Modes of convergence

Q35 The following examples were given in the lecture:

(1) $f_n = n^{-1}\chi_{(0,n)}$,
(2) $f_n = \chi_{(n,n+1)}$,
(3) $f_n = n\chi_{[0,1/n]}$,
(4) $f_n = \chi_{[i2^{-k},(i+1)2^{-k}]}$, where $n = 2^k + i$ and $0 \le i < 2^k$.

Verify that the convergence $f_n \to 0$ is (a) uniform but not in L^1 in (1), (b) pointwise but not uniform or L^1 in (2), (c) almost everywhere but not everywhere or L_1 in (3), and (d) in L^1 but not almost everywhere in (4).
Then show that the sequence (f_n) in (1), (3) and (4) satisfies $f_n \to 0$ in measure. Also show that the sequence (f_n) in (2) is not Cauchy in measure.

Q36 Show that $f_n \to f$ almost uniformly implies that $f_n \to f$ almost everwhere.

Q37 Show that $f_n \to f$ almost uniformly implies that $f_n \to f$ in measure.

Q38 Show that $f_n \to f$ in measure implies that a subsequence of (f_n) converges to f almost uniformly.

Q39 Suppose that $|f_n| \le g \in L^1(\mu)$ and $f_n \to f$ in measure. Show:

(a) $\int f = \lim \int f_n$,
(b) $f_n \to f$ in L^1.

Q40 If μ is σ–finite and $f_n \to f$ almost everywhere, then there exist measurable sets $E_k \subseteq X$, such that $\mu\Big(\big(\bigcup_{k=1}^{\infty} E_k\big)^c \Big) = 0$ and $f_n \to f$ uniformly on each E_k.

Q41 Let μ be counting measure on $\mathbb{N}$, and $f_n, f\colon \mathbb{N} \to \mathbb{C}$ be measurable functions. Then $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

Q42 Suppose (X, Σ, μ) is a measure space with $\mu(X) < \infty$. For measurable functions $f, g\colon X \to \mathbb{C}$, define

$$d(f,g) = \int \frac{|f-g|}{1+|f-g|}\, d\mu.$$

(a) Show that d defines a metric on the set of equivalence classes of complex-valued, measurable functions on X, where two functions are considered equivalent if they are equal almost everywhere.
(b) Show that $[f_n] \to [f]$ with respect to this metric if and only if $f_n \to f$ in measure.