Ricci flow on quasiprojective manifolds II

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Abstract

We study the Ricci flow on complete Kähler metrics that live on the complement of a divisor in a compact complex manifold. In earlier work, we considered finite-volume metrics which, at spatial infinity, are transversely hyperbolic. In the present paper we consider three different types of spatial asymptotics: cylindrical, bulging and conical. We show that in each case, the asymptotics are preserved by the Kähler-Ricci flow. We address long-time existence, parabolic blowdown limits and the role of the Kähler-Ricci flow on the divisor.

1 Introduction

Let $\overline{X}$ be a compact complex manifold of complex dimension n. Let D be a divisor in $\overline{X}$ with normal crossings. In a series of papers, Tian and Yau gave sufficient conditions for the quasiprojective manifold $X = \overline{X} - D$ to admit a complete Kähler-Einstein metric. The papers differed in the kind of spatial asymptotics that were considered. The paper [22] gave sufficient conditions for X to admit a finite-volume Kähler-Einstein metric with negative Ricci tensor, having cuspidal asymptotics. The papers [23] and [24] dealt with Ricci-flat Kähler metrics. The paper [23] considered two types of spatial asymptotics, which we call cylindrical and bulging. The paper [24] considered asymptotically conical metrics.

In [15], we looked at the Kähler-Ricci flow on quasiprojective manifolds with cuspidal asymptotics. That is, the initial Kähler metric on X was assumed to be complete, finite-volume and asymptotic to a hyperbolic cusp in directions transverse to D. Unlike in [22], no assumptions were made on the divisor class, since we were not necessarily looking for Kähler-Einstein metrics. We defined two flavors of spatial asymptotics: "standard"

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spatial asymptotics, which describes the leading behavior of the Kähler form at spatial infinity, and "superstandard" spatial asymptotics, in which the leading behavior of the Kähler potential is also prescribed. We showed that standard and superstandard spatial asymptotics are both preserved by the Kähler-Ricci flow (taking into account the Kähler-Ricci flow on D). We gave a formula for the first singularity time, if there is one, in terms of the cohomology of $\overline{X}$.

In the present paper we look at the technically more challenging case of the Kähler-Ricci flow on quasiprojective manifolds with spatial asymptotics similar to those considered in [23] and [24]. As in our earlier paper, the goal is to define classes of spatial asymptotics that are as general as possible while being preserved under the Kähler-Ricci flow, and about which one can say something nontrivial. We consider "cylindrical", "bulging" and "conical" spatial asymptotics. In each case, there are notions of "standard" asymptotics and "superstandard" asymptotics.

Theorem 1.1. *If the initial Kähler metric has (cylindrical, bulging or conical) (standard or superstandard) spatial asymptotics then so do the evolving Kähler metrics of the Kähler-Ricci flow. With superstandard asymptotics,*

(i) *In the cylindrical or bulging case, if $K_{\overline{X}} + D \geq 0$ in $H^{(1,1)}(\overline{X}; \mathbb{R})$ then the Kähler-Ricci flow exists for all positive time.*

(ii) *In the conical case, if $K_{\overline{X}} + (n+1)D \geq 0$ in $H^{(1,1)}(\overline{X}; \mathbb{R})$ then the Kähler-Ricci flow exists for all positive time.*

One theme of this paper is the relationship between the Kähler-Ricci flow on X and the Kähler-Ricci flow on the divisor D. The result depends on what kind of spatial asymptotics we are considering, so we address them separately.

1.1 Cylindrical spatial asymptotics

(c.f. [23, Section 5]) In this case, each component $\{D_i\}_{i=1}^k$ of D is assumed to have a trivial normal bundle. More precisely, we assume that there are holomorphic fiberings $f_i : \overline{X} \to C_i$ from $\overline{X}$ to complex curves so that $D_i = f^{-1}(s_i)$ for some $s_i \in C_i$. If z_i is a local coordinate for C_i near s_i, let z_i also denote its pullback to $\overline{X}$. If $I = (i_1, \ldots, i_m)$ is a multi-index, put $D_I = D_{i_1} \cap \ldots \cap D_{i_m}$. Let D_I^{int} be the smooth manifold consisting of points in D_I that do not lie in lower-dimensional strata. Let $\omega_{D_I^{\text{int}}}$ be a complete Kähler metric on D_I^{int}.

Given $\overline{x} \in D$, let I be such that $\overline{x} \in D_I$ but $\overline{x} \notin D_i$ for $i \notin I$. If a Kähler metric ω_X on X has cylindrical standard spatial asymptotics, associated to positive numbers $\{c_i\}_{i=1}^k$

and the metrics $\{\omega_{D_I^{\mathrm{int}}}\}$, then for all $\overline{x} \in D$, the asymptotics of ω_X "near" $\overline{x}$ are

$$\omega_X \sim \sum_{i \in I} 2c_i \sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z_i|^2} + \omega_{D_I^{\mathrm{int}}}. \tag{1.1}$$

That is, in the way of approaching spatial infinity specified by $\overline{x}$, the metric ω_X looks like a product of Euclidean cylinders with $\omega_{D_I^{\mathrm{int}}}$. The Kähler-Ricci flow on the divisor enters into the Kähler-Ricci flow on X in the following way.

Theorem 1.2. *Suppose that the initial metric $\omega_X(0)$ has cylindrical standard spatial asymptotics associated to $\{c_i\}_{i=1}^k$ and $\{\omega_{D_I^{\mathrm{int}}}(0)\}$. Then under the Kähler-Ricci flow on X, the metric $\omega_X(t)$ has standard spatial asymptotics associated to $\{c_i\}_{i=1}^k$ and $\{\omega_{D_I^{\mathrm{int}}}(t)\}$, where $\omega_{D_I^{\mathrm{int}}}(\cdot)$ is the Kähler-Ricci flow on D_I^{int} starting from $\omega_{D_I^{\mathrm{int}}}(0)$.*

1.2 Bulging spatial asymptotics

(c.f. [23, Section 4]) Suppose that D is a smooth divisor, connected for simplicity. Let x_0 be a basepoint in $X = \overline{X} - D$. Suppose that $n > 1$ and let ω_D be a Kähler metric on D. If X has bulging standard spatial asymptotics, associated to ω_D, then a sphere of large distance R from x_0 is the total space of a circle bundle over D. As $R \to \infty$, the lengths of the circle fibers are $O\left(R^{-\frac{n-1}{n+1}}\right)$. The metric on the base of the bundle is comparable to $R^{\frac{2}{n+1}}\omega_D$. The sectional curvatures are $O\left(R^{-\frac{2}{n+1}}\right)$.

If $\omega_X(0)$ has bulging standard spatial asymptotics associated to $\omega_D(0)$, and $\omega_X(\cdot)$ is the ensuing Kähler-Ricci flow, then it turns out that $\omega_X(t)$ has bulging standard spatial asymnptotics associated to $\omega_D(0)$. That is, the divisor flow does not enter into the finite-time spatial asymptotics on X. The proof of this uses results from Appendix B about the preservation, under Kähler-Ricci flow, of a power law decay in the curvature. Intuitively, in order to see a significant change in the geometry at a point $x \in X$, there must be an elapsed time comparable to $d_0(x, x_0)^{\frac{2}{n+1}}$, where d_0 is the time-zero distance.

This suggests that to see the divisor flow, one should take a sequence of time-zero points $\{x_i\}_{i=1}^\infty$ in X, going to spatial infinity, and perform a parabolic rescaling around $(x_i, 0)$ by a factor of $d_0(x_i, x_0)^{-\frac{2}{n+1}}$, i.e.

$$\omega_i(t) = d_0(x_i, x_0)^{-\frac{2}{n+1}} \, \omega_X\left(d_0(x_i, x_0)^{\frac{2}{n+1}} t\right). \tag{1.2}$$

The ensuing pointed limit should see the divisor flow.

There is a technical issue in taking such a blowdown limit. Namely, one needs uniform sectional curvature bounds on forward parabolic balls in the rescaled metrics. Such curvature bounds would follow from Perelman's pseudolocality result [16, 17], if it were applicable. Unfortunately, pseudolocality does not apply in this case, because of the shrinking circle fibers.

To get uniform sectional curvature bounds, we instead use a remarkable feature of Kähler-Ricci flow: a local biLipschitz bound implies a local curvature bound, independent of the elapsed time. The proof of this statement is given in Appendix A, building on work of Sherman-Weinkove [20]. To get biLipschitz estimates, it suffices to have a Ricci curvature bound.

Theorem 1.3. *Suppose that the initial metric $\omega_X(0)$ has bulging standard spatial asymptotics associated to $\omega_D(0)$. Let $(X, \omega_X(\cdot))$ be the Kähler-Ricci flow starting from $\omega_X(0)$.*
(a) For any time $t \geq 0$, the Kähler metric $\omega_X(t)$ also has bulging standard spatial asymptotics associated to $\omega_D(0)$.
(b) Suppose that the flow $(X, \omega_X(\cdot))$ exists for all positive time. Given $A < \infty$, suppose that $|\operatorname{Ric}(x,t)| = O\left(d_0(x,x_0)^{-\frac{2}{n+1}}\right)$ on the spacetime region $\left\{(x,t) \, : \, t \leq A \, d_0(x,x_0)^{\frac{2}{n+1}}\right\}$. Let $\{x_i\}_{i=1}^\infty$ be a sequence of points in X going to spatial infinity. Define $\omega_i(\cdot)$ by (1.2). Then the pointed Ricci flow limit $\lim_{i\to\infty} (X, (x_i, 0), \omega_i(\cdot)) = (X_\infty, (x_\infty, 0), \omega_\infty(\cdot))$ exists on the time interval $[0, A]$ and is given by

$$\omega_\infty(t) = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + n\omega_D(t)\right). \tag{1.3}$$

Here X_∞ is interpreted as an étale groupoid of complex dimension n (because of the collapsing circle fibers) with unit space $\mathbb{C} \times D$, so u lies in $\mathbb{C}$. Alternatively, one can express the convergence in terms of local covers; see Proposition 4.15.

1.3 Conical asymptotics

(c.f. [24]) Suppose that D is a smooth divisor, connected for simplicity. Let ω_D be a Kähler metric on D. If X has conical standard spatial asymptotics, associated to ω_D, then it has quadratic curvature decay. The asymptotic cone of X is a metric cone (CY, ω_{CY}), where Y is the total space of a circle bundle over D, the latter having metric ω_D. More precisely, the conical metric ω_{CY} is defined on $CY - \star_{CY}$, where $\star_{CY} \in CY$ is the vertex.

If $\omega_X(0)$ has conical standard spatial asymptotics, associated to ω_D, then $\omega_X(t)$ also has conical standard spatial asymptotics associated to ω_D. To go beyond this, it is natural to look at parabolic blowdowns. Suppose that the flow $(X, \omega_X(\cdot))$ exists for all positive time. Using pseudolocality, we show that one can extract a blowdown Ricci flow limit $\omega_\infty(\cdot)$ that exists on $\{(x,t) \in (CY - \star_{CY}) \times [0,\infty) \, : \, 0 \leq t \leq \epsilon d_{CY}(x, \star_{CY})^2\}$, for some $\epsilon > 0$. (One can also form such blowdown limits in the nonKähler case.) Its initial metric $\omega_\infty(0)$ is the conical metric ω_{CY}.

One may hope to see dynamics on D entering into the asymptotic cone of the blowdown limit. However, one finds that the asymptotic cone of a time slice $(CY - \star_{CY}, \omega_\infty(t))$ of the blowdown limit is still the asymptotic cone (CY, ω_{CY}) of $(X, \omega_X(0))$, constructed using the initial metric ω_D on D. In this conical setting, the dynamical object associated to D is no longer a Kähler-Ricci flow on D. Rather, we show that there is a formal gradient expanding soliton $\omega_{\mathrm{sol}}(\cdot)$ on $CY - \star_{CY}$, which is uniquely determined by ω_D. The time-one slice of this soliton takes the form

$$\omega_{\mathrm{sol}}(1) = \omega_{CY} - \mathrm{Ric}(\omega_{CY}) + \sqrt{-1}\partial\overline{\partial} \sum_{k>0} u_{(k)}. \tag{1.4}$$

The summation is a formal power series in a holomorphic coordinate z transverse to D (and its complex conjugate) or, equivalently, in inverse powers of the distance to $\star_{CY}$. The vector field associated to the soliton is the radial vector field on CY.

Theorem 1.4. *Suppose that the initial metric $\omega_X(0)$ has conical standard spatial asymptotics associated to ω_D. Let $(X, \omega_X(\cdot))$ be the Kähler-Ricci flow starting from $\omega_X(0)$.*
(a) For any time $t \geq 0$, the Kähler metric $\omega_X(t)$ has conical standard spatial asymptotics associated to ω_D.
(b) Suppose that the flow $(X, \omega_X(\cdot))$ exists for all positive time. Let $\omega_\infty(\cdot)$ be a parabolic blowdown limit. Then any asymptotic expansion of $\omega_\infty(\cdot)$, in powers of z and $\overline{z}$, equals the gradient expanding soliton $\omega_{\mathrm{sol}}(\cdot)$ associated to ω_D.

1.4 Structure of the paper

In Section 2 we recall some results from [15]. In Section 3 we define cylindrical standard spatial asymptotics and cylindrical superstandard spatial asymptotics. We show that they are preserved by the Kähler-Ricci flow, when taking into account the flow on the divisor. We give a formula for the first singularity time, if there is one.

In Section 4 we define bulging standard spatial asymptotics and bulging superstandard spatial asymptotics. We show that they are preserved by the Kähler-Ricci flow. We give a formula for the first singularity time, if there is one. With a decay assumption on the Ricci curvature, we show that there is a parabolic blowdown limit based on a sequence of points on the time-zero slice which go to spatial infinity. We prove that the blowdown limit is a product flow in which the divisor flow appears.

In Section 5 we define conical standard spatial asymptotics and conical superstandard spatial asymptotics. We show that they are preserved by the Kähler-Ricci flow. We give a formula for the first singularity time, if there is one. We show that one can take parabolic blowdowns of asymptotically conical Ricci flows. We construct the formal asymptotic expansion of a gradient expanding soliton on the asymptotic cone. We show that any formal asymptotic expansion of the parabolic blowdown equals the expanding soliton.

There are two appendices which may be of independent interest. In Appendix A we prove a local curvature estimate in Kähler-Ricci flow, showing that a biLipschitz bound implies a curvature bound, independent of the elapsed time. The proof is along the lines of Sherman-Weinkove [20], with some modifications. The result of Appendix A is used in Section 4 and Appendix B.

In Appendix B we discuss the preservation, under Ricci flow, of a power law decay of the curvature tensor. For general Ricci flow, we show that such a decay is preserved whenever the initial geometry has a reasonable end structure. More precisely, we need a smooth distance-like function ϕ so that $\frac{\mathrm{Hess}(\phi)}{\phi}$ is bounded below. The proof is along the lines of Dai-Ma [8]. Using Appendix A, we show that a power law decay of the curvature is preserved under Kähler-Ricci flow, provided that the first derivatives of the initial curvature tensor also have the right decay. The results of Appendix B are used in Sections 4 and 5.

More detailed descriptions are given at the beginnings of the sections.

1.5 Further directions

One can ask about more refined asymptotics for the Kähler-Ricci flow in the cases being considered. In the cuspidal case discussed in [15], such asymptotics were developed in Rochon-Zhang [18].

One can also ask about long-time behavior, at least when $K_{\overline{X}}+D \geq 0$ in the cylindrical and bulging cases, or when $K_{\overline{X}} + (n+1)D \geq 0$ in the conical case. Chau [2] and the authors [15, Theorem 5.1] gave sufficient conditions in the noncompact case to ensure convergence to a Kähler-Einstein metric of negative Ricci curvature. For convergence to a noncompact Ricci-flat Kähler metric, the only result that we know is by Chau and Tam [3], who prove convergence to a Ricci-flat Kähler metric if a Sobolev inequality holds for the initial metric (relevant to conical asymptotics) and if the Ricci potential of the initial metric has a faster-than-quadratic decay.

2 Background material

Throughout this paper, we use the notation and conventions of [15, Section 2]. In particular, Δ is the open unit ball in $\mathbb{C}$ and $\Delta^* = \Delta - \{0\}$. We let H denote the upper half plane $\{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0\}$. If (Z, d_Z) is a metric space then for $\lambda > 0$, we let $\frac{1}{\lambda}Z$ denote Z with the metric $\frac{d_Z}{\lambda}$. Thus if (Z, g_Z) is a Riemannian manifold then $\frac{1}{\lambda}Z$ denotes Z with the Riemannian metric $\frac{g_Z}{\lambda^2}$.

Let X be a connected complex manifold of complex dimension n. Suppose that ω_0 is a smooth complete Kähler metric on X with bounded curvature. The Kähler-Ricci flow equation is

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\,\mathrm{Ric}(\widetilde{\omega}_t), \qquad \widetilde{\omega}_0 = \omega_0. \tag{2.1}$$

There is some $T > 0$ so that there is a solution of (2.1) on the time interval $[0, T]$ having complete time slices and uniformly bounded curvature on $[0, T]$. Furthermore, such a solution is unique.

Put

$$\omega_t = \omega_0 - t \operatorname{Ric}(\omega_0). \tag{2.2}$$

Consider the equation

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}u)^n}{\omega_0^n} \tag{2.3}$$

with the initial condition $u(0, \cdot) = 0$. It is implicit that we only consider solutions u of (2.3) on time intervals so that $\omega_t + \sqrt{-1}\partial\overline{\partial}u > 0$. The following results are from [15].

Proposition 2.4. *Suppose that there is a smooth solution to (2.1) on a time interval $[0, T]$, with complete time slices and uniformly bounded curvature. Then there is a smooth solution for u in (2.3) on the time interval $[0, T]$ so that*

(i) *For each $t \in [0, T]$, $\omega_t + \sqrt{-1}\partial\overline{\partial}u$ is a Kähler metric which is biLipschitz-equivalent to ω_0, and*

(ii) *For each k, the kth covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded.*

Also, $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}u$.

Conversely, suppose that there is a smooth solution to (2.3) on a time interval $[0, T]$ so that

(i) *For each $t \in [0, T]$, $\omega_t + \sqrt{-1}\partial\overline{\partial}u$ is a Kähler metric which is biLipschitz-equivalent to ω_0, and*

(ii) *For each k, the kth covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded.*

Then $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}u$ is a solution to (2.1) on $[0, T]$, with complete time slices and uniformly bounded curvature.

Theorem 2.1. *Suppose that ω_0 is a complete Kähler metric on a complex manifold X, with bounded curvature.*

Let T_1 be the supremum (possibly infinite) of the numbers $T' \geq 0$ so that there is a smooth solution for (2.3) on the time interval $[0, T']$ such that

(i) *For each $t \in [0, T']$, $\omega_t + \sqrt{-1}\partial\overline{\partial}u$ is a Kähler metric which is biLipschitz-equivalent to ω_0, and*

(ii) *For each k, the kth covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded on $[0, T']$.*

Let T_2 be the supremum (possibly infinite) of the numbers $T \geq 0$ for which there is a function $F_T \in C^\infty(X)$ such that

(i) $\omega_T + \sqrt{-1}\partial\overline{\partial}F_T$ *is a Kähler metric which is biLipschitz-equivalent to* ω_0*, and*

(ii) *For each k, the kth covariant derivatives of F_T (with respect to the initial metric ω_0) are uniformly bounded.*

Then $T_1 = T_2$.

3 Cylindrical Kähler-Ricci flows

This section deals with cylindrical spatial asymptotics. In Subsection 3.1 we define the notion of cylindrical standard spatial asymptotics, and show that it is preserved under the Kähler-Ricci flow, taking into account the divisor flow. In Subsection 3.2 we introduce cylindrical superstandard spatial asymptotics. We show that if $\overline{X}$ admits a Kähler metric then X admits a metric with cylindrical superstandard spatial asymptotics. We prove that having cylindrical superstandard spatial asymptotics is preserved under the Kähler-Ricci flow. Given a metric with cylindrical superstandard spatial asymptotics, we define a certain renormalized cohomology class on the compactification $\overline{X}$. We use this cohomology class to characterize the first singularity time, if there is one.

Let $\overline{X}$ be a compact connected n-dimensional complex manifold. For $1 \leq i \leq k$, let $f_i : \overline{X} \to C_i$ be a holomorphic fibering over a complex curve C_i. Given points $s_i \in C_i$, let $g_{s_i} : \Delta \to C_i$ be a local parametrization of C_i with $g_{s_i}(0) = s_i$. Given an increasing multi-index $I = (i_1, \ldots, i_m)$, put

(i) $C_I = C_{i_1} \times \ldots \times C_{i_m}$,

(ii) $s_I = (s_{i_1}, \ldots, s_{i_m})$, and

(iii) Let $f_I : \overline{X} \to C_I$ be the product map $f_I = (f_{i_1}, \ldots f_{i_m})$.

Suppose that for each multi-index I, the point s_I is a regular value for f_I. Put $D_I = f_I^{-1}(s_I)$ and $D = \bigcup_{i=1}^k D_i$. Then D is an effective divisor with simple normal crossings. Put $D_I^{\mathrm{int}} = D_I - \bigcup_{I':|I'|>|I|} D_{I'}$. Then D_I^{int} is a smooth complex manifold of dimension $n - |I|$, possibly noncompact. Put $X = \overline{X} - D$.

Let L_i be the holomorphic line bundle on C_i associated to s_i. Then $L_D = \bigotimes_{i=1}^k f_i^* L_i$ is the holomorphic line bundle on $\overline{X}$ associated to D. There is a holomorphic section σ_i of L_i with zero set s_i, which is nondegenerate at s_i. It is unique up to multiplication by a nonzero complex number.

Remark 3.1. In what follows, we could replace the point s_i by a finite subset of C_i, without any essential change.

Remark 3.2. In [23, Section 5], Tian and Yau considered the case of a smooth divisor, i.e. when $k = 1$. They proved that if $K_{\overline{X}} + D = 0$ then there is a complete Ricci-flat Kähler metric on X.

3.1 Cylindrical standard spatial asymptotics

Suppose that $\overline{x} \in D_I^{\text{int}}$. After permutation of indices, we can assume that $\overline{x} \in (D_1 \cap D_2 \cap \ldots \cap D_m) - (D_{m+1} \cup D_{m+2} \cup \ldots \cup D_k)$. We write 0 for $(0, \ldots, 0) \in \Delta^n$. Let $p_i : \Delta^n \to \Delta$ be projection onto the i-th factor. Then there are a neighborhood U of $\overline{x}$ in $\overline{X}$ and a biholomorphic map $F_{\overline{x}} : \Delta^n \to U$ so that

(i) For $i > m$, $U \cap D_i = \emptyset$,

(ii) $F_{\overline{x}}(0) = \overline{x}$, and

(iii) For $1 \le i \le m$, we have $f_i \circ F_{\overline{x}} = g_{s_i} \circ p_i$.

In particular, $F_{\overline{x}}\left((\Delta^*)^m \times \Delta^{n-m}\right) = U \cap X$. The map $G_{\overline{x}}$ on Δ^{n-m}, given by $G_{\overline{x}}(w) = F_{\overline{x}}(0, w)$, is a biholomorphic map from Δ^{n-m} to a neighborhood of $\overline{x}$ in D_I^{int}. Let z^i be a local coordinate on C_i around s_i, which is the local inverse of g_{s_i}. We also write z^i for its pullback under f_i to a function on a neighborhood of $\overline{x}$.

Given $r \in (\mathbb{R}^+)^m$, let $\alpha_r : (\Delta^*)^m \to (\Delta^*)^m$ be multiplication by r. If Z is an auxiliary space then we also write α_r for $(\alpha_r, \mathrm{Id}) : (\Delta^*)^m \times Z \to (\Delta^*)^m \times Z$.

Definition 3.1. Let $\left\{\omega_{D_I^{\text{int}}}\right\}$ be complete Kähler metrics on $\left\{D_I^{\text{int}}\right\}$. Let $\{c_i\}_{i=1}^k$ be positive numbers. Then ω_X has *cylindrical standard spatial asymptotics* associated to $\left\{\omega_{D_I^{\text{int}}}\right\}$ and $\{c_i\}_{i=1}^k$ if for every $\overline{x} \in D_I^{\text{int}}$ and every local parametrization $F_{\overline{x}}$,

$$\lim_{r \to 0} \alpha_r^* F_{\overline{x}}^* \omega_X = \sum_{i \in I} 2c_i \sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z_i|^2} + G_{\overline{x}}^* \omega_{D_I^{\text{int}}}. \tag{3.3}$$

The limit in (3.3) is taken in the pointed C^∞-topology around the basepoint $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \times 0 \in (\Delta^*)^m \times \Delta^{n-m}$.

Proposition 3.4. *The notion of cylindrical standard spatial asymptotics in Definition 3.1 is consistent under change of local coordinate.*

Proof. Let $\{w^i\}_{i=1}^{n-m}$ be local coordinates for $D_{(1,\ldots,m)}$ around $\overline{x}$. If $\{\widehat{z}^i, \widehat{w}^i\}$ is a different choice of coordinates then we can write $\widehat{z}^i = \widehat{z}^i(z^i)$ and $\widehat{w}^i = \widehat{w}^i(z, w)$. Let α_r be the operation of multiplying z by r and let $\widehat{\alpha}_r$ be the operation of multiplying $\widehat{z}$ by r. Then

$$\alpha_r^* F_{\overline{x}}^* \omega_X = \alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{\alpha}_r^* \widehat{F}_{\overline{x}}^* \omega_X. \tag{3.5}$$

Suppose that ω_X has cylindrical standard spatial asymptotics with respect to $(\widehat{z}, \widehat{w})$. Then

$$\lim_{r\to 0} \widehat{\alpha}_r^* \widehat{F}_{\overline{x}}^* \omega_X = \sum_{i=1}^m 2c_i \sqrt{-1} \frac{d\widehat{z}^i \wedge d\overline{\widehat{z}}^i}{|\widehat{z}_i|^2} + \widehat{G}_{\overline{x}}^* \omega_{D_I^{\mathrm{int}}}. \tag{3.6}$$

Now

$$(\widehat{\alpha}_r^*)^{-1} \widehat{z}^i = r^{-1} \widehat{z}^i. \tag{3.7}$$

Expanding

$$\widehat{z}^i = a_{1,i} z^i + a_{2,i} \left(z^i\right)^2 + a_{3,i} \left(z^i\right)^3 + \dots, \tag{3.8}$$

with $a_{1,i} \neq 0$, we have

$$\left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{z}^i = r^{-1} \left(a_{1,i} z^i + a_{2,i} \left(z^i\right)^2 + a_{3,i} \left(z^i\right)^3 + \dots\right) \tag{3.9}$$

Then

$$\alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{z}^i = a_{1,i} z^i + a_{2,i} r \left(z^i\right)^2 + a_{3,i} r^2 \left(z^i\right)^3 + \dots \tag{3.10}$$

It follows that

$$\lim_{r\to 0} \alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{z}^i = a_{1,i} z^i. \tag{3.11}$$

Hence

$$\lim_{r\to 0} \alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \frac{d\widehat{z}^i \wedge d\overline{\widehat{z}^i}}{|\widehat{z}^i|^2} = \frac{dz^i \wedge d\overline{z^i}}{|z^i|^2}, \tag{3.12}$$

with smooth pointed convergence around $\frac{1}{2} \in \Delta^*$.

Similarly, writing $\widehat{w} = \widehat{w}(z, w)$, we have

$$\lim_{r\to 0} \alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{w} = \widehat{w}(0, w) \tag{3.13}$$

Then

$$\lim_{r\to 0} \alpha_r^* \left(\widehat{F}_{\overline{x}}^{-1} \circ F_{\overline{x}}\right)^* (\widehat{\alpha}_r^*)^{-1} \widehat{G}_{\overline{x}}^* \omega_{D_I^{\mathrm{int}}} = G_{\overline{x}}^* \omega_{D_I^{\mathrm{int}}}. \tag{3.14}$$

In view of (3.5) and (3.6), the proposition follows. □

Remark 3.15. In the proof of Proposition 3.4, if we allowed more general coordinate changes, of the form $\widehat{z}^i = \widehat{z}^i(z, w)$, then the result of the proposition would definitely fail. This explains why we assume that $\overline{X}$ fibers over curves, so that it makes sense to consider coordinate changes of the form $\widehat{z}^i = \widehat{z}^i(z)$. At the least, we need to assume that D_i has a trivial normal bundle.

Proposition 3.16. *If $\overline{X}$ admits a Kähler metric then X admits a complete Kähler metric with cylindrical standard spatial asymptotics.*

Proof. This will follow from Proposition 3.22. □

We now prove Theorem 1.2, showing that the property of having cylindrical standard spatial asymptotics is preserved under the Kähler-Ricci flow.

Proposition 3.17. *Suppose that $\omega_X(0)$ has cylindrical standard spatial asymptotics associated to $\{\omega_{D_I^{\mathrm{int}}}(0)\}$ and $\{c_i\}_{i=1}^k$. Suppose that the Kähler-Ricci flow $\omega_X(t)$, with initial Kähler form $\omega_X(0)$, exists on a maximal time interval $[0,T)$ in the sense of Theorem 2.1. Then for all $t \in [0,T)$, the metric $\omega_X(t)$ has cylindrical standard spatial asymptotics associated to $\{\omega_{D_I^{\mathrm{int}}}(t)\}$ and $\{c_i\}_{i=1}^k$, where $\omega_{D_I^{\mathrm{int}}}(t)$ is the Kähler-Ricci flow on D_I^{int} with initial Kähler form $\omega_{D_I^{\mathrm{int}}}(0)$.*

Proof. Take $\overline{x} \in D_I^{\mathrm{int}}$. After a change of labels we can assume that $I = (1,\ldots,m)$. Let $F_{\overline{x}} : (\Delta^*)^m \times \Delta^{n-m} \to \overline{X}$ be a local parametrization. Let $r_j \to 0$ be any sequence.

From our assumptions, there is a uniform positive lower bound on the injectivity radius of $F_{\overline{x}}^*\omega_X(0)$ at $\alpha_{r_j}\left(\frac{1}{2},\ldots,\frac{1}{2},0\right)$ or, equivalently, on $\alpha_{r_j}^* F_{\overline{x}}^*\omega_X(0)$ at $\left(\frac{1}{2},\ldots,\frac{1}{2},0\right)$. Using the curvature bounds, we can apply Hamilton's compactness theorem [11] to extract a subsequence of $\left\{\alpha_{r_j}^* F_{\overline{x}}^*\omega_X(\cdot)\right\}$ that converges to a Ricci flow solution $(X_\infty, x_\infty, \omega_{X_\infty}(\cdot))$, defined on the time interval $[0,T)$. There is a technical issue that *a priori*, the solution is only smooth on $(0,T)$, where we can apply Shi's local derivative estimates. In order to get smoothness on $[0,T)$, we need uniform bounds on the derivatives of the curvature tensor of $\alpha_{r_j}^* F_{\overline{x}}^*\omega_X(t)$ for t in some interval $[0,\delta]$. Since we have uniform bounds on the derivatives of the curvature tensor of $\alpha_{r_j}^* F_{\overline{x}}^*\omega_X(0)$ (on time-0 metric balls around $\left(\frac{1}{2},\ldots,\frac{1}{2},0\right)$), the local derivative estimate of [14, Appendix D] gives the needed uniform bounds on the k-th derivatives of the curvature tensor for small but positive time. Then from the proof of [11], we can say that after passing to a subsequence, there is a smooth pointed limit

$$\lim_{j\to\infty}\left(\alpha_{r_j^{-1}}\left((\Delta^*)^m\right)\times\Delta^{n-m},\left(\frac{1}{2},\ldots,\frac{1}{2},0\right),\alpha_{r_j}^* F_{\overline{x}}^*\omega_X(\cdot)\right) = \left((\mathbb{C}^*)^m\times\Delta^{n-m},\left(\frac{1}{2},\ldots,\frac{1}{2},0\right),\omega_{\infty,\overline{x}}(\cdot)\right) \tag{3.18}$$

for some Kähler-Ricci flow solution $\omega_{\infty,\overline{x}}(\cdot)$ that exists on $(\mathbb{C}^*)^m \times \Delta^{n-m}$ for the time interval $[0,T)$.

Covering D_I^{int} by a locally finite collection $\{G_{\overline{x}_k}(\Delta^{n-m})\}$ of charts, we can assume that the Ricci flow solutions $\left((\mathbb{C}^*)^m\times\Delta^{n-m},\left(\frac{1}{2},\ldots,\frac{1}{2},0\right),\omega_{\infty,\overline{x}_k}(\cdot)\right)$ glue together to give a Ricci flow solution $\left((\mathbb{C}^*)^m\times D_I^{\mathrm{int}},\omega_{X_\infty}(\cdot)\right)$ with complete time slices and bounded curvature on compact time intervals; c.f. [15, Proof of Theorem 7.1]. From the assumption of cylindrical standard spatial asymptotics,

$$\omega_{X_\infty}(0) = \sum_{i=1}^m 2c_i\sqrt{-1}\frac{dz^i\wedge d\overline{z}^i}{|z_i|^2} + \omega_{D_I^{\mathrm{int}}}(0). \tag{3.19}$$

From the uniqueness of complete Ricci flow solutions with bounded curvature on compact time intervals [4], it follows that

$$\omega_{X_\infty}(t) = \sum_{i=1}^{m} 2c_i \sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z_i|^2} + \omega_{D_I^{\text{int}}}(t). \tag{3.20}$$

To return to the proof of the proposition, suppose that its conclusion is not true. Then for some $\overline{x} \in D_I^{\text{int}}$ and some $t \in [0, T)$, there is a sequence $r_j \to 0$ with the property that even after passing to any subsequence, $\alpha_{r_j}^* F_{\overline{x}}^* \omega_X(t)$ does not converge to $\sum_{i=1}^m 2c_i \sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z_i|^2} + G_{\overline{x}}^* \omega_{D_I^{\text{int}}}(t)$ in the pointed C^∞-topology. Here the basepoint is $\left(\frac{1}{2}, \ldots, \frac{1}{2}, 0\right) \in (\Delta^*)^m \times \Delta^{n-m}$. However, taking $\overline{x}_1 = \overline{x}$ in the above construction, we have shown that after passing to a subsequence, $\lim_{j \to \infty} \alpha_{r_j}^* F_{\overline{x}}^* \omega_X(t) = \sum_{i=1}^m 2c_i \sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z_i|^2} + G_{\overline{x}}^* \omega_{D_I^{\text{int}}}(t)$ in the pointed C^∞-topology. This is a contradiction, thereby proving the proposition. □

3.2 Cylindrical superstandard spatial asymptotics

Let h_i be a Hermitian metric on L_i.

Definition 3.2. A Kähler metric ω_X on X has *cylindrical superstandard spatial asymptotics* associated to $\{h_i\}_{i=1}^k$ if it has cylindrical standard spatial asymptotics (associated to $\{\omega_{D_I^{Int}}\}$ and $\{c_i\}_{i=1}^k$) and

$$\omega_X = \eta_{\overline{X}} + \sqrt{-1}\partial\overline{\partial} \left(\sum_{i=1}^{k} c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + H \right), \tag{3.21}$$

where

(i) $\eta_{\overline{X}}$ is a smooth closed $(1,1)$-form on $\overline{X}$, and

(ii) $H \in C^\infty(X) \cap L^\infty(X)$.

Note that in Definition 3.2, the choice of $\{h_i\}_{i=1}^k$ does matter.

Proposition 3.22. *If $\overline{X}$ admits a Kähler metric then X admits a complete Kähler metric with cylindrical superstandard spatial asymptotics.*

Proof. We have

$$\sqrt{-1}\partial\overline{\partial} \log^2 |\sigma_i|_{h_i}^{-2} = 2\sqrt{-1}\partial \log |\sigma_i|_{h_i}^{-2} \wedge \overline{\partial} |\sigma_i|_{h_i}^{-2} + 2 \left(\log |\sigma_i|_{h_i}^{-2} \right) F_{h_i}, \tag{3.23}$$

where F_{h_i} is the curvature 2-form associated to h_i. In terms of a local coordinate z^i around s^i, the right-hand side is asymptotic to $2\sqrt{-1} \frac{dz^i \wedge d\overline{z}^i}{|z^i|^2}$ as $z^i \to 0$.

Let $\omega_{\overline{X}}$ be a Kähler metric on $\overline{X}$. Given a parameter $K < \infty$ and positive constants $\{c_i\}_{i=1}^k$, put

$$\omega_X = \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2}\right) + K\omega_{\overline{X}}. \tag{3.24}$$

Taking K sufficiently large, ω_X is a complete Kähler metric on X with cylindrical standard spatial asymptotics. Then it clearly also has cylindrical superstandard spatial asymptotics. □

We now show that the property of having cylindrical superstandard spatial asymptotics is preserved under the Kähler-Ricci flow.

Proposition 3.25. *Suppose that $\omega_X(0)$ has cylindrical superstandard spatial asymptotics associated to $\{h_i\}_{i=1}^k$. Suppose that the Kähler-Ricci flow $\omega_X(t)$, with initial Kähler metric $\omega_X(0)$, exists on a maximal time interval $[0,T)$ in the sense of Theorem 2.1. Then for all $t \in [0,T)$, $\omega_X(t)$ has cylindrical superstandard spatial asymptotics, associated to $\{h_i\}_{i=1}^k$.*

Proof. Choose a Hermitian metric $h_{K_{\overline{X}} \otimes L_D}$ on $K_{\overline{X}} \otimes L_D$. Along with $\{h_i\}_{i=1}^k$, we obtain a Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Then

$$\mathrm{Ric}(\omega_X(0)) = -\sqrt{-1}F\left(h_{K_{\overline{X}} \otimes L_D}\right) - \sqrt{-1}\partial\overline{\partial}\left(\log \frac{h_{K_{\overline{X}}} \prod_{i=1}^k f_i^* |\sigma_i|_{h_i}^2}{h_{K_X}}\right) \tag{3.26}$$

on X.

Put $\eta'_{\overline{X}} = -\sqrt{-1}F\left(h_{K_{\overline{X}} \otimes L_D}\right)$ and $H' = \log \frac{h_{K_{\overline{X}}} \prod_{i=1}^k f_i^* |\sigma_i|_{h_i}^2}{h_{K_X}}$. By the cylindrical standard spatial asymptotics, $H' \in C^\infty(X) \cap L^\infty(X)$.

Recall the definition of ω_t from (2.2). We can write

$$\begin{aligned} \omega_X(t) &= \omega_t + \sqrt{-1}\partial\overline{\partial}u(t) \\ &= \eta_{\overline{X}} - t\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + H + tH' + u(t)\right). \end{aligned} \tag{3.27}$$

Since $u(t) \in C^\infty(X) \cap L^\infty(X)$, the proposition follows. □

If $\omega_X(0)$ has cylindrical superstandard spatial asymptotics then we would like to give a characterization of the first singularity time, if there is one, by making Theorem 2.1 more explicit. For the "cuspidal" asymptotics considered in [15], the manifold $(X, \omega_X(0))$ had finite volume. Transplanting $\omega_X(0)$ to $\overline{X}$, we obtained a closed $(1,1)$-current, which represented a cohomology class on $\overline{X}$. In the present case, $(X, \omega_X(0))$ has infinite volume and we cannot directly obtain a cohomology class on $\overline{X}$. However, we can subtract the leading singularity, which is $\sqrt{-1}\partial\overline{\partial}$ of a function, and thereby define a renormalized cohomology class in $\overline{X}$.

With reference to Definition 3.2, since H is a smooth bounded function on X, it extends by zero to an integrable function on $\overline{X}$. Then $\sqrt{-1}\partial\overline{\partial}H$ is a closed $(1,1)$-current on $\overline{X}$ (which is cohomologically trivial). Hence the form on X given by $\omega_X(0)-\sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2}\right)$, which equals $\eta_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}H$, has a natural extension to a closed $(1,1)$-current on $\overline{X}$.

The relevant ring of functions, for cylindrical asymptotics, can be characterized in the following way.

Definition 3.3. The ring $C^\infty_{\mathrm{cyl}}(X)$ consists of the smooth functions f on $X = \overline{X} - D$ so that for every $\overline{x} \in D$ and every local parametrization $F_{\overline{x}}$, the pullback $F_{\overline{x}}^* f \in C^\infty((\Delta^*)^m \times \Delta^{n-m})$ has the property that for any multi-index $(l_1, \bar{l}_1, \ldots, l_n, \bar{l}_n)$, the function

$$\left(z^1 \frac{\partial}{\partial z^1}\right)^{l_1} \left(\overline{z}^1 \frac{\partial}{\partial \overline{z}^1}\right)^{\bar{l}_1} \cdots \left(z^m \frac{\partial}{\partial z^m}\right)^{l_m} \left(\overline{z}^m \frac{\partial}{\partial \overline{z}^m}\right)^{\bar{l}_m} \tag{3.28}$$
$$\left(\frac{\partial}{\partial w^1}\right)^{l_{m+1}} \left(\frac{\partial}{\partial \overline{w}^1}\right)^{\bar{l}_{m+1}} \cdots \left(\frac{\partial}{\partial w^{n-m}}\right)^{l_n} \left(\frac{\partial}{\partial \overline{w}^{n-m}}\right)^{\bar{l}_n} F_{\overline{x}}^* f$$

is uniformly bounded.

Proposition 3.29. *Suppose that $\omega_X(0)$ has cylindrical superstandard spatial asymptotics associated to $\{h_i\}_{i=1}^k$. Let $\eta_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of the cohomology class represented by the closed current*

$$\omega_X(0) - \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2}\right) \tag{3.30}$$

on $\overline{X}$. Let $\eta'_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of $-2\pi[K_{\overline{X}} + D] \in \mathrm{H}^{(1,1)}(\overline{X})$. Let T_3 be the supremum (possibly infinite) of the numbers T' for which there is some $f_{T'} \in C^\infty_{\mathrm{cyl}}(X)$ so that

$$\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + f_{T'}\right) \tag{3.31}$$

is a Kähler form on X which is biLipschitz to $\omega_X(0)$. Then T_3 equals the numbers $T_1 = T_2$ of Theorem 2.1.

Proof. Let $\omega_X(0)$ and T' be as in the statement of the proposition. Since the present $\eta_{\overline{X}}$ and the $\eta_{\overline{X}}$ of Definition 3.2 differ by $\sqrt{-1}\partial\overline{\partial}$ of a smooth function on $\overline{X}$, we can still write

$$\omega_X(0) = \eta_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + H\right) \tag{3.32}$$

for some $H \in C^\infty(X) \cap L^\infty(X)$. With reference to the proof of Proposition 3.25, as the present $\eta'_{\overline{X}}$ differs from $-\sqrt{-1}F\left(h_{K_{\overline{X}}\otimes L_D}\right)$ by $\sqrt{-1}\partial\overline{\partial}$ of a smooth function on $\overline{X}$, we can still write

$$\omega_{T'} = \eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^{k} c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + H + T'H'\right) \tag{3.33}$$

for some $H' \in C^\infty(X) \cap L^\infty(X)$. Then

$$\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^{k} c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + f_{T'}\right) = \omega_{T'} + \sqrt{-1}\partial\overline{\partial}\left(f_{T'} - H - T'H'\right). \tag{3.34}$$

Put $F_{T'} = f_{T'} - H - T'H'$. By assumption, the left-hand side of (3.34), and hence also $\omega_{T'} + \sqrt{-1}\partial\overline{\partial}F_{T'}$, is a Kähler metric which is biLipschitz to $\omega_X(0)$. Since $\omega_X(0)$ has cylindrical standard spatial asymptotics, it follows that

$$-\operatorname{const.}\omega_X(0) \leq \sqrt{-1}\partial\overline{\partial}F_{T'} \leq \operatorname{const.}\omega_X(0). \tag{3.35}$$

Hence

$$|\triangle_{\omega_X(0)} F_{T'}| \leq \operatorname{const.} \tag{3.36}$$

Since $\omega_X(0)$ has bounded geometry (including a positive injectivity radius), elliptic regularity implies that for each k, the kth covariant derivatives of $F_{T'}$ (with respect to $\omega_X(0)$) are uniformly bounded. It follows that $T_3 \leq T_2$.

Now suppose that T' is as in the definition of T_1 in Theorem 2.1. As $\omega_X(0)$ has cylindrical superstandard spatial asymptotics, we can write

$$\omega_{T'} + \sqrt{-1}\partial\overline{\partial}u(T') = \eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^{k} c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + H + T'H' + u(T')\right) \tag{3.37}$$

for some $H, H' \in C^\infty(X) \cap L^\infty(X)$. Put $f_{T'} = H + T'H' + u(T')$, so

$$\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^{k} c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + f_{T'}\right) = \omega_{T'} + \sqrt{-1}\partial\overline{\partial}u(T'). \tag{3.38}$$

From Proposition 3.25, $\omega_{T'} + \sqrt{-1}\partial\overline{\partial}u(T')$ has cylindrical superstandard spatial asymptotics. As before, using elliptic regularity we conclude that for each k, the kth covariant derivatives of $f_{T'}$ (with respect to $\omega_X(0)$) are uniformly bounded. This is equivalent to saying that $f_{T'} \in C^\infty_{\mathrm{cyl}}(X)$.

Thus $T_1 \leq T_3$. This proves the proposition. □

Corollary 3.39. *Suppose that $\omega_X(0)$ has cylindrical superstandard spatial asymptotics. If $[K_{\overline{X}} + D] \geq 0$ then the flow exists for all positive time.*

Proof. With reference to Proposition 3.29, we can choose $-\eta'_{\overline{X}}$ to be a nonnegative closed $(1,1)$-form. Since $T_3 > 0$, there is some $f_0 \in C^\infty_{\mathrm{cyl}}(X)$ so that $\eta_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + f_0\right)$ is a Kähler form on X which is biLipschitz to $\omega_X(0)$. Then for any $T' > 0$, $\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^k c_i f_i^* \log^2 |\sigma_i|_{h_i}^{-2} + f_0\right)$ is also a Kähler form on X which is biLipschitz to $\omega_X(0)$. Using Proposition 3.29, this proves the corollary. □

Remark 3.40. Proposition 3.29 is only partly a statement about $\overline{X}$, since the definition of T_3 is a statement about Kähler metrics on X with certain properties. In this sense, the proposition is not as definitive as the corresponding statement about cuspidal asymptotics in [15, Theorem 8.19].

4 Bulging Kähler-Ricci flows

This section deals with bulging spatial asymptotics. In Subsection 4.1 we define the notion of bulging standard spatial asymptotics, and show that it is preserved under the Kähler-Ricci flow, with no change in the divisor metric. In Subsection 4.2 we consider parabolic rescalings around a sequence of points that go to spatial infinity in the time-zero slice. Under a decay assumption on the Ricci curvature, we show that the limit is a product flow which exhibits the Kähler-Ricci flow on the divisor.

In Subsection 4.3 we assume that D is ample. We introduce bulging superstandard spatial asymptotics. We show that if $\overline{X}$ admits a Kähler metric then X admits a metric with bulging superstandard spatial asymptotics. We prove that having bulging superstandard spatial asymptotics is preserved under the Kähler-Ricci flow. We characterize the first singularity time, if there is one.

Let $\overline{X}$ be a compact connected n-dimensional complex manifold. Let D be a smooth effective divisor in $\overline{X}$. Let L_D be the holomorphic line bundle on $\overline{X}$ associated to D. There is a holomorphic section σ of L_D with zero set D, which is nondegenerate at D. It is unique up to multiplication by a nonzero complex number.

4.1 Bulging standard spatial asymptotics

Given $\overline{x} \in D$, there are a neighborhood U of $\overline{x}$ in $\overline{X}$ and a biholomorphic map $F_{\overline{x}} : \Delta^n \to U$ so that

(i) $F_{\overline{x}}(0) = \overline{x}$, and

(ii) $F_{\overline{x}}(\Delta^* \times \Delta^{n-1}) = U \cap X$.

The map $G_{\overline{x}}$ on Δ^{n-1}, given by $G_{\overline{x}}(w) = F_{\overline{x}}(0, w)$, is a biholomorphic map from Δ^{n-1} to a neighborhood of $\overline{x}$ in D. Let z be the local coordinate on U coresponding to the first factor in Δ^n.

Using the covering map $H \to \Delta^*$, given by $u \to e^{\sqrt{-1}u}$, let $\widetilde{F}_{\overline{x}} : H \times \Delta^{n-1} \to U \cap X$ be the lift of $F_{\overline{x}}\Big|_{\Delta^* \times \Delta^{n-1}}$.

Given $r \in \mathbb{R}^+$, let $\alpha_r : \mathbb{C} \to \mathbb{C}$ be the map $\alpha_r(u) = ru + \sqrt{-1}\frac{r^2}{2}$. If Z is an auxiliary space then we also write α_r for $(\alpha_r, \mathrm{Id}) : \mathbb{C} \times Z \to \mathbb{C} \times Z$.

Definition 4.1. Let ω_D be a Kähler metric on D. Given $N > 0$, ω_X has *bulging standard spatial asymptotics* associated to (ω_D, N) if for every $\overline{x} \in D$ and every local parametrization $F_{\overline{x}}$,

$$\lim_{r \to \infty} r^{-\frac{2}{N}} \alpha_r^* \widetilde{F}_{\overline{x}}^* \omega_X = \frac{1}{2} \left(\frac{N+1}{N} \right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + N G_{\overline{x}}^* \omega_D \right). \tag{4.1}$$

The limit in (4.1) means smooth convergence on any subset $\{z \in \mathbb{C} : |z| < S\} \times \Delta^{n-1}$ of $\mathbb{C} \times \Delta^{n-1}$.

Remark 4.2. Note that although $\widetilde{F}_{\overline{x}}$ is originally defined on $H \times \Delta^{n-1}$, the limit is taken around the basepoint $(0, 0)$ in $\mathbb{C} \times \Delta^{n-1}$. This makes sense because $\alpha_r(0) = \sqrt{-1}\frac{r}{2} \in H$.

Proposition 4.3. *If ω_X has bulging standard spatial asymptotics then to leading order, as $z \to 0$, the metric in local coordinates has the form*

$$\omega_X \sim \frac{1}{2} \left(\frac{N+1}{N} \right)^2 \left(\log |z|^{-2} \right)^{\frac{1}{N}} \left(\sqrt{-1} \frac{dz \wedge d\overline{z}}{|z|^2 \log |z|^{-2}} + N \omega_D \right). \tag{4.4}$$

Proof. Relating the local function z on U to the local function u on $\mathbb{C}$ by $\alpha_r^* \widetilde{F}_{\overline{x}}^*$, we have

$$z = e^{\sqrt{-1}\left(ru + \sqrt{-1}\frac{r^2}{2}\right)}, \tag{4.5}$$

so

$$\frac{dz}{z} = \sqrt{-1}\, r\, du \tag{4.6}$$

and

$$\log |z|^{-2} = r^2 - \sqrt{-1} r (u - \overline{u}). \tag{4.7}$$

Then

$$\begin{aligned} &\frac{1}{2} \left(\frac{N+1}{N} \right)^2 r^{\frac{2}{N}} \left(\sqrt{-1} du \wedge d\overline{u} + N G_{\overline{x}}^* \omega_D \right) = \\ &\frac{1}{2} \left(\frac{N+1}{N} \right)^2 r^{\frac{2}{N}} \left(\sqrt{-1} \frac{dz \wedge d\overline{z}}{r^2 |z|^2} + N G_{\overline{x}}^* \omega_D \right) \sim \\ &\frac{1}{2} \left(\frac{N+1}{N} \right)^2 (\log |z|^{-2})^{\frac{1}{N}} \left(\sqrt{-1} \frac{dz \wedge d\overline{z}}{|z|^2 \log |z|^{-2}} + N \omega_D \right). \end{aligned} \tag{4.8}$$

This proves the proposition. ∎

One can check that the notion of bulging standard spatial asymptotics in Definition 4.1 is consistent under change of local coordinate.

Going out the end of X corresponds to taking $z \to 0$. The distance from a basepoint in X is asymptotic to

$$R \sim \left(\log |z|^{-2}\right)^{\frac{N+1}{2N}}. \tag{4.9}$$

Fix $\overline{x} \in D$ and let θ be the angular coordinate of z. Then as $z \to 0$, i.e. as $R \to \infty$, the metric on X is asymptotic to

$$g_X \sim dR^2 + \left(\frac{N+1}{N}\right)^2 R^{-2\frac{N-1}{N+1}} d\theta^2 + \frac{(N+1)^2}{2N} R^{\frac{2}{N+1}} g_D(\overline{x}). \tag{4.10}$$

The sectional curvatures decay as

$$|\operatorname{Rm}| = O\left(R^{-\frac{2}{N+1}}\right). \tag{4.11}$$

For a given $\overline{x} \in D$, as $z \to 0$, the geometry comes closer and closer to having a product structure.

We now show that the property of having bulging standard spatial asymptotics is preserved under the Kähler-Ricci flow, with no change in the divisor metric.

Proposition 4.12. *Let $\omega_X(\cdot)$ be a Kähler-Ricci flow defined for $t \in [0, T)$, with bounded curvature on compact time intervals. Suppose that $\omega_X(0)$ has bulging standard spatial asymptotics associated to (ω_D, N). Then for all $t \in [0, T)$, the metric $\omega_X(t)$ has bulging standard spatial asymptotics associated to (ω_D, N).*

Proof. Suppose that the conclusion of the proposition is not true. Then there are some $t \in [0, T)$, $\overline{x} \in D$, $F_{\overline{x}} : \Delta^* \times \Delta^{n-1} \to \overline{X}$, $S < \infty$ and $r_i \to \infty$ with the property that even after passing to any subsequence of the r_i's, the metrics $r_i^{-\frac{2}{N}} \alpha_{r_i}^* \widetilde{F}_{\overline{x}}^* \omega_X(t)$ do not smoothly converge to $\frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + N G_{\overline{x}}^* \omega_D\right)$ on $\{z \in \mathbb{C} : |z| < S\} \times \Delta^{n-1}$.

From the bulging standard spatial asymptotics of $\omega_X(0)$, given a multi-index k, we have $|\nabla^k \operatorname{Rm}|(x, 0) = O\left(d_0(x, x_0)^{-\frac{k+2}{N+1}}\right)$. We now want to show that

$$|\nabla^k \operatorname{Rm}|(x, t) = O\left(d_0(x, x_0)^{-\frac{k+2}{N+1}}\right). \tag{4.13}$$

There are two arguments for this. First, given a large positive number A, put

$$\phi(x) = \left(A + \log |\sigma(x)|^{-2}\right)^{\frac{N+1}{2N}}. \tag{4.14}$$

The definition of ϕ is motivated by (4.9). Then ϕ is a distance-like function that satisfies (B.4). Proposition B.12 implies that (4.13) holds.

Alternatively, we can apply Proposition B.15 to see that (4.13) holds. Either way, we obtain uniform bounds on the curvature of $r_i^{-\frac{2}{N}} \alpha_{r_i}^* \widetilde{F}_{\overline{x}}^* \omega_X(t)$ and its covariant derivatives, when considered on any fixed $\{z \in \mathbb{C} : |z| \leq S'\} \times \Delta^{n-1}$, as $i \to \infty$. After passing to a subsequence, we can assume that there is a Riemannian metric $\omega_\infty(t)$ on $\mathbb{C} \times \Delta^{n-1}$ so that the Riemannian metrics $\left\{ r_i^{-\frac{2}{N}} \alpha_{r_i}^* \widetilde{F}_{\overline{x}}^* \omega_X(t) \right\}_{i=1}^\infty$ converge smoothly to $\omega_\infty(t)$ on each $\{z \in \mathbb{C} : |z| < S'\} \times \Delta^{n-1}$.

On the time interval $[0,t]$, the curvature of X decays uniformly as $O\left(d_0(x,x_0)^{-\frac{2}{N+1}}\right)$. It follows from the Ricci flow equation that at a point $x \in X$, the metrics $\omega_X(t)$ and $\omega_X(0)$ are $e^{\mathrm{const.}\, t d_0(x,x_0)^{-\frac{2}{N+1}}}$-biLipschitz to each other. Applying this to small neighborhoods of points $x_i = \widetilde{F}_{\overline{x}}(\alpha_{r_i}(0,w)) = F_{\overline{x}}\left(e^{-\frac{r_i^2}{2}}, w\right)$, since $d_0(x_i,x_0) \to \infty$, it follows that $\omega_\infty(t)$ is isometric to the corresponding $\omega_\infty(0)$. The latter is the product metric $\frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + NG_{\overline{x}}^*\omega_D\right)$ on $\mathbb{C} \times \Delta^{n-1}$. Hence as $i \to \infty$, the metrics $r_i^{-\frac{2}{N}} \alpha_{r_i}^* \widetilde{F}_{\overline{x}}^* \omega_X(t)$ smoothly converge to $\frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + NG_{\overline{x}}^*\omega_D\right)$ on $\{z \in \mathbb{C} : |z| < S\} \times \Delta^{n-1}$. This is a contradiction. □

4.2 Parabolic rescaling of bulging asymptotics

As mentioned in the introduction and seen in Proposition 4.12, if the initial metric has bulging standard spatial asymptotics then the divisor flow does not enter into the asymptotics of the Kähler-Ricci flow on X for any finite time interval. In order to see the divisor flow, we must rather do parabolic rescalings around points x_i that tend to spatial infinity in the initial time slice. The sectional curvature at x_i decays like $O\left(d_0(x_i,x_0)^{-\frac{2}{N+1}}\right)$, so the relevant time scale increases like $d_0(x_i,x_0)^{\frac{2}{N+1}}$.

In order to take a limit of the parabolic rescalings, we need uniform curvature estimates on balls centered around x_i of radius comparable to $d_0(x_i,x_0)^{\frac{1}{N+1}}$, and on a time interval comparable to $d_0(x_i,x_0)^{\frac{2}{N+1}}$. Such estimates would follow from pseudolocality, if we could apply it. Unfortunately, because of the shrinking circle fiber, the injectivity radius on such time-zero balls goes to zero as $i \to \infty$ even before rescaling, and even more so after rescaling. The curvature estimates from pseudolocality can definitely fail in such a situation [16, Section 4].

Because one cannot apply pseudolocality, it appears that one must make some assumption to ensure the needed sectional curvature bounds. There is some flexibility in the precise assumption made. We assume a decay of the Ricci curvature on the relevant spacetime region and show that it implies the needed sectional curvature bound. The Ricci curvature assumption implies a uniform biLipschitz bound within the parabolic ball. Using Appendix A, this gives the sectional curvature bound.

Proposition 4.15. *Let $\omega_X(\cdot)$ be a Kähler-Ricci flow defined for $t \in [0,\infty)$, with bounded curvature on compact time intervals. Suppose that $\omega_X(0)$ has bulging standard spatial asymptotics associated to $(\omega_D(0), N)$. Let $\omega_D(\cdot)$ denote the Ricci flow on D.*

Given $A < \infty$, suppose that

$$|\operatorname{Ric}(x,t)| = O\left(d_0(x,x_0)^{-\frac{2}{N+1}}\right) \tag{4.16}$$

uniformly on the spacetime region $\left\{(x,t) \;:\; t \;\leq\; A\, d_0(x,x_0)^{\frac{2}{N+1}}\right\}$. *Given* $\overline{x} \in D$, $F_{\overline{x}} : \Delta^* \times \Delta^{n-1} \to \overline{X}$ *and* $r > 0$, *consider the Ricci flow* $\omega_r(t) = r^{-\frac{2}{N}} \alpha_r^* \widetilde{F}_{\overline{x}}^* \omega_X(r^{\frac{2}{N}} t)$. *Then*

$$\lim_{r\to\infty} \omega_r(\cdot) = \frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + N G_{\overline{x}}^* \omega_D(\cdot)\right), \tag{4.17}$$

with smooth convergence on the product of the time interval $[0,A]$ *with any* $\{z \in \mathbb{C} : |z| < S\} \times \Delta^{n-1}$.

Proof. Let x_0 be the basepoint in X. As before, the Ricci flow equation and (4.11) imply that at any $x \in X$, the metrics $\omega_X(t)$ and $\omega_X(0)$ are $e^{\text{const.}\; t\, d_0(x,x_0)^{-\frac{2}{N+1}}}$-biLipschitz. Using the bulging standard spatial asymptotics of $\omega_X(0)$ and Proposition A.1 of the appendix (with the parameter r of the proposition equal to $d_0(x,x_0)^{\frac{1}{N+1}}$), the biLipschitz bounds imply a uniform curvature bound

$$|\operatorname{Rm}(x,t)| = O\left(d_0(x,x_0)^{-\frac{2}{N+1}}\right) \tag{4.18}$$

on $\left\{(x,t) \;:\; t \;\leq\; A\, d_0(x,x_0)^{\frac{2}{n+1}}\right\}$. Similarly, Proposition A.37 gives

$$|\nabla^k \operatorname{Rm}|(x,t) = O\left(d_0(x,x_0)^{-\frac{2(k+1)}{N+1}}\right). \tag{4.19}$$

Let $\overline{x} \in D$ and $F_{\overline{x}}$ be as in the statement of the proposition. The curvature bounds imply that if $r_i \to \infty$ then we can take a subsequence of $\{\omega_{r_i}(\cdot)\}_{i=1}^\infty$ that converges smoothly to a Ricci flow solution defined on $\mathbb{C} \times \Delta^{n-1} \times [0,A]$.

This process can be globalized with respect to the divisor D; c.f. [15, Proof of Theorem 7.1]. The result is that for an arbitrary $\overline{x} \in D$, there is a subsequence of the pointed Ricci flows $\left\{\left(r_i^{-\frac{2}{N}} \omega_X(r_i^{\frac{2}{N}} \cdot), F_{\overline{x}}(e^{-\frac{r_i^2}{2}}, 0)\right)\right\}_{i=1}^\infty$ that converges, in the sense of Ricci flows on étale groupoids, to a limiting Ricci flow $\omega_\infty(\cdot)$ on the étale groupoid $(\mathbb{R} \rtimes \mathbb{R}) \times \mathbb{R} \times D$, defined for $t \in [0,A]$. Here the $(\mathbb{R} \rtimes \mathbb{R})$-factor comes from the real factor in $\mathbb{C}$ and represents the fact that the rescaled circle factor in X collapses as $R \to \infty$. The $\mathbb{R}$-factor is the imaginary factor in $\mathbb{C}$ and represents the radial direction on X. The D-factor is the divisor. (Because D is compact, the choice of basepoint $\overline{x} \in D$ is irrelevant.) This limiting Ricci flow will have bounded curvature.

From the bulging standard spatial asymptotics of $\omega_X(0)$, on the unit space $\mathbb{C} \times D$ of the groupoid we have

$$\omega_\infty(0) = \frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + N \omega_D(0)\right). \tag{4.20}$$

From the uniqueness of Ricci flow solutions on étale groupoids with bounded curvature on compact time intervals [13], we conclude that

$$\omega_\infty(t) = \frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + N\omega_D(t)\right). \tag{4.21}$$

To return to the proof of the proposition, if the conclusion of the proposition is not true then there are some $\overline{x} \in D$, $F_{\overline{x}} : \Delta^* \times \Delta^{n-1} \to \overline{X}$, $S < \infty$ and a sequence $r_i \to 0$ with the property that even after passing to any subsequence of the r_i's, the Ricci flows $r_i^{-\frac{2}{N}} \alpha_{r_i}^* \widetilde{F}_{\overline{x}}^* \omega_X(r_i^{\frac{2}{N}} \cdot)$ do not converge smoothly to $\frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + NG_{\overline{x}}^*\omega_D(\cdot)\right)$ on the spacetime region $\{z \in \mathbb{C} : |z| < S\} \times \Delta^{n-1} \times [0, A]$. However, this contradicts the fact there is a subsequence of the r_i's so that the pointed Ricci flows $\left\{\left(r_i^{-\frac{2}{N}} \omega_X(r_i^{\frac{2}{N}} \cdot), F_{\overline{x}}(e^{-\frac{r_i^2}{2}}, 0)\right)\right\}_{i=1}^\infty$ smoothly converge in the pointed sense to the Ricci flow $\frac{1}{2}\left(\frac{N+1}{N}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + N\omega_D(\cdot)\right)$ on the unit space $\mathbb{C} \times D$ of the étale groupoid $(\mathbb{R} \rtimes \mathbb{R}) \times \mathbb{R} \times D$, for the time interval $[0, A]$. □

Proof of Theorem 1.3 : If the theorem is not true then after replacing $\{x_i\}_{i=1}^\infty$ by a subsequence, we can assume that no subsequence $\{x_{i_j}\}_{j=1}^\infty$ is such that $\left\{\left(X, (x_{i_j}, 0), \omega_{i_j}(\cdot)\right)\right\}_{j=1}^\infty$ has a limit given by (1.3).

Thinking of x_i as an element of $\overline{X}$, after passing to a subsequence, we can assume that $\lim_{i\to\infty} \overline{x}_i = \overline{x}$ for some $\overline{x} \in D$. Then for large i, we can find $(z_i, w_i) \in \Delta^* \times \Delta^{n-1}$ so that $x_i = F_{\overline{x}}(z_i, w_i)$, with $\lim_{i\to\infty} |z_i| = 0$.

Define $r_i \in \mathbb{R}^+$ by $e^{-\frac{r_i^2}{2}} = |z_i|$. Then $\lim_{i\to\infty} r_i = \infty$ and $\widetilde{F}_{\overline{x}}(\alpha_{r_i}(0), w_i) = F_{\overline{x}}(|z_i|, w_i)$. Applying Proposition 4.15 with $N = n$ gives,

$$\lim_{i\to\infty} \omega_{r_i}(\cdot) = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \left(\sqrt{-1}du \wedge d\overline{u} + nG_{\overline{x}}^*\omega_D(\cdot)\right). \tag{4.22}$$

This is a contradiction. □

Remark 4.23. The only role of the Ricci curvature bound (4.16) is to ensure an appropriate biLipschitz condition between $\omega_X(x, t)$ and $\omega_X(x, 0)$, in order to apply Proposition A.1 of the appendix. Other curvature conditions imply this. For example, given any continuous function $f : [0, \infty) \to [0, \infty)$, it would be enough to assume that

$$|\operatorname{Ric}(x, t)| \leq d_0(x_0, x)^{-\frac{2}{N+1}} f(d_0(x_0, x)^{-\frac{2}{N+1}} t). \tag{4.24}$$

Equation (4.16) is the special case when f is a constant function.

Remark 4.25. One can ask how generally the Ricci curvature assumption in Proposition 4.15 holds. It obviously holds if the initial metric is Ricci-flat, as in the work of Tian and Yau [23]. We expect that it also holds, at least, if the initial metric is a perturbation of the Ricci-flat metric.

4.3 Bulging superstandard spatial asymptotics

We now specialize to the case when D is ample. Let h be a Hermitian metric on L_D with positive curvature form. Let ω_D be the restriction, to D, of the curvature form associated to h. As before, σ is a holomorphic section of L_D with zero-set D. Let L_D^1 denote the unit circle bundle of L_D.

Definition 4.2. A Kähler metric ω_X on X has *bulging superstandard spatial asymptotics* associated to h if it has bulging standard spatial asymptotics, associated to (ω_D, N), and

$$\omega_X = \eta_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\frac{N+1}{2}\left(\log|\sigma|_h^{-2}\right)^{\frac{N+1}{N}} + H\right), \tag{4.26}$$

where

(i) $\eta_{\overline{X}}$ is a smooth closed $(1,1)$-form on $\overline{X}$, and

(ii) $H \in C^\infty(X) \cap L^\infty(X)$.

Note that in Definition 4.2, the choice of h does matter.

Proposition 4.27. *If $\overline{X}$ admits a Kähler metric then X admits a complete Kähler metric with bulging superstandard spatial asymptotics.*

Proof. Let $\omega_{\overline{X}}$ be a Kähler metric on $\overline{X}$. By assumption, $-\sqrt{-1}\partial\overline{\partial}\left(\log|\sigma|_h^2\right)$ is a positive $(1,1)$-form on $\overline{X}$. Put

$$\omega_X = \frac{N+1}{2}\sqrt{-1}\partial\overline{\partial}\left(\log|\sigma|_h^{-2}\right)^{\frac{N+1}{N}}. \tag{4.28}$$

We first claim that ω_X is a complete Kähler metric on X with bulging standard spatial asymptotics. To see this, we have

$$\begin{aligned}\omega_X =& \frac{(N+1)^2}{2N}\sqrt{-1}\left(\log|\sigma|_h^{-2}\right)^{\frac{1}{N}}\partial\overline{\partial}\left(\log|\sigma|_h^{-2}\right) + \\ & \frac{1}{2}\left(\frac{N+1}{N}\right)^2\left(\log|\sigma|_h^{-2}\right)^{\frac{1-N}{N}}\sqrt{-1}\partial\left(\log|\sigma|_h^{-2}\right)\wedge\overline{\partial}\left(\log|\sigma|_h^{-2}\right).\end{aligned} \tag{4.29}$$

Since $-\sqrt{-1}\partial\bar{\partial}\left(\log|\sigma|_h^2\right)$ is positive, it follows that ω_X is a Kähler form. One can check that it is complete.

To see that ω_X has bulging standard spatial asymptotics, given $\overline{x} \in D$, let $(z, w^1, \ldots, w^{n-1})$ be the local holomorphic coordinates for a neighborhood of $\overline{x}$ in $\overline{X}$ coming from $F_{\overline{x}}$. In this coordinate system, $|\sigma|_h^2 = az\overline{z}$ for some smooth positive function $a(z, w^1, \ldots, w^{n-1})$. On $H \times \Delta^{n-1}$, we have

$$\log|\sigma|_h^{-2} = \log\frac{1}{az\overline{z}} = \log\left(\frac{1}{ae^{\sqrt{-1}u}e^{-\sqrt{-1}\overline{u}}}\right) = \log a^{-1} - \sqrt{-1}u + \sqrt{-1}\overline{u}. \tag{4.30}$$

Then

$$\alpha_r^* \log |\sigma|_h^{-2} = \alpha_r^* \log a^{-1} - \sqrt{-1} r u + \sqrt{-1} r \overline{u} + r^2, \tag{4.31}$$

$$\alpha_r^* \partial \left(\log |\sigma|_h^{-2}\right) = \partial \alpha_r^* \log a^{-1} - \sqrt{-1} r du \tag{4.32}$$

and

$$\alpha_r^* \overline{\partial} \left(\log |\sigma|_h^{-2}\right) = \overline{\partial} \alpha_r^* \log a^{-1} + \sqrt{-1} r d\overline{u}. \tag{4.33}$$

It follows that

$$\lim_{r \to \infty} r^{-\frac{2}{N}} \alpha_r^* \widetilde{F}_{\overline{x}}^* \omega_X = \frac{1}{2} \left(\frac{N+1}{N} \right)^2 \left(\sqrt{-1} du \wedge d\overline{u} + N G_{\overline{x}}^* \omega_D \right). \tag{4.34}$$

Hence ω_X has bulging standard spatial asymptotics.

From (4.28), it is now clear that ω_X also has bulging superstandard spatial asymptotics. This proves the proposition. □

In the rest of this section, we assume that $N = n$, the complex dimension of X, The next proposition shows that the property of having bulging superstandard spatial asymptotics is preserved under the Kähler-Ricci flow.

Proposition 4.35. *Suppose that $\omega_X(0)$ has bulging superstandard spatial asymptotics associated to h. Suppose that the Kähler-Ricci flow $\omega_X(t)$, with initial Kähler metric $\omega_X(0)$, exists on a maximal time interval $[0,T)$ in the sense of Theorem 2.1. Then for all $t \in [0,T)$, $\omega_X(t)$ has bulging superstandard spatial asymptotics, associated to h.*

Proof. Choose a Hermitian metric $h_{K_{\overline{X}} \otimes L_D}$ on $K_{\overline{X}} \otimes L_D$. Along with h, we obtain a Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Then

$$\mathrm{Ric}(\omega_X(0)) = -\sqrt{-1} F \left(h_{K_{\overline{X}} \otimes L_D}\right) - \sqrt{-1} \partial \overline{\partial} \left(\log \frac{h_{K_{\overline{X}}} |\sigma|_h^2}{h_{K_X}} \right) \tag{4.36}$$

on X.

Put $\eta'_{\overline{X}} = -\sqrt{-1} F \left(h_{K_{\overline{X}} \otimes L_D}\right)$ and $H' = \log \frac{h_{K_{\overline{X}}} |\sigma|_h^2}{h_{K_X}}$. By equation (4.4), the bulging standard spatial asymptotics imply that $H' \in C^\infty(X) \cap L^\infty(X)$. This is the place where we use that $N = n$.

Recall the definition of ω_t from (2.2). We can write

$$\begin{aligned} \omega_X(t) =& \omega_t + \sqrt{-1} \partial \overline{\partial} u(t) \\ =& \eta_{\overline{X}} - t \eta'_{\overline{X}} + \sqrt{-1} \partial \overline{\partial} \left(\frac{n+1}{2} \left(\log |\sigma|_h^{-2}\right)^{\frac{n+1}{n}} + H + tH' + u(t) \right). \end{aligned} \tag{4.37}$$

The proposition follows. □

Proposition 4.38. *Suppose that $\omega_X(0)$ has bulging superstandard spatial asymptotics associated to h. Let $\eta_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of the cohomology class represented by the closed current*

$$\omega_X(0) - \sqrt{-1}\partial\overline{\partial}\left(\frac{n+1}{2}\left(\log|\sigma|_h^{-2}\right)^{\frac{n+1}{n}}\right) \tag{4.39}$$

on $\overline{X}$. Let $\eta'_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of $-2\pi[K_{\overline{X}} + D] \in \mathrm{H}^{(1,1)}(\overline{X})$. Let T_3 be the supremum (possibly infinite) of the numbers T' for which there is some $f_{T'} \in C^\infty(X) \cap L^\infty(X)$ with bounded covariant derivatives (with respect to $\omega_X(0)$) so that

$$\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(\frac{n+1}{2}\left(\log|\sigma|_h^{-2}\right)^{\frac{n+1}{n}} + f_{T'}\right) \tag{4.40}$$

is a Kähler form on X which is biLipschitz to $\omega_X(0)$. Then T_3 equals the numbers $T_1 = T_2$ of Theorem 2.1.

Proof. The proof is similar to that of Proposition 3.29. We omit the details. □

Corollary 4.41. *Suppose that $\omega_X(0)$ has bulging superstandard spatial asymptotics. If $[K_{\overline{X}} + D] \geq 0$ then the flow exists for all positive time.*

Proof. The proof is similar to that of Corollary 3.39. We omit the details. □

Remark 4.42. In [23, Section 4], Tian and Yau showed that if $K_{\overline{X}} + D = 0$ then there is a complete Ricci-flat Kähler metric on X.

5 Conical Kähler-Ricci flows

This section deals with conical spatial asymptotics. We begin by defining asymptotically conical Riemannian metrics. In Subsection 5.1 we show that this property is preserved under Ricci flow, with no change in the asymptotic cone. In Subsection 5.2 we use pseudolocality to show that if the initial metric of an immortal solution is asymptotically conical then a parabolic blowdown limit always exists, defined on the complement of the vertex in a cone.

Passing to the Kähler case, in Subsection 5.3 we define conical standard spatial asymptotics and show that this property is preserved under the Kähler-Ricci flow. Starting with Subsection 5.4, we assume that D is ample. We introduce conical superstandard spatial asymptotics. We show that if $\overline{X}$ admits a Kähler metric then X admits a metric with conical superstandard spatial asymptotics. We prove that having conical superstandard spatial asymptotics is preserved under the Kähler-Ricci flow. We characterize the first singularity time, if there is one.

In Subsection 5.5 we consider an ample line bundle E over a compact complex manifold D. We show that there is a unique formal asymptotic expansion for an expanding Kähler soliton on the complement of the zero-section of E, whose associated vector field generates rescaling of the line bundle. The soliton turns out to be a gradient soliton. Given an asymptotic expansion for a Kähler-Ricci flow on the complement of the zero-section, we show that its blowdown limit is the expanding soliton. We apply this to the case of conical asymptotics.

5.1 Asymptotically conical metrics

Let (Y, g_Y) be a compact Riemannian manifold. Let CY denote the cone over Y, i.e. $CY = ((0,\infty) \times Y) \cup \{\star\}$, with the metric $g_{CY} = dR^2 + R^2 g_Y$ on $CY - \star$.

Let X be a smooth manifold with a basepoint x_0. Let g_X be a complete Riemannian metric on X. Given $0 < R_1 < R_2 < \infty$, put $A(R_1, R_2) = \overline{B(x_0, R_2)} - B(x_0, R_1)$.

Definition 5.1. We say that (X, x_0, g_X) is *asymptotically conical*, with asymptotic cone (CY, g_{CY}), if there is a pointed Gromov-Hausdorff limit $\lim_{\lambda\to\infty} \left(\frac{1}{\lambda}X, x_0, g\right) = (CY, \star, g_{CY})$ so that for $0 < R_1 < R_2 < \infty$, there is a smooth limit

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} A(\lambda R_1, \lambda R_2) = \overline{B(\star, R_2)} - B(\star, R_1). \tag{5.1}$$

In Definition 5.1, it is implicit that the same maps are used to define the Gromov-Hausdorff limit and the smooth limits. From the definition, (X, g_X) automatically has quadratic curvature decay.

Remark 5.2. If we instead assumed that (X, g_X) has quadratic curvature decay, Y is a compact metric space whose Hausdorff dimension is $\dim(X) - 1$, and that there is a pointed Gromov-Hausdorff limit $\lim_{\lambda\to\infty} \left(\frac{1}{\lambda}X, x_0, g\right) = (CY, \star, g_{CY})$, then we would conclude that Y is a smooth manifold with a $C^{1,\alpha}$-regular Riemannian metric g_Y. For simplicity, in Definition 5.1 we will just assume that g_Y is smooth.

Proposition 5.3. *Let $g_X(\cdot)$ be a Ricci flow that exists for $t \in [0, T)$ and whose initial condition $g_X(0)$ is asymptotically conical with asymptotic cone CY. Suppose that $g_X(\cdot)$ has complete time slices and bounded curvature on compact time intervals. Then for all $t \in [0, T)$, the time slice $(X, g(t))$ is asymptotically conical with asymptotic cone CY.*

Proof. Suppose that the conclusion of the proposition is not true. Then for some $t \in [0, T)$, there is a sequence $\lambda_i \to \infty$ with the property that even after passing to any subsequence of the λ_i's, either

(i) The sequence $\left\{ \left(\frac{1}{\lambda_i} X, x_0, g(t) \right) \right\}_{i=1}^{\infty}$ does not have a pointed Gromov-Hausdorff limit isometric to CY, or

(ii) There are some $0 < R_1 < R_2 < \infty$ so that the sequence $\left\{ \left(\frac{1}{\lambda_i} A(\lambda_i R_1, \lambda_i R_2) \right) \right\}_{i=1}^{\infty}$ does not have a smooth limit.

Let ϕ be a slight smoothing of the function $1+d_0(\cdot, x_0)$. Because of the conical asymptotics, we can assume that the Hessian of ϕ is bounded in norm. Then ϕ is a distance-like function that satisfies (B.4). From Proposition B.10, on the time interval $[0,t]$, the curvature of X decays uniformly as $O\left(d_0(x,x_0)^{-2}\right)$. It follows from the Ricci flow equation that at a point $x \in X$, the metrics $g_X(t)$ and $g_X(0)$ are $e^{\mathrm{const.}\, t d_0(x,x_0)^{-2}}$-biLipschitz to each other. Hence $\lim_{i\to\infty} \left(\frac{1}{\lambda_i} X, x_0, g(t) \right) = CY$ in the pointed Gromov-Hausdorff topology, so we can assume that (ii) holds.

Since $g_X(0)$ is asymptotically conical, given a multi-index k, we have $|\nabla^k \mathrm{Rm}|(x,0) = O\left(d_0(x,x_0)^{-2-k}\right)$. From Proposition B.12, $|\nabla^k \mathrm{Rm}|(x,t) = O\left(d_0(x,x_0)^{-2-k}\right)$. This implies that there are uniform bounds for $|\nabla^k \mathrm{Rm}|_{\frac{1}{\lambda_i^2} g_X(t)}$ on $A\left(\frac{1}{2}R_1, 2R_2\right) \subset \left(\frac{1}{\lambda_i} X, x_0\right)$. After passing to a subsequence and using the noncollapsing, we can assume that there is smooth convergence as $i \to \infty$ of the metrics $\frac{1}{\lambda_i^2} g_X(t)$ on $A\left(R_1, R_2\right) \subset \left(\frac{1}{\lambda_i} X, x_0\right)$. This is a contradiction. □

5.2 Parabolic blowdowns of asymptotically conical Ricci flows

We now use pseudolocality to show that we have the curvature bounds needed to take a blowdown limit of an immortal Ricci flow solution whose initial metric is asymptotically conical.

Proposition 5.4. *Suppose that the Ricci flow $g_X(\cdot)$ of Proposition 5.3 is defined for all $t \geq 0$. Then there are $R < \infty$ and $\epsilon > 0$ so that $|\mathrm{Rm}(x,t)| \leq (\epsilon d_0(x,x_0))^{-2}$ whenever $d_0(x,x_0) \geq R$ and $t \leq \epsilon d_0(x,x_0)^2$.*

Proof. From Definition 5.1, there are $C, R' < \infty$ so that $|\mathrm{Rm}(x,0)| \leq C d_0(x,x_0)^{-2}$ whenever $d_0(x,x_0) \geq R'$. Hence we can find $\alpha > 0$ so that whenever $d_0(x,x_0) \geq 2R'$, we have $|\mathrm{Rm}| \leq (\alpha d_0(x,x_0))^{-2}$ on $B_0(x, \alpha d_0(x,x_0))$. From [16, Proposition 1], there is some $\epsilon_0 > 0$ so that $|\mathrm{Rm}(x,t)| \leq (\epsilon_0 \alpha d_0(x,x_0))^{-2}$ whenever $d_0(x,x_0) \geq 2R'$ and $t \leq (\epsilon_0 \alpha d_0(x,x_0))^2$. After redefining the constants, the proposition follows. □

Proposition 5.5. *There are $R < \infty$ and $\epsilon > 0$ such that for each $k > 0$, there is some $C_k < \infty$ so that $|\nabla^k \mathrm{Rm}| \leq C_k d_0(x,x_0)^{-2-k}$ whenever $d_0(x,x_0) \geq R$ and $t \leq \epsilon d_0(x,x_0)^2$.*

Proof. Given $k > 0$, Proposition 5.4 and Shi's local derivative estimates imply that for any $\epsilon_k \in (0,\epsilon)$, there is a bound $|\nabla^k \mathrm{Rm}| \leq C'_k(\epsilon_k) d_0(x,x_0)^{-2-k}$ whenever $d_0(x,x_0) \geq 2R$ and $\epsilon_k d_0(x,x_0)^2 \leq t \leq \epsilon d_0(x,x_0)^2$. Now by the smooth convergence in Definition 5.1, we know that there is some $C'_k < \infty$ so that $|\nabla^k \mathrm{Rm}|(x,0) \leq C'_k d_0(x,x_0)^{-2-k}$ whenever

$d_0(x, x_0) \geq R$. From the local derivative estimate in [14, Appendix D], there are some $\epsilon_k > 0$ and $C''_k < \infty$ so that $|\nabla^k \operatorname{Rm}| \leq C''_k d_0(x, x_0)^{-2-k}$ whenever $d_0(x, x_0) \geq 2R$ and $t \leq \epsilon_k d_0(x, x_0)^2$. The proposition follows. □

Proposition 5.6. *Let ϵ be as in Proposition 5.5. For any sequence $r_i \to \infty$, after passing to a subsequence there is a smooth Ricci flow solution g_∞ defined on $\{(x, t) \in (CY - \star) \times [0, \infty) : 0 \leq t \leq \epsilon d(x, \star)^2\}$ so that*

(i) $g_\infty(0) = g_{CY}$ *and*

(ii) $\lim_{i\to\infty} \frac{1}{r_i^2} g(r_i^2 t) = g_\infty(t)$, *with smooth convergence on annuli.*

More precisely, given $0 < R_1 < R_2 < \infty$, for large i there are smooth embeddings $\phi_{R_1,R_2,i} : (R_1, R_2) \times Y \to X$ so that $\lim_{i\to\infty} \frac{1}{r_i^2} \phi^*_{R_1,R_2,i} g(r_i^2 t) = g_\infty(t)$, provided that $t \leq \epsilon R_2^2$.

Proof. This follows from Proposition 5.5 and the proof of Hamilton's compactness theorem [11]. In our case, we apply a diagonal argument to annuli; c.f. [11, Section 2]. Note that $g_\infty(0)$ is defined on the complement of the vertex in the asymptotic cone of $(X, g_X(0))$. The asymptotic cone is CY. □

Proposition 5.7. *For any $t \geq 0$, $g_\infty(t)$ is asymptotically conical with asymptotic cone CY.*

Proof. We first remark that the estimate $|\nabla^k \operatorname{Rm}| \leq C_k d_0(x, x_0)^{-2-k}$ on $g_X(0)$, for $k \geq 0$, passes to g_∞. Hence for any $t \geq 0$ and any sequence $s_j \to \infty$, after passing to a subsequence the limit $\lim_{j\to\infty} \left(X, x_0, \frac{1}{s_j^2} g_\infty(t)\right) = CY$ is smooth away from $\star \in CY$. At $x \in CY - \star$, the metrics $g_\infty(t)$ and $g_\infty(0)$ are $e^{\text{const.}\, t d_{CY}(x,\star)^{-2}}$-biLipschitz equivalent. It follows that any asymptotic cone of $g_\infty(t)$ is isometric to the asymptotic cone of $g_\infty(0)$, which is CY. □

Example 5.8. Suppose that (X, g_0) is an asymptotically conical Ricci-flat manifold. Its asymptotic cone is the Ricci-flat cone CY. The Ricci flow starting from g_0 is the static Ricci flow $g_X(t) = g_0$. The blowdown limit is the static Ricci flow $g_\infty(t) = g_{CY}$. This is an expanding gradient soliton with respect to the function $f = \frac{R^2}{4t}$.

Example 5.9. Consider the metrics constructed in [9, Section 5]. They live on a n-dimensional complex manifold X, $n \geq 2$, which is a complex line bundle over $\mathbb{C}P^{n-1}$. Given $k > n$ and $p \in \mathbb{R}^+$, there is a Ricci flow solution whose metric on $X - \mathbb{C}P^{n-1}$ is the $\mathbb{Z}_k$-quotient of the following metric on $\mathbb{C}^n$:

$$\begin{aligned} g_X(t) = &\left\{ |z|^{-2+2p} B\left(4(t+t_0)|z|^{-2p}\right) \delta_{\alpha\overline{\beta}} + \right. \\ &\left. \left[(p-1)|z|^{2p} B\left(4(t+t_0)|z|^{-2p}\right) - 4(t+t_0) p B'\left(4(t+t_0)|z|^{-2p}\right)\right] |z|^{-4} z^{\overline{\alpha}} z^{\beta} \right\} dz^\alpha \, dz^{\overline{\beta}}. \end{aligned} \tag{5.10}$$

Here $t_0 > 0$ and B is a certain smooth function with $B(0) > 0$. Going out the end of X corresponds to taking $z \to \infty$. For all $t \geq 0$, the metric $g_X(t)$ is asymptotically conical with asymptotic cone $CY = \mathbb{C}^n/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ action on $\mathbb{C}^n$ is multiplication by scalars, and the metric on the asymptotic cone is

$$g_{CY} = B(0) \left\{ |z|^{-2+2p} \delta_{\alpha\overline{\beta}} + (p-1)|z|^{2p}|z|^{-4} z^{\overline{\alpha}} z^{\beta} \right\} dz^{\alpha}\, dz^{\overline{\beta}}. \tag{5.11}$$

Given a sequence $r_i \to \infty$, put $\phi_i(z) = r_i^{\frac{1}{p}} z$. Then the blowdown limit is

$$\lim_{i\to\infty} \frac{1}{r_i^2} \phi_i^* g_X(r_i^2 t) = g_\infty(t), \tag{5.12}$$

where

$$\begin{aligned} g_\infty(t) = &\left\{ |z|^{-2+2p} B\left(4t|z|^{-2p}\right) \delta_{\alpha\overline{\beta}} + \right. \\ &\left. \left[(p-1)|z|^{2p} B\left(4t|z|^{-2p}\right) - 4tpB'\left(4t|z|^{-2p}\right) \right] |z|^{-4} z^{\overline{\alpha}} z^{\beta} \right\} dz^{\alpha}\, dz^{\overline{\beta}}. \end{aligned} \tag{5.13}$$

For $t > 0$, this is an gradient expanding soliton solution. At $t = 0$, we have $g_\infty(0) = g_{CY}$.

Example 5.14. With reference to Example 5.8, in the case $n = 2$ let g_0 be a k-center Eguchi-Hanson metric [10]. Then $CY = \mathbb{C}^2/\mathbb{Z}_k$, where a generator of $\mathbb{Z}_k$ acts on $\mathbb{C}^2$ by $\begin{pmatrix} e^{\frac{2\pi\sqrt{-1}}{k}} & 0 \\ 0 & e^{\frac{-2\pi\sqrt{-1}}{k}} \end{pmatrix}$, and $g_\infty(\cdot)$ is the static flow on $CY - \star$.

Example 5.15. In [19], it was shown that if $g_X(0)$ has nonnegative curvature operator then any blowdown limit is a *smooth* gradient expanding soliton. In [1, Remark 7.3], this result was extended to nonnegative complex sectional curvture. In [5], it was shown that in the Kähler case, if one assumes positive holomorphic bisectional curvature then the soliton is $U(n)$-invariant.

5.3 Conical standard spatial asymptotics

We now specialize to the Kähler case. Let $\overline{X}$ be a compact connected n-dimensional complex manifold. Let D be a smooth effective divisor in $\overline{X}$. Let L_D be the holomorphic line bundle on $\overline{X}$ associated to D. There is a holomorphic section σ of L_D with zero set D, which is nondegenerate at D. It is unique up to multiplication by a nonzero complex number. Let h be a Hermitian metric on L_D. Let ∇ be the corresponding Hermitian holomorphic connection. Let ∇_D be its restriction to D. Let L_D^1 denote the unit circle bundle of L_D.

Suppose that $\overline{x} \in D$. There are a neighborhood U of $\overline{x}$ in $\overline{X}$ and a biholomorphic map $F_{\overline{x}} : \Delta^n \to U$ so that

(i) $F_{\overline{x}}(0) = \overline{x}$, and

(ii) $F_{\overline{x}}(\Delta^* \times \Delta^{n-1}) = U \cap X$.

The map $G_{\overline{x}}$ on Δ^{n-1}, given by $G_{\overline{x}}(w) = F_{\overline{x}}(0, w)$, is a biholomorphic map from Δ^{n-1} to a neighborhood of $\overline{x}$ in D. Let z the local coordinate on U coresponding to the first factor in Δ^n.

Given $r \in \mathbb{R}^+$, let $\alpha_r : \Delta^* \to \Delta^*$ be multiplication by r. If Z is an auxiliary space then we also write α_r for $(\alpha_r, \mathrm{Id}) : \Delta^* \times Z \to \mathbb{C} \times Z$.

Definition 5.2. Let ω_D be a Kähler metric on D. Let h_D be the restriction of h to $L_D\big|_D$. We identify z with a local multiple of σ. We say that ω_X has *conical standard spatial asymptotics* associated to (ω_D, h_D) if for every $\overline{x} \in D$ and every local parametrization $F_{\overline{x}}$, after possibly multiplying z by a constant, we have

$$\lim_{r \to 0} \frac{1}{r^2} \alpha_r^* F_{\overline{x}}^* \omega_X = \sqrt{-1} \frac{h_D(\nabla_D z \wedge \overline{\nabla_D z})}{|z|_h^4} + \frac{G_{\overline{x}}^* \omega_D}{|z|_h^2}. \tag{5.16}$$

The limit in (5.16) means smooth convergence on any subset $\{z \in \Delta^* : 0 < R_1 < |z| < R_2 < 1\} \times \Delta^{n-1}$ of $\Delta^* \times \Delta^{n-1}$.

Going out the end of X corresponds to taking $z \to 0$. One can check that the notion of conical standard spatial asymptotics in Definition 5.2 is consistent under change of local coordinate.

Proposition 5.17. *If (X, ω_X) has conical standard spatial asymptotics then it is asymptotically conical in the sense of Definition 5.1. The asymptotic cone is CY, where Y is the unit circle bundle L^1_D.*

Proof. In terms of the local coordinates $(w^1, \ldots, w^{n-1})$ on Δ^{n-1}, there is a function $h(w)$ so that $h_D(z, z) = h(w)|z|^2$. Then

$$\begin{aligned} \sqrt{-1} \frac{h_D(\nabla_D z \wedge \overline{\nabla_D z})}{|z|_h^4} + \frac{G_{\overline{x}}^* \omega_D}{|z|_h^2} &= \sqrt{-1} \frac{h(dz + z h^{-1} \partial_w h) \wedge (d\overline{z} + \overline{z} h^{-1} \partial_{\overline{w}} h)}{(h|z|^2)^2} + \frac{\omega_D}{h|z|^2} \\ &= \sqrt{-1} \frac{\left(\frac{dz}{z} + \frac{\partial_w h}{h}\right) \wedge \left(\frac{d\overline{z}}{\overline{z}} + \frac{\partial_{\overline{w}} h}{h}\right)}{h|z|^2} + \frac{\omega_D}{h|z|^2}. \end{aligned} \tag{5.18}$$

Write $z = \alpha e^{\sqrt{-1}\theta}$, so

$$\frac{dz}{z} = \frac{d\alpha}{\alpha} + \sqrt{-1} d\theta. \tag{5.19}$$

Define R by

$$R^2 = \frac{1}{h|z|^2} = \frac{1}{h\alpha^2}. \tag{5.20}$$

Then

$$\frac{dR}{R} = -\frac{d\alpha}{\alpha} - \frac{1}{2} \frac{\partial_w h}{h} - \frac{1}{2} \frac{\partial_{\overline{w}} h}{h}. \tag{5.21}$$

It follows that

$$\frac{\left(\frac{dz}{z}+\frac{\partial_w h}{h}\right)\left(\frac{d\overline{z}}{\overline{z}}+\frac{\partial_{\overline{w}} h}{h}\right)}{h|z|^2}+\frac{g_D}{h|z|^2}=dR^2+R^2\left\{\left[d\theta+\frac{1}{2\sqrt{-1}}\left(\frac{\partial_w h}{h}-\frac{\partial_{\overline{w}} h}{h}\right)\right]^2+g_D\right\}. \tag{5.22}$$

The right-hand side of (5.22) is the conical metric on $CY-\star$, where $Y=L^1_D$ has the Sasaski metric. □

Proposition 5.23. *Let $\omega_X(\cdot)$ be a Kähler-Ricci flow defined for $t\in[0,T)$, with bounded curvature on compact time intervals. Suppose that $\omega_X(0)$ has conical standard spatial asymptotics associated to (ω_D,h_D). Then for all $t\in[0,T)$, the metric $\omega_X(t)$ has conical standard spatial asymptotics associated to (ω_D,h_D).*

Proof. The proof is similar to that of Proposition 4.12. We omit the details. □

5.4 Conical superstandard spatial asymptotics

We now assume that D is ample. Let h be a Hermitian metric on L_D with positive curvature form. Let ω_D be the restriction, to D, of the curvature form associated to h. As before, σ is a holomorphic section of L_D with zero-set D.

Definition 5.3. A Kähler metric ω_X on X has *conical superstandard spatial asymptotics* associated to h and a number $k\in\mathbb{R}$ if it has conical standard spatial asymptotics (associated to (ω_D,h_D)) and

$$\omega_X=\eta_{\overline{X}}+\sqrt{-1}\partial\overline{\partial}\left(|\sigma|_h^{-2}+k\log|\sigma|_h^{-2}+H\right), \tag{5.24}$$

where

(i) $\eta_{\overline{X}}$ is a smooth closed $(1,1)$-form on $\overline{X}$, and

(ii) $H\in C^\infty(X)\cap L^\infty(X)$.

Remark 5.25. In [24], in order to construct Kähler-Einstein metrics, a class $\sqrt{-1}\partial\overline{\partial}|\sigma|_h^{-2\alpha}$ of model metrics was considered. These metrics are also asymptotically conical. In terms of the asymptotic cone CY, the manifold Y is again a circle bundle over D and the parameter α determines the length of the circle fiber in Y. For simplicity, we only consider the case $\alpha=1$. The discussion below can be easily extended to general $\alpha>0$.

Specializing the results of [24] to the case when $\alpha=1$, they showed that if D admits a Kähler metric ω_D with $\mathrm{Ric}(\omega_D)=n\omega_D$ then there is a complete Ricci-flat Kähler metric on X [24].

Note that $\sqrt{-1}\partial\overline{\partial}\log|\sigma|_h^{-2}$ is bounded with respect to $g_{\overline{X}}$.

Proposition 5.26. *If $\overline{X}$ admits a Kähler metric then X admits a complete Kähler metric with conical superstandard spatial asymptotics.*

Proof. Let $\omega_{\overline{X}}$ be a Kähler metric on $\overline{X}$. By assumption, $\sqrt{-1}F_h = -\sqrt{-1}\partial\bar{\partial}\left(\log|\sigma|_h^2\right)$ is a positive $(1,1)$-form on $\overline{X}$. Put

$$\omega_X = \sqrt{-1}\partial\overline{\partial}|\sigma|_h^{-2} = \sqrt{-1}\frac{h(\nabla\sigma \wedge \overline{\nabla\sigma})}{|\sigma|_h^4} + \frac{\sqrt{-1}F_h}{|\sigma|_h^2}. \tag{5.27}$$

Taking the local coordinate z to be σ,

$$\lim_{r\to 0}\frac{1}{r^2}\alpha_r^* F_{\overline{x}}^*\omega_X = \sqrt{-1}\frac{h_D(\nabla_D z \wedge \overline{\nabla_D z})}{|z|_h^4} + \frac{\omega_D}{|z|_h^2} \tag{5.28}$$

Hence ω_X has conical standard spatial asymptotics. It clearly also has conical superstandard spatial asymptotics. □

We now show that the property of having conical superstandard spatial asymptotics is preserved under the Kähler-Ricci flow.

Proposition 5.29. *Suppose that $\omega_X(0)$ has conical superstandard spatial asymptotics associated to (h,k). Suppose that the Kähler-Ricci flow $\omega_X(t)$, with initial Kähler metric $\omega_X(0)$, exists on a maximal time interval $[0,T)$ in the sense of Theorem 2.1. Then for all $t\in[0,T)$, $\omega_X(t)$ has conical superstandard spatial asymptotics, associated to $(h,k+nt)$.*

Proof. Choose a Hermitian metric $h_{K_{\overline{X}}\otimes L_D}$ on $K_{\overline{X}}\otimes L_D$. Along with h, we obtain a Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Then

$$\mathrm{Ric}(\omega_X(0)) = -\sqrt{-1}F\left(h_{K_{\overline{X}}\otimes L_D}\right) - \sqrt{-1}\partial\overline{\partial}\left(n\log|\sigma|_h^{-2} + \log\frac{h_{K_{\overline{X}}}|\sigma|_h^{2(n+1)}}{h_{K_X}}\right) \tag{5.30}$$

on X.

Put $\eta'_{\overline{X}} = -\sqrt{-1}F\left(h_{K_{\overline{X}}\otimes L_D}\right)$ and $H' = \log\frac{h_{K_{\overline{X}}}|\sigma|_h^{2(n+1)}}{h_{K_X}}$. By equation (5.16), the conical standard spatial asymptotics imply that $H'\in C^\infty(X)\cap L^\infty(X)$.

Recall the definition of ω_t from (2.2). We can write

$$\begin{aligned}\omega_X(t) &= \omega_t + \sqrt{-1}\partial\overline{\partial}u(t) \qquad (5.31)\\ &= \eta_{\overline{X}} - t\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(|\sigma|_h^{-2} + (k+nt)\log|\sigma|_h^{-2} + H + tH' + u(t)\right).\end{aligned}$$

The proposition follows. □

We now give a characterization of the first singularity time, if there is one. The relevant ring of functions, for conical asymptotics, can be characterized in the following way.

Definition 5.4. The ring $C^\infty_{\mathrm{cone}}(X)$ consists of the smooth functions f on $X = \overline{X} - D$ so that for every $\overline{x} \in D$ and every local parametrization $F_{\overline{x}}$, the pullback $F^*_{\overline{x}} f \in C^\infty(\Delta^* \times \Delta^{n-1})$ is such that for any multi-index $(l_1, \bar{l}_1, \dots, l_n, \bar{l}_n)$, the function

$$\left(z^2 \frac{\partial}{\partial z}\right)^{l_1} \left(\overline{z}^2 \frac{\partial}{\partial \overline{z}}\right)^{\bar{l}_1} \left(z \frac{\partial}{\partial w^1}\right)^{l_2} \left(\overline{z} \frac{\partial}{\partial \overline{w}^1}\right)^{\bar{l}_2} \dots \left(z \frac{\partial}{\partial w^{n-1}}\right)^{l_n} \left(\overline{z} \frac{\partial}{\partial \overline{w}^{n-1}}\right)^{\bar{l}_n} F^*_{\overline{x}} f$$

is uniformly bounded.

Proposition 5.32. *Suppose that $\omega_X(0)$ has conical superstandard spatial asymptotics associated to (h, k). Let $\eta_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of the cohomology class represented by the closed current*

$$\omega_X(0) - \sqrt{-1}\partial\overline{\partial}\left(|\sigma|_h^{-2} + k \log |\sigma|_h^{-2}\right) \tag{5.33}$$

on $\overline{X}$. Let $\eta'_{\overline{X}} \in \Omega^{(1,1)}(\overline{X})$ be a smooth representative of $-2\pi[K_{\overline{X}} + D] \in \mathrm{H}^{(1,1)}(\overline{X})$. Let T_3 be the supremum (possibly infinite) of the numbers T' for which there is some $f_{T'} \in C^\infty_{\mathrm{cone}}(X)$ so that

$$\eta_{\overline{X}} - T'\eta'_{\overline{X}} + \sqrt{-1}\partial\overline{\partial}\left(|\sigma|_h^{-2} + (k + nT') \log |\sigma|_h^{-2} + f_{T'}\right) \tag{5.34}$$

is a Kähler form on X which is biLipschitz to $\omega_X(0)$. Then T_3 equals the numbers $T_1 = T_2$ of Theorem 2.1.

Proof. The proof is similar to that of Proposition 3.29. We omit the details. □

Corollary 5.35. *Suppose that $\omega_X(0)$ has conical superstandard spatial asymptotics. If $[K_{\overline{X}} + (n+1)D] \geq 0$ then the flow exists for all positive time.*

Proof. The proof is similar to that of Corollary 3.39. We omit the details. □

5.5 Formal asymptotics

In this subsection we discuss the asymptotics of the Kähler-Ricci flow on the complement of the zero-section in the total space of a line bundle. We then apply this to the quasiprojective case, where the relevant line bundle is the normal bundle to the divisor.

Let $\pi : E \to D$ be an ample holomorphic line bundle E over a complex manifold D. Let h be a Hermitian metric on E with curvature 2-form F, so that the representative $\sqrt{-1}F$ of $2\pi c_1(E)$ is a Kähler form ω_D. In terms of a local holomorphic section σ of E, we have $\omega_D = -\sqrt{-1}\partial\overline{\partial} \log |\sigma|_h^2$.

There is a canonical section S of the bundle π^*E over E so that for $e \in E$, we have $S(e) = e \in (\pi^*E)_e \cong E_{\pi(e)}$. Let E' be the complement of the zero-section of E. Define $\omega_0 \in \Omega^2(E')$ by $\omega_0 = \sqrt{-1}\partial\overline{\partial}|S|^{-2}_{\pi^*h}$. In terms of a local holomorphic trivialization

$E\big|_U = U \times \mathbb{C}$ of E over a coordinate chart U, let $\{w^\alpha\}_{\alpha=1}^{n-1}$ denote coordinates on U and let z denote the coordinate of the $\mathbb{C}$-factor. Then $|S|^2_{\pi^* h} = h(w) z\overline{z}$ and

$$\begin{aligned}\omega_0 &= \sqrt{-1}\partial\overline{\partial}\left(\frac{1}{hz\overline{z}}\right) \\ &= \frac{\omega_D}{hz\overline{z}} + \sqrt{-1}\frac{1}{(hz\overline{z})^2} h(dz + zh^{-1}\partial_w h) \wedge (d\overline{z} + \overline{z}h^{-1}\overline{\partial}_w h).\end{aligned} \tag{5.36}$$

There is a universal constant $C = C(n)$ so that

$$\omega_0^n = Ch^{-n}(z\overline{z})^{-(n+1)} dz \wedge d\overline{z} \wedge \omega_D^{n-1}. \tag{5.37}$$

Writing the metric in terms of the local coordinates $\{w^1, \ldots, w^{n-1}, z\}$ as $g_{i\overline{j}}$, the Ricci curvature of ω_0 is

$$\mathrm{Ric}(\omega_0) = -\sqrt{-1}\partial\overline{\partial}\log\det(g_{i\overline{j}}) = -\sqrt{-1}\partial\overline{\partial}\log\left(h^{-n}\det(g^D_{\alpha\overline{\beta}})\right) = \pi^*(\mathrm{Ric}(\omega_D) - n\omega_D). \tag{5.38}$$

Put $V = z\partial_z + \overline{z}\partial_{\overline{z}}$, which is globally defined on E'. One can check that $\mathcal{L}_V\omega_0 = -2\omega_0$ and $\mathcal{L}_V \mathrm{Ric}(\omega_0) = 0$. We use the notion of formal weight with respect to $\mathcal{L}_V$, which is the same as the grading in the Taylor series expansion of a function in terms of z and $\overline{z}$. Note that going out the conical end of E' corresponds to taking $z \to 0$. An expansion in z (and $\overline{z}$) is effectively an expansion in inverse powers of the distance from the basepoint.

The expanding soliton equation for ω, with respect to the vector field V, is

$$\mathrm{Ric}(\omega) + \frac{1}{2}\mathcal{L}_V\omega = -\omega. \tag{5.39}$$

Proposition 5.40. *Given ω_0, put $\omega = \omega_0 - \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}u$. There is a unique asymptotic expansion*

$$u \sim \sum_{k>0} u_{(k)}, \tag{5.41}$$

with $\mathcal{L}_V u_{(k)} = k u_{(k)}$, so that ω formally satisfies (5.39).

Proof. We have

$$\begin{aligned}\mathcal{L}_V\omega &= -2\omega_0 + \sqrt{-1}\sum_{k>0}\mathcal{L}_V\partial\overline{\partial}u_{(k)} \\ &= -2\omega_0 + \sqrt{-1}\sum_{k>0}\partial\overline{\partial}\mathcal{L}_V u_{(k)} \\ &= -2\omega_0 + \sqrt{-1}\sum_{k>0}k\partial\overline{\partial}u_{(k)}.\end{aligned} \tag{5.42}$$

Substituting into (5.39) gives

$$\begin{aligned}&\mathrm{Ric}\left(\omega_0 - \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}u\right) - \omega_0 + \sqrt{-1}\sum_{k>0}\frac{k}{2}\partial\overline{\partial}u_{(k)} = \\ &-\omega_0 + \mathrm{Ric}(\omega_0) - \sqrt{-1}\sum_{k>0}\partial\overline{\partial}u_{(k)},\end{aligned} \tag{5.43}$$

or

$$-\partial\overline{\partial}\log\frac{(\omega_0-\mathrm{Ric}(\omega_0)+\partial\overline{\partial}\sum_{k>0}u_{(k)})^n}{\omega_0^n}+\sum_{k>0}\frac{k}{2}\partial\overline{\partial}u_{(k)}= \tag{5.44}$$
$$-\sum_{k>0}\partial\overline{\partial}u_{(k)}.$$

Hence it suffices to solve

$$\log\frac{(\omega_0-\mathrm{Ric}(\omega_0)+\sqrt{-1}\partial\overline{\partial}\sum_{k>0}u_{(k)})^n}{\omega_0^n}=\sum_{k>0}\left(\frac{k}{2}+1\right)u_{(k)}. \tag{5.45}$$

Equivalently,

$$\mathrm{Tr}\log\left(I-\omega_0^{-1}\mathrm{Ric}(\omega_0)+\sqrt{-1}\omega_0^{-1}\partial\overline{\partial}\sum_{k>0}u_{(k)}\right)=\sum_{k>0}\left(\frac{k}{2}+1\right)u_{(k)}. \tag{5.46}$$

The term $\omega_0^{-1}\mathrm{Ric}(\omega_0)$ has formal weight 2 with respect to $\mathcal{L}_V$ and $\omega_0^{-1}\partial\overline{\partial}u_{(k)}$ has formal weight $k+2$. We can expand the left-hand side of (5.46) with respect to the $\mathcal{L}_V$-weighting, as

$$-\mathrm{Tr}\left(\omega_0^{-1}\mathrm{Ric}(\omega_0)\right)+\left[-\frac{1}{2}\mathrm{Tr}\left(\left(\omega_0^{-1}\mathrm{Ric}(\omega_0)\right)^2\right)+\mathrm{Tr}\left(\sqrt{-1}\omega_0^{-1}\partial\overline{\partial}u_{(2)}\right)\right]+\cdots. \tag{5.47}$$

It is easy to see directly that $u_{(2k-1)}=0$ for $k=1,2,\cdots$. Also

$$u_{(2)}=-\frac{1}{2}\mathrm{Tr}\left(\omega_0^{-1}\mathrm{Ric}(\omega_0)\right). \tag{5.48}$$

For $k=2,\cdots$, the term of weight $2k$ on the left-hand side of (5.46) can be expressed in terms of $u_{(2)},\ldots,u_{(2k-2)}$. Equating it with the term $(k+1)\,u_{(2k)}$ of weight $2k$ on the right-hand side determines $u_{(2k)}$ inductively in terms of $u_{(2)},\ldots,u_{(2k-2)}$. For example,

$$u_{(4)}=\frac{1}{3}\left[-\frac{1}{2}\mathrm{Tr}\left(\left(\omega_0^{-1}\mathrm{Ric}(\omega_0)\right)^2\right)+\mathrm{Tr}\left(\sqrt{-1}\omega_0^{-1}\partial\overline{\partial}u_{(2)}\right)\right]. \tag{5.49}$$

That gives the existence and uniqueness. □

Proposition 5.50. *The formal expanding soliton of Proposition 5.40 is a gradient expanding soliton.*

Proof. Let $V=V^{(1,0)}+V^{(0,1)}$ be the splitting of V into its $(1,0)$ and $(0,1)$ components, i.e. $V^{(1,0)}=z\partial_z$. We need to find a function F so that

$$i_{V^{(1,0)}}\omega \;=\; \sqrt{-1}\,\overline{\partial}F. \tag{5.51}$$

We first claim that ω is invariant under the $U(1)$-action given by multiplying z by complex numbers of norm one. To see this, note that ω_0 and $\mathrm{Ric}(\omega_0)$ are $U(1)$-invariant. Then from the inductive procedure to construct $u_{(k)}$ in the proof of Proposition 5.40, it follows that $u_{(k)}$ is $U(1)$-invariant. Hence ω is $U(1)$-invariant.

From the $U(1)$-invariance,

$$i_{V^{(1,0)}}\partial u_{(k)} = i_{V^{(0,1)}}\overline{\partial} u_{(k)}. \tag{5.52}$$

We know that

$$\mathcal{L}_V u_{(k)} = i_V du_{(k)} = i_{V^{(1,0)}}\partial u_{(k)} + i_{V^{(0,1)}}\overline{\partial} u_{(k)} = k u_{(k)}, \tag{5.53}$$

so

$$i_{V^{(1,0)}}\partial u_{(k)} = \frac{k}{2} u_{(k)}. \tag{5.54}$$

Then

$$i_{V^{(1,0)}}\left(\partial\overline{\partial} u_{(k)}\right) = \overline{\partial}\left(i_{V^{(1,0)}}\partial u_{(k)}\right) = \frac{k}{2}\overline{\partial} u_{(k)} \tag{5.55}$$

From the definition of ω_0,

$$i_{V^{(1,0)}}\omega_0 = \sqrt{-1} i_{V^{(1,0)}}\partial\overline{\partial}|S|^{-2}_{\pi^* h} = \sqrt{-1}\overline{\partial}\left(i_{V^{(1,0)}}\partial|S|^{-2}_{\pi^* h}\right) = \sqrt{-1}\overline{\partial}\left(-|S|^{-2}_{\pi^* h}\right). \tag{5.56}$$

From (5.38), we have $i_{V^{(1,0)}}\mathrm{Ric}(\omega_0) = 0$. Hence (5.51) is satisfied with

$$F = -|S|^{-2}_{\pi^* h} + \sum_{k>0} \frac{k}{2} u_{(k)}. \tag{5.57}$$

This proves the proposition. □

Now consider the Kähler-Ricci flow on E'. Put $\omega(t) = \omega_0 - t\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial} u(t)$. The flow equation for the potential function u is

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_0 - t\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial} u)^n}{\omega_0^n} = \mathrm{Tr}\log\left(I - t\omega_0^{-1}\mathrm{Ric}(\omega_0) + \sqrt{-1}\omega_0^{-1}\partial\overline{\partial} u\right). \tag{5.58}$$

Proposition 5.59. *There is a unique asymptotic expansion*

$$u(t) \sim \sum_{k=0}^{\infty} u_{(k)}(t), \tag{5.60}$$

where $\mathcal{L}_V u_{(k)} = k u_{(k)}$, so that $u(\cdot)$ formally satisfies (5.58). The blowdown limit $u^\infty(w,z,t) = \lim_{s\to\infty} s^{-2} u(w, s^{-1}z, s^2 t)$ exists and equals the Ricci flow generated by the gradient expanding soliton of Proposition 5.40.

Proof. Substituting (5.60) into (5.58) and equating the terms of various weights gives

$$\begin{aligned}\frac{\partial u_{(0)}}{\partial t} =&0,\\ \frac{\partial u_{(1)}}{\partial t} =&0,\\ \frac{\partial u_{(2)}}{\partial t} =&- t \operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right) + \sqrt{-1} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} u_{(0)}\right),\\ \frac{\partial u_{(3)}}{\partial t} =&\sqrt{-1} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} u_{(1)}\right),\\ \frac{\partial u_{(4)}}{\partial t} =&- \frac{1}{2}t^2 \operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right)^2 + \sqrt{-1}t \operatorname{Tr}\left[\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right)\left(\omega_0^{-1}\partial\overline{\partial} u_{(0)}\right)\right] +\\ &\frac{1}{2} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} u_{(0)}\right)^2 + \sqrt{-1} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} u_{(2)}\right),\\ &\vdots\end{aligned} \tag{5.61}$$

The solution is

$$\begin{aligned}u_{(0)} =&c_0,\\ u_{(1)} =&c_1,\\ u_{(2)} =&- \frac{1}{2}t^2 \operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right) + \sqrt{-1}t \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} c_0\right) + c_2,\\ u_{(3)} =&\sqrt{-1}t \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} c_1\right) + c_3,\\ u_{(4)} =&- \frac{1}{6}t^3 \left[\operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right)^2 + \sqrt{-1} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} \operatorname{Tr}(\omega_0^{-1} \operatorname{Ric}(\omega_0))\right)\right] + \ldots + c_4,\\ &\vdots\end{aligned} \tag{5.62}$$

where $\mathcal{L}_V c_k = k c_k$.

If $u^\infty(w, z, t) = \lim_{s\to\infty} s^{-2} u\left(w, s^{-1}z, s^2 t\right)$, then u^∞ has an asymptotic expansion

$$u^\infty(t) \sim \sum_{k=1}^{n} u^\infty_{(k)}(t), \tag{5.63}$$

with

$$\begin{aligned}u^\infty_{(0)} =&0,\\ u^\infty_{(1)} =&0,\\ u^\infty_{(2)} =&- \frac{1}{2}t^2 \operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right),\\ u^\infty_{(3)} =&0,\\ u^\infty_{(4)} =&- \frac{1}{6}t^3 \left[\operatorname{Tr}\left(\omega_0^{-1} \operatorname{Ric}(\omega_0)\right)^2 + \sqrt{-1} \operatorname{Tr}\left(\omega_0^{-1}\partial\overline{\partial} \operatorname{Tr}(\omega_0^{-1} \operatorname{Ric}(\omega_0))\right)\right],\\ &\vdots\end{aligned} \tag{5.64}$$

Note that the construction of $u^\infty(w,z,t)$ amounts to keeping only the terms in u with the highest power of t, i.e. $t^{\frac{k}{2}+1}$ for $u_{(k)}$. That is, we remove the terms involving the c_i's. Then one sees that $\omega_0 - \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}u^\infty(1)$ is the formal gradient expanding soliton of Proposition 5.40. Hence $\omega_0 - \mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}u^\infty(t)$ is the time-t solution for the flow of the expanding soliton. □

Proposition 5.65. *In the setting of Subsection 5.3, suppose that $(X, \omega_X(\cdot))$ is a Kähler-Ricci flow with conical standard spatial asymptotics, that exists on the time interval $[0,\infty)$. Suppose that there is an asymptotic expansion*

$$\omega_X(t) \sim \omega_0 - t\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\sum_{k=0}^{\infty}\partial\overline{\partial}u_{(k)}(t), \tag{5.66}$$

where $\mathcal{L}_V u_{(k)} = k u_{(k)}$, with $V = z\partial_z + \overline{z}\partial_{\overline{z}}$. Let $(CY - \star, \omega_{X_\infty}(\cdot))$ be a parabolic blowdown limit of $\omega_X(\cdot)$. Then the asymptotic expansion of $\omega_{X_\infty}(\cdot)$ is the formal gradient expanding soliton of Proposition 5.40.

Proof. Let E be the restriction of L_D to D, i.e. the normal bundle of D in $\overline{X}$. Then the proposition follows from Proposition 5.59. □

We have now proved Theorem 1.4.

Example 5.67. With reference to Example 5.9, consider the case when $p = 1$. The asymptotic cone is flat. The corresponding formal expanding soliton is also flat. This is consistent with the explicit solution of Example 5.9, which approaches the asymptotic cone exponentially fast. Note that in this case, the parabolic blowdown limit is also an expanding soliton, which differs from the formal expanding soliton of its asymptotic expansion.

If we took $p \neq 1$ in Example 5.9 (see Remark 5.25) then the asymptotic cone, and also the formal expanding soliton, would be nonflat.

A Local curvature estimates in Kähler-Ricci flow

In this section we prove a curvature estimate for the Kähler-Ricci flow. The assumptions are that the curvature and its first covariant derivative are bounded on an initial ball, and that on the given time interval, the metric on the ball is uniformly biLipschitz to the initial metric.

Proposition A.1. *Let $(X, p, \omega(\cdot))$ be a pointed Kähler-Ricci flow on a complex n-dimensional manifold X, defined on a time interval $[0,T]$, with possibly incomplete time slices. Given $C_1 < \infty$, there is some $C_2 = C_2(C_1, n) < \infty$ with the following property. If $r > 0$, suppose that the time-zero ball $B_0(p,r)$ has compact closure in X, and at time zero we have*

(i) $|\operatorname{Rm}| \leq C_1 r^{-2}$ *on* $B_0(p,r)$*, and*

(ii) $|\nabla \operatorname{Rm}| \leq C_1 r^{-3}$ *on* $B_0(p,r)$*.*

Furthermore, assume that for all $t \in [0,T]$*,*

$$C_1^{-1}\omega(0) \leq \omega(t) \leq C_1 \omega(0). \tag{A.2}$$

Then $|\operatorname{Rm}(x,t)| \leq C_2$ *for all* $x \in B_0\left(p, \frac{1}{4}r\right)$ *and* $t \in [0,T]$*.*

Remark A.3. In [20], Sherman and Weinkove prove a related result in which C_2 depends, in an unspecified way, on $\omega(0)$. Since we will apply Proposition A.1 to a compactness theorem, we need a uniformity result for C_2.

Remark A.4. The point of Proposition A.1 is that C_2 does not depend on T. We do not know whether the analog of Proposition A.1 holds for non-Kähler Ricci flows.

Proof. After rescaling, we can assume that $r = 1$.

In the following, we use $\widetilde{\cdot}$ for notations including g, $|\cdot|$, ∇, Rm and Δ to indicate if they are with respect to the initial metric $\omega(0)$. Otherwise, they are with respect to the evolving flow metric $\omega(\cdot)$. Indices are in terms of any local holomorphic coordinates on X. We let $\langle\cdot,\cdot\rangle$ denote a Euclidean inner product and $(\cdot,\cdot)$ denote a Hermitian inner product. We let ∇ denote the $(1,0)$ component of the covariant derivative, and $\overline{\nabla}$ denote a $(0,1)$ component of the covariant derivative. For example, if f_1 and f_2 are smooth real functions then $(\nabla f_1, \nabla f_2) = g^{\bar{j}i}\partial_i f_1 \partial_{\bar{j}} f_2$, $\langle df_1, df_2\rangle = \operatorname{Re}(\nabla f_1, \nabla f_2)$ and $\langle df_1, df_1\rangle = |\nabla f_1|^2 = |\overline{\nabla} f_1|^2$.

The letter C will denote a positive constant that can depend on C_1 and n, but does not depend in any other way on $\omega(0)$. The value of C is allowed to change from place to place.

Define the tensor Ψ by

$$\Psi_{ij}^k = (\nabla - \widetilde{\nabla})_{ij}^k = g^{\bar{l}k}\widetilde{\nabla}_i g_{j\bar{l}}. \tag{A.5}$$

In view of (A.2), a bound on Ψ is equivalent to a first derivative bound on $\omega(t)$, relative to $\omega(0)$. Put $S = |\Psi|^2$. From [21, Proposition 2.8],

$$\left(\frac{\partial}{\partial t} - \Delta\right) S = -|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 - 2\operatorname{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \nabla^{\bar{b}} \widetilde{R}_{i\bar{b}p}{}^k \overline{\Psi_{jq}^l}\right). \tag{A.6}$$

In view of (A.2) and (A.5),

$$-2\operatorname{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \nabla^{\bar{b}} \widetilde{R}_{i\bar{b}p}{}^k \overline{\Psi_{jq}^l}\right) \leq C|\widetilde{\nabla}\widetilde{\operatorname{Rm}}| \cdot |\Psi| + C|\widetilde{\operatorname{Rm}}| \cdot |\Psi|^2. \tag{A.7}$$

Then using assumptions (i) and (ii) of the proposition,

$$\left(\frac{\partial}{\partial t} - \Delta\right) S \leq -|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + C \cdot S + C. \tag{A.8}$$

From [21, (2.43)],

$$\nabla_{\bar{b}}\Psi^k_{ip} = \widetilde{R}_{i\bar{b}p}{}^k - R_{i\bar{b}p}{}^k. \tag{A.9}$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) S \leq -|\overline{\nabla}\Psi|^2 + C \cdot S + C \leq -\frac{1}{2}|\operatorname{Rm}|^2 + C \cdot S + C. \tag{A.10}$$

We will need a cutoff function. Let $\Phi : [0,\infty) \to [0,\infty)$ be a smooth nonincreasing function so that $\Phi\Big|_{[0,\frac{1}{2}]} = 1$ and $\Phi\Big|_{[1,\infty]} = 0$. We can and will assume that where $\Phi \neq 0$,

$$\frac{(\Phi')^2}{\Phi} = 4\left(\left(\Phi^{\frac{1}{2}}\right)'\right)^2 \leq C. \tag{A.11}$$

Put

$$\phi(x) = \Phi(d_0(x,p)), \tag{A.12}$$

a time-independent Lipschitz function. Let d_p denote the time-zero distance function from p, i.e. $d_p(x) = d_0(x,p)$ At time zero,

$$|\widetilde{\nabla}\phi|^2 = \widetilde{g}^{\bar{j}i}\partial_i\phi\partial_{\bar{j}}\phi = (\Phi')^2 \circ d_p. \tag{A.13}$$

At time t, $|\nabla\phi|^2 = g^{\bar{j}i}\partial_i\phi\partial_{\bar{j}}\phi$. Then by (A.2), $|\nabla\phi| \leq C$ on $B_0(p,1) \times [0,T]$.

Next, the $(1,1)$-component of $\operatorname{Hess}(\phi)$ is given by

$$\sqrt{-1}\partial\overline{\partial}\phi = \sqrt{-1}\left(\Phi'' \circ d_p\right)\partial d_p \wedge \overline{\partial} d_p + \sqrt{-1}\left(\Phi' \circ d_p\right)\partial\overline{\partial} d_p. \tag{A.14}$$

Assumption (i) of the proposition and Hessian comparison imply there is an estimate

$$\sqrt{-1}\partial\overline{\partial} d_p \leq C\omega(0) \tag{A.15}$$

in the barrier sense. As $\Phi' \leq 0$, using (A.2) and (A.14) we obtain

$$\triangle\phi = g^{\bar{j}i}\partial_i\partial_{\bar{j}}\phi \geq -C. \tag{A.16}$$

Then on $B_0(p,1) \times [0,T]$, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\phi \cdot S) \qquad\qquad (A.17)\\
&= \phi\left(\frac{\partial}{\partial t} - \Delta\right) S - S\Delta\phi - 2\operatorname{Re}(\nabla\phi, \nabla S)\\
&\leq \phi\left(-|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + C \cdot S + C\right) + C \cdot S + 2|\nabla\phi| \cdot |\nabla S|\\
&\leq \phi\left(-|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + C \cdot S + C\right) + C \cdot S + 4|\nabla\phi| \cdot (|\nabla\Psi| + |\overline{\nabla}\Psi|) \cdot |\Psi|
\end{aligned}$$

Let $\epsilon > 0$ be small enough that $\epsilon|\nabla\phi|^2 - \phi \leq 0$, which is possible from (A.11). Since

$$\begin{aligned}&\phi\left(-|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + C \cdot S + C\right) + C \cdot S + 4|\nabla\phi| \cdot (|\nabla\Psi| + |\overline{\nabla}\Psi|) \cdot |\Psi| \\ &\leq \phi\left(-|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + C \cdot S + C\right) + C \cdot S + \epsilon|\nabla\phi|^2 \cdot (|\nabla\Psi|^2 + |\overline{\nabla}\Psi|^2) + C(\epsilon) \cdot S,\end{aligned} \tag{A.18}$$

we conclude that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\phi \cdot S) \leq C \cdot S + C. \tag{A.19}$$

From assumption (i) of the proposition, (A.2) and [21, (2.26) and (2.27)], we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{Tr}\left(\omega_0^{-1}\omega\right) \leq C - C \cdot S. \tag{A.20}$$

Choosing A large enough, we have the following differential inequality on $B_0(p,1) \times [0,T]$:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\phi \cdot S + A \operatorname{Tr}\left(\omega_0^{-1}\omega\right)\right) \leq -S + C. \tag{A.21}$$

On $\partial B_0(p,1) \times [0,T]$, the function ϕ vanishes. On $B_0(p,1) \times \{0\}$, the function S vanishes. By (A.2),

$$A\,|\operatorname{Tr}\left(\omega_0^{-1}\omega\right)| \leq C' \tag{A.22}$$

for some constant $C' < \infty$. In particular,

$$\phi \cdot S + A \operatorname{Tr}\left(\omega_0^{-1}\omega\right) \leq C' \tag{A.23}$$

on the parabolic boundary $(\partial B_0(p,1) \times [0,T]) \cup (B_0(p,1) \times \{0\})$.

With reference to the constant C of the right-hand side of (A.21), suppose that

$$\phi(x,t) \cdot S(x,t) + A \operatorname{Tr}\left(\omega_0^{-1}\omega\right)(x,t) \geq C + C' \tag{A.24}$$

for some (x,t). Then $\phi(x,t) \cdot S(x,t) \geq C$, so $S(x,t) \geq C$, so $-S(x,t) + C \leq 0$. The reasoning in the proof of the parabolic maximum principle, applied to (A.21), now implies that

$$\phi \cdot S + A \operatorname{Tr}\left(\omega_0^{-1}\omega\right) \leq C + C' \tag{A.25}$$

on $B_0(p,1) \times [0,T]$. Reverting to the use of C to denote a generic positive constant, since ϕ is one on $B_0\left(p, \frac{1}{2}\right) \times [0,T]$, we conclude that

$$S \leq C \tag{A.26}$$

on $B_0\left(p, \frac{1}{2}\right) \times [0,T]$, independent of T.

Hereafter we work on $B_0\left(p, \frac{1}{2}\right) \times [0,T]$. From (A.10) and (A.26),

$$\left(\frac{\partial}{\partial t} - \Delta\right) S \leq -\frac{1}{2}|\operatorname{Rm}|^2 + C. \tag{A.27}$$

From [21, (2.58)],

$$\left(\frac{\partial}{\partial t}-\Delta\right)|\operatorname{Rm}|^2 \leq -|\nabla \operatorname{Rm}|^2 - |\overline{\nabla}\operatorname{Rm}|^2 + C|\operatorname{Rm}|^3. \tag{A.28}$$

Put $H = \frac{|\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}}$ where C is large enough so that $C - S \geq 1$ in $B_0\left(p, \frac{1}{2}\right) \times [0, T]$. Since $1 \leq C - S \leq C$, a bound on H is equivalent to a bound on $|\operatorname{Rm}|^2$. Then

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)H \qquad (A.29)\\
&= \frac{1}{(C-S)^{\frac{1}{2}}}\left(\frac{\partial}{\partial t}-\Delta\right)|\operatorname{Rm}|^2 + |\operatorname{Rm}|^2\left(\frac{\partial}{\partial t}-\Delta\right)\frac{1}{(C-S)^{\frac{1}{2}}} - 2\operatorname{Re}\left(\nabla|\operatorname{Rm}|^2, \nabla\frac{1}{(C-S)^{\frac{1}{2}}}\right)\\
&= \frac{1}{(C-S)^{\frac{1}{2}}}\left(\frac{\partial}{\partial t}-\Delta\right)|\operatorname{Rm}|^2 + \frac{1}{2}\frac{|\operatorname{Rm}|^2}{(C-S)^{\frac{3}{2}}}\left(\frac{\partial}{\partial t}-\Delta\right)S -\\
&\quad \frac{3}{8}\frac{|\operatorname{Rm}|^2(|\nabla S|^2 + |\overline{\nabla} S|^2)}{(C-S)^{\frac{5}{2}}} - \frac{\operatorname{Re}(\nabla|\operatorname{Rm}|^2, \nabla S)}{(C-S)^{\frac{3}{2}}}\\
&\leq \frac{1}{(C-S)^{\frac{1}{2}}}\left(-|\nabla\operatorname{Rm}|^2 - |\overline{\nabla}\operatorname{Rm}|^2 + C|\operatorname{Rm}|^3\right) + \frac{1}{2}\frac{|\operatorname{Rm}|^2}{(C-S)^{\frac{3}{2}}}\left(-\frac{1}{2}|\operatorname{Rm}|^2 + C\right) -\\
&\quad \frac{3}{8}\frac{|\operatorname{Rm}|^2(|\nabla S|^2 + |\overline{\nabla} S|^2)}{(C-S)^{\frac{5}{2}}} + \frac{\sqrt{|\nabla\operatorname{Rm}|^2 + |\overline{\nabla}\operatorname{Rm}|^2}\cdot|\operatorname{Rm}|\cdot\sqrt{|\nabla S|^2 + |\overline{\nabla} S|^2}}{(C-S)^{\frac{3}{2}}}\\
&= -\frac{11}{36}\frac{|\nabla\operatorname{Rm}|^2 + |\overline{\nabla}\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}} - \frac{3}{200}\frac{|\operatorname{Rm}|^2(|\nabla S|^2 + |\overline{\nabla} S|^2)}{(C-S)^{\frac{5}{2}}} -\\
&\quad \left(\frac{5}{6}\frac{\sqrt{|\nabla\operatorname{Rm}|^2 + |\overline{\nabla}\operatorname{Rm}|^2}}{(C-S)^{\frac{1}{4}}} - \frac{3}{5}\frac{|\operatorname{Rm}|\cdot\sqrt{|\nabla S|^2 + |\overline{\nabla} S|^2}}{(C-S)^{\frac{5}{4}}}\right)^2 +\\
&\quad \frac{1}{2}\frac{|\operatorname{Rm}|^2}{(C-S)^{\frac{3}{2}}}\left(-\frac{1}{2}|\operatorname{Rm}|^2 + 2|\operatorname{Rm}|(C-S) + C\right)\\
&\leq -\frac{1}{100}\frac{|\nabla\operatorname{Rm}|^2 + |\overline{\nabla}\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}} - \frac{1}{100}\frac{|\operatorname{Rm}|^2(|\nabla S|^2 + |\overline{\nabla} S|^2)}{(C-S)^{\frac{5}{2}}} - CH^2 + C.
\end{aligned}$$

Define $\widehat{\phi} \in C(X)$ by

$$\widehat{\phi}(x) = \Phi(2d_0(x,p)), \tag{A.30}$$

so $\operatorname{supp}(\widehat{\phi}) \subset \overline{B_0\left(p, \frac{1}{2}\right)}$. Then

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)(\widehat{\phi}\cdot H) &= \widehat{\phi}\left(\frac{\partial}{\partial t}-\Delta\right)H - H\Delta\widehat{\phi} - 2\operatorname{Re}(\nabla\widehat{\phi}, \nabla H) \qquad (A.31)\\
&\leq \widehat{\phi}\left(\frac{\partial}{\partial t}-\Delta\right)H + C\cdot H + 2|\nabla\widehat{\phi}|\cdot|\nabla H|.
\end{aligned}$$

Now

$$\nabla H = \frac{\nabla|\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}} + \frac{1}{2}\frac{|\operatorname{Rm}|^2\nabla S}{(C-S)^{\frac{3}{2}}}, \tag{A.32}$$

so

$$\begin{aligned}
&2|\nabla\widehat{\phi}|\cdot|\nabla H| \\
&\leq 2|\nabla\widehat{\phi}|\cdot\left(\sqrt{2}\frac{|\operatorname{Rm}|\sqrt{|\nabla\operatorname{Rm}|^2+|\overline{\nabla}\operatorname{Rm}|^2}}{(C-S)^{\frac{1}{2}}}+\frac{1}{2\sqrt{2}}\frac{|\operatorname{Rm}|^2\sqrt{|\nabla S|^2+|\overline{\nabla}S|^2}}{(C-S)^{\frac{3}{2}}}\right) \\
&\leq \frac{\epsilon}{100}|\nabla\widehat{\phi}|^2\left(\frac{|\nabla\operatorname{Rm}|^2+|\overline{\nabla}\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}}+\frac{|\operatorname{Rm}|^2(|\nabla S|^2+|\overline{\nabla}S|^2)}{(C-S)^{\frac{5}{2}}}\right)+ \\
&\quad \frac{100}{\epsilon}\left(\frac{2|\operatorname{Rm}|^2}{(C-S)^{\frac{1}{2}}}+\frac{|\operatorname{Rm}|^2}{8(C-S)^{\frac{1}{2}}}\right).
\end{aligned} \tag{A.33}$$

for any $\epsilon > 0$. From (A.11), we can choose ϵ so that $\epsilon|\nabla\widehat{\phi}|^2 - \widehat{\phi} \leq 0$. Using (A.29), (A.31) and (A.33), we arrive at

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\widehat{\phi}\cdot H) \leq \widehat{\phi}\left(-CH^2+C\right)+C\cdot H \leq C|\operatorname{Rm}|^2+C. \tag{A.34}$$

For large $B < \infty$, equation (A.10) now gives

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\widehat{\phi}\cdot H+B\cdot S) \leq -|\operatorname{Rm}|^2+C. \tag{A.35}$$

Using the parabolic maximum principle as in the proof of (A.26), we conclude that

$$|\operatorname{Rm}| \leq C \tag{A.36}$$

on $B_0\left(p,\frac{1}{4}\right)\times[0,T]$. This proves the proposition. □

We now extend the preceding proposition to include higher derivative curvature bounds.

Proposition A.37. *Let $(X,p,\omega(\cdot))$ be a pointed Kähler-Ricci flow on a complex n-dimensional manifold X, defined on a time interval $[0,T]$, with possibly incomplete time slices. Given $l \geq 1$ and $\widetilde{C}_l < \infty$, there is some $\widehat{C}_l = \widehat{C}_l(\widetilde{C}_l, n) < \infty$ with the following property. If $r > 0$, suppose that the time-zero ball $B_0(p,r)$ has compact closure in X, and at time zero, for all $0 \leq k \leq l$ we have*

$$\left|\nabla^k\operatorname{Rm}\right| \leq \frac{\widetilde{C}_l}{r^{k+2}} \tag{A.38}$$

on $B_0(p,r)$. Further assume that for all $t \in [0,T]$,

$$\widetilde{C}_l^{-1}\omega(0) \leq \omega(t) \leq \widetilde{C}_l\omega(0). \tag{A.39}$$

Then

$$\left|\nabla^k\operatorname{Rm}\right|(x,t) \leq \frac{\widehat{C}_l}{r^{k+2}} \tag{A.40}$$

for all $k \leq l$, $x \in B_0\left(p,\frac{1}{8}r\right)$ and $t \in [0,T]$.

Proof. After rescaling, we can assume that $r = 1$. Proposition A.1 gives a uniform curvature bound on $B_0\left(p, \frac{1}{4}\right) \times [0, T]$. For any $k \geq 1$ and $\epsilon > 0$, Shi's local derivative estimate implies a bound

$$\left|\nabla^k \operatorname{Rm}\right|(x, t) \leq \widehat{C}''_k \tag{A.41}$$

on $B_0\left(p, \frac{1}{8}\right) \times [\epsilon, T]$, where $\widehat{C}''_k$ depends on n, ϵ and the parameter C_2 in the conclusion of Proposition A.1. Given $\alpha > 0$, the local derivative estimate in [14, Appendix D] gives a bound

$$\left|\nabla^k \operatorname{Rm}\right|(x, t) \leq \widehat{C}'_l \tag{A.42}$$

on $B_0\left(p, \frac{1}{8}\right) \times [0, \alpha]$, where $1 \leq k \leq l$ and $\widehat{C}'_l$ depends on n, α, C_2 and $\widetilde{C}_l$. Taking $\epsilon < \alpha$, the proposition follows. □

B Power law decay of curvature and Ricci flow

In this section we give sufficient conditions for Ricci flow to preserve a power law decay of curvature. In Subsection B.1 we show that this is true under a technical condition, which will be satisfied in the cases of interest. The proof is along the lines of Dai-Ma [8]. In Subsection B.2 we show it is always true for the Kähler-Ricci flow.

We remark that Hamilton showed in [12, Theorem 18.2] that Ricci flow preserves the property that the curvature decays to zero at spatial infinity.

B.1 Power law decay in Ricci flow

Suppose that (X, x_0, g_X) is a complete pointed n-dimensional Riemannian manifold with $|\operatorname{Rm}| \leq k_0$. From [7, Lemma 12.30], there are $C = C(n, k_0) < \infty$ and a function $\phi \in C^\infty(X)$ so that

$$\begin{aligned} C^{-1}(d(x, x_0) + 1) \leq & \phi(x) \leq C(d(x, x_0) + 1), \\ & |d\phi|_{g_X} \leq C, \\ & \operatorname{Hess}_{g_X}(\phi) \leq C g_X. \end{aligned} \tag{B.1}$$

Note that there is an upper bound on $\operatorname{Hess}(\phi)$ but generally not a lower bound.

Now let $(X, g_X(\cdot))$ be a Ricci flow defined on a time interval $[0, T)$, with complete time slices and bounded curvature on compact time intervals. Let $d_t(\cdot, \cdot)$ denote the time-t distance function. For any $T' \in [0, T)$, the identity map from $(X, g_X(0))$ to $(X, g_X(t))$ is uniformly biLipschitz for $t \in [0, T']$.

Let ϕ be a distance-like function as above, relative to the time-zero metric $g_X(0)$. From [7, Lemma 12.5], given $T' \in [0, T)$, there is some $C_{T'} < \infty$ so that for all $t \in [0, T']$, we have

$$
\begin{aligned}
C_{T'}^{-1}(d_t(x,x_0)+1) \leq & \phi(x) \leq C_{T'}(d_t(x,x_0)+1), \\
|d\phi|_{g_X(t)} & \leq C_{T'}, \\
\mathrm{Hess}_{g_X(t)}(\phi) & \leq C_{T'} g_X.
\end{aligned} \tag{B.2}
$$

In particular, the notion of power law decay is the same as measured with ϕ, d_0 or d_t. For future convenience, we will only state results in terms of d_0.

Lemma B.3. *If there is some $B_0 \in \mathbb{R}$ so that at time zero, we have*

$$
\phi^{-1} \, \mathrm{Hess}_{g_x(0)}(\phi) \geq B_0 g_X(0), \tag{B.4}
$$

then for any $T' \in [0,T)$, there is some $B_{T'} \in \mathbb{R}$ so that for all $t \in [0,T']$,

$$
\phi^{-1} \, \mathrm{Hess}_{g_X(t)}(\phi) \geq B_{T'} g_X(t). \tag{B.5}
$$

Proof. By the same argument as in [7, Part (iii) of proof of Lemma 12.5], there is some $C_{T'} < \infty$ so that for all $t \in [0,T']$,

$$
\mathrm{Hess}_{g_X(t)}(\phi) - \mathrm{Hess}_{g_X(0)}(\phi) \geq -C_{T'} \cdot g_X(0). \tag{B.6}
$$

The lemma follows. □

Proposition B.7. *(c.f. [8, Theorem 4]) Let $(X, g(\cdot))$ be a Ricci flow solution on a connected manifold X, defined for $t \in [0,T]$, with uniformly bounded curvature and complete time slices. Let $F(x,t)$ and $u(x,t)$ be smooth bounded functions, with u nonnegative. Given $C < \infty$, suppose that*

$$
\left(\frac{\partial}{\partial t} - \triangle_{g(t)} \right) u \leq F\sqrt{u} + Cu. \tag{B.8}
$$

Let $x_0 \in X$ be a basepoint. For some $\beta > 0$, suppose that as $x \to \infty$, $F(x,t) = O\left(d_0(x,x_0)^{-\frac{\beta}{2}}\right)$, uniformly in t. Suppose that $u(x,0) = O\left(d_0(x,x_0)^{-\beta}\right)$. Suppose that the time-zero distance-like function ϕ can be chosen to satisfy (B.4) for some $B_0 \in \mathbb{R}$. Then we have $u(x,t) = O\left(d_0(x,x_0)^{-\beta}\right)$ uniformly in $t \in [0,T]$.

Proof. One calculates that

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t} - \triangle_{g(t)} \right) (\phi^\beta u) \leq \\
& \left(-\beta \frac{\triangle \phi}{\phi} + \beta(\beta+1) \frac{|\nabla\phi|^2}{\phi^2} \right) \phi^\beta u - 2\beta \left\langle \frac{\nabla \phi}{\phi}, \nabla(\phi^\beta u) \right\rangle + \phi^{\frac{\beta}{2}} F \sqrt{\phi^\beta u} + C\phi^\beta u.
\end{aligned} \tag{B.9}
$$

By the weak maximum principle [7, Theorem 12.14], $\phi^\beta u$ is uniformly bounded above for all $t \in [0,T]$. This proves the proposition. □

Proposition B.10. *(c.f. [8, Theorem 1A]) Let $(X, g(\cdot))$ be a Ricci flow solution on a connected manifold X, defined for $t \in [0, T]$, with uniformly bounded curvature and complete time slices.*

Let $x_0 \in X$ be a basepoint. For some $\alpha > 0$, suppose that as $x \to \infty$, $|\operatorname{Rm}(x, 0)| = O\left(d_0(x, x_0)^{-2\alpha}\right)$. Suppose that the time-zero distance-like function ϕ can be chosen to satisfy (B.4) for some $B_0 \in \mathbb{R}$. Then we have $|\operatorname{Rm}(x, t)| = O\left(d_0(x, x_0)^{-2\alpha}\right)$ uniformly in $t \in [0, T]$.

Proof. From [6, (6.1)],

$$\left(\frac{\partial}{\partial t} - \triangle_{g(t)}\right) |\operatorname{Rm}|^2 \leq 16 |\operatorname{Rm}| \cdot |\operatorname{Rm}|^2. \tag{B.11}$$

By assumption, there is some $K < \infty$ so that $|\operatorname{Rm}(x, t)| \leq K$ for all (x, t). Put $u(x, t) = |\operatorname{Rm}(x, t)|^2$. Applying Proposition B.7 with $F = 0$ and $C = 16K$, the claim follows. □

Proposition B.12. *Under the hypotheses of Proposition B.10, suppose that $|\nabla^k \operatorname{Rm}|(x, 0) = O\left(d_0(x, x_0)^{-(k+2)\alpha}\right)$, for all $0 \leq k \leq l$. Then for all $0 \leq k \leq l$ and t, we have $|\nabla^k \operatorname{Rm}|(x, t) = O\left(d_0(x, x_0)^{-(k+2)\alpha}\right)$, uniformly in $t \in [0, T]$.*

Proof. Given $1 \leq k \leq l$, put $u(x, t) = |\nabla^k \operatorname{Rm}(x, t)|^2$.

By assumption, $u(x, 0) = O\left(d_0(x, x_0)^{-2(k+2)\alpha}\right)$. From [6, (6.24)],

$$\left(\frac{\partial}{\partial t} - \triangle_{g(t)}\right) u \leq c(n) \sum_{l=1}^{k-1} |\nabla^l \operatorname{Rm}| \cdot |\nabla^{k-l} \operatorname{Rm}| \sqrt{u} + c(n) |\operatorname{Rm}| u. \tag{B.13}$$

By induction, we can assume that

$$|\nabla^l \operatorname{Rm}| \cdot |\nabla^{k-l} \operatorname{Rm}| = O\left(d_0(x, x_0)^{-(k+4)\alpha}\right), \tag{B.14}$$

uniformly in $t \in [0, T]$. The claim now follows from Proposition B.7. □

B.2 Power law decay in Kähler-Ricci flow

Proposition B.15. *Let X be a complex manifold. Let $(X, g(\cdot))$ be a Kähler-Ricci flow on X, defined for $t \in [0, T)$, with complete time slices and bounded curvature on compact time intervals. Let $x_0 \in X$ be a basepoint. For some $\alpha \in (0, 1]$ and $l \geq 1$, suppose that $|\nabla^k \operatorname{Rm}|(x, 0) = O\left(d_0(x, x_0)^{-(k+2)\alpha}\right)$, for all $0 \leq k \leq l$. Then for all $0 \leq k \leq l$ and t, we have $|\nabla^k \operatorname{Rm}|(x, t) = O\left(d_0(x, x_0)^{-(k+2)\alpha}\right)$, uniformly in $t \in [0, T]$.*

Proof. Given $p \in X - B_0(x_0, 1)$, put $r = \frac{1}{2} d(p, x_0)^\alpha$. By assumption, there is some $\widetilde{C}_l < \infty$ so that the hypotheses of Proposition A.37 are satisfied for all such p. (Hypothesis A.39 is satisfied because the bounded curvature assumption implies biLipschitzness on a finite time interval.) Then Proposition A.37, applied at the center of the ball $B_0\left(p, \frac{1}{8} r\right)$, implies the proposition. □

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