Problem Set 6

Q53 Let $(X_\alpha, \mathcal{O}_\alpha)$ be a topological space for each $\alpha \in I$, where I is an arbitrary index set. Let

$$(X = \prod_{\alpha \in I} X_\alpha, \mathcal{O}_{prod})$$

be the product space. Show that $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$ if and only if $(p_\alpha(x_n))_{n\in\mathbb{N}} \subseteq X_\alpha$ congerges to $p_\alpha(x)$ for each $\alpha \in I$. In words: Convergence in X is the same as componentwise convergence.

Q54 (a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and give $X \times Y$ the product topology. If $A \subseteq X$ and $B \subseteq Y$ are closed subsets, show that $A \times B \subseteq X \times Y$ is closed.

(b) Give an example of a closed subset of $\mathbb{R} \times \mathbb{R}$ such that the projection of the set onto the first factor is not closed.

Q55 Give an example of a non-Hausdorff space containing a compact subset that is not closed.

Q56 Let $(X, \mathcal{O}_X)$ be a topological space and $K_1, \ldots K_n$ be compact subsets of X. Show that their union, $\bigcup_{i=1}^{n} K_i$, is also compact.

Q57 Let $X = (0, 1)$ and let

$$\mathcal{O} = \{A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } A = (0, 1) \text{ or } A = (0, 1 - 1/n) \text{ for } n \geq 2\}.$$

Show that every proper open subset of X is compact. Is X compact?

Q58 Let $X = \mathbb{R}$ and consider the co-countable topology:

$$\mathcal{O} = \{A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } \mathbb{R} \setminus A \text{ is countable}\}.$$

Is $[0, 1]$ compact in $(X, \mathcal{O})$? What are the compact sets in $(X, \mathcal{O})$?