Assignment 3

Your solutions should be submitted by the beginning of the lecture on

Wednesday, 23 September 2009.

Q1 Let X be a set and $\mathcal{F}_b(X)$ be the real vector space of all bounded, real-valued functions on X. For each $f \in \mathcal{F}_b(X)$, let

$$||f|| = \sup_{x \in X} |f(x)|.$$

You may assume that $(\mathcal{F}_b(X), ||\cdot||)$ is a normed space.

(a) Show that $(\mathcal{F}_b(X), ||\cdot||)$ is complete.

(b) Suppose that (X, d) is a metric space and fix $a \in X$. For $x \in X$, define the function $f_x \colon X \to \mathbb{R}$ by:

$$f_x(t) = d(x,t) - d(a,t).$$

Show that $f_x \in \mathcal{F}_b(X)$, and that the map $x \to f_x$ is an isometry onto its image (where $\mathcal{F}_b(X)$ is given the metric induced by the norm). Conclude that X is homeomorphic to a subset of $\mathcal{F}_b(X)$, where both spaces are given the induced topologies.

(c) Discuss differences and analogies of (b) with the result that every normed space is isometric to a subspace of its double-dual.

Q2 (a) Let X be a normed space and $S \subseteq X$. Show that if $\{f(x) \mid x \in S\}$ is bounded for each $f \in X^*$, then S is bounded. (Yes, this is Q9 on Set 4.) Is the converse also true?

(b) Use the previous part to show that if two norms on a vector space V are not equivalent, then there is a linear functional on V which is continuous with respect to one of the norms and discontinuous with respect to the other.

Q3 Let X be a normed space over Λ ($\mathbb{R}$ or $\mathbb{C}$), and $f \colon X \to \Lambda$ be a non-zero linear functional on X. Show that the following are equivalent:

(a) $f \notin X^*$,

(b) $f(B(X)) = \Lambda$,

(c) $\ker f$ is dense in X.

Q4 Let K be a subset of l_p, where $1 \leq p < \infty$. Show that the following are equivalent:

(a) K is compact;

(b) K is closed and for each $\varepsilon > 0$, there exist $y_1, \ldots, y_{n(\varepsilon)} \in K$, such that

$$K \subseteq \bigcup_{k=1}^{n(\varepsilon)} B(y_k; \varepsilon);$$

(c) K is closed, bounded and for each $\varepsilon > 0$, there exists $m = m(\varepsilon)$, such that

$$\sum_{k=m+1}^{\infty} |x_k|^p < \varepsilon$$

for all $x = (x_k)_{k=1}^{\infty} \in K$.

Hints:

–Show "(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)."

–Since l_p is complete, K is compact if and only if it is closed and every sequence in K contains a Cauchy sequence.