Odd reflections in the Yangian associated with $\mathfrak{gl}(m|n)$

A. I. Molev

Abstract

The odd reflections is an effective tool in the Lie superalgebra representation theory, as they relate non-conjugate Borel subalgebras. We introduce analogues of the odd reflections for the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and use them to produce a transition rule for the parameters of the highest weight modules corresponding to a change of the parity sequence. This leads to a description of the finite-dimensional irreducible representations of the Yangian associated with an arbitrary parity sequence.

1 Introduction

The *Yangian* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n}[u])$ in the class of Hopf algebras. The original definition is due to Nazarov [12], where the Yangian was introduced via an R-matrix presentation. A Drinfeld-type presentation corresponding to a standard Borel subalgebra was obtained by Gow [7], extending the results of Drinfeld [5] and Brundan and Kleshchev [3] on the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. An earlier note by Stukopin [16] contains an alternative approach to the Yangian presentations (with a correction in the set of defining relations pointed out in [7]). More recently, Drinfeld-type parabolic presentations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, corresponding to arbitrary Borel subalgebras in $\mathfrak{gl}_{m|n}$ were given by Peng [14]. In a particular case, such a presentation was reproduced by Tsymbaliuk [19] who also gave a description of the center of the Yangian by using the *quantum Berezinian*, thus generalizing the results of [12] and [6].

It was shown by Zhang [22], that the finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ are described in a way similar to the representations of the Yangians associated with the simple Lie algebras, as given by Tarasov [18] and Drinfeld [5]. The classification in [22] relies on the presentation of the Yangian corresponding to the standard Borel subalgebra in $\mathfrak{gl}_{m|n}$ (or the standard parity sequence) and leaves open the question as to how the finite-dimensionality conditions on irreducible highest weight representations can be stated for an arbitrary Borel subalgebra. For the Lie superalgebra $\mathfrak{gl}_{m|n}$ itself (along with other basic Lie superalgebras), the same question can be answered with the use of the *odd reflections*; see e.g. [4, Secs 1.3 & 2.4] and [11, Sec. 3.5]. They originate in the work of Serganova [8, Appendix]; some quantum versions (as *super-reflections*) were considered by Yamane [20].

Our goal in this paper is to introduce analogues of the odd reflections for the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. More precisely, we derive a transition rule for the parameters of the irreducible highest weight representations corresponding to a change of the parity sequence. Accordingly, necessary and sufficient conditions for such representations to be finite-dimensional can be obtained by applying a chain of transitions. As this relies on such conditions for the standard parity sequence, we will give a brief review (in Section 3) of the classification results for the finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ following [21] and [22]; cf. [17] where an approach based on the Drinfeld presentation was used.

The transition rule is associated with an odd simple isotropic root for a given Borel subalgebra of $\mathfrak{gl}_{m|n}$. This determines a subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ isomorphic to $\mathrm{Y}(\mathfrak{gl}_{1|1})$. Therefore, the arguments are essentially reduced to the analysis of the irreducible highest weight representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_{1|1})$. The key step is to describe how the odd reflections effect the highest weights. The calculations rely on the properties of the quantum Berezinian and they lead to the general transition rules as given in Theorem 4.3 and Corollary 4.5.

The version of the transition rule stated in Corollary 4.5 applies in a similar form to the orthosymplectic Yangians introduced in [1]. However, apart from the particular case of $\mathfrak{osp}_{1|2n}$ considered in [10], the classification problem for their finite-dimensional irreducible representations is still open.

2 Definition and basic properties of the Yangian

For given nonnegative integers m and n consider the *parity sequences* $\mathfrak{s} = \mathfrak{s}_1 \dots \mathfrak{s}_{m+n}$ of length $m+n$, where each term $\mathfrak{s}_i$ is 0 or 1, and the total number of zeros is m. The *standard sequence* $\mathfrak{s}^{\mathrm{st}} = 0 \dots 01 \dots 1$ is defined by $\mathfrak{s}_i = 0$ for $i = 1, \dots, m$ and $\mathfrak{s}_i = 1$ for $i = m+1, \dots, m+n$.

When a parity sequence $\mathfrak{s}$ is fixed, we will simply write $\bar{\imath}$ to denote its i-th term $\mathfrak{s}_i$. For a fixed sequence introduce the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{m|n}$ over $\mathbb{C}$ with the basis $e_1, e_2, \dots, e_{m+n}$, where the parity of the basis vector e_i is defined to be $\bar{\imath} \mod 2$. Accordingly, equip the endomorphism algebra $\operatorname{End} \mathbb{C}^{m|n}$ with a $\mathbb{Z}_2$-gradation, where the parity of the matrix unit e_{ij} is found by $\bar{\imath} + \bar{\jmath} \mod 2$.

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ is formed by elements E_{ij} of the parity $\bar{\imath} + \bar{\jmath} \mod 2$ for $1 \leqslant i, j \leqslant m+n$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

The *Yang R-matrix* associated with $\mathfrak{gl}_{m|n}$ is the rational function in u given by $R(u) = 1 - P u^{-1}$, where P is the permutation operator,

$$P = \sum_{i,j=1}^{m+n} e_{ij} \otimes e_{ji} (-1)^{\bar{\jmath}} \in \operatorname{End} \mathbb{C}^{m|n} \otimes \operatorname{End} \mathbb{C}^{m|n}.$$

Following [12], define the *Yangian* $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ associated with $\mathfrak{s}$ (written simply as $\mathrm{Y}(\mathfrak{gl}_{m|n})$, if the sequence $\mathfrak{s}$ is fixed), as the $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath} + \bar{\jmath} \mod 2$, where $1 \leqslant i, j \leqslant m+n$ and $r = 1, 2, \dots$, satisfying the following quadratic relations. To write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]] \tag{2.1}$$

and combine them into the matrix $T(u)=[t_{ij}(u)]$ so that

$$T(u)=\sum_{i,j=1}^{m+n} e_{ij}\otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}\in \operatorname{End}\mathbb{C}^{m|n}\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]].$$

Consider the algebra $\operatorname{End}\mathbb{C}^{m|n}\otimes\operatorname{End}\mathbb{C}^{m|n}\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u)=\sum_{i,j=1}^{m+n} e_{ij}\otimes 1\otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}},\qquad T_2(u)=\sum_{i,j=1}^{m+n} 1\otimes e_{ij}\otimes t_{ij}(u)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}.$$

The defining relations for the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ take the form of the *RTT-relation*

$$R(u-v)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,R(u-v). \tag{2.2}$$

More explicitly, the defining relations can be written with the use of super-commutator in terms of the series (2.1) as

$$\Big[t_{ij}(u),t_{kl}(v)\Big]=\frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v)-t_{kj}(v)\,t_{il}(u)\Big)(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}}. \tag{2.3}$$

It is clear from the defining relations that the re-labelling map $t_{ij}(u)\mapsto t_{\sigma(i)\,\sigma(j)}(u)$ for any permutation σ of the set $\{1,\dots,m+n\}$ extends to an isomorphism between the respective Yangians. Therefore, all algebras $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ associated with different parity sequences $\mathfrak{s}$ are isomorphic to each other.

The *Yangian* $\mathrm{Y}(\mathfrak{sl}_{m|n})$ is the subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ which consists of the elements stable under the automorphisms

$$t_{ij}(u)\mapsto f(u)\,t_{ij}(u) \tag{2.4}$$

for all series $f(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$. If $m\neq n$, then we have the tensor product decomposition

$$\mathrm{Y}(\mathfrak{gl}_{m|n})=\mathrm{ZY}(\mathfrak{gl}_{m|n})\otimes \mathrm{Y}(\mathfrak{sl}_{m|n}), \tag{2.5}$$

where $\mathrm{ZY}(\mathfrak{gl}_{m|n})$ denotes the center of $\mathrm{Y}(\mathfrak{gl}_{m|n})$. The center is freely generated by the coefficients of the *quantum Berezinian*. Its definition in the case of standard parity sequence $\mathfrak{s}=\mathfrak{s}^{\mathrm{st}}$ goes back to [12]; see also [6] and [7] for more details on the properties of the Berezinian, including a formula in terms of the Gaussian generators. Its generalization for an arbitrary parity sequence $\mathfrak{s}$ was found in [19].

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ can be regarded as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ via the embedding

$$E_{ij}\mapsto t_{ij}^{(1)}(-1)^{\bar{\imath}}, \tag{2.6}$$

while the mapping

$$t_{ij}(u)\mapsto \delta_{ij}+E_{ij}(-1)^{\bar{\imath}}\,u^{-1} \tag{2.7}$$

defines the *evaluation homomorphism* $\mathrm{ev}\colon \mathrm{Y}(\mathfrak{gl}_{m|n})\to \mathrm{U}(\mathfrak{gl}_{m|n})$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{m+n} t_{ik}(u) \otimes t_{kj}(u). \tag{2.8}$$

For any parity sequence $\mathfrak{s}$ and a nonnegative integer p introduce the *Yangian of level* p, denoted by $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ (or simply by $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ for a fixed $\mathfrak{s}$), as the quotient of the algebra $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ by the two-sided ideal generated by all elements $t_{ij}^{(r)}$ with $r > p$. For a given p, all algebras $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ associated with different parity sequences $\mathfrak{s}$ are isomorphic to each other.

Note that by the results of [2] and [13], the Yangians $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ are isomorphic to certain finite $\mathcal{W}$-algebras corresponding to the general linear Lie superalgebras.

3 Finite-dimensional irreducible representations

A representation V of the algebra $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ associated with a parity sequence $\mathfrak{s}$, is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\,\xi &= 0 && \text{for} \quad 1 \leqslant i < j \leqslant m+n, \qquad \text{and} \\ t_{ii}(u)\,\xi &= \lambda_i(u)\,\xi && \text{for} \quad i = 1, \dots, m+n, \end{aligned}$$

for some formal series

$$\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]. \tag{3.1}$$

The vector ξ is called the *highest vector* of V and the $(m+n)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{m+n}(u))$ is called its *highest weight*.

Given an arbitrary tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{m+n}(u))$ of formal series of the form (3.1), the *Verma module* $M(\lambda(u))$ is defined as the quotient of the algebra $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant m+n$, and $t_{ii}(u) - \lambda_i(u)$ for $i = 1, \dots, m+n$. We will denote by $L(\lambda(u))$ its irreducible quotient and will say that $L(\lambda(u))$ is associated with the parity sequence $\mathfrak{s}$. It is clear that the isomorphism class of $L(\lambda(u))$ is determined by $\lambda(u)$.

Proposition 3.1. *Every finite-dimensional irreducible representation of the algebra* $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ *is isomorphic to* $L(\lambda(u))$ *for a certain highest weight* $\lambda(u) = (\lambda_1(u), \dots, \lambda_{m+n}(u))$.

Proof. The argument is essentially the same as for the proof of the counterpart of the property for the Yangians associated with Lie algebras as in [22]; cf. [9, Sec. 3.2]. □

Note a property of the Yangians associated with sequences of length two. It is easy to check that we have the isomorphisms

$$\mathrm{Y}(\mathfrak{gl}^{01}_{1|1}) \to \mathrm{Y}(\mathfrak{gl}^{10}_{1|1}), \qquad t^{01}_{ij}(u) \mapsto t^{10}_{ij}(-u), \tag{3.2}$$

and

$$\mathrm{Y}(\mathfrak{gl}^{00}_{2|0}) \to \mathrm{Y}(\mathfrak{gl}^{11}_{0|2}), \qquad t^{00}_{ij}(u) \mapsto t^{11}_{ij}(-u), \tag{3.3}$$

where the superscripts indicate which algebras the generating series correspond to.

Proposition 3.2. *Suppose that the representation $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ is finite-dimensional. Then for each subsequence of $\mathfrak{s}$ of the form $\mathfrak{s}_i\mathfrak{s}_{i+1} = 01$ or $\mathfrak{s}_i\mathfrak{s}_{i+1} = 10$ the ratio $\lambda_i(u)/\lambda_{i+1}(u)$ is the expansion into a series in u^{-1} of a rational function in u,*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\overline{Q}_i(u)}{Q_i(u)},$$

for some monic polynomials $\overline{Q}_i(u)$ and $Q_i(u)$ in u of the same degree.

Moreover, for each subsequence of $\mathfrak{s}$ of the form $\mathfrak{s}_i\mathfrak{s}_{i+1} = 00$ or $\mathfrak{s}_i\mathfrak{s}_{i+1} = 11$ we have

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)} \qquad \textit{and} \qquad \frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)},$$

respectively, for a monic polynomial $P_i(u)$ in u.

Proof. Consider the subalgebra Y_i of $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ generated by the coefficients of the series $t_{ii}(u)$, $t_{i\,i+1}(u)$, $t_{i+1\,i}(u)$ and $t_{i+1,i+1}(u)$. This subalgebra is isomorphic to one of the four algebras occurring in (3.2) and (3.3). The cyclic span $\mathrm{Y}_i\,\xi$ is a finite-dimensional module with the highest weight $(\lambda_i(u), \lambda_{i+1}(u))$. In view of the isomorphisms (3.2) and (3.3), we only need to derive the required conditions for the highest weight representations of the algebras $\mathrm{Y}(\mathfrak{gl}^{01}_{1|1})$ and $\mathrm{Y}(\mathfrak{gl}^{00}_{2|0})$. The derivation is the same in both cases following the original approach of [18] as shown in [21]; see also [9, Sec. 3.3]. □

The necessary conditions of Proposition 3.2 need not be sufficient in general. However, they are sufficient in the case of the standard parity sequence $\mathfrak{s} = \mathfrak{s}^{\mathrm{st}}$. The next theorem, whose proof we outline below, is due to Zhang [22]; see also [17]. We will write $\mathrm{Y}(\mathfrak{gl}_{m|n})$ for the Yangian associated with the sequence $\mathfrak{s}^{\mathrm{st}} = 0\dots01\dots1$.

Theorem 3.3. *The representation $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is finite-dimensional if and only if there exist monic polynomials $P_1(u), \dots, P_{m-1}(u), \overline{Q}_m(u), Q_m(u), P_{m+1}(u), \dots, P_{m+n-1}(u)$ in the variable u such that*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)} \qquad \textit{for} \quad i = 1, \dots, m-1,$$

$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)} \qquad \textit{for} \quad i = m+1, \dots, m+n-1,$$

and

$$\frac{\lambda_m(u)}{\lambda_{m+1}(u)} = \frac{\overline{Q}_m(u)}{Q_m(u)}.$$

Proof. The necessity of the conditions follows from Proposition 3.2, so we only need to show that they are sufficient. The conditions determine the highest weight $\lambda(u)$ up to a simultaneous multiplication of all components $\lambda_i(u)$ by a series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. This operation corresponds to twisting the action of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on $L(\lambda(u))$ by the automorphism

(2.4). Therefore, it is enough to prove that a particular module $L(\lambda(u))$ corresponding to a given set of polynomials is finite-dimensional. Moreover, all components $\lambda_i(u)$ of the highest weight of such a module can be assumed to be polynomials in u^{-1}.

Observe that if $L(\mu(u))$ and $L(\nu(u))$ are the irreducible highest weight modules with the highest weights

$$\mu(u) = \big(\mu_1(u), \dots, \mu_{m+n}(u)\big) \qquad \text{and} \qquad \nu(u) = \big(\nu_1(u), \dots, \nu_{m+n}(u)\big),$$

then the coproduct rule (2.8) implies that the cyclic span $\mathrm{Y}(\mathfrak{gl}_{m|n})(\xi \otimes \xi')$ of the tensor product of the respective highest vectors of $L(\mu(u))$ and $L(\nu(u))$ is a highest weight module with the highest weight

$$\big(\mu_1(u)\,\nu_1(u), \dots, \mu_{m+n}(u)\,\nu_{m+n}(u)\big).$$

Consider the irreducible highest weight representation $L(\pi)$ of the Lie superalgebra $\mathfrak{gl}_{m|n}$ associated with the standard Borel subalgebra. This representation is finite-dimensional if and only if the highest weight π has the form

$$\pi = (\pi_1, \dots, \pi_m, \pi_{m+1}, \dots, \pi_{m+n}) \qquad \text{with} \quad \pi_i - \pi_{i+1} \in \mathbb{Z}_+ \quad \text{for all} \quad i \neq m. \tag{3.4}$$

Use the evaluation homomorphism (2.7) to equip $L(\pi)$ with an $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module structure. The evaluation module $L(\pi)$ is the Yangian highest weight module, where the components of the highest weight are

$$\pi_i(u) = 1 + \pi_i\,(-1)^{\bar{\imath}}\,u^{-1}, \qquad i = 1, \dots, m+n.$$

Now factorize the polynomials $\lambda_i(u)$ to write

$$\lambda_i(u) = (1 + \lambda_i^{(1)}\,(-1)^{\bar{\imath}}\,u^{-1}) \dots (1 + \lambda_i^{(p)}\,(-1)^{\bar{\imath}}\,u^{-1}), \qquad i = 1, \dots, m+n,$$

for a certain positive integer p. By the assumptions on the highest weight $\lambda(u)$, we can renumber the factors in these decompositions to ensure that all tuples

$$\lambda^{(r)} = (\lambda_1^{(r)}, \dots, \lambda_{m+n}^{(r)}), \qquad r = 1, \dots, p,$$

satisfy conditions (3.4). The coproduct formula (2.8) implies that the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda(u))$ is isomorphic to a subquotient of the tensor product of finite-dimensional evaluation modules

$$L(\lambda^{(1)}) \otimes \dots \otimes L(\lambda^{(p)}), \tag{3.5}$$

thus proving that $L(\lambda(u))$ is finite-dimensional. □

4 Odd reflections

The proof of Theorem 3.3 shows, that after twisting with a suitable automorphism of the form (2.4), every finite-dimensional irreducible representation of the algebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ factors through a representation of the Yangian $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ of some level p. This is implied by the evaluation and coproduct formulas (2.7) and (2.8) which show that all generators $t_{ij}^{(r)}$ of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ with $r > p$ act as the zero operators in the modules (3.5). We will now be concerned with representations of the Yangians $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$.

We will use the same notation $t_{ij}^{(r)}$ for the generators of $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ and extend the definitions of highest weight representations to $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ accordingly. Introduce the polynomials

$$T_{ij}(u) = u^p\, t_{ij}(u) = \delta_{ij}u^p + t_{ij}^{(1)}\, u^{p-1} + \dots + t_{ij}^{(p)}, \qquad i, j = 1, \dots, m+n.$$

The defining relations for $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ in terms of the polynomials $T_{ij}(u)$ take the same form (2.3), with the replacement $t_{ij}(u) \mapsto T_{ij}(u)$.

We will now concentrate on the particular case $m = n = 1$ and introduce a suitable version of the odd reflections for the corresponding Yangian of level p. We will work with the Yangian $\mathrm{Y}_p(\mathfrak{gl}_{1|1})$ associated with the standard parity sequence $\mathfrak{s} = 01$; the results in the opposite case $\mathfrak{s} = 10$ will then follow by the application of the isomorphism (3.2). Note the relation

$$(u - v + 1)\, T_{21}(u)\, T_{21}(v) = -(u - v - 1)\, T_{21}(v)\, T_{21}(u) \tag{4.1}$$

implied by (2.3). By setting $v = u + 1$ we get

$$T_{21}(u + 1)\, T_{21}(u) = 0. \tag{4.2}$$

We will keep the same notation $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ for the highest weight of the representation $L(\lambda(u))$ of $\mathrm{Y}_p(\mathfrak{gl}_{1|1})$, but now $\lambda_1(u)$ and $\lambda_2(u)$ will be regarded as monic polynomials in u of degree p with $T_{ii}(u)\, \xi = \lambda_i(u)\, \xi$. To state the next proposition, write the expansions

$$\lambda_1(u) = (u + \alpha_1) \dots (u + \alpha_p) \qquad \text{and} \qquad \lambda_2(u) = (u + \beta_1) \dots (u + \beta_p) \tag{4.3}$$

for some $\alpha_i, \beta_i \in \mathbb{C}$. Assume that $\alpha_i \neq \beta_j$ for all $i, j \in \{1, \dots, p\}$. Furthermore, to avoid vanishing of some elements below due to (4.2), we will impose the following *anti-string condition* on the parameters $\alpha_1, \dots, \alpha_p$:

$$\alpha_i - \alpha_j + 1 \neq 0 \qquad \text{for all} \quad 1 \leqslant i < j \leqslant p. \tag{4.4}$$

This condition is not restrictive; it can be achieved by a renumbering of the roots of $\lambda_1(u)$. Keeping these assumptions, we can now state a key result underlying the use of odd reflections in the Yangian context.

Proposition 4.1. *The vector*

$$\zeta = T_{21}(-\alpha_1) \dots T_{21}(-\alpha_p)\, \xi \in L(\lambda(u))$$

is nonzero. Moreover, the following relations hold:

$$T_{11}(u)\, \zeta = (u + \alpha_1 - 1) \dots (u + \alpha_p - 1)\, \zeta, \tag{4.5}$$

$$T_{22}(u)\, \zeta = (u + \beta_1 - 1) \dots (u + \beta_p - 1)\, \zeta, \tag{4.6}$$

and

$$T_{21}(u)\, \zeta = 0. \tag{4.7}$$

Proof. We have

$$T_{12}(u)\,\zeta = \sum_{i=1}^{p} (-1)^{i-1}\, T_{21}(-\alpha_1)\dots\Big[T_{12}(u), T_{21}(-\alpha_i)\Big]\dots T_{21}(-\alpha_p)\,\xi.$$

By the defining relations,

$$\Big[T_{12}(u), T_{21}(-\alpha_i)\Big] = -\frac{1}{u+\alpha_i}\Big(T_{22}(u)\,T_{11}(-\alpha_i) - T_{22}(-\alpha_i)\,T_{11}(u)\Big)$$

and

$$T_{11}(u)\,T_{21}(-\alpha_j) = \frac{u+\alpha_j-1}{u+\alpha_j}\,T_{21}(-\alpha_j)\,T_{11}(u) + \frac{1}{u+\alpha_j}\,T_{21}(u)\,T_{11}(-\alpha_j). \tag{4.8}$$

Since

$$T_{11}(u)\,\xi = (u+\alpha_1)\dots(u+\alpha_p)\,\xi, \tag{4.9}$$

an easy induction yields

$$T_{12}(u)\,\zeta = \sum_{i=1}^{p} (-1)^{i} \prod_{l=1}^{i-1}(u+\alpha_l) \prod_{l=i+1}^{p}(u+\alpha_l-1)\,T_{21}(-\alpha_1)\dots T_{22}(-\alpha_i)\dots T_{21}(-\alpha_p)\,\xi.$$

Now set $u = -\alpha_p + 1$ to get

$$T_{12}(-\alpha_p+1)\,\zeta = (-1)^p \prod_{i=1}^{p}(\beta_i - \alpha_p) \prod_{l=1}^{p-1}(\alpha_l - \alpha_p + 1)\,T_{21}(-\alpha_1)\dots T_{21}(-\alpha_{p-1})\,\xi.$$

The assumptions on the parameters and the anti-string condition (4.4) imply that the numerical coefficient of the vector on the right hand side is nonzero. Therefore, by applying the operators $T_{12}(-\alpha_{p-1}+1),\dots,T_{12}(-\alpha_1+1)$ to the vector on the right hand side consecutively, we obtain the highest vector ξ with a nonzero coefficient, thus proving that the vector ζ is nonzero.

To prove the second part of the proposition, note that relation (4.5) is implied by (4.8) and (4.9), as was pointed out in the above calculation. Furthermore, relation (4.7) is a consequence of the identity in $\mathrm{Y}_p(\mathfrak{gl}_{1|1})$ which holds for independent variables $u_1,\dots,u_{p+1}$:

$$T_{21}(u_1)\dots T_{21}(u_{p+1}) = 0. \tag{4.10}$$

Indeed, the expression on the left hand side is a polynomial in these variables with the degree with respect to each variable not exceeding $p-1$. However, the expression vanishes at the evaluations $u_i = u_j + 1$ for all $1 \leqslant i < j \leqslant p+1$ due to (4.1) and (4.2), and therefore must be identically zero. Alternatively, an equivalent form of (4.10) reads

$$t_{21}^{(r_1)}\dots t_{21}^{(r_{p+1})} = 0$$

for all $r_i \in \{1,\dots,p\}$, which holds by (4.1) as the generators $t_{21}^{(r)}$ and $t_{21}^{(s)}$ anticommute modulo lower degree terms.

Finally, for the proof of (4.6) we use the quantum Berezinian for the Yangian $\mathrm{Y}(\mathfrak{gl}_{1|1})$, as introduced in [12]; see also [6]. We will regard $L(\lambda(u))$ as the $\mathrm{Y}(\mathfrak{gl}_{1|1})$-module, obtained via the composition with the natural epimorphism $\mathrm{Y}(\mathfrak{gl}_{1|1}) \to \mathrm{Y}_p(\mathfrak{gl}_{1|1})$. All coefficients of the series $b(u)$ defined by

$$b(u) = \Big(t_{22}(u) - t_{21}(u)\, t_{11}(u)^{-1} t_{12}(u)\Big)\, t_{11}(u)^{-1}$$

belong to the center of the algebra $\mathrm{Y}(\mathfrak{gl}_{1|1})$. They act as multiplications by scalars in the module $L(\lambda(u))$ which are found by $b(u) \mapsto \lambda_2(u)/\lambda_1(u)$. On the other hand, the defining relations imply that

$$b(u)\, t_{11}(u)\, t_{11}(u+1) = t_{22}(u+1)\, t_{11}(u) + t_{12}(u+1)\, t_{21}(u).$$

Apply the series on both sides to the vector $\zeta \in L(\lambda(u))$ by using the already verified relations (4.5) and (4.7) to get

$$T_{22}(u)\, \zeta = (u + \beta_1 - 1) \dots (u + \beta_k - 1)\, \zeta,$$

completing the proof. $\square$

We will also state a generalization of Proposition 4.1 to the case where the polynomials $\lambda_1(u)$ and $\lambda_2(u)$ have common roots. Consider the expansions (4.3) and renumber the parameters if necessary, to assume that for some $k \in \{0, 1, \dots, p\}$ we have $\alpha_i = \beta_i$ for $i = k+1, \dots, p$, whereas $\alpha_i \neq \beta_j$ for all $i, j \in \{1, \dots, k\}$. The anti-string condition (4.4) will now be assumed for the parameters $\alpha_1, \dots, \alpha_k$.

Corollary 4.2. *The fraction*

$$\overline{T}_{21}(u) = \frac{T_{21}(u)}{\gamma(u)} \qquad \textit{with} \quad \gamma(u) = (u + \alpha_{k+1}) \dots (u + \alpha_p) \tag{4.11}$$

is a polynomial in u, as an operator in $L(\lambda(u))$. Moreover, the vector

$$\zeta = \overline{T}_{21}(-\alpha_1) \dots \overline{T}_{21}(-\alpha_k)\, \xi \in L(\lambda(u))$$

is nonzero and satisfies $T_{21}(u)\, \zeta = 0$ and

$$T_{11}(u)\, \zeta = (u + \alpha_1 - 1) \dots (u + \alpha_k - 1)\, \gamma(u)\, \zeta,$$

$$T_{22}(u)\, \zeta = (u + \beta_1 - 1) \dots (u + \beta_k - 1)\, \gamma(u)\, \zeta.$$

Proof. Consider the representation $L(\bar{\lambda}(u))$ of the Yangian $\mathrm{Y}_k(\mathfrak{gl}_{1|1})$ of level k, where the components of the highest weight have the form

$$\bar{\lambda}_1(u) = (u + \alpha_1) \dots (u + \alpha_k) \qquad \text{and} \qquad \bar{\lambda}_2(u) = (u + \beta_1) \dots (u + \beta_k).$$

Denote the generator polynomials for the Yangian $\mathrm{Y}_k(\mathfrak{gl}_{1|1})$ by $\overline{T}_{ij}(u)$ and use the same symbols for the corresponding operators on $L(\bar{\lambda}(u))$. It is immediate from the defining relations (2.3) that the assignment

$$T_{ij}(u) \mapsto \gamma(u)\, \overline{T}_{ij}(u) \qquad \text{for all} \quad i, j \in \{1, 2\}$$

defines a representation of the Yangian $\mathrm{Y}_p(\mathfrak{gl}_{1|1})$ on the vector space $L(\bar{\lambda}(u))$. This representation is clearly irreducible and isomorphic to $L(\lambda(u))$. This proves the first part of the proposition, because the action of the operator in $L(\lambda(u))$ defined in (4.11) corresponds to the action of the operator $\overline{T}_{21}(u)$ in $L(\bar{\lambda}(u))$. The remaining parts of the corollary are now immediate from Proposition 4.1. □

Now consider the Yangian $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ associated with an arbitrary parity sequence $\mathfrak{s}$ and introduce some notation to state the main results. Let $L(\lambda(u))$ be the irreducible highest weight representation of $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ with a certain highest weight $\lambda(u)$, whose components, regarded as polynomials in u with $T_{jj}(u)\,\xi = \lambda_j(u)\,\xi$, are given by

$$\lambda_j(u) = (u + \lambda_j^{(1)}) \dots (u + \lambda_j^{(p)}), \qquad j = 1, \dots, m+n, \quad \lambda_j^{(p)} \in \mathbb{C}.$$

Suppose that the parity sequence $\mathfrak{s}$ has a subsequence of the form $\mathfrak{s}_i\mathfrak{s}_{i+1} = 10$. Then we let $\mathfrak{s}^+$ denote the sequence obtained from $\mathfrak{s}$ by replacing this subsequence with 01 and leaving the remaining terms unchanged. Similarly, if $\mathfrak{s}$ has a subsequence $\mathfrak{s}_i\mathfrak{s}_{i+1} = 01$, we let $\mathfrak{s}^-$ denote the sequence obtained from $\mathfrak{s}$ by replacing this subsequence with 10 and leaving the remaining terms unchanged.

With a chosen value of i, we will assume that the roots of the polynomials $\lambda_i(u)$ and $\lambda_{i+1}(u)$ are numbered in such a way that

$$\lambda_i^{(r)} = \lambda_{i+1}^{(r)} \qquad \text{for all} \quad r = k+1, \dots, p \tag{4.12}$$

for certain $k \in \{0, 1, \dots, p\}$, while $\lambda_i^{(r)} \neq \lambda_{i+1}^{(s)}$ for all $1 \leqslant r, s \leqslant k$. Then define new polynomials $\lambda_j^{\pm}(u)$ for $j = 1, \dots, m+n$ by

$$\lambda_i^{\pm}(u) = (u + \lambda_{i+1}^{(1)} \pm 1) \dots (u + \lambda_{i+1}^{(k)} \pm 1)(u + \lambda_{i+1}^{(k+1)}) \dots (u + \lambda_{i+1}^{(p)}),$$

$$\lambda_{i+1}^{\pm}(u) = (u + \lambda_i^{(1)} \pm 1) \dots (u + \lambda_i^{(k)} \pm 1)(u + \lambda_i^{(k+1)}) \dots (u + \lambda_i^{(p)}),$$

and $\lambda_j^{\pm}(u) = \lambda_j(u)$ for $j \neq i, i+1$. These polynomials form the respective highest weights $\lambda^+(u)$ and $\lambda^-(u)$ for the Yangian of level p.

Theorem 4.3. *With the above assumptions, the following holds.*

1. *If the parity sequence $\mathfrak{s}$ has a subsequence $\mathfrak{s}_i\mathfrak{s}_{i+1} = 01$ then the $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$-module $L(\lambda(u))$ associated with $\mathfrak{s}$ is isomorphic to the $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$-module $L(\lambda^-(u))$ associated with $\mathfrak{s}^-$.*

2. *If the parity sequence $\mathfrak{s}$ has a subsequence $\mathfrak{s}_i\mathfrak{s}_{i+1} = 10$ then the $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$-module $L(\lambda(u))$ associated with $\mathfrak{s}$ is isomorphic to the $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$-module $L(\lambda^+(u))$ associated with $\mathfrak{s}^+$.*

Proof. The two parts of the theorem are equivalent, and so it is sufficient to prove the first part. The subalgebra Y_i of $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ generated by the coefficients of the polynomials $T_{ii}(u)$, $T_{i\,i+1}(u)$, $T_{i+1\,i}(u)$ and $T_{i+1,i+1}(u)$ is isomorphic to $\mathrm{Y}_p(\mathfrak{gl}_{1|1})$. The cyclic span $\mathrm{Y}_i\xi$ of the highest vector ξ of $L(\lambda(u))$ is a highest weight module associated with the parity sequence 01, with the highest weight $(\lambda_i(u), \lambda_{i+1}(u))$. Now use the assumptions (4.12) and renumber the parameters

$\lambda_i^{(1)}, \dots, \lambda_i^{(k)}$, if necessary, to satisfy the anti-string condition (4.4). Apply Corollary 4.2 to get the nonzero vector

$$\zeta_i = \overline{T}_{i+1\,i}(-\lambda_i^{(1)}) \dots \overline{T}_{i+1\,i}(-\lambda_i^{(k)})\,\xi \in L(\lambda(u)),$$

where

$$\overline{T}_{i+1\,i}(u) = \frac{T_{i+1\,i}(u)}{\gamma_i(u)} \qquad \text{with} \quad \gamma_i(u) = (u + \lambda_i^{(k+1)}) \dots (u + \lambda_i^{(p)}).$$

An easy induction with the use of the defining relations (2.3) shows that for the vector ζ_i we have

$$T_{ab}(u)\,\zeta_i = 0 \tag{4.13}$$

for all $a < b$ with $a < i$ or $b > i+1$. Twist the action of $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ on $L(\lambda(u))$ with the re-labelling isomorphism $T_{ab}(u) \mapsto T_{\sigma_i(a)\,\sigma_i(b)}(u)$ for the transposition $\sigma_i = (i, i+1)$ to get a representation of $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}^-}_{m|n})$ on the same vector space. Corollary 4.2 and relations (4.13) imply that the resulting $\mathrm{Y}_p(\mathfrak{gl}^{\mathfrak{s}^-}_{m|n})$-module is an irreducible highest weight representation with the highest vector ζ_i and the highest weight $\lambda^-(u)$, so it is isomorphic to $L(\lambda^-(u))$. □

Due to the homomorphisms (2.6) and (2.7), the Yangian $\mathrm{Y}_1(\mathfrak{gl}_{m|n})$ of level 1 is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$. Therefore Theorem 4.3 for $p = 1$ is a particular case of the well-known properties of odd reflections; see e.g. [15].

Any two parity sequences are related by a chain of transitions $01 \leftrightarrow 10$ involving two adjacent entries. Therefore, as a consequence of Theorem 4.3 and the criterion of Theorem 3.3, we can get necessary and sufficient conditions for the irreducible highest weight representation $L(\lambda(u))$ of $\mathrm{Y}_p(\mathfrak{gl}_{m|n})$ associated with an arbitrary parity sequence $\mathfrak{s}$ to be finite-dimensional. As with the case $p = 1$, an explicit form of such conditions should look quite complicated; cf. [4, Sec. 2.4], where extremal weights in polynomial modules are discussed.

Example 4.4. Take the degree sequence $\mathfrak{s} = 101$ for $\mathrm{Y}_p(\mathfrak{gl}_{1|2})$ and consider the irreducible highest weight representation $L(\lambda(u))$ with

$$\begin{aligned}
\lambda_1(u) &= (u + \alpha_1) \dots (u + \alpha_p),\\
\lambda_2(u) &= (u + \beta_1) \dots (u + \beta_p),\\
\lambda_3(u) &= (u + \gamma_1) \dots (u + \gamma_p).
\end{aligned}$$

Renumber the roots of the polynomials $\lambda_1(u)$ and $\lambda_2(u)$ if necessary, and introduce the parameter $k \in \{0, 1, \dots, p\}$ to have $\alpha_i = \beta_i$ for $i = k+1, \dots, p$ and $\alpha_i \neq \beta_j$ for all $i, j \in \{1, \dots, k\}$. By Part 2 of Theorem 4.3, the representation $L(\lambda(u))$ is isomorphic to the $\mathrm{Y}_p(\mathfrak{gl}_{1|2})$-module $L(\lambda^+(u))$ associated with the standard parity sequence $\mathfrak{s}^+ = 011$, where

$$\begin{aligned}
\lambda_1^+(u) &= (u + \beta_1 + 1) \dots (u + \beta_k + 1)(u + \beta_{k+1}) \dots (u + \beta_p),\\
\lambda_2^+(u) &= (u + \alpha_1 + 1) \dots (u + \alpha_k + 1)(u + \alpha_{k+1}) \dots (u + \alpha_p),\\
\lambda_3^+(u) &= (u + \gamma_1) \dots (u + \gamma_p).
\end{aligned}$$

Hence, by Theorem 3.3, the representation $L(\lambda(u))$ is finite-dimensional if and only if

$$\frac{\lambda_3^+(u)}{\lambda_2^+(u)} = \frac{P(u+1)}{P(u)}$$

for a monic polynomial $P(u)$ in u.

This particular case allows for an alternative formulation of the criterion which also applies to representations of $\mathrm{Y}(\mathfrak{gl}_{1|2})$ without referring to Yangians of a finite level. Namely, by considering the cyclic span of the highest vector $\xi \in L(\lambda(u))$ with respect to the subalgebra of $\mathrm{Y}(\mathfrak{gl}_{1|2})$ generated by the coefficients of the series $t_{11}(u), t_{13}(u), t_{31}(u)$ and $t_{33}(u)$, we find that if $\dim L(\lambda(u)) < \infty$, then there exists a monic polynomial $Q(u)$ in u such that

$$\frac{\lambda_3(u)}{\lambda_1(u)} = \frac{Q(u+1)}{Q(u)}.$$

Write the ratio of $\lambda_1(u)$ and $\lambda_2(u)$ in the reduced form

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{(u+\alpha_1)\dots(u+\alpha_k)}{(u+\beta_1)\dots(u+\beta_k)}.$$

Then the above criterion can be seen to be equivalent to the following: the representation $L(\lambda(u))$ is finite-dimensional if and only if

$$Q(-\alpha_1) = \dots = Q(-\alpha_k) = 0,$$

where possible equal values of the $-\alpha_i$ are understood as multiple roots of $Q(u)$. $\square$

Since the odd reflections of Theorem 4.3 only depend on one subsequence $\mathfrak{s}_i\mathfrak{s}_{i+1}$, we can derive a slightly more general transition rule which applies to representations of the Yangian $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$. Consider the irreducible highest weight $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$-module $L(\lambda(u))$, where the components of the highest weight are series of the form (3.1). Suppose that the parity sequence $\mathfrak{s}$ has a subsequence $\mathfrak{s}_i\mathfrak{s}_{i+1} = 01$ or $\mathfrak{s}_i\mathfrak{s}_{i+1} = 10$, and that the components $\lambda_i(u)$ and $\lambda_{i+1}(u)$ are polynomials in u^{-1},

$$\lambda_i(u) = (1 + \lambda_i^{(1)}u^{-1})\dots(1 + \lambda_i^{(k)}u^{-1}),$$

$$\lambda_{i+1}(u) = (1 + \lambda_{i+1}^{(1)}u^{-1})\dots(1 + \lambda_{i+1}^{(k)}u^{-1}),$$

without common roots. Transform the highest weight $\lambda(u)$ by setting

$$\lambda_i^{\pm}(u) = \big(1 + (\lambda_{i+1}^{(1)} \pm 1)u^{-1}\big)\dots\big(1 + (\lambda_{i+1}^{(k)} \pm 1)u^{-1}\big),$$

$$\lambda_{i+1}^{\pm}(u) = \big(1 + (\lambda_i^{(1)} \pm 1\big)u^{-1})\dots\big(1 + (\lambda_i^{(k)} \pm 1)u^{-1}\big),$$

and $\lambda_j^{\pm}(u) = \lambda_j(u)$ for $j \neq i, i+1$, and keep the notation $\mathfrak{s}^-$ and $\mathfrak{s}^+$ used in Theorem 4.3.

Corollary 4.5. *The $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda(u))$ associated with the parity sequence $\mathfrak{s}$ is isomorphic to the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda^{\pm}(u))$ associated with $\mathfrak{s}^{\pm}$, where the plus and minus signs are chosen for $\mathfrak{s}_i\mathfrak{s}_{i+1} = 10$ and $\mathfrak{s}_i\mathfrak{s}_{i+1} = 01$, respectively.*

Proof. Consider the $\mathrm{Y}(\mathfrak{gl}_{1|1})$-subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by the coefficients of the series $t_{ii}(u)$, $t_{i\,i+1}(u)$, $t_{i+1\,i}(u)$ and $t_{i+1,i+1}(u)$. The cyclic span $\overline{L} = \mathrm{Y}(\mathfrak{gl}_{1|1})\xi$ is a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{1|1})$ with the highest weight $(\lambda_i(u), \lambda_{i+1}(u))$. It is easy to verify using relations (2.3), that all vectors $t_{ab}^{(r)}\,\xi$ with $a, b \in \{i, i+1\}$ and $r > k$ are zero in $L(\lambda(u))$, and that the representation $\overline{L}$ factors through the representation of the Yangian $\mathrm{Y}_k(\mathfrak{gl}_{1|1})$ of level k. The corollary then follows by the application of the relations provided in Proposition 4.1, as in the proof of Theorem 4.3. □

Note that the transition rule of Corollary 4.5 is applicable in a more general case, where the ratio $\lambda_i(u)/\lambda_{i+1}(u)$ is a rational function. By twisting the module $L(\lambda(u))$ with a suitable automorphism (2.4), we can ensure that the components $f(u)\lambda_i(u)$ and $f(u)\lambda_{i+1}(u)$ of the new highest weight satisfy the required assumptions.

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School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au