Cowan's Identity

My website would not be complete without special advertisement of my 2007 research on convex hulls. There are some original identities arising in that research — and I regard them as potentially-useful contributions to the mathematical literature. One identity, in particular, has been named after me (as *Cowan's Identity* in the 2008 book by Schneider and Weil [1] and *Cowan's Formula* in [2]).

Whether the identity/formula is important remains to be seen, but it is quite stunning and deserving better publicity than has occurred to date (with my almost–total neglect of it).

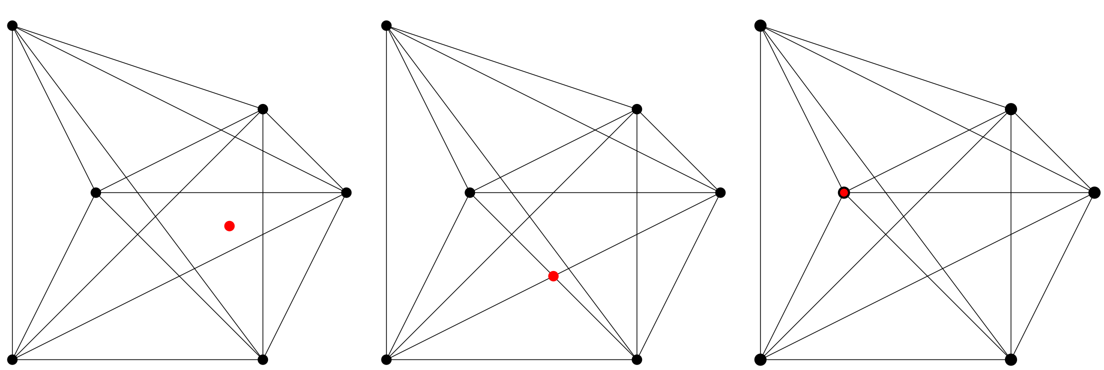
I describe, in Theorem 1 below, a simple planar version of the identity. Some pictures to illustrate the identity follow. Later, in Theorem 2, the identity is discussed in d dimensions.

The identity in $\mathbb{R}^2$

Theorem 1: For $n \geq 3$, place n points $P_1, P_2, ..., P_n$ in the plane, positioned so that H_n, the convex hull of the points, has dimension 2. In other words, we require that there is no line in $\mathbb{R}^2$ which contains all n points. Nor can it be that all points coincide. Now, add a point P at any position in $\mathring{H}_n$, the interior of H_n. Define $c_j(P)$, for $1 \leq j \leq n$, as the number of sub-collections of j points from $\{P_1, P_2, ...P_n\}$ whose convex-hull contains the point P. Then

$$\Psi(P) := c_1(P) - c_2(P) + ...(-1)^{n-1}c_n(P) = 1 \qquad \text{for all } P \in \mathring{H}_n. \qquad \square$$

Example 1: See Figure 1, which has three cases (a), (b) and (c), each with $n = 6$ and the same six points $P_1, P_2, ...P_6$. Consider cases (a) and (b), which show the six points in black. All 2-hulls are shown, these being line-segments joining pairs of points. The reference point P (coloured red) is also shown. In case (b), P lies on two of the 2-hulls. Case (c) also shows the six points and the 2-hulls, but it has the additional feature that P is positioned on one of the P_j (see the black dot with a red centre). Theorem 1 allows this feature. .



(a) $\Psi = 0 - 0 + 8 - 12 + 6 - 1 = 1$

(b) $\Psi = 0 - 2 + 10 - 12 + 6 - 1 = 1$

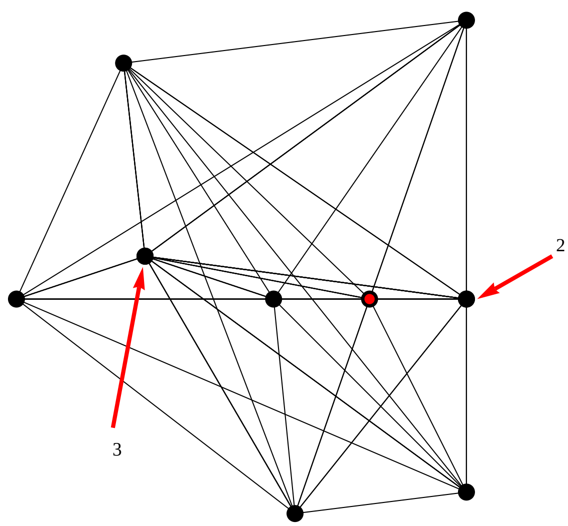
(c) $\Psi = 1 - 5 + 13 - 13 + 6 - 1 = 1$

Figure 1: Here $n = 6$. As described in Example 1, points P_1, P_2, P_3, P_4, P_5 and P_6 are in the same *general position* in all three drawings. $\Psi(P) = 1$ as shown under each case. So Theorem 1 holds in the three cases.

Theorem 1 holds even when the points $P_1, P_2, ..., P_n$ are not all distinct. We show an example where P_7 coincides with P_6.

Example 1 (continued): Let $n = 7$. The first six of the points are the black dots of Figure 1(a). The seventh, P_7, coincides with P_6, which is the black point furthest to the right of Figure 1(a). Clearly $c_1(P) = c_2(P) = 0$ and $c_7 = 1$. With considerably more effort, if working with just a pencil and paper, the following results emerge: $c_3(P) = 13$; $c_4(P) = 26$; $c_5(P) = 20$; $c_6(P) = 7$. Therefore $\Psi(P) = 1$.

Example 2: One point (black with red centre) lies on P. The red arrows indicate a multiplicity at the indicated black dot. Each 0f the six remaining black dots accommodates one point. So we see that $n = 12$. The $c_j(P)$ counts and the resulting value of $\Psi(P)$ are displayed below Figure 2. Thus, for this quite complicated example (with large numbers involved in the calculation of Ψ), Theorem 1 is validated.



$$\Psi = 1 - 16 + 118 - 373 + 701 - 883 + 782 - 494 + 220 - 66 + 12 - 1 = 1$$

Figure 2: Note here a few colinearities: of three points, and also one of four points. Theorem 1 handles these complications. See Example 2.

These examples do not prove Theorem 1, of course. The proof can be found in my paper [3]. That paper also contains the trivial one-dimensional case. We now deal with the remaining situation: the d-dimensional case (with $d \geq 3$).

The identity in $\mathbb{R}^d, d \geq 3$

Theorem 2: With $n \geq d+1$, we place n points $P_1, P_2, ..., P_n$ in $\mathbb{R}^d, d \geq 3$, positioned so that H_n, the convex hull of the points, has dimension d. Add the point P at any position in $\mathring{H}_n$, the interior of H_n. Define $c_j(P)$, for $1 \leq j \leq n$, as before: the number of sub-collections of j points from $\{P_1, P_2, ... P_n\}$ whose convex-hull contains the point P. Then, there exists a set $\mathcal{E} \subset \mathbb{R}^d$ of dimension at most $(d-2)$, such that

$$\Psi(P) := c_1(P) - c_2(P) + ...(-1)^{n-1} c_n(P) = (-1)^d \qquad \text{for all } P \in \mathring{H}_n \setminus \mathcal{E}. \qquad \square \tag{1}$$

Theorem 2 was proved in [3]-[4] and (by a more streamlined approach) in [1].

Remark 1: *Formula (1) has been proved with a different exceptional set $\mathcal{E}$, namely "$\mathcal{E}$* is a set of d-dimensional Lebesgue measure zero*" instead of "$\mathcal{E}$* is a set of dimension $\leq (d-2)$". *Which version gives the stronger mathematical result? This question is now unimportant, because* Kabluchko et al *[2] have proved formula (1) for $\mathcal{E} = \emptyset$: that is, for all positions of $P \in \mathring{H}_n$.*

I was very pleased to find the Euler-like alternating sum, to prove it for $d \leq 2$ and also prove it (albeit with a small exceptional set of P positions) when $d \geq 3$. I am also pleased that others ([2], [5]) have removed the exceptional set and extended the scope of the Identity.

References:

[1] Schneider, R. and Weil, W. *Stochastic and Integral Geometry.* Springer, Berlin Heidelberg. 2008.
[2] Kabluchko, Z. et al. Inclusion-exclusion principles for convex hulls and the Euler relation. *Discrete and Computational Geometry*, **58**, 417-434 (2017)
[3] Cowan, R. Identities linking volumes of convex hulls. *Advances in Applied Probability*. **39**, 630–644 (2007).
[4] Cowan, R. Recurrence relationships for the mean number of faces and vertices for random convex hulls. *Discrete and Computational Geometry* **43**(2), 209-220 (2010)
[5] Hug, D. and Kabluchko, Z. An inclusion-exclusion identity for normal cones of polyhedral sets. *Mathematika* **64**(1) 124–136 (2018).

Note: Personal-archive versions of [3] and [4] are available on this website. See papers 77 and 78 on

www.maths.usyd.edu.au/u/richardc/randomgeom.html