Finally, we take

$$V(w(t)) = V_1(w(t))+\alpha V_2(w(t))$$

with $\alpha = ay^*(1+\tau_1 x^*)/(bx^*)$. Then V is a degenerate Lyapunov functional satisfying

$$\dot{V}(w(t)) \leq -x^*[1-\bar{\Delta}][e^{u(t)}-1]^2, \qquad (6.21)$$

where $\bar{\Delta}$ is defined by Theorem 3.2. Then, under the assumption $\bar{\Delta}<1$, (6.1) implies the boundedness of $u(t), z(t)$ and hence of $v(t)$. Thus $u(t)$ and $v(t)$ are uniformly continuous on $[T,\infty)$. Also, (6.1) lead to $[e^{u(t)}-1]^2 \in L_1[T,\infty)$. Applying Barbalat's lemma (Lemma 1.2.2, Gopalsamy [10]), we conclude that $[e^{u(t)}-1]^2 \to 0$ as $t\to\infty$, that is $u(t)\to 0$ as

$t \to \infty$. Using Barbalat's lemma (Lemma 1.2.3, Gopalsamy [10]), we also have $\dot{u}(t) \to 0$. This, together with the first equation of (2.7), leads to $v(t) \to 0$ as $t \to \infty$. Therefore we conclude that $(u(t), v(t)) \to 0$ as $t \to \infty$. Note that $M_1 \geq x^*$, $M_2 \geq y^*$ and hence $\bar{\Delta} < 1$ implies $\Delta < 1$. The trivial solution of (2.7) is thus locally asymptotically stable. Therefore, the positive equilibrium (x^*, y^*) of (3.5) is globally asymptotically stable. This completes the proof.

7. Remarks

It is well known that the both Lyapunov functionals and Razumikhin arguments are very powerful methods on the study of the stability of functional differential equations. As pointed by Hale and Verduyn Lunel [16], it is certainly worthwhile to discuss the advantages and disadvantages of those two different approaches. It appears that for scalar delay equations, when the limit of the equations is a point, it is easier to use Razumikhin arguments than to construct Lyapunov functionals, in particular, when the involved delayes are functions of t. This is because of the simplicity of the required function V, which can be fitted by the condition $\dot{V}(t, \phi(0)) \leq -w(|\phi(0)|)$ for $V(t+\theta, \phi(\theta)) \leq V(t, \phi(0))$ (see Theorem 5.4.1 in [16]) under some weaker conditions (we refer to the examples in Chapter 5 of [16] for more detailed discussion). However, for the non-scalar equations, we would loss, in general, the simplicity of the function V and we may have to construct the required Lyapunov function(al)s and this is often difficult. In the following, as an example, we consider the delayed autonomous Lotka-Volterra equations, from which we try to indicate that we do have some ideas to follow on the construction of the required Lyapunov functionals, although it is not easy to achieve those ideas in certain circumstances.

Consider

$$\dot{x}_i(t) = x_i(t)[r_i - \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij})], \qquad i = 1, 2, \cdots, n, \tag{7.1}$$

where $r_i, a_{ij}, \tau_{ij} (\geq 0)$ are constants for $i, j = 1, \cdots, n$. We assume that the system (7.1) has a positive equilibrium, say $x^* = (x_1^*, \cdots, x_n^*)$ and it is globally asymptotically stable for the corresponding ODE systems

$$\dot{x}_i = x_i[r_i - \sum_{j=1}^{n} a_{ij} x_j], \qquad i = 1, 2, \cdots, n. \tag{7.2}$$

Suppose there exists a Lyapunov function $V(x)$ with negative rates of change along the solutions of the system (7.2), that is $\dot{V}(x(t)) \leq -W(|x(t)|)$. Then W is either positively definite or positively semi-definite.

1. If the stability conditions for (7.2) are of diagonal dominated (see Hofbauer and Sigmand [14]), then the same Lyapunov function V for (7.2) can be used for (7.1). If, in addition, the system (7.1) is of "no-pure-delay-type", then one can use the inequality

$ab \leq (a^2+b^2)/2$ to seperate the delay and non-delay terms and then use the same technique used in this paper to convert the delay terms into non-delay terms. The stability conditions are often delay independent (see, for example, Kuang [25]). If, on the other hand, the system (7.1) is of "pure-delay-type", one can treat (7.1) as a perturbation of the ODE system (7.2) and use the same techniques to construct the required functionals (apart from the V for (7.2)). In this case, the stability conditions are often delay dependent (see, for example, He and Gopalsamy [20] and He [19]). Also, the estimates on the (ultimate) upper bounds of the solutions are needed.

2. If the stability conditions for (7.2) are of Volterra-Lyapunov stability and the function W for the system (7.2) is positively definite, we can still treat (7.1) as a perturbation of (7.2) and use the same V for (7.2) and the techniques mentioned above to construct the required (weakly degenerate) Lyapunov functionals. The corresponding stability conditions would be delay dependent (see, for example, He [18]).

3. If the stability conditions for (7.2) are neither 1 nor 2 mentioned above and the function W is positively semi-definite, we may have to use the idea of this paper to construct the required strongly degenerate Lyapunov functionals for the system (7.1). In this case, as we have seen, the information provided by the functions V and W for the ODE system (7.1) is very useful.

8. Appendix. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the characteristic equation

$$D(\lambda) = \lambda^2 + A\lambda e^{-\lambda\tau_1} + Be^{-(\tau_2+\tau_3)\lambda} = 0, \tag{8.1}$$

where $A = x^*, B = x^*y^*ab$. We prove Theorem 2.1 case by case.

CASE 1. $\tau_1 = 0,\ \tau_2 + \tau_3 = \tau > 0.$

In this case, (8.1) has the form

$$D(\lambda) = \lambda^2 + A\lambda e^{-\lambda\tau} + B = 0. \tag{8.2}$$

Let $\lambda = i\omega$ with $\omega > 0$ in (8.2). Then we have

$$D(i\omega) = -\omega^2 + B + iA\omega[\cos\omega\tau - i\sin\omega\tau] = 0$$

and which leads to

$$A\omega\sin\omega\tau = \omega^2 - B, \qquad A\omega\cos\omega\tau = 0.$$

Thus,

$$\omega^4 - (2B + A^2)\omega^2 + B^2 = 0.$$

Its root are

$$\omega_\pm^2 = \frac{1}{2}[(A^2 + 2B) \pm A\sqrt{A^2 + 4B}] > 0.$$

Obviousely, $\omega_+ > \omega_- > 0$. For ω_+, $\omega_+^2 - B > 0$ and hence $\sin\omega\tau > 0$. For ω_-, $\omega_-^2 - B < 0$ and hence $\sin\omega\tau < 0$. These, together with $\cos\omega\tau = 0$, imply that

$$\tau_{n,1} = \frac{\theta_1}{\omega_+} + \frac{2n\pi}{\omega_+},\ n = 0, 1, 2, \cdots ,$$

$$\tau_{n,2} = \frac{\theta_2}{\omega_-} + \frac{2n\pi}{\omega_-},\ n = 0, 1, 2, \cdots ,$$

where $\theta_1 = \pi/2$ and $\theta_2 = 3\pi/2$. Obviousely, $\tau_o = \tau_{0,1} < \tau_{0,2}$.

We now check the sign of $d(Re\lambda)/(d\tau)$ at $\tau = \tau_o$. From (8.2),

$$[2\lambda + A(1-\lambda\tau)e^{-\lambda\tau}]\frac{d\lambda}{d\tau} = A\lambda^2 e^{-\lambda\tau},$$

and it leads to

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda e^{\lambda\tau} + A}{A\lambda^2} - \frac{\tau}{\lambda}.$$

Therefore,

$$\begin{aligned} sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_+} &= sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_+} \\ &= sign\left\{Re\left[\frac{2\lambda e^{\lambda\tau}+A}{A\lambda^2}\right]\right\}_{\lambda=i\omega_+} \\ &= sign\left[\frac{2\sin(\omega_+\tau_{0,1})}{A\omega_+}\right] = +1. \end{aligned}$$

Then the conclusion follows from the Hopf bifurcation theorem.

CASE 2. $\tau_1 = 0$, $\tau_2 + \tau_3 = \tau > 0$.

In this case, (8.1) has the form

$$D(\lambda) = \lambda^2 + A\lambda + Be^{-\lambda\tau} = 0. \tag{8.3}$$

Let $\lambda = i\omega$ with $\omega > 0$, then we have

$$B\cos\omega\tau = \omega^2, \qquad B\sin\omega\tau = A\omega^2.$$

Solving the equations, we obtain only one positive root for ω

$$w^2 = \frac{1}{2}[-A^2 + \sqrt{A^4 + 4B^2}] > 0.$$

Correspondingly,

$$\sin\omega\tau = \frac{A\omega}{B} > 0, \qquad \cos\omega\tau = \frac{\omega^2}{B} > 0$$

and

$$\tau_n = \frac{\theta}{\omega} + \frac{2n\pi}{\omega},\ n = 0, 1, \cdots$$

with $\theta \in (0, \pi/2]$ satisfing $\sin\theta = A\omega/B$. Checking the sign of the real part of the $d\lambda/d\tau$ at $\tau_0 = \theta/\omega$, we have from (8.3) that

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\tau=\tau_o} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_o}$$
$$= sign\left\{Re\left[\frac{(2\lambda+A)e^{\lambda\tau}}{B\lambda}\right]\right\}_{\tau=\tau_o} = sign\left[\frac{2\omega^2 + A^2}{B^2}\right] = +1.$$

CASE 3. $\tau_1 = \tau > 0$, $\tau_2 + \tau_3 = \tau > 0$

In this case, (8.1) has the form

$$D(\lambda) = \lambda^2 + (Al + B)e^{-\lambda\tau} = 0. \tag{8.4}$$

Let $\lambda = i\omega$ with $\omega > 0$, then $D(i\omega) = 0$ implies

$$A\omega \sin\omega\tau + B\cos\omega\tau = \omega^2, \qquad A\omega\cos\omega\tau - B\sin\omega\tau = 0$$

Solving these equations, we derive that

$$\omega^2 = \frac{1}{2}[A^2 + \sqrt{A^4 + 4B^2}]$$

and

$$\sin\omega\tau = \frac{a\omega^3}{A^2\omega^2 + B^2}, \qquad \cos\omega\tau = \frac{B\omega^2}{A^2\omega^2 + B^2}.$$

Hence,

$$\tau_n = \frac{1}{\omega}(\theta + 2n\pi), \qquad n = 0, 1, 2, \cdots$$

with $\theta \in (0, \pi/2]$ satisfying $\tan\theta = (A\omega/B)$. On the other hand, from (8.4), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda e^{\lambda\tau} + A}{\lambda(A\lambda + B)} - \frac{\tau}{\lambda},$$

from which, it is found that $d(Re\lambda)/d\tau > 0$ at $\tau = \tau_o$.

Case 4. $\tau_1 = \tau_2 = \tau_3 = \tau > 0$.

In this case, (8.1) has the form

$$D(\lambda) = \lambda^2 + A\lambda e^{-\lambda\tau} + Be^{-2\lambda\tau} = 0. \tag{8.5}$$

Let $\lambda = i\omega$ with $\omega > 0$ real. Then $D(i\omega) = 0$ implies

$$\begin{aligned} A\omega\sin\omega\tau + B\cos 2\omega\tau &= \omega^2, \\ A\omega\cos\omega\tau - B\sin 2\omega\tau &= 0. \end{aligned} \tag{8.6}$$

Squaring both sides of (8.6), we obtain

$$A^2\omega^2 + B^2 - 2AB\omega\sin\omega\tau = \omega^4. \tag{8.7}$$

(i) The case when $A^2 < 4B$. In this case, we claim that $\cos\omega\tau \neq 0$. Otherwise, we have $\cos\omega\tau = 0$, $\sin\omega\tau = 1$. Then, from the first equation in (8.6), $A\omega - B^2 = \omega^2$, which has no real solution. Thus, $\cos\omega\tau \neq 0$. Using the second equation in (8.6), one has $\sin\omega\tau = A\omega/(2B)$. Then, it turns out from (8.7) that $\omega = \sqrt{B}$. Therefore,

$$\tau_n = \tau_o + \frac{2n\pi}{\sqrt{B}}, \qquad n = 0, 1, \cdots$$

with τ_o defined in the theorem.

(ii) The case when $A^2 > 4B$. We claim that $\cos\omega\tau = 0$. Otherwise, from the part (i), we have $\sin\omega\tau = A/(2\sqrt{B}) > 1$, which is impossible. Also, from the part (i), we know that $\omega_{\pm} = [A \pm \sqrt{A^2 - 4B}]/2$ and $\tau_n = (\pi/2 + 2n\pi)/\omega_{\pm}\,(n = 0, 1, 2, \cdots)$. Since $\omega_+ > \omega_-$, $\tau_o = \frac{\pi}{2\omega_+}$.

We now check the sign of $d(Re\lambda)/(d\tau)$ at τ_o. It follows from (8.5) that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda e^{2\lambda\tau} + Ae^{\lambda\tau}}{A\lambda^2 e^{\lambda\tau} + 2B\lambda} - \frac{\tau}{\lambda}.$$

Denote the above function by $F(\lambda)$. Then one can verify that

$$Re(F(i\omega)) = [2\omega^4 - (A^2 - 2B)\omega^2 - 2B(B - \omega^2)\cos 2\omega\tau]/C$$

for some constant $C > 0$. Firstly, if $A^2 < 4B$, then $\omega^2 = B$. Thus,

$$Re(F(i\omega)) = B[4B - A^2]/C > 0.$$

Now, if $A^2 > 4B$, then $\cos\omega\tau = 0$ and

$$Re(F(i\omega)) = \omega^2[2\omega^2 - (A^2 - 2B)]/C.$$

At ω_+, $Re(F(i\omega_+)) = 2A\omega^2\sqrt{A^2 - 2B}/C > 0$. Therefore, in both cases, we have verified that $d(Re\lambda)/(d\tau) > 0$ at $\tau = \tau_o$. This completes the proof.

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