Signed measures and all that

Q27 Let (X, Σ, μ) be a measurable space and $f\colon X \to \mathbb{R}^*$ be a measurable function with the property that $\int f$ is defined. Show that

$$\nu(E) = \int_E f d\mu$$

defines a signed measure on (X, Σ).

Q28 Let μ_1 and μ_2 be measures on (X, Σ) and suppose that at least one of them is finite. Show that $\mu_1 - \mu_2$ is a signed measure on (X, Σ).

Q29 Let μ be a signed measure on (X, Σ).

(a) If $(A_k) \subseteq \Sigma$ is an increasing sequence, then $\mu(\bigcup_{k\in\mathbb{N}} A_k) = \lim_{k\to\infty} \mu(A_k)$.

(b) If $(A_k) \subseteq \Sigma$ is a decreasing sequence and $\mu(A_1) < \infty$, then $\mu(\bigcap_{k\in\mathbb{N}} A_k) = \lim_{k\to\infty} \mu(A_k)$.

Q30 Let μ and ν be signed measures on (X, Σ).

(a) $A \in \Sigma$ is null for μ if and only if $|\mu|(A) = 0$.

(b) $\mu \perp \nu$ if and only if $|\mu| \perp \nu$ if and only if $(\mu^+ \perp \nu$ and $\mu^- \perp \nu)$

Q31 Let μ be a signed measure on (X, Σ), and $X = P \cup N$ be a Hahn decomposition for μ. Let $f = \chi_P - \chi_N$. Show that for all $A \in \Sigma$,

$$\mu(A) = \int_A f d|\mu|.$$

Q32 Let ν be a signed measure on (X, Σ) and μ be a (positive) measure.

(a) $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $(\nu^+ \ll \mu$ and $\nu^- \ll \mu)$

(b) $\nu \ll \mu$ and $\nu \perp \mu$ implies $\nu = 0$.

Q33 Let (ν_k) be a sequence of (positive) measures on (X, Σ) and μ be a (positive) measure.

(a) If $\nu_k \perp \mu$ for all k, then $(\sum \nu_k) \perp \mu$.

(b) If $\nu_k \ll \mu$ for all k, then $(\sum \nu_k) \ll \mu$.

Q34 Lemma 3.15 states: "If ν is a finite signed measure and μ a positive measure, then $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $|\nu(E)| < \varepsilon$."
It was stated that finiteness is only needed for the forward direction. Show that it fails when ν is not finite, by considering the following two examples:

(a) $d\nu(x) = \frac{dx}{x}$ and $d\mu(x) = dx$ on $(0, 1)$ with the Borel σ–algebra;

(b) ν =counting measure and $\mu(A) = \sum_{n\in A} 2^{-n}$ on $\mathbb{N}$ with $\mathcal{P}(\mathbb{N})$.