# LINEAR PARABOLIC EQUATION WITH DIRICHLET WHITE NOISE BOUNDARY CONDITIONS

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Abstract. We study inhomogeneous Dirichlet boundary value problems associated to a linear parabolic equation $\frac{du}{dt} = Au$ with strongly elliptic operator $A$ on bounded and unbounded domains with white noise boundary data. Our main assumption is that the heat kernel of the corresponding homogeneous problem enjoys the Gaussian type estimates taking into account the distance to the boundary. Under mild assumptions about the domain, we show that $A$ generates a $C_0$-semigroup in weighted $L^p$-spaces where the weight is a proper power of the distance from the boundary. We also prove some smoothing properties and exponential stability of the semigroup. Finally, we reformulate the Cauchy-Dirichlet problem with white noise boundary data as an evolution equation in the weighted space and prove the existence of Markovian solutions.

## Contents



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## 1. Introduction

The aim of this paper is to study the following linear stochastic boundary value problem

$$
(1.1)\qquad \begin{cases} \dfrac{\partial X}{\partial t}(t,x) = \mathcal{A}X(t,x)\,, & x\in\mathcal{O}, \quad t>0\,, \\[2ex] X(t,x) = \dfrac{\partial W}{\partial t}(t,x)\,, & x\in\partial\mathcal{O}, \quad t>0\,, \\[2ex] X(0,x) = X_0(x) & x\in\mathcal{O}\,. \end{cases}
$$

where $\mathcal{O}\subset\mathbb{R}^d$ is an open, possibly unobunded, domain, $W$ is a Wiener process taking values in a space of distributions on $\partial\mathcal{O}$, and $\mathcal{A}$ is a second order, strongly elliptic operator in $\mathcal{O}$. Let us note that solutions to (1.1) are Markovian if and only if $W$ is a process with independent increments.

There exists vast literature on the non-homogeneous Dirichlet boundary value problem for deterministic linear parabolic equations, for a classical exposition see the fundamental monograph [24] or more recent [28]. Extension of the classical results to rough boundaries and rough boundary conditions is still a subject of ongoing research, see for example [23] and references therein. In this paper we study equation (1.1) in a relatively regular domain, see Section 2 for details, but the boundary condition $\frac{\partial W}{\partial t}$ can be very irregular, including space-time white noise. Apart from purely mathematical motivations, such an extension is important in non-equlibrium statistical mechanics and optimal control theory, see for example [17], [26], [15] and a recent book [29].

Stochastic equations with boundary noise were usually studied in the case of Neumann boundary conditions that are more tractable, see [18], [37], [30] and also aforementioned papers [26] and [15]. Much less is known about stochastic equations with Dirichlet boundary noise. Equation (1.1) was proposed in the seminal work [11], where it was shown that it has no $L^2(\mathcal{O},\mathrm{d}x)$-valued solutions. In [1] and [2] solutions to a nonlinear equation in $\mathcal{O}=(0,+\infty)$ for $\mathcal{A}=\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ are studied and proved to have trajectories in $L^2\left((0,+\infty);\,x^{1+\theta}\mathrm{d}x\right)$. In [4] a similar approach is used to consider a very general formulation of the stochastic boundary value problem in multidimensional domains for a large class of elliptic operators $\mathcal{A}$ and distribution-valued Gaussian noises, see also [6] for the case of stochastic wave equation.

In [1, 2] the problem was not stated as an evolution equation in $L^2\left((0,+\infty);\,x^{1+\theta}\mathrm{d}x\right)$. Such a formulation was introduced and exploited in [17]. The main ingredient was a result by Krylov [20], who proved that Laplacian generates a strongly continuous analytic semigroup in the space $L^2\left(\mathbb{R}^d_+;\rho^{1+\theta}(x)\,\mathrm{d}x\right)$, where $\rho$ stands for the distance of a point $x\in\mathbb{R}^d_+$ to the boundary. It seems that the method used in [20] to prove this generation result does not extend to more general domains and more general elliptic operators. One of our goals in this paper is to show that equation (1.1) can be reformulated as a stochastic evolution equation

$$
(1.2)\qquad \mathrm{d}X = AX\mathrm{d}t + B\mathrm{d}W, \quad X(0)=X_0,
$$

on a state space $E=L^p\left(\mathcal{O},\rho^\theta(x)\,\mathrm{d}x\right)$ with an appropriately chosen operator $B$. The operator $A$ is an abstract realisation of $\mathcal{A}$ as a generator of the $C_0$-semigroup in $E$. This will ensure the Markov property of the solution and since $E$ is a function space, it will open the way to study nonlinear perturbations of (1.1).

## 2. Formulation of the problem

The boundary noise $W$ is a Wiener process taking values in a space of distributions on $\partial\mathcal{O}$. More precisely, we will assume that $W$ can be represented as a formal series

$$W(t,x)=\sum_k e_k(x)W_k(t), \tag{2.1}$$

where $W_k$ are independent real-valued Wiener processes defined on a filtered probability space $(\Omega,\mathfrak{F},(\mathfrak{F}_t),\mathbb{P})$ and $(e_k)$ is a finite or infinite sequence of functions on $\partial\mathcal{O}$. In order to simplify the presentation we assume that $\{e_k\}\subset L^2(\partial\mathcal{O},\mathrm{ds})$, where s is the surface measure and that

$$\sum_k\left(\int_{\partial\mathcal{O}}e_k(y)\psi(y)\mathrm{ds}(y)\right)^2<+\infty,\qquad\forall\,\psi\in L^2(\partial\mathcal{O},\mathrm{ds}).$$

However, our framework can be easily adapted to the case where $e_k$ are distributions on $\partial\mathcal{O}$, see however Remark 8.8.

*Remark* 2.1. Let us recall, see e.g. [12], that there exists a Hilbert space $H_W$ called the *Reproducing Kernel Hilbert Space* of $W$ such that

$$W(t,x)=\sum_k \tilde{e}_k(x)\tilde{W}_k(t),$$

where $\tilde{W}_k$ are independent real-valued Wiener processes defined on a filtered probability space $(\Omega,\mathfrak{F},(\mathfrak{F}_t),\mathbb{P})$ and $(\tilde{e}_k)$ is an orthonormal basis of $H_W$. It can be shown that linspan$\{e_k\}$ is a dense subspace of $H_W$. In a particular case of the so-called space white noise $H_W=L^2(\partial\mathcal{O},\mathrm{ds})$.

In Section 3 we derive the concept of a *formal mild solution* to (1.1). Briefly it is given by the formula

$$X(t)=S(t)X(0)+\int_0^t(\lambda-A)\,S(t-s)D_\lambda\mathrm{d}W(s),\qquad t\geq 0, \tag{2.2}$$

where $S$ is the semigroup generated by the realization $A$ of $\mathcal{A}$ with homogeneous boundary conditions, and $D_\lambda$ is the *Dirichlet map*. Let us recall that given $\lambda\geq 0$ and a function $\gamma$ on $\partial\mathcal{O}$, $u=D_\lambda\gamma$ is, the possibly weak, see Section 6, unique solution to the Poisson equation

$$\mathcal{A}u(x)=\lambda u(x),\quad x\in\mathcal{O},\qquad u(x)=\gamma(x),\quad x\in\partial\mathcal{O}. \tag{2.3}$$

To the best of our knowledge, equation (1.1) with the solution defined by (2.2) has been introduced in [11].

*Remark* 2.2. The stochastic integral appearing in (2.2) is not well defined in a space $L^p(\mathcal{O})$ but it does exist in a certain space $E$ such that $L^p(\mathcal{O})\hookrightarrow E$. In fact, see [11], Example 2.3, Popositions 8.1, 8.2, 8.12, 8.17, the solution to the problem on a bounded interval, or half line or half-space lives in a Sobolev space of negative order or on weighted $L^p(\mathcal{O},w(x)\mathrm{d}x)$-space.

It turns out, see [1, 2, 4], that under mild assumptions on $\mathcal{O}$, $\mathcal{A}$, and $W$, the solution $X$ to (1.1) is a smooth ($C^\infty$ in time and space variables) random field on $(0,+\infty)\times\mathcal{O}$ and that there is a $\kappa>0$ such that for $t>0$,

$$\mathbb{E}\,|X(t,x)|^p\leq C(t)\,(\mathrm{dist}\,(x,\partial\mathcal{O}))^{-\kappa}\,.$$

Therefore, $X(t)$ takes values in $L^p(\mathcal{O},w)$-space with an appropriate weight function $w$.

One of our main goals is to show that the problem (1.1) can be written equivalently as the stochastic partial differential equation

$$\mathrm{d}X = AX\mathrm{d}t + B\mathrm{d}W, \quad X(0) = X_0, \tag{2.4}$$

on an appropriately chosen state space $E$ with $B = (\lambda - A)D_\lambda$. This ensures the Markov property of the solution and if $E$ is a function-valued space, it will enable us to study nonlinear perturbations of (1.1). We have to face, however, the problem with the interpretation of $(\lambda - A)D_\lambda$. In fact $A$, as the generator of the heat semigroup with homogeneous Dirichlet boundary condition, is defined on regular functions vanishing on the boundary $\partial\mathcal{O}$, whereas the restriction of $D_\lambda \frac{\partial W}{\partial t}$ to $\partial\mathcal{O}$ equals $\frac{\partial W}{\partial t}$! Therefore one needs to consider $A$ as the generator of the extension of the semigroup $S$ on a suitable Sobolev space of negative-order (subspace of the space of distributions). The following example is taken from [4].

**Example 2.3.** Assume that $\mathcal{O} = (0,1)$, $\mathcal{A} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and $\lambda = 0$. Then any function $\gamma\colon \partial\mathcal{O} = \{0,1\} \mapsto \mathbb{R}$ can be identified with a pair $(\gamma_0, \gamma_1) \in \mathbb{R}^2$. We have

$$D_0(\gamma_0,\gamma_1)(x) = \gamma_0 + (\gamma_1 - \gamma_0)x, \qquad x \in (0,1),$$

and

$$AD_0(\gamma_0,\gamma_1) = \gamma_0\delta'_0 - \gamma_1\delta'_1,$$

where $\delta'_a$ is the derivative of the Dirac delta distribution et $a$, and $A$ is the generator of the heat semigroup considered, for example, on the Sobolev space $W^{2,-2}(0,1)$.

**Hypothesis 2.4.** *There are $\lambda \geq 0$, $p > 1$ and $s_0 \geq 0$ such that the Dirichlet map $D_\lambda$ is a well defined bounded linear operator acting from linspan$\{e_k\}$ into the Sobolev space $W^{-s_0,p}(\mathcal{O})$.*

**Hypothesis 2.5.** *Operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions generates an analytic $C_0$-semigroup $S$ on each $W^{s,p}(\mathcal{O})$-spaces. For all $s, s' \in \mathbb{R}$, $p > 1$ and $t > 0$, $S(t)\colon W^{s,p}(\mathcal{O}) \mapsto W^{s',p}(\mathcal{O})$. Moreover, if $A_{s,p}$[1] denotes the generator of $S$ on $W^{s,p}(\mathcal{O})$, then we assume that there is an $s_1$ such that $W^{-s_0,p}(\mathcal{O}) \hookrightarrow D(A_{-s_1,p})$.*

*Remark* 2.6. It is well known that Hypotheses 2.4 and 2.5 hold in a number of cases. By Theorem 4.10 in [28] if $\mathcal{O}$ is a bounded Lipschitz domain and the operator $A$ has Lipschitz coefficients, then $D_0\colon H^{1/2}(\partial\mathcal{O}) \to H^1(\mathcal{O})$ is well defined and bounded. In that case it is enough to assume that linspan $\{e_k\} \subset H^{1/2}(\partial\mathcal{O})$.

If $\mathcal{O}$ is a bounded $C^\infty$ domain and the operator $A$ has $C^\infty$ coefficients, then

$$D_0\colon H^{-s-\frac{3}{2}}(\partial\mathcal{O}) \to H^{-s}(\mathcal{O})$$

is well defined and bounded for any $s \geq 0$, see Sections 6 and 7 in Chapter 2 of [24]. In particular, if $s \leq -\frac{3}{2}$ then linspan $\{e_k\} \subset H^{-s-\frac{3}{2}}(\partial\mathcal{O}) \subset L^2(\mathcal{O})$.

Very general conditions given in terms of capacities of $\mathcal{O}$ can be found in Chapter 15.7 of [27].

Hypotheses 2.4 and 2.5 enable us to reformulate problem (2.2) into problem (2.4) considered on the state space $W^{-s_1,p}(\mathcal{O})$. In fact the map

$$B = (\lambda - A)\, D_\lambda := (\lambda - A_{-s_1,p})\, D_\lambda$$

is a bounded linear operator from linspan $\{e_k\}$ into $W^{-s_1,p}(\mathcal{O})$ and $A = A_{-s_1,p}$ generates a $C_0$-semigroup $S = S_{-s_1,p}$ on $W^{-s_1,p}(\mathcal{O})$. Therefore, by our Proposition 7.2 we have the following result.

[1]Later we will skip the subscripts $s$ and $p$ and we will write $A$ instead of $A_{s,p}$.

**Theorem 2.7.** *Under Hypotheses 2.4 and 2.5, problem (2.4) has the mild solution solution*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)B\mathrm{d}W(s) \tag{2.5}$$

*in $W^{-s_1,p}(\mathcal{O})$-space if and only if*

$$\int_{\mathcal{O}} \left[ \sum_k \int_0^T \left( (I-\Delta)^{-s_1/2} S(t)Be_k \right)^2 (x)\mathrm{d}t \right]^{p/2} \mathrm{d}x < +\infty \tag{2.6}$$

*for a certain or equivalently for any $T \in (0,+\infty)$. Moreover, if there is an $\alpha > 0$ such that*

$$\int_{\mathcal{O}} \left[ \sum_k \int_0^T t^{-\alpha} \left( (I-\Delta)^{-s_1/2} S(t)Be_k \right)^2 (x)\mathrm{d}t \right]^{p/2} \mathrm{d}x < +\infty$$

*then the mild solution has continuous trajectories*[2] *in $W^{-s_1,p}(\mathcal{O})$,*

*Remark* 2.8. In Section 7 we will show that condition (2.6) guarantees that for any $t \geq 0$, stochastic integral $\int_0^t S(t-s)B\mathrm{d}W(s)$ is well defined in $W^{-s_1,p}(\mathcal{O})$. Note that, if $p=2$, than $W^{-s_1,p}(\mathcal{O})$ is Hilbert space, and (2.6) can be equivalently written as

$$\int_0^T \|S(t)B\|^2_{L_{(HS)}(H_W, W^{-s_1,2}(\mathcal{O}))} \mathrm{d}t < +\infty,$$

where$\| \cdot \|_{L_{(HS)}(H_W, W^{-s_1,2}(\mathcal{O}))}$ is the Hilbert–Schmidt norm and $H_W$ is the Reproducing Kernel Hilbert Space of $W$, see Remark 2.1.

Since

$$(\lambda - A_{0,p})\, S_{0,p}(s) D_\lambda = S_{-s_1,p}(s)\, (\lambda - A_{-s_1,p})\, D_\lambda$$

the formal mild solution and the mild solution defined by (2.5) coincide.

In order to obtain the function-valued solutions we need the following assumption.

**Hypothesis 2.9.** *The semigroup $S$ can be extended to a $C_0$-semigroup on the weighted space $L^p_{\theta,\delta} := L^p(\mathcal{O}, w_{\theta,\delta}(x)\mathrm{d}x)$, where*

$$w_{\theta,\delta}(x) = \min \left\{ \mathrm{dist}\,(x, \partial\mathcal{O})^\theta, \left(1+|x|^2\right)^{-\delta} \right\}, \tag{2.7}$$

*$p \in (1,+\infty)$, $\theta \in [0, 2p-1)$ and $\delta \geq 0$.*

In Section 4 we will show that Hypothesis 2.9 is fulfilled under very mild assumptions on $\mathcal{O}$ and $\mathcal{A}$.

Under the above three hypotheses the operator $B$ acts from linspan $\{e_k\}$ into $W^{-s_0,p}(\mathcal{O})$. For any $t > 0$, the $C_0$-semigroup $S(t)$ on $L^p_{\theta,\delta}$ has a unique continuous extension

$$S(t) \colon W^{-s_0,p}(\mathcal{O}) \mapsto W^{0,p}(\mathcal{O}) = L^p(\mathcal{O}, \mathrm{d}x) \hookrightarrow L^p_{\theta,\delta}.$$

Therefore, as a consequence of our Proposition 7.2 and the classical theory of SPDEs (see e.g. [11]) we have the following general result.

**Theorem 2.10.** *Assume Hypotheses 2.4, 2.5, 2.9. Problem (2.4) has the mild solution solution in $L^p_{\theta,\delta}$ if and only if*

$$\mathcal{J}_T(\{e_k\}, p, \theta, \delta) := \int_{\mathcal{O}} \left[ \sum_k \int_0^T \left(S(t)Be_k\right)^2 (x)\mathrm{d}t \right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x < +\infty \tag{2.8}$$

[2] In fact Hölder continuous with arbitrary exponent $< \alpha/2$.

*for a certain or equivalently for any $T \in (0,+\infty)$. Moreover, (2.8) guarantees that problem (2.4) equivalently (1.1), defines a Markov family on the state space $L^p_{\theta,\delta}$. If for a certain $\alpha > 0$,*

$$\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) := \int_{\mathcal{O}} \left[ \sum_k \int_0^T t^{-\alpha} \left(S(t)Be_k\right)^2(x)\mathrm{d}t \right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x < +\infty, \tag{2.9}$$

*then the mild solution has continuous trajectories in $L^p_{\theta,\delta}$.*

*Finally, the existence of an invariant measure is equivalent to the integrability condition*

$$\mathcal{J}_{+\infty}(\{e_k\},p,\theta,\delta) := \int_{\mathcal{O}} \left[ \sum_k \int_0^{+\infty} \left(S(t)Be_k\right)^2 \mathrm{d}t \right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x < +\infty. \tag{2.10}$$

*Remark* 2.11. If the semigroup $S$ is exponentially stable, i.e. for a certain $\alpha > 0$,

$$\|S(t)\|_{L(L^p_{\theta,\delta},L^p_{\theta,\delta})} \le C\mathrm{e}^{-\alpha t}, \qquad t \ge 0,$$

then condition (2.10) follows from (2.8). In Theorem 5.2, we will show that the semigroup $S$ is exponential stable on $L^p_{\theta,\delta}$ if it is exponentially stable on $L^p_{0,\delta}$. Obviously if the domain $\mathcal{O}$ is bounded then for all $p$, $\theta$ and $\delta$, the spaces $L^p_{\theta,\delta}$ and $L^p_{\theta,0}$ are equivalent. Therefore, if $\mathcal{O}$ is bounded then we can always take $\delta = 0$. Note that if $\mathcal{O}$ is bounded and $\mathcal{A}$ equals Laplace operator $\Delta$, then the corresponding semigroup is exponentially stable on $L^p_{0,0}$ and consequently on $L^p_{\theta,0}$ for any $p > 1$.

*Remark* 2.12. Assume (2.8), Then for any $X_0 \in L^p_{\theta,\delta}$ and for any $t > 0$, $X(t)$ is a gaussian element in $L^p_{\theta,\delta}$. Therefore, by the Fernique theorem there is a $\beta > 0$ such that

$$\mathbb{E}\,\exp\left\{\beta\,|X(t)|^2_{L^p_{\theta,\delta}}\right\} < +\infty.$$

If (2.9) is satisfied for an $\alpha > 0$, then for any $T \in (0,+\infty)$, and for any $X_0 \in L^p_{\theta,\delta}$, $X(\cdot)$ is a gaussian random element in $C([0,T];L^p_{\theta,\delta})$. Thus there is a $\beta > 0$ such that

$$\mathbb{E}\,\exp\left\{\beta \sup_{t\in[0,T]} |X(t)|^2_{L^p_{\theta,\delta}}\right\} < +\infty.$$

Our framework enables us to study nonlinear problems.

**Theorem 2.13.** *Assume (2.8), and Hypotheses 2.4, 2.5, 2.9. Then for any Lipschitz continuous function $f\colon \mathbb{R} \mapsto \mathbb{R}$, and any $X_0 \in L^p_{\theta,\delta}$, the boundary problem*

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = \mathcal{A}X(t,x) + f(X(t,x))\,, & x \in \mathcal{O}, \quad t > 0\,,\\[2ex] X(t,x) = \dfrac{\partial W}{\partial t}(t,x)\,, & x \in \partial\mathcal{O}, \quad t > 0\,,\\[2ex] X(0,x) = X_0(x), & x \in \mathcal{O}, \end{cases}$$

*has a unique solution in $L^p_{\theta,\delta}$, and*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)B\mathrm{d}W(s),$$

*where $F(X(t))(x) = f(X(t,x))$. Finally, if the semigroup is exponentially stable with exponent $L$ and the Lipschitz constant of $f$ strictly less than $L$, then there is a unique invariant measure on $L^p_{\theta,\delta}$ for the nonlinear problem.*

The paper is organized as follows. In the next section we will heuristically derive the concept of the formal mild solution. Then, in Section 4, we will show that Hypothesis 2.9 about the $C_0$-property of the semigroup on weighted $L^p$-spaces, is fulfilled under rather mild assumptions. The main difficulty is that the weight of the above form is not an $\mathcal{A}^*$-excessive function. Therefore the semigroup is not of contraction type. In Section 5 we will study properties of the semigroup on weighted spaces. In our opinion the results of Sections 4 and 5 are of independent interest.

In Section 6 we will derive some useful point estimates for $S(t)Be$ for $e \in L^2(\partial\mathcal{O}, ds)$. In Section 7 we outline the concept of stochastic integration in $L^p$-spaces. Section 8 is devoted to particular examples. We relay on estimates established in Section 6 and on results from the previous section.

In Section **??**, we give a sufficient condition for the strong Feller property of the Markov family $\mathcal{X}$ defined in Theorem 2.10.

## 3. Formal mild solution

In our derivation of the concept of the *formal mild solution to* (1.1) we follow [11]. Recall that $D_\lambda$ denotes the Dirichlet map (see (2.3)). Assume Hypothesis 2.4 and 2.5.

Assume temporally that the boundary perturbation is of the form

$$W(t,x) = \sum_k e_k(x)\beta_k(t),$$

where series is finite, $e_k$ are functions or distributions on $\partial\mathcal{O}$, and $\beta_k \in C^1([0,+\infty))$. We assume that for any $k$, $e_k$ belongs to the domain of the Dirichlet map $D_\lambda$ and that $D_\lambda e_k \in W^{-s_0,p}(\mathcal{O})$.

Note that if $X$ is a solution to (1.1) with $W$ as above, then

$$Y(t,x) := X(t,x) - D_\lambda \frac{\partial W}{\partial t}(t,x)$$

satisfies the homogeneous Dirichlet boundary conditions. Moreover, at least formally, for $t > 0$ and $x \in \mathcal{O}$ we have

$$\begin{aligned}\frac{\partial Y}{\partial t}(t,x) &= AX(t,x) - \frac{\partial}{\partial t} D_\lambda \frac{\partial W}{\partial t}(t,x)\\ &= AY(t,x) + \lambda D_\lambda \frac{\partial W}{\partial t}(t,x) - \frac{\partial}{\partial t} D_\lambda \frac{\partial W}{\partial t}(t,x).\end{aligned}$$

Therefore

$$\begin{aligned}Y(t,x) &= S(t)Y(0,x) + \int_0^t S(t-s)\left[\lambda D_\lambda \frac{\partial W}{\partial s}(s,x) - \frac{\partial}{\partial s} D_\lambda \frac{\partial W}{\partial s}(s,x)\right] \mathrm{d}s\\ &= S(t)Y(0,x) + \int_0^t S(t-s)\lambda D_\lambda \frac{\partial W}{\partial s}(s,x)\mathrm{d}s\\ &\quad - \left[D_\lambda \frac{\partial W}{\partial t}(t,x) - S(t)D_\lambda \frac{\partial W}{\partial t}(0,x) + \int_0^t AS(t-s)D_\lambda \frac{\partial W}{\partial s}(s,x)\mathrm{d}s\right]\\ &= S(t)X(0,x) - D_\lambda \frac{\partial W}{\partial t}(t,x) + \int_0^t (\lambda - A)\, S(t-s)D_\lambda \frac{\partial W}{\partial s}(s,x)\mathrm{d}s.\end{aligned}$$

Hence we infer that

$$X(t) = S(t)X(0) + \int_0^t (\lambda - A)\, S(t-s)D_\lambda \mathrm{d}W(s). \tag{3.1}$$

Let us recall that the space $W^{-s_1,p}(\mathcal{O})$ appears in Hypotheses 2.5. Note that for any $k$ the stochastic integral

$$\int_0^t (\lambda - A)\, S(t-s) D_\lambda e_k \mathrm{d}W_k(s), \qquad t \geq 0,$$

takes valued in $W^{-s_1,p}(\mathcal{O})$.

**Definition 3.1.** Let $X(0) \in W^{-s_1,p}(\mathcal{O})$, and let the noise $W$ in (1.1) have the form (2.1). If the series

$$\sum_k \int_0^t (\lambda - A)\, S(t-s) D_\lambda e_k \mathrm{d}W_k(s) =: \int_0^t (\lambda - A)\, S(t-s) D_\lambda \mathrm{d}W(s)$$

converges in $L^p(\Omega, \mathfrak{F}, \mathbb{P}; W^{-s_1,p}(\mathcal{O}))$, then we call the proces defined by formula (3.1) the *formal mild solution* to (1.1) in $W^{-s_1,p}(\mathcal{O})$.

## 4. Semigroup in weighted spaces

Let $\mathcal{O} \subset \mathbb{R}^d$, $\mathcal{O} \neq \mathbb{R}^d$, be an open connected domain. From now on the following two assumptions will be satisfied.

**Assumption 4.1.** *We will assume that $\mathcal{O}$ is a $C^{1,\alpha}$-domain with $\alpha \in (0,1)$, satisfying the connected line condition, see e.g. [8] for a precise definition. Let us recall here that the connected line condition holds in many important cases including:*

- *bounded $C^{1,\alpha}$ domain,*
- *graph above $C^{1,\alpha}$ function,*
- *$\mathcal{O} = \mathbb{R}^d_+$ or*

$$\mathcal{O} = \left\{ (x_i) \in \mathbb{R}^{d+1} \colon a < x_{d+1} < b \right\}.$$

Let us consider the following second order differential operator

$$\mathcal{A}\phi(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left( a_{ij}(x) \frac{\partial \phi}{\partial x_j}(x) \right) + \sum_{i=1}^d \mu^i(x) \frac{\partial \phi}{\partial x_i}(x).$$

**Assumption 4.2.** *We assume that the homogeneous Dirichlet boundary problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathcal{A}u(t,x)\,, & x \in \mathcal{O}, \quad t > 0\,, \\ u(t,x) = 0\,, & x \in \partial\mathcal{O}, \quad t > 0\,, \\ u(0,x) = f(x) & x \in \mathcal{O}\,, \end{cases} \tag{4.1}$$

*generates a $C_0$-semigroup $(S(t))$ on $L^2(\mathcal{O}, \mathrm{d}x)$. The generator of this semigroup will be denoted by $A$. Next, we assume that the semigroup can be represented by a Green kernel $G$,*

$$S(t)\psi(x) = \int_{\mathcal{O}} G(t,x,y)\psi(y)\mathrm{d}y, \quad x \in \mathcal{O}. \tag{4.2}$$

*Finally we assume that there exists a constant $\lambda > 0$ such that*

$$\lambda |h|^2 \leq \langle a(x) a^T(x) h, h \rangle \leq \lambda^{-1} |h|^2, \quad x, h \in \mathbb{R}^d,$$

$$G(t,x,y) \leq C m_t(y) g_{ct}(x-y), \qquad t \leq 1, \qquad x, y \in \mathcal{O}, \tag{4.3}$$

*and*

$$|\nabla_x G(t,x,y)| \le C\frac{m_t(y)}{\sqrt{t}} g_{ct}(x-y), \qquad t\le 1, \qquad x,y\in\mathcal{O}, \tag{4.4}$$

*where*

$$m_t(z) := \min\left\{1, \frac{\rho(z)}{\sqrt{t}}\right\}, \qquad \rho(z) := \mathrm{dist}\,(z,\partial\mathcal{O})$$

*and*

$$g_t(z) = (2\pi t)^{-\frac{d}{2}}\,\mathrm{e}^{-\frac{|z|^2}{2t}}.$$

*Remark* 4.3. Assumption 4.2 is fulfilled if Assumption 4.1 holds, the operator $\mathcal{A}$ is uniformly elliptic, the coefficients $a_{ij}$ are Dini continuous, and $\mu^i$ are sign measures of the parabolic Kato class. In general $a_{i,j}$ and $\mu^i$ may depend on $t$ and $x$ variables. For more details see [8]. In fact in [8] the following stronger estimate has beed obtained

$$G(t,x,y) \le Cm_t(x)m_t(y)g_{ct}(x-y), \qquad t\le 1, \qquad x,y\in\mathcal{O}.$$

In the main theorem of this section we require the following assumption

**Assumption 4.4.** *For any $c>0$ and $\alpha\in(-1,0)$ there is a constant $C<+\infty$ such that*

$$\sup_{x\in\mathcal{O}}\int_{\mathcal{O}} \rho^\alpha(y) g_{ct}(x-y)\mathrm{d}y \le Ct^{\frac{\alpha}{2}}, \qquad \forall\, t\in(0,1].$$

A proof of the following lemma is postponed to Appendix A.

**Lemma 4.5.** *Assumption 4.4 is satisfied if $\mathcal{O}$ is a half space or if $\mathcal{O}$ is a bounded $C^{1,\alpha}$-domain.*

Recall that the family of weights $w_{\theta,\delta}$, $\theta\ge 0$, $\delta\ge 0$, were introduced in (2.7). We will use the notations

$$L^p_{\theta,\delta} := L^p\left(\mathcal{O},\mathcal{B}(\mathcal{O}), w_{\theta,\delta}(x)\mathrm{d}x\right), \qquad L^p := L^p_{0,0} = L^p(\mathcal{O},\mathrm{d}x), \qquad p\ge 1,\ \theta,\delta\ge 0.$$

Let $S=(S(t))$ be the $C_0$ semigroup on $L^2$ corresponding to (4.1). By Assumption 4.2, for each $t>0$, $S(t)$ is defined by (4.3) at least on compactly supported functions $\psi$.

The main result of this section is the following theorem. Its proof is given in Section 4.2.

**Theorem 4.6.** *Let $p\in[1,+\infty)$, $\theta\in[0,2p-1)$ and $\delta\ge 0$. Under Assumptions 4.1, 4.2, and 4.4 we have:*

- (i) *For each $t$, $S(t)$ defined on compactly supported functions by (4.2) has a unique extension to a bounded linear operator, denoted still by $S(t)$, acting from $L^p_{\theta,\delta}$ into $L^p_{\theta,\delta}$. Moreover, $S=(S(t))$ forms a $C_0$-semigroup on $L^p_{\theta,\delta}$.*
- (ii) *There exists a constant $C>0$ such that for all $t\in(0,1]$ and $\psi\in L^p_{\theta,\delta}$, $S(t)\psi(x)$ is differentiable for each $x\in\mathcal{O}$ and*

$$\left|\frac{\partial}{\partial x_i}S(t)\psi\right|_{L^p_{\theta,\delta}} \le \frac{C}{\sqrt{t}}|\psi|_{L^p_{\theta,\delta}}, \qquad i=1,\ldots,d.$$

*Remark* 4.7. If $\theta>2p-1$, then $L^p_\theta$ contains functions $f$ with growth $\rho^{-2}(y)$ at vicinity of some point of $\partial\mathcal{O}$. On the other hand, the integral $\int_{\mathcal{O}} G(t,x,y)f(y)\mathrm{d}y$ does not converge as $G(t,x,y)$ decays only at rate $\rho(y)$ at the boundary. Therefore, for $t>0$, $S(t)$ cannot be extended to $L^p_\theta$.

*Remark* 4.8. For $0\le\theta<p$ we are able to show the $C_0$-property and gradient estimates without Assumption 4.4, for details see Appendix B.

4.1. **Preliminaries.** Let $w^i\colon \mathcal{O}\mapsto(0,+\infty)$, $i=1,2$, be measurable weights. Let

$$\mathcal{L}^p_i := L^p\left(\mathcal{O},\mathcal{B}(\mathcal{O}), w^i(x)\mathrm{d}x\right), \qquad i=1,2,$$

and let

$$\mathcal{L}^p := L^p(\mathcal{O},\mathcal{B}(\mathcal{O}), w(x)\,\mathrm{d}x),$$

where $w(x)=\min\{w^1(x),w^2(x)\}$. We will need the following elementary result.

**Lemma 4.9.** *Assume that $T$ is a bounded linear operator from $\mathcal{L}^p_i$ to $\mathcal{L}^p_i$ for $i=1,2$. Then it is bounded from $\mathcal{L}^p$ to $\mathcal{L}^p$ and the operator norm satisfies the estimate*

$$\|T\|_{L(\mathcal{L}^p)} \le N := 2^{(p-1)/p}\max\left\{\|T\|_{L(\mathcal{L}^p_1)}, \|T\|_{L(\mathcal{L}^p_2)}\right\}.$$

*Proof.* Let $\psi\in\mathcal{L}^p_1\cap\mathcal{L}^p_2$ and

$$\mathcal{D}:=\{x\in\mathcal{O}\colon w_1(x)<w_2(x)\},\qquad \mathcal{D}^c:=\mathcal{O}\setminus\mathcal{D}.$$

Then

$$\begin{aligned}
\int_{\mathcal{O}}|T\psi(x)|^p\, w(x)\mathrm{d}x &= \int_{\mathcal{O}}|T(\chi_{\mathcal{D}}\psi)(x)+T(\chi_{\mathcal{D}^c}\psi)(x)|^p\, w(x)\mathrm{d}x\\
&\le 2^{p-1}\left[\int_{\mathcal{O}}|T(\chi_{\mathcal{D}}\psi)(x)|^p\, w(x)\mathrm{d}x+\int_{\mathcal{O}}|T(\chi_{\mathcal{D}^c}\psi)(x)|^p\, w(x)\mathrm{d}x\right]\\
&\le 2^{p-1}\left[\int_{\mathcal{O}}|T(\chi_{\mathcal{D}}\psi)(x)|^p\, w^1(x)\mathrm{d}x+\int_{\mathcal{O}}|T(\chi_{\mathcal{D}^c}\psi)(x)|^p\, w^2(x)\mathrm{d}x\right]\\
&\le N^p\left[\int_{\mathcal{O}}|\chi_{\mathcal{D}}(x)\psi(x)|^p\, w^1(x)\mathrm{d}x+\int_{\mathcal{O}}|\chi_{\mathcal{D}^c}(x)\psi(x)|^p\, w^2(x)\mathrm{d}x\right].
\end{aligned}$$

Since

$$\int_{\mathcal{O}}|\chi_{\mathcal{D}}(x)\psi(x)|^p\, w^1(x)\mathrm{d}x+\int_{\mathcal{O}}|\chi_{\mathcal{D}^c}(x)\psi(x)|^p\, w^2(x)\mathrm{d}x=|\psi|^p_{\mathcal{L}^p},$$

we have the desired conclusion. □

4.2. **Proof of Theorem 4.6.** Let

$$L^p_\theta := L^p\left(\mathcal{O},\rho^\theta(x)\mathrm{d}x\right)\quad\text{and}\quad \mathcal{L}^p_\delta := L^p\left(\mathcal{O},\left(1+|x|^2\right)^{-\delta}\mathrm{d}x\right).$$

By Lemma 4.9 it is enough to prove that $S=(S(t))$ extends to a $C_0$-semigroup in the spaces $\mathcal{L}^p_\delta$ and $L^p_\theta$ separately. The $C_0$-property of the semigroup $(S(t))$ in $\mathcal{L}^p_\delta$ can be shown using the method from [33]. Therefore, it remains to prove the semigroup property in $L^p_\theta$. We have

$$\begin{aligned}
\int_{\mathcal{O}}|S(t)\varphi(x)|^p\,\rho^\theta(x)\mathrm{d}x &= \int_{\mathcal{O}}\left|\rho^{\frac{\theta+1}{p}}(x)S(t)\varphi(x)\right|^p\frac{\mathrm{d}x}{\rho(x)}\\
&=\int_{\mathcal{O}}\left|\int_{\mathcal{O}}\rho^{\frac{\theta+1}{p}}(x)G(t,x,y)\rho(y)\varphi(y)\frac{\mathrm{d}y}{\rho(y)}\right|^p\frac{\mathrm{d}x}{\rho(x)}\\
&=\int_{\mathcal{O}}\left|\int_{\mathcal{O}}\rho^{\frac{\theta+1}{p}}(x)G(t,x,y)\rho^{1-\frac{\theta+1}{p}}(y)\rho^{\frac{\theta+1}{p}}(y)\varphi(y)\frac{\mathrm{d}y}{\rho(y)}\right|^p\frac{\mathrm{d}x}{\rho(x)}\\
&\le C\int_{\mathcal{O}}\left|\int_{\mathcal{O}}\rho^{\frac{\theta+1}{p}}(x)m_t(y)g_{ct}(x-y)\rho^{1-\frac{\theta+1}{p}}(y)\psi(y)\frac{\mathrm{d}y}{\rho(y)}\right|^p\frac{\mathrm{d}x}{\rho(x)},
\end{aligned}$$

with $\psi(y)=\rho^{\frac{\theta+1}{p}}(y)\varphi(y)$. Note that the last inequality follows from (4.3). In other words

$$|S(t)\varphi|^p_{L^p_\theta}\le C\,|K_t\psi|^p_{L^p\left(\mathcal{O},\frac{\mathrm{d}y}{\rho(y)}\right)},$$

where

$$K_t\psi(x) := \int_{\mathcal{O}} k_t(x,y)\psi(y)\,\frac{\mathrm{d}y}{\rho(y)},$$
$$k_t(x,y) := \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} m_t(y) g_{ct}(x-y)\rho(y),$$

and $\psi$ is as above.

Since $\varphi \mapsto \psi = \rho^{\frac{\theta+1}{p}}\varphi$ is an isometry between $L^p_\theta$ and $L^p\left(\mathcal{O}, \frac{\mathrm{d}y}{\rho(y)}\right)$, the proof of a $C_0$-property will be completed as soon as we show that for each $0<t\leq 1$, $K_t$ is a bounded linear operator from $L^p\left(\mathcal{O}, \frac{\mathrm{d}y}{\rho(y)}\right)$ into $L^p\left(\mathcal{O}, \frac{\mathrm{d}y}{\rho(y)}\right)$ and that $\sup_{0<t\leq 1}\|K_t\| < +\infty$, where $\|\cdot\|$ is the operator norm on $L\left(L^p\left(\mathcal{O}, \frac{\mathrm{d}y}{\rho(y)}\right), L^p\left(\mathcal{O}, \frac{\mathrm{d}y}{\rho(y)}\right)\right)$. The second part of the theorem follows since, by (4.4),

$$\left|\frac{\partial S(t)\varphi}{\partial x_i}\right|^p_{L^p_\theta} \leq C t^{-\frac{p}{2}} \left|K_t\psi\right|^p_{L^p\left(\mathcal{O},\frac{\mathrm{d}y}{\rho(y)}\right)}.$$

Taking into account the Schur test, see e.g. Theorem 5.9.2 in [19], it is enough to show that

$$\sup_{0<t\leq 1}\sup_{x\in\mathcal{O}}\int_{\mathcal{O}} k_t(x,y)\,\frac{\mathrm{d}y}{\rho(y)} + \sup_{0<t\leq 1}\sup_{y\in\mathcal{O}}\int_{\mathcal{O}} k_t(x,y)\,\frac{\mathrm{d}x}{\rho(x)} < +\infty\,.$$

Note that our assumption $\theta < 2p-1$ is necessary for the application of the Schur test.

Given $t\in(0,1]$, let $\mathcal{O}_t := \left\{x\in\mathcal{O}\colon \rho(x) < \sqrt{t}\right\}$ and $(\mathcal{O}_t)^{\mathrm{c}} := \mathcal{O}\setminus\mathcal{O}_t$. Write

$$k_1 := \sup_{0<t\leq 1}\sup_{x\in\mathcal{O}_t}\int_{\mathcal{O}_t} k_t(x,y)\,\frac{\mathrm{d}y}{\rho(y)}, \qquad k_2 := \sup_{0<t\leq 1}\sup_{y\in\mathcal{O}_t}\int_{\mathcal{O}_t} k_t(x,y)\,\frac{\mathrm{d}x}{\rho(x)},$$
$$k_3 := \sup_{0<t\leq 1}\sup_{x\in(\mathcal{O}_t)^{\mathrm{c}}}\int_{\mathcal{O}_t} k_t(x,y)\,\frac{\mathrm{d}y}{\rho(y)}, \qquad k_4 := \sup_{0<t\leq 1}\sup_{y\in\mathcal{O}_t}\int_{(\mathcal{O}_t)^{\mathrm{c}}} k_t(x,y)\,\frac{\mathrm{d}x}{\rho(x)},$$
$$k_5 := \sup_{0<t\leq 1}\sup_{x\in\mathcal{O}_t}\int_{(\mathcal{O}_t)^{\mathrm{c}}} k_t(x,y)\,\frac{\mathrm{d}y}{\rho(y)}, \qquad k_6 := \sup_{0<t\leq 1}\sup_{y\in(\mathcal{O}_t)^{\mathrm{c}}}\int_{\mathcal{O}_t} k_t(x,y)\,\frac{\mathrm{d}x}{\rho(x)},$$
$$k_7 := \sup_{0<t\leq 1}\sup_{x\in(\mathcal{O}_t)^{\mathrm{c}}}\int_{(\mathcal{O}_t)^{\mathrm{c}}} k_t(x,y)\,\frac{\mathrm{d}y}{\rho(y)}, \qquad k_8 := \sup_{0<t\leq 1}\sup_{y\in(\mathcal{O}_t)^{\mathrm{c}}}\int_{(\mathcal{O}_t)^{\mathrm{c}}} k_t(x,y)\,\frac{\mathrm{d}x}{\rho(x)}.$$

Note that the proof will be completed as soon as we show that all $k_j$ are finite.

To estimate $k_5$ to $k_8$ where $y\in(\mathcal{O}_t)^{\mathrm{c}}$ we use the Lipchitz continuity of the distance function $\rho$ and the estimate $\rho(y)\geq\sqrt{t}$ for $y\in(\mathcal{O}_t)^{\mathrm{c}}$. Namely, for any $\alpha\geq 0$, we have

$$\left(\frac{\rho(x)}{\rho(y)}\right)^{\alpha} \leq \left(\frac{|\rho(x)-\rho(y)|}{\rho(y)}+1\right)^{\alpha} \leq C\left(\frac{|x-y|}{\sqrt{t}}+1\right)^{\alpha}.$$

Since $m_t(y)\leq 1$, we have

$$\begin{aligned} k_5 &\leq \sup_{0<t\leq 1}\sup_{x\in\mathcal{O}_t}\int_{(\mathcal{O}_t)^{\mathrm{c}}}\left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y)\mathrm{d}y \\ &\leq \sup_{0<t\leq 1}\sup_{x\in\mathcal{O}}\int_{(\mathcal{O}_t)^{\mathrm{c}}}\left(\frac{|x-y|}{\sqrt{t}}+1\right)^{\frac{\theta+1}{p}} g_{ct}(x-y)\mathrm{d}y \\ &\leq \sup_{0<t\leq 1}\int_{\mathbb{R}^d}\left(\frac{|z|}{\sqrt{t}}+1\right)^{\frac{\theta+1}{p}} g_{ct}(z)\mathrm{d}z \leq \int_{\mathbb{R}^d}\left(|z|+1\right)^{\frac{\theta+1}{p}-1} g_c(z)\mathrm{d}z < +\infty. \end{aligned}$$

To estimate $k_7$ note that

$$k_7 \le \sup_{0<t\le 1} \sup_{x\in(\mathcal{O}_t)^c} \int_{(\mathcal{O}_t)^c} \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y)\mathrm{d}y \le C\int_{\mathbb{R}^d} (|z|+1)^{\frac{\theta+1}{p}} g_c(z)\mathrm{d}z < +\infty.$$

In the case of $\frac{\theta+1}{p}-1>0$, equivalently of $\theta>p-1$, one can use the same arguments to evaluate $k_6$ and $k_8$. Namely, we have

$$\begin{aligned} k_6 &\le \sup_{0<t\le 1} \sup_{y\in(\mathcal{O}_t)^c} \int_{\mathcal{O}_t} \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y) \left(\frac{\rho(x)}{\rho(y)}\right)^{-1} \mathrm{d}x \\ &\le C \sup_{0<t\le 1} \sup_{y\in(\mathcal{O}_t)^c} \int_{\mathcal{O}_t} \left(\frac{|x-y|}{\sqrt{t}}+1\right)^{\frac{\theta+1}{p}-1} (x) g_{ct}(x-y)\mathrm{d}x < +\infty \end{aligned}$$

and

$$\begin{aligned} k_8 &\le \sup_{0<t\le 1} \sup_{y\in(\mathcal{O}_t)^c} \int_{(\mathcal{O}_t)^c} \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y) \left(\frac{\rho(x)}{\rho(y)}\right)^{-1} \mathrm{d}x \\ &\le C \sup_{0<t\le 1} \sup_{y\in(\mathcal{O}_t)^c} \int_{(\mathcal{O}_t)^c} \left(\frac{|x-y|}{\sqrt{t}}+1\right)^{\frac{\theta+1}{p}-1} (x) g_{ct}(x-y)\mathrm{d}x < +\infty \end{aligned}$$

The case of $\frac{\theta+1}{p}-1<0$ can be treated as follows

$$\begin{aligned} k_6 &\le \sup_{0<t\le 1} \sup_{y\in(\mathcal{O}_t)^c} \int_{\mathcal{O}_t} \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y) \left(\frac{\rho(x)}{\rho(y)}\right)^{-1} \mathrm{d}x \\ &\le \sup_{0<t\le 1} \sup_{\sqrt{t}\le u\le 1} \sup_{y\colon \rho(y)=u} u^{1-\frac{\theta+1}{p}} \int_{\mathcal{O}_t} \rho^{\frac{\theta+1}{p}-1}(x) g_{ct}(x-y)\mathrm{d}x. \end{aligned}$$

Note that for any $\sqrt{t}\le u\le 1$, we have

$$\inf\left\{|x-y|^2 \colon y\in(\mathcal{O}_t)^c, \rho(y)=u,\ x\in\mathcal{O}_t\right\} = |u-\sqrt{t}|^2.$$

Thus, by Assumption 4.4,

$$\begin{aligned} k_6 &\le (2\pi c)^{\frac{d}{2}} \sup_{0<t\le 1} \sup_{\sqrt{t}\le u\le 1} \sup_{y\colon \rho(y)=u} u^{1-\frac{\theta+1}{p}} \mathrm{e}^{-\frac{|u-\sqrt{t}|}{4ct}} \int_{\mathcal{O}_t} \rho^{\frac{\theta+1}{p}-1}(x) g_{2ct}(x-y)\mathrm{d}x \\ &\le (2\pi c)^{\frac{d}{2}} \sup_{0<t\le 1} \sup_{\sqrt{t}\le u\le 1} \sup_{y\colon \rho(y)=u} u^{1-\frac{\theta+1}{p}} \mathrm{e}^{-\frac{|u-\sqrt{t}|}{4ct}} t^{\frac{\theta+1}{2p}-\frac{1}{2}} < +\infty. \end{aligned}$$

In the same way, if $\frac{\theta+1}{p}-1<0$, then

$$\begin{aligned} k_8 &\le \sup_{0<t\le 1} \sup_{\sqrt{t}\le u\le 1} \sup_{y\colon \rho(y)=u} \int_{(\mathcal{O}_t)^c} \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} g_{ct}(x-y) \left(\frac{\rho(x)}{\rho(y)}\right)^{-1} \mathrm{d}x \\ &\le (2\pi c)^{\frac{d}{2}} \sup_{0<t\le 1} \sup_{\sqrt{t}\le u\le 1} \sup_{y\colon \rho(y)=u} u^{1-\frac{\theta+1}{p}} \mathrm{e}^{-\frac{|u-\sqrt{t}|}{4ct}} \int_{\mathcal{O}} \rho^{\frac{\theta+1}{p}-1}(x) g_{2ct}(x-y)\mathrm{d}x < +\infty. \end{aligned}$$

We use Assumption 4.4 to evaluate $k_4$. Namely, since $2-\frac{\theta+1}{p}>0$, we have

$$\begin{aligned} k_4 &= \sup_{0<t\le 1} \sup_{y\in\mathcal{O}_t} \rho^{2-\frac{\theta+1}{p}}(y) t^{-\frac{1}{2}} \int_{(\mathcal{O}_t)^c} \rho^{\frac{\theta+1}{p}-1}(x) g_{ct}(x-y)\,\mathrm{d}x \\ &\le \sup_{0<t\le 1} t^{\frac{1}{2}-\frac{\theta+1}{2p}} \sup_{y\in\mathcal{O}_t} \int_{(\mathcal{O}_t)^c} \rho^{\frac{\theta+1}{p}-1}(x) g_{ct}(x-y)\,\mathrm{d}x < +\infty. \end{aligned}$$

The same argument can be used to evaluate $k_1$. Namely since for $x\in\mathcal{O}_t$, $\rho(x)\le\sqrt{t}$, we have

$$k_1=\sup_{0<t\le 1}\sup_{x\in\mathcal{O}_t}\int_{\mathcal{O}_t}\rho^{\frac{\theta+1}{p}}(x)\frac{\rho(y)}{\sqrt{t}}g_{ct}(x-y)\rho^{-\frac{\theta+1}{p}}(y)\,dy$$
$$\le C_1\sup_{0<t\le 1}t^{\frac{\theta+1}{2p}-\frac12}\sup_{x\in\mathcal{O}_t}\int_{\mathcal{O}_t}\rho^{1-\frac{\theta+1}{p}}(y)g_{ct}(x-y)dy<+\infty.$$

Above we used Assumption 4.4 and the fact that $1-\frac{\theta+1}{p}>-1$ as $\theta<2p-1$.

To estimate $k_2$ we need $2-\frac{\theta+1}{p}>0$, that is $\theta<2p-1$. Since $\rho(x)\le\sqrt{t}$ and $\rho(y)\le\sqrt{t}$ for $x,y\in\mathcal{O}_t$, we have

$$k_2=\sup_{0<t\le 1}\sup_{y\in\mathcal{O}_t}\int_{\mathcal{O}_t}\rho^{\frac{\theta+1}{p}}(x)t^{-1}g_{ct}(x-y)\rho^{2-\frac{\theta+1}{p}}(y)\,dx$$
$$\le\sup_{0<t\le 1}t^{\frac{\theta+1}{2p}-1+1-\frac{\theta+1}{2p}}\sup_{y\in\mathcal{O}_t}\int_{\mathcal{O}_t}g_{ct}(x-y)dx<+\infty.$$

It remains to evaluate $k_3$. We have

$$k_3=\sup_{0<t\le 1}\sup_{x\in(\mathcal{O}_t)^c}\rho^{\frac{\theta+1}{p}}(x)t^{-\frac12}\int_{\mathbb{B}^d_t}g_{ct}(x-y)\rho^{1-\frac{\theta+1}{p}}(y)\,dy.$$

Note that for any $\sqrt{t}\le u\le 1$, we have

$$\inf\left\{|x-y|^2\colon x\in(\mathcal{O}_t)^c,\rho(x)=u,\ y\in\mathcal{O}_t\right\}=|u-\sqrt{t}|^2.$$

Thus, by Assumption 4.4,

$$k_3\le C_1\sup_{0<t\le 1}\sup_{\sqrt{t}0\le u\le 1}\sup_{x\in\mathcal{O}_t\colon\rho(x)=u}u^{\frac{\theta+1}{p}}t^{-\frac12}e^{-\frac{|u-\sqrt{t}|^2}{4ct}}\int_{\mathcal{O}_t}\rho^{1-\frac{\theta+1}{p}}(y)g_{2ct}(x-y)dy$$
$$\le C_2\sup_{0<t\le 1}\sup_{\sqrt{0}\le u\le 1}u^{\frac{\theta+1}{p}}t^{-\frac12}e^{-\frac{|u-\sqrt{t}|^2}{4ct}}t^{\frac12-\frac{\theta+1}{2p}}<+\infty.$$

□

## 4.3. **Analiticity.**

*Remark* 4.10. Assume that the derivatives $\frac{\partial}{\partial x_i}$ commute with the semigroup in the following sense

$$\frac{\partial}{\partial x_i}S(t)=S(t/2)\frac{\partial}{\partial x_i}S(t/2)+R_i(t)S(t/2),$$

where $R_i(t)$, $t>0$ are bounded linear operator satisfying

$$\|R_i(t)\|_{L(L^p_{\theta,\delta},L^p_{\theta,\delta})}\le C_1t^{-1/2}.$$

Then, by second part of Theorem 4.6,

$$\left|\frac{\partial^2}{\partial x_i^2}S(t)\psi\right|_{L^p_{\theta,\delta}}=\left|\frac{\partial}{\partial x_i}S(t/2)\frac{\partial}{\partial x_i}S(t/2)\psi+\frac{\partial}{\partial x_i}S(t/2)R_i(t)S(t/2)\psi\right|_{L^p_{\theta,\delta}}\le\frac{C^2}{t}|\psi|_{L^p_{\theta,\delta}}.$$

This leads to the analiticity of $S$ on $L^p_{\theta,\delta}$ in the case of $\mathcal{A}$ of the form $\sum_{i,j}a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b_i\frac{\partial}{\partial x_i}$ with bounded $a_{i,j}$ and $b_j$.

The classical Aronson estimates for the Green kernel, see e.g. [16, 36, 31] for required assumptions on $\mathcal{A}$ and $\mathcal{O}$, yield that $G$ is of class $C^\infty((0,+\infty)\times\mathcal{O}\times\mathcal{O})$ and for any non-negative integer $n$, multi-indices $\alpha$, $\beta$, and time $T>0$, there are constants $C,c>0$ such that for all $t\in(0,T]$ and $x,y\in\mathcal{O}$,

$$\left|\frac{\partial^n}{\partial t^n}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}G(t,x,y)\right|\le Ct^{-\frac{|\alpha|+|\beta|+2n}{2}}g_{ct}(x-y).$$

In our proofs of the $C_0$-property and gradient estimate we needed something different, namely estimates (4.3) and (4.4) which guarantee that $G(t,x,y)$ and $\nabla_x G(t,x,y)$ decay for $y$ near the boundary of $\mathcal{O}$ at rate $\rho(y)/\sqrt{t}$ uniformly in $x$. Clearly, our proof yelds the following.

**Proposition 4.11.** *If for a certain multi index $\alpha$, there are constants $C,c>0$ such that*

$$\left|\frac{\partial^{|\alpha|}G}{\partial x^\alpha}(t,x,y)\right|\le Ct^{-\frac{|\alpha|}{2}}m_t(y)g_{ct}(x-y),\qquad\forall\, x,y\in\mathcal{O},\ \forall\, t\in(0,1],\tag{4.5}$$

*then, for all $p\ge1$, $\theta\in[0,2p-1)$, $\delta\ge0$, and $T>0$, there is a constant $C_1$ such that*

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}S(t)\psi\right|_{L^p_{\theta,\delta}}\le C_1t^{-\frac{|\alpha|}{2}}\left|\psi\right|_{L^p_{\theta,\delta}},\qquad\forall\,\psi\in L^p_{\theta,\delta},\ t\in(0,T].$$

**Corollary 4.12.** *If $\mathcal{A}=\Delta$ and $\mathcal{O}$ is a half space, then for all $1\le p<+\infty$, $\theta\in[0,2p-1)$ and $\delta\ge0$, the semigroup $S$ is analytical on $L^p_{\theta,\delta}$.*

*Proof.* We need to show that there is a constant $C$ such that

$$\left|\Delta S(t)\psi\right|_{L^p_{\theta,\delta}}\le\frac{C}{t}\left|\psi\right|_{L^p_{\theta,\delta}},\qquad\forall\,\psi\in L^p_{\theta,\delta},\ t\in(0,1].$$

We may assume that $\mathcal{O}=\{x\in\mathbb{R}^d\colon x_1>0\}$. Then $m_t(y)=(y_1/\sqrt{t})\wedge1$, and the Green kernel in known, namely

$$G(t,x,y)=g_{2t}(x-y)-g_{2t}(\overline{x}-y),\tag{4.6}$$

where

$$\overline{x}=\overline{(x_1,x_2,\ldots,x_d)}=\overline{(x_1,\mathbf{x})}=(-x_0,\mathbf{x}).\tag{4.7}$$

By elementary calculation one can verified estimate (4.5) for any second order derivative $\frac{\partial^2}{\partial x_j^2}$. Indeed, given $a>0$, $z\in\mathbb{R}$ and $\mathbf{z}\in\mathbb{R}^{d-1}$ write

$$g^1_a(z):=(2\pi a)^{-\frac12}\,\mathrm{e}^{-\frac{z^2}{2a}},\qquad g^{d-1}_a(\mathbf{z}):=(2\pi a)^{-\frac{d-1}{2}}\,\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2a}}.$$

Note that there is a constant $C$ such that for all $x_1,y_1\ge0$, $t\in(0,1]$,

$$\left|g^1_{2t}(x_1-y_1)-g^1_{2t}(x_1+y_1)\right|\le Cm_t(y)g^1_{4t}(x_1-y_1).\tag{4.8}$$

For, (4.8) can be reformulated equivalently as

$$\left|\mathrm{e}^{-z^2}-\mathrm{e}^{-(z+v)^2}\right|\le C\,v\wedge1\,\mathrm{e}^{-\frac{z^2}{2}},\qquad\forall\,z\in\mathbb{R},v\ge0.$$

or

$$\mathrm{e}^{-\frac{z^2}{2}}\left|1-\mathrm{e}^{-(z+v)^2+z^2}\right|\le C\,v\wedge1,\qquad\forall\,z\in\mathbb{R},v\ge0.$$

We have

$$\left|\frac{\partial^2G}{\partial x_1^2}(t,x,y)\right|=\left|\left[\frac{(x_1-y_1)^2}{4t^2}-\frac{1}{2t}\right]g_{2t}(x-y)+\left[-\frac{(x_1+y_1)^2}{4t^2}+\frac{1}{2t}\right]g_{2t}(\overline{x}-y)\right|.$$

Therefore, by (4.8), it is enough to show that for all $u, v \geq 0$,

$$\left|(u-v)^2 \mathrm{e}^{-\frac{(u-v)^2}{4}} - (u+v)^2 \mathrm{e}^{-\frac{(u+v)^2}{4}}\right| \leq C\, v \wedge 1\, \mathrm{e}^{-\frac{(u-v)^2}{8}}.$$

For $j > 1$ we have

$$\begin{aligned}\left|\frac{\partial^2 G}{\partial x_j^2}(t,x,y)\right| &= \left|g^1_{2t}(x_1-y_1) - g^1_{2t}(x_1+y_1)\right| \left|\frac{(x_j-y_j)^2}{4t^2} - \frac{1}{2t}\right| g^{d-1}_{2t}(\mathbf{x}-\mathbf{y}) \\ &\leq \frac{C_1}{t}\left|g^1_{2t}(x_1-y_1) - g^1_{2t}(x_1+y_1)\right| g^{d-1}_{4t}(\mathbf{x}-\mathbf{y}) \\ &\leq \frac{CC_1}{t} m_t(y) g_{4t}(x-y).\end{aligned}$$

□

4.4. **Related results.** In this section we comment some recent results of Krylov [20] and [20] and Lindemulder and Veraar [23] concerning heat semigroup on weighted spaces.

4.4.1. *Krylov's result.* Let $P = \left\{x \in \mathbb{R}^d \colon x_1 > 0\right\}$ be a half space in $\mathbb{R}^d$. Let $\rho(x) = x_1$ be the distance of $x \in P$ from the boundary. Let $S$ be the semigroup generated by the Laplace operator $A = \Delta$ on $P$ with homogeneous Dirichlet boundary conditions. By $\nabla$ we denote the gradient operator and by $\nabla^2$ the Hessian. Given $\theta \in \mathbb{R}$ let $L^p_\theta = L^p(P, x_1^\theta \mathrm{d}x)$. The following result follows directly from the Krylov Theorem 2.5 ([20]). In the original Krylov theorem $p = q$, $\alpha = 2 = \hat{\alpha} = a_+$, $\gamma = \overline{\gamma} = 0$.

**Theorem 4.13.** *Let $p \in (1, +\infty)$. Then for every $\theta \in (-2p, p)$, $S$ is a $C_0$-semigroup on $L^p_\theta$. Moreover, there is a constant $N$ such that for any $t > 0$,*

$$\|S(t)u\|_{L^p_\theta} \leq N\|u\|_{L^p_\theta} \qquad \textit{and} \qquad \|\nabla^2 S(t)u\|_{L^p_\theta} \leq N t^{-1}\|u\|_{L^p_\theta}.$$

Given a vector $a \in \mathbb{R}^d$ and a number $\delta \in \mathbb{R}$ let us denote by $P(a,\delta)$ the half space

$$P(a,\delta) := \left\{x \in \mathbb{R}^d \colon \langle x, a\rangle > \delta\right\}.$$

Let $\rho_{P(a,\delta)}(x)$ be the distances of $x \in \mathbb{R}^d$ from the boundary $\partial P(a,\delta)$. Obviously the Krylov result can be extended to any of half space $P(a,\delta)$. The $L^p_\theta$ space should be replaced by

$$L^p_\theta(P(a,\delta)) := L^p(P(a,\delta), \rho^\theta_{P(a,\delta)}(x)\mathrm{d}x).$$

Note that the constant $N$ appearing in the theorem is universal for any half space.

Let $\mathcal{O}$ be a not necessarily bounded domain in $\mathbb{R}^d$. Let $\rho_{\mathcal{O}}(x)$ be the distance of $x \in \mathcal{O}$ from the boundary $\partial\mathcal{O}$. Given $\theta \in \mathbb{R}$ write $L^p_\theta = L^p(\mathcal{O}, \rho^\theta_{\mathcal{O}}(x)\mathrm{d}x)$. Let $S$ be the heat semigroup on $\mathcal{O}$ with homogeneous Dirichlet boundary conditions.

**Theorem 4.14.** *Assume that $\mathcal{O}$ is a convex domain in $\mathbb{R}^d$. Let $p \in (1, +\infty)$. Then for every $\theta \in (-2p, p)$, $S$ is a $C_0$-semigroup on $L^p_\theta$. Moreover, there is an independent of $\mathcal{O}$ constant $N$ such that for any $t > 0$,*

$$|S(t)\psi|_{L^p_\theta} \leq N|\psi|_{L^p_\theta}, \qquad \psi \in L^p_\theta. \tag{4.9}$$

*Proof.* Since $\mathcal{O}$ is convex then there is a family of subspaces $P(a_j,\delta_j)$, $j \in J$, such that

$$\mathcal{O} = \bigcap_{j\in J} P(a_j,\delta_j). \tag{4.10}$$

Let $j \in J$, and let $\psi \in C_0^\infty(\mathcal{O})$. Let $T_{a_j,\delta_j}$ be the heat semigroup on $P(a_j,\delta_j)$ with homogeneous Dirichlet boundary conditions. Let us observe that

$$|S(t)\psi(x)| \leq T_{a_j,\delta_j}(t)|\psi|(x), \qquad x \in \mathcal{O}. \tag{4.11}$$

For, (4.11) follows immediately for example from the following probabilistic representations

$$S(t)\psi(x) = \mathbb{E}\left(\psi(x+W(t)); t < \tau_x(\mathcal{O})\right),$$
$$T_{a_j,\delta_j}(t)\psi(x) = \mathbb{E}\left(\psi(x+W(t)); t < \tau_x\left(P(a_j,\delta_j)\right)\right),$$

where $\tau_x(\mathcal{O})$ and $\tau_x\left(P(a_j,\delta_j)\right)$ are exit times

$$\tau_x(\mathcal{O}) := \inf\{s > 0 \colon x + W(s) \notin \mathcal{O}\}, \qquad \tau_x\left(P(a_j,\delta_j)\right) := \inf\{s > 0 \colon x + W(s) \notin P(a_j,\delta_j)\}.$$

Thus

$$|S(t)\psi(x)|^p \rho^\theta_{\mathcal{O}}(x) \le \left|T_{a_j,\delta_j}(t)\psi(x)\right|^p \rho^\theta_{P(a_j,\delta_j)}(x), \qquad \forall\, x \in \mathcal{O}.$$

Therefore after integration we obtain

$$|S(t)\psi|_{L^p_\theta} \le |T_{a_j,\delta_j}(t)\psi|_{L^p_\theta(P(a_j,\delta_j))} \le N|\psi|_{L^p_\theta(P(a_j,\delta_j))},$$

where the constant $N$ does not depend on $j$. Hence

$$|S(t)\psi|_{L^p_\theta} \le N \inf_{j\in J} |\psi|_{L^p_\theta(P(a_j,\delta_j))}.$$

Taking into account (4.10) we have

$$\inf_{j\in J} |\psi|_{L^p_\theta(P(a_j,\delta_j))} = |\psi|_{L^p_\theta},$$

which gives (4.9) and obviously $C_0$-property of $S$. □

*Remark* 4.15. Unfortunately, since we do not have the estimate for the gradient

$$|\nabla S(t)\psi(x)| \le |\nabla T_{a_j,\delta_j}(t)\psi(x)|, \qquad x \in \mathcal{O}.$$

the derivation of the estimate $|\nabla S(t)\psi|_{L^p_\theta} \le N t^{-1/2}|\psi|_{L^p_\theta}$ needs some different arguments!

4.4.2. *Lindemulder and Veraar results.* As in the Krylov papers, paper [23] of Lindemulder and Veraar deals with the Laplace operator $\mathcal{A} = \Delta$. It is shown that $\Delta$ with Dirichlet boundary conditions admits a bounded $H^\infty$-calculus on weighted spaces $L^p_\theta := L^p\left(\mathcal{O}, \rho^\theta(x)\mathrm{d}x\right)$, where $\rho(x) = \mathrm{dist}\,(x, \partial\mathcal{O})$, $p \in (1,+\infty)$ and $\theta \in (-1, 2p-1) \setminus \{p-1\}$. Therefore, the corresponding heat semigroup is not only $C_0$ but also analytical on $L^p_\theta$. In [23], $\mathcal{O}$ is a halfspace or a bounded $C^2$-domain.

## 5. Properties of the semigroup on weighted spaces

In this section, Assumptions 4.1 and 4.2 are satisfied.

**Lemma 5.1.** *There exists a $C > 0$ such that*

$$|S(t)\psi|_{\mathcal{L}^p_\delta} \le C t^{-\frac{\theta}{2p}} |\psi|_{L^p_{\theta,\delta}} \qquad \textit{for } t \in (0,1].$$

*Proof.* Let

$$w_\delta(x) = w_{0,\delta}(x) = \left(1 + |x|^2\right)^{-\delta} \tag{5.1}$$

be the weight on $\mathcal{L}^p_\delta$. Then, by Assumption 4.2, we have

$$|S(t)\psi|^p_{\mathcal{L}^p_\delta} \le C^p \int_{\mathcal{O}} w_\delta(x) \left(\int_{\mathcal{O}} m_t(y) g_{ct}(x-y)\,|\psi(y)|\,\mathrm{d}y\right)^p \mathrm{d}x$$
$$\le \tilde{C} t^{d/2}(I_1 + I_2),$$

where

$$I_1 := \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \left( \int_{\left(\mathcal{O}/\sqrt{t}\right)_1} m_t(y\sqrt{t}) g_c(x-y)\, |\phi(y)|\, \mathrm{d}y \right)^p \mathrm{d}x,$$

$$I_2 := \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \left( \int_{\left(\mathcal{O}/\sqrt{t}\right)_1^c} m_t(y\sqrt{t}) g_c(x-y)\, |\phi(y)|\, \mathrm{d}y \right)^p \mathrm{d}x,$$

$\phi(z) = \psi(z\sqrt{t})$, and

$$(\mathcal{O}/\sqrt{t})_1 := \left\{ x \in \mathcal{O}/\sqrt{t} \colon m_t(x\sqrt{t}) = \rho(x\sqrt{t})/\sqrt{t} < 1 \right\},$$

and

$$(\mathcal{O}/\sqrt{t})_1^c := \left\{ x \in \mathcal{O}/\sqrt{t} \colon m_t(x\sqrt{t}) = 1 \right\}.$$

We have

$$\begin{aligned} I_1 &\leq \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \int_{\left(\mathcal{O}/\sqrt{t}\right)_1} m_t^p(y\sqrt{t}) g_c(x-y) |\phi(y)|^p \mathrm{d}y\mathrm{d}x \\ &\leq \int_{\left(\mathcal{O}/\sqrt{t}\right)_1} \left( \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \frac{m_t^p(y\sqrt{t})}{w_{\theta,\delta}(y\sqrt{t})} g_c(x-y)\mathrm{d}x \right) |\phi(y)|^p\, w_{\theta,\delta}(y\sqrt{t})\mathrm{d}y \\ &\leq \int_{\left(\mathcal{O}/\sqrt{t}\right)_1} F(t,y)\, |\phi(y)|^p\, w_{\theta,\delta}(y\sqrt{t})\mathrm{d}y, \end{aligned}$$

where

$$\begin{aligned} F(t,y) &= \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \frac{m_t^p(y\sqrt{t})}{w_{\theta,\delta}(y\sqrt{t})} g_c(x-y)\mathrm{d}x, \\ &= \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \frac{\rho^p(y\sqrt{t})}{t^{p/2} w_{\theta,\delta}(y\sqrt{t})} g_c(x-y)\mathrm{d}x, \qquad y \in \left(\mathcal{O}/\sqrt{t}\right)_1. \end{aligned}$$

Recall that $w_{\theta,\delta}(z) = \min\left\{\rho^\theta(z), w_\delta(z)\right\}$. Thus, if $\rho^\theta(y\sqrt{t}) \leq w_\delta(y\sqrt{t})$ then since

$$\rho^{p-\theta}(y\sqrt{t}) \leq t^{(p-\theta)/2} \quad \text{for} \quad y \in \left(\mathcal{O}/\sqrt{t}\right)_1,$$

we have

$$F(t,y) \leq \int_{\mathcal{O}/\sqrt{t}} \frac{\rho^{p-\theta}(y\sqrt{t})}{t^{p/2}} g_c(x-y) w_\delta(x\sqrt{t})\mathrm{d}x \leq \frac{C_1}{t^{\theta/2}}.$$

If $w_\delta(y\sqrt{t}) < \rho^\theta(y\sqrt{t})$ then

$$F(t,y) \leq \int_{\mathcal{O}/\sqrt{t}} \frac{w_\delta(x\sqrt{t})}{w_\delta(y\sqrt{t})} g_c(x-y)\mathrm{d}x.$$

Putting $a = \sqrt{t}$ we find that

$$\begin{aligned} \frac{1+a|y|^2}{1+a|x|^2} &\leq \frac{1+2a|y-x|^2+2a|x|^2}{1+a|x|^2} \\ &\leq 1+2a|x-y|^2 + \frac{2a|x|^2}{1+a|x|^2} \\ &\leq 3+2a|x-y|^2. \end{aligned}$$

Therefore

$$\sup_{y\in\left(\mathcal{O}/\sqrt{t}\right)_1} F(t,y) \leq \frac{C_2}{t^{\theta/2}},$$

and hence

$$t^{\frac{d}{2}} I_1 \le C_2 t^{-\theta/2} |\psi|^p_{L^p_{\theta,\delta}}.$$

For $I_2$ we obtain

$$\begin{aligned}
I_2 &= \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \left( \int_{\left(\mathcal{O}/\sqrt{t}\right)^c_1} m_t(y\sqrt{t}) g_c(x-y)\phi(y)\mathrm{d}y \right)^p \mathrm{d}x \\
&\le \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \int_{\left(\mathcal{O}/\sqrt{t}\right)^c_1} \frac{g_c(x-y)}{w_{\theta,\delta}(y\sqrt{t})} |\phi(y)|^p w_{\theta,\delta}(y\sqrt{t}) \mathrm{d}y\mathrm{d}x \\
&\le \int_{\left(\mathcal{O}/\sqrt{t}\right)^c_1} H(t,y) |\phi(y)|^p w_{\theta,\delta}(y\sqrt{t}) \mathrm{d}y,
\end{aligned}$$

where

$$H(t,y) := \int_{\mathcal{O}/\sqrt{t}} w_\delta(x\sqrt{t}) \frac{g_c(x-y)}{w_{\theta,\delta}(y\sqrt{t})} \mathrm{d}x, \qquad y \in \left(\mathcal{O}/\sqrt{t}\right)^c_1.$$

Note that

$$\frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}} \ge 1.$$

Thus

$$1 \ge w_{\theta,\delta}(y\sqrt{t}) = \min\left\{ \rho^\theta(y\sqrt{t}), w_\delta(y\sqrt{t}) \right\} \ge \min\left\{ t^{\theta/2}, w_\delta(y\sqrt{t}) \right\}.$$

Therefore

$$\begin{aligned}
H(t,y) &\le \int_{\mathcal{O}/\sqrt{t}} \left[ \frac{w_\delta(x\sqrt{t})}{w_\delta(y\sqrt{t})} + w_\delta(x\sqrt{t}) t^{-\theta/2} \right] g_c(x-y)\mathrm{d}x \\
&\le \int_{\mathcal{O}/\sqrt{t}} \left[ \frac{w_\delta(x\sqrt{t})}{w_\delta(y\sqrt{t})} + t^{-\theta/2} \right] g_c(x-y)\mathrm{d}x \\
&\le C t^{-\theta/2}.
\end{aligned}$$

□

In what follows we denote by $L(E,V)$ the space of all linear bounded operators from a Banach space $E$ to a Banach space $V$, equipped with the operator norm $\|\cdot\|_{L(E,V)}$.

**Theorem 5.2.** *Assume that there are positive constants $M, \alpha$ such that for every $t > 0$,*

$$\|S(t)\|_{L(\mathcal{L}^p_\delta, \mathcal{L}^p_\delta)} \le M \mathrm{e}^{-\alpha t}.$$

*Then there exist $C > 0$, such that for every $t > 0$,*

$$\|S(t)\|_{L(L^p_{\theta,\delta}, L^p_{\theta,\delta})} \le C \mathrm{e}^{-\alpha t}.$$

*Proof.* Clearly

$$\|S(t)\psi\|_{L^p_{\theta,\delta}} \le \|S(t)\psi\|_{\mathcal{L}^p_\delta} = \|S(t-1)S(1)\psi\|_{\mathcal{L}^p_\delta} \le M\mathrm{e}^{-\alpha(t-1)} \|S(1)\psi\|_{\mathcal{L}^p_\delta}.$$

By Lemma 5.1,

$$\|S(1)\psi\|_{\mathcal{L}^p_\delta} \le C_2 \|\psi\|_{L^p_{\theta,\delta}}.$$

□

## 6. Dirichlet map

In this section we derive useful estimates for

$$(\lambda - A)\, S(t) D_\lambda e = S(t) B e, \qquad t \geq 0, e \in L^2(\partial\mathcal{O}, \mathrm{ds}).$$

We will need Assumptions 4.1 and 4.2, and additionally the following assumption, which is satisfied, see Remark 4.3 if the drift coefficients $\nu^i$ of $\mathcal{A}$ are of the class $C_b^1$.

**Assumption 6.1.** *Assume that the coefficients $a_{i,j}$ are bounded, and that for any $T > 0$ there is a constant $C$ such that*

$$|\nabla_y G(t,x,y)| \leq \frac{C}{\sqrt{t}} g_{ct}(x-y), \qquad t \leq T, \qquad x, y \in \mathcal{O}.$$

Let $\mathcal{S}_0(\overline{\mathcal{O}})$ be the set of all $\psi$ of tempered test functions such that $\psi(x) = 0$ for $x \in \partial\mathcal{O}$. Let $\gamma$ be a continuous compactly supported function on $\partial\mathcal{O}$. Recall that $u = D_\lambda \gamma$ is the solution to the non-homogeneous Poisson problem (2.2). Let $\psi \in \mathcal{S}_0(\overline{\mathcal{O}})$. Then applying Gauss–Green integration by parts formula we obtain

$$\int_{\mathcal{O}} \mathcal{A}u(x)\psi(x)\mathrm{d}x = \int_{\mathcal{O}} u(x)\mathcal{A}^*\psi(x)\mathrm{d}x - \int_{\partial\mathcal{O}} \gamma(x) \sum_{i,j} a_{ij}(x) \frac{\partial \psi}{\partial x_i}(x)\mathbf{n}^j(x)\mathrm{ds}(x),$$

where $\mathcal{A}^*$ is the formal adjoint operator;

$$\mathcal{A}^*\psi(x) = \sum_{i,j} \frac{\partial}{\partial x_j}\left( a_{i,j}(x) \frac{\partial \psi}{\partial x_i}(x)\right) - \sum_i \frac{\partial}{\partial x_i}\left(\mu^i(x)\psi(x)\right),$$

$\mathbf{n} = \left(\mathbf{n}^1, \ldots, \mathbf{n}^d\right)$ is the outward pointing unit normal vector to the boundary $\partial\mathcal{O}$, and s is the surface measure.

Therefore we have

$$\int_{\mathcal{O}} u(x)\mathcal{A}^*\psi(x)\mathrm{d}x = \lambda \int_{\mathcal{O}} u(x)\psi(x)\mathrm{d}x + \int_{\partial\mathcal{O}} \gamma(x) \sum_{i,j} a_{ij}(x)\frac{\partial \psi}{\partial x_i}(x)\mathbf{n}^j(x)\mathrm{ds}(x). \tag{6.1}$$

In fact (6.1) can be treated as the definition of the weak solution to (2.3), see e.g. [4, 24].

Let

$$\mathbf{n}^a = \left( \sum_j a_{1,j}\mathbf{n}^j, \ldots, \sum_j a_{d,j}\mathbf{n}^j \right).$$

Then

$$\sum_{i,j} a_{ij}(x) \frac{\partial \psi}{\partial x_i}(x)\mathbf{n}_j(x) = \frac{\partial \psi}{\partial \mathbf{n}^a}(x),$$

and (6.1) has the form

$$\int_{\mathcal{O}} D_\lambda\gamma(x)\left(\mathcal{A}^* - \lambda\right)\psi(x)\mathrm{d}x = \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial \psi}{\partial \mathbf{n}^a}(x)\mathrm{ds}(x).$$

In what follows $A$ is the generator of the semigroup $S$ on $L^2(\mathcal{O})$, $S^*$ is the adjoint semigroup and $A^*$ is its generator. Note that $\mathcal{A}^* \subset A^*$.

**Proposition 6.2.** *Let $\lambda$ be in the resolvent of $A^*$. Then the Dirichlet map is uniquely characterized by the relation*

$$\int_{\mathcal{O}} D_\lambda\gamma(x)\psi(x)\mathrm{d}x = \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\left(A^* - \lambda\right)^{-1}\psi(x)\mathrm{ds}(x), \qquad \psi \in \mathcal{S}_0(\overline{\mathcal{O}}). \tag{6.2}$$

*Proof.* Assume (6.2). Let $\psi \in \mathcal{S}_0(\overline{\mathcal{O}})$. Then

$$\begin{aligned}\int_{\mathcal{O}} D_\lambda\gamma(x)\left(\mathcal{A}^*-\lambda\right)\psi(x)\mathrm{d}x &= \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\left(A^*-\lambda\right)^{-1}\left(\mathcal{A}^*-\lambda\right)\psi(x)\mathrm{d}s(x)\\ &= \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\psi(x)\mathrm{d}s(x).\end{aligned}$$

□

**Corollary 6.3.** *We have*

$$\int_{\mathcal{O}} \left(\lambda - A\right) S(t) D_\lambda\gamma(x)\psi(x)\mathrm{d}x = -\int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\left[S^*(t)\psi(x)\right]\mathrm{d}s(x). \tag{6.3}$$

Let $G$ be the Green kernel corresponding to the heat semigroup $S$ generated by $\mathcal{A}$ with homogeneous boundary conditions. Let

$$\mathcal{G}_\lambda(x,y) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} G(t,x,y)\mathrm{d}t, \tag{6.4}$$

where $\lambda$ is from the resolvent set.

**Theorem 6.4.** *We have:*

$$D_\lambda\gamma(x) = \int_{\partial\mathcal{O}} \gamma(y)\frac{\partial}{\partial \mathbf{n}^a(y)}\mathcal{G}_\lambda(x,y)\mathrm{d}s(y) \tag{6.5}$$

*and*

$$(\lambda - A)S(t)D_\lambda\gamma(x) = -\int_{\partial\mathcal{O}} \gamma(y)\frac{\partial}{\partial \mathbf{n}^a(y)}G(t,x,y)\mathrm{d}s(y). \tag{6.6}$$

*Proof.* Let $\psi \in \mathcal{S}_0(\overline{\mathcal{O}})$. Then, by (6.1),

$$\begin{aligned}\int_{\mathcal{O}} D_\lambda\gamma(x)\psi(x)\mathrm{d}x &= \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\left(A^*-\lambda\right)^{-1}\psi(x)\mathrm{d}s(x)\\ &= \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a(x)}\int_{\mathcal{O}} \mathcal{G}_\lambda(y,x)\psi(y)\mathrm{d}y\mathrm{d}s(x)\\ &= \int_{\mathcal{O}}\int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a(x)}\mathcal{G}_\lambda(y,x)\mathrm{d}s(x)\psi(y)\mathrm{d}y\\ &= \int_{\mathcal{O}}\int_{\partial\mathcal{O}} \gamma(y)\frac{\partial}{\partial \mathbf{n}^a(y)}\mathcal{G}_\lambda(x,y)\mathrm{d}s(y)\psi(x)\mathrm{d}x.\end{aligned}$$

To see (6.6) note that

$$\begin{aligned}\int_{\mathcal{O}} (\lambda - A)S(t)D_\lambda\gamma(x)\psi(x)\mathrm{d}x &= \int_{\mathcal{O}} D_\lambda\gamma(x)\left(\lambda - A^*\right)S^*(t)\psi(x)\mathrm{d}x\\ &= \int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}\left(A^*-\lambda\right)^{-1}\left(\lambda - A^*\right)S^*(t)\psi(x)\mathrm{d}s(x)\\ &= -\int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a}S^*(t)\psi(x)\mathrm{d}s(x)\\ &= -\int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a(x)}\int_{\mathcal{O}} G(t,y,x)\psi(y)\mathrm{d}y\mathrm{d}s(x)\\ &= -\int_{\mathcal{O}}\int_{\partial\mathcal{O}} \gamma(x)\frac{\partial}{\partial \mathbf{n}^a(x)}G(t,y,x)\mathrm{d}s(x)\psi(y)\mathrm{d}y\\ &= -\int_{\mathcal{O}}\int_{\partial\mathcal{O}} \gamma(y)\frac{\partial}{\partial \mathbf{n}^a(y)}G(t,x,y)\mathrm{d}s(y)\psi(x)\mathrm{d}x.\end{aligned}$$

□

Note that by Assumption 6.1, the coefficients $a_{i,j}$ are bounded. Therefore we have the following consequence of the theorem above.

**Corollary 6.5.** *Under Assumptions 4.1, 4.2, and 6.1, for any* $T>0$ *there is a constant* $C>0$*, such that for* $t\in(0,T]$*,* $\psi\in L^2(\partial\mathcal{O},\mathrm{ds})$*, and* $x\in\mathcal{O}$*,*

$$|S(t)B\psi(x)| = |(\lambda-A)S(t)D_\lambda\psi(x)| \le \frac{C}{\sqrt{t}}\left|\int_{\partial\mathcal{O}} g_{ct}(x-y)\psi(y)\mathrm{ds}(y)\right|.$$

## 7. Stochastic integration in $L^p$-spaces

In this paper we need only very naive theory of stochastic integration in $L^p$-spaces. Namely, for $B:=(\lambda-A)D_\lambda$ set

$$\psi_k(t,x) := (S(t)Be_k)(x), \qquad x\in\mathcal{O}.$$

By Theorem 6.4,

$$\psi_k(t,x) = -\int_{\partial\mathcal{O}}\frac{\partial}{\partial\mathbf{n}^a(y)}G(t,x,y)e_k(y)\mathrm{ds}(y).$$

We need to define

$$M(t,x) := \sum_k\int_0^t \psi_k(t-s,x)\mathrm{d}W_k(s), \quad x\in\mathcal{O},\ t\in[0,T],$$

and

$$M_\alpha(t,x) := \sum_k\int_0^t (t-s)^{-\alpha}\psi_k(t-s,x)\mathrm{d}W_k(s), \quad x\in\mathcal{O},\ t\in[0,T],$$

as $L^p_{\theta,\delta}$-valued provesses.

The following result from [7] stated there as Proposition A.1, enables us to define rigorously each component of the sums above. Below $W$ is a real valued Wiener process defined on a filtered probability space $(\Omega,\mathfrak{F},(\mathfrak{F}_t),\mathbb{P})$.

**Proposition 7.1.** *Let* $(\mathfrak{D},\mathfrak{G},\nu)$ *be a* $\sigma$*-finite measurable space. Let* $p,q\in(1,+\infty)$*,* $T\in(0,+\infty)$*. For any adapted and strongly measurable process* $\phi\colon[0,T]\times\Omega\mapsto L^p(\mathfrak{D})$ *the following there assertions are equivalent.*

(1) *There exists a sequence of adapted step processes* $(\phi_n)$ *such that*

$$\lim_{n\to+\infty}\|\phi-\phi_n\|_{L^q(\Omega,L^p(\mathfrak{D},L^2(0,T)))}=0,$$

$$\left(\int_0^T\phi_n(t)\mathrm{d}W(t)\right) \quad \textit{is a Cauchy sequence in } L^q(\Omega;L^p(\mathfrak{D})).$$

(2) *There exists a random variable* $\eta\in L^q(\Omega;L^p(\mathfrak{D}))$ *such that for all sets* $A\in\mathfrak{G}$ *with finite measure one has* $(t,\omega)\mapsto\int_A\phi(t,\omega)\mathrm{d}\nu\in L^q(\Omega;L^2(0,T))$*, and*

$$\int_A\eta\mathrm{d}\nu = \int_0^T\int_A\phi(t)\mathrm{d}\nu\mathrm{d}W(t) \quad \textit{in } L^q(\Omega).$$

(3) $\|\phi\|_{L^q(\Omega;L^p(\mathfrak{D};L^2(0,T)))}<+\infty$.

*Moreover, in this situation one has*

$$\lim_{N\to+\infty}\int_0^T\phi_n(t)\mathrm{d}W(t)=\eta,$$

*and there is a constant $C_{p,q} \in (0,+\infty)$ such that*

$$C_{p,q}^{-1}\|\phi\|_{L^q(\Omega;L^p(\mathfrak{O};L^2(0,T)))} \leq \|\eta\|_{L^q(\Omega;L^p(\mathfrak{O}))} \leq C_{p,q}\|\phi\|_{L^q(\Omega;L^p(\mathfrak{O};L^2(0,T)))}.$$

Process $\phi$ which satisfies ony of these conditions is called *$L^q$-stochastically integrable in $L^p(\mathfrak{O})$ on* $[0,T]$ and we write

$$\int_0^T \phi(t)\mathrm{d}W(t) := \eta.$$

Given $\alpha \geq 0$ let

$$\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) := \int_{\mathcal{O}} \left(\sum_k \int_0^T t^{-\alpha}\psi_k^2(t,x)\mathrm{d}t\right)^{p/2} w_{\theta,\delta}(x)\mathrm{d}x. \tag{7.1}$$

By the Burkholder–Davis–Gundy inequality, for every $p \in [1,+\infty)$ there exist positive constants $c_p$ and $C_p$ such that for all $t \in [0,T]$

$$c_p\mathcal{J}_{T,0}(\{e_k\},p,\theta,\delta) \leq \mathbb{E}\int_{\mathcal{O}} |M(t,x)|^p\, w_{\theta,\delta}(x)\mathrm{d}x \leq C_p\mathcal{J}_{T,0}(\{e_k\},p,\theta,\delta)\,,$$

see for example [39]. We have thus the following result.

**Proposition 7.2.** *Given $T$, $p$, $\theta$ and $\delta$, the process*

$$\sum_k \int_0^T S(T-t)Be_k\mathrm{d}W_k(t)$$

*takes values in $L^p_{\theta,\delta}$ if and only if $\mathcal{J}_{T,0}(\{e_k\},p,\theta,\delta) < +\infty$. Moreover, if for a certain $\alpha > 0$, $\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) < +\infty$, then the process has continuous trajectories in $L^p_{\theta,\delta}$.*

The simple idea above can be made rigorous and much more general (see e.g. [5, 7]).

## 8. Examples

Let in the whole section Assumptions 4.1, 4.2, and 6.1 be satisfied.

### 8.1. **One dimensional case.**

Consider the simplest cases of $\mathcal{O} = (0,1)$ and $\mathcal{O} = (0,+\infty)$. In the first case the surface measure $\mathrm{s} = \delta_0 + \delta_1$ and $L^2(\partial\mathcal{O},\mathrm{ds}) \equiv \mathbb{R}^2$, whereas in the second case $\mathrm{s} = \delta_0$ and $L^2(\partial\mathcal{O},\mathrm{d}\mathbf{s}) \equiv \mathbb{R}^1$.

**Proposition 8.1.** *Let $p \in (1,+\infty)$ and $\theta \in (p-1,2p-1)$. Then the boundary problem*

$$\begin{aligned}
\frac{\partial X}{\partial t}(t,x) &= \mathcal{A}X(t,x), \quad x \in (0,1),\\
X(t,0) &= \frac{\mathrm{d}W_0}{\mathrm{d}t}(t),\\
X(t,1) &= \frac{\mathrm{d}W_1}{\mathrm{d}t}(t),
\end{aligned}$$

*defines Markov family with continuous trajectories in the space $L^p_{\theta,0}$.*

*Proof.* We are in a framework of Theorem 2.10. By Theorem 4.6 the heat semigroup can be extended to $L^p_{\theta,0}$ for $\theta \in [0,2p-1)$. Clearly $H_W$ is 2-dimensional with $e_1 = \chi_{\{0\}}$ and $e_2 = \chi_{\{1\}}$. Taking into account Proposition 7.2, it is enough to verify whether for

$\theta \in (p-1, 2p-1)$ and $T \in (0, +\infty)$, there is an $\alpha > 0$ such that $\mathcal{J} := \mathcal{J}_{T,\alpha}(\chi_0, \chi_1, p, \theta, 0) < +\infty$. Let $\alpha > 0$. By Corollary 6.5, we have

$$
\begin{aligned}
\mathcal{J} &\le c_1 \int_0^1 \left[\int_0^T t^{-1-\alpha}\left(g_{ct}^2(x) + g_{ct}^2(x-1)\right) \mathrm{d}t\right]^{p/2} \min\{x^\theta, (1-x)^\theta\}\mathrm{d}x \\
&\le c_2 \int_0^1 \left[\int_0^T t^{-1-\alpha} g_{ct}^2(t)(x)\mathrm{d}t\right]^{p/2} x^\theta \mathrm{d}x \\
&\le c_3 \int_0^1 \left[\int_0^T t^{-2-\alpha}\mathrm{e}^{-\frac{x^2}{2ct}}\mathrm{d}t\right]^{p/2} x^\theta \mathrm{d}x = c_3 \int_0^1 \left[\int_0^{T/x^2} t^{-2-\alpha}\mathrm{e}^{-\frac{1}{2ct}}\mathrm{d}t x^{-2-2\alpha}\right]^{p/2} x^\theta \mathrm{d}x \\
&\le c_3 \int_0^1 x^{\theta-p-\alpha p}\mathrm{d}x.
\end{aligned}
$$

□

Similar calculation can be done in the case of half-line $\mathcal{O} = (0, +\infty)$.

**Proposition 8.2.** *Assume that $\delta > 1/2$ and $p \in (1, +\infty)$ and $\theta \in (p-1, 2p-1)$. Let $\mathcal{A}$ be a second order defined as above. Then the boundary problem*

$$
\begin{aligned}
\frac{\partial X}{\partial t}(t,x) &= \mathcal{A}X(t,x), \qquad x \in (0,+\infty), \\
X(t,0) &= \frac{\mathrm{d}W_0}{\mathrm{d}t}(t),
\end{aligned}
$$

*defines Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$.*

*Proof.* We have

$$
\int_0^{+\infty} \left[\int_0^T t^{-2-\alpha}\mathrm{e}^{-\frac{x^2}{ct}}\mathrm{d}t\right]^{p/2} x^\theta (1+|x|^2)^{-\delta}\mathrm{d}x \le c_1 \int_0^{+\infty} x^{\theta-p-\alpha p}(1+|x|^2)^{-\delta}\mathrm{d}x.
$$

□

*Remark* 8.3. Assume that $A$ is equal to the Laplace operator $\Delta$. Let $p \in (1, +\infty)$ and $\theta \in (p-1, 2p-1)$. Then the Markov family defined by boundary problem on $(0,1)$ or $(0,+\infty)$ has a unique invariant measure. For, in the case of interval we can use Theorem 5.2, whereas in the case of problem on half-line we can use a direct approach. Namely, we have, see Remark 8.13 of Section 8.4,

$$
\frac{\partial G}{\partial \mathbf{n}_y}(t,x,0) = -\frac{x}{t} g_{2t}(x) = -\frac{x}{2t\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4t}}.
$$

Clearly, there are constant $C, c$ such that for all $t, x \ge 0$,

$$
\left|\frac{\partial G}{\partial \mathbf{n}_y}(t,x,0)\right| \le \frac{C}{\sqrt{t}} g_{ct}(x).
$$

Therefore, from the proof of the proposition it follows that $\sup_{T>0} \mathcal{J}_T(\chi_0, p, \theta, 0) < +\infty$, and the desired conclusion holds, see Theorem 2.10.

8.2. **Equation on a ball.** Let $\mathcal{O} = \mathbb{B}^d$ be a unite ball in $\mathbb{R}^d$ with center at 0. We assume that $d \ge 2$, otherwise we have the case of equations on an interval studied in the previous section. Then $\partial\mathcal{O} = \mathbb{S}^{d-1}$. Assume that the boundary noise has the form

$$
W(t,y) = \sum_k e_k(y) W_k(t), \qquad t \ge 0,\ y \in \mathbb{S}^{d-1}, \tag{8.1}
$$

where $(e_k)$ is a sequence of functions on $\mathbb{S}^{d-1}$ and $(W_k)$ are independent real-valued Wiener processes.

**Proposition 8.4.** *If*

$$A := \sum_k \sup_{y\in\mathbb{S}^{d-1}} e_k^2(y) < +\infty. \tag{8.2}$$

*then for any $1 < p$ and $\theta \in (p-1, 2p-1)$, boundary problem (1.1) defines a Markov family with continuous trajectories in $L^p_{\theta,0}$.*

*Proof.* Let $\mathcal{J} := \mathcal{J}_{T,\alpha}(\{e_k\}, p, \theta, 0)$. Using, Corollary 6.5, (7.1), (8.2), we obtain

$$\begin{aligned}
\mathcal{J} &\le c_1 \int_{\mathbb{B}^d} \left[\sum_k \int_0^T t^{-1-\alpha} \left(\int_{\mathbb{S}^{d-1}} g_{ct}(x-y)|e_k(y)|\mathrm{d}s(y)\right)^2 \mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x \\
&\le c_1 A^{p/2} \int_{\mathbb{B}^d} \left[\int_0^T t^{-1-\alpha} \left(\int_{\mathbb{S}^{d-1}} g_{ct}(x-y)\mathrm{d}s(y)\right)^2 \mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x \qquad (8.3)\\
&\le c_2 \int_{\mathbb{B}^d} \left[\int_0^T t^{-1-\alpha-d} \left(\int_{\mathbb{S}^{d-1}} \mathrm{e}^{-\frac{|x-y|^2}{ct}}\mathrm{d}s(y)\right)^2 \mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x.
\end{aligned}$$

We have to evaluate

$$I(t,x) := \int_{\mathbb{S}^{d-1}} \mathrm{e}^{-\frac{|x-y|^2}{ct}}\mathrm{d}s(y).$$

Let $\rho(x) := \mathrm{dist}\left(x, \mathbb{S}^{d-1}\right)$. We are showing that there is a constant $C_1 > 0$ such that

$$I(t,x) \le C_1 \mathrm{e}^{-\frac{\rho^2(x)}{C_1 t}} t^{\frac{d-1}{2}}, \qquad \forall\, t\in(0,T],\ x\in\mathbb{B}^d. \tag{8.4}$$

In fact our method leads also to the following lower bound estimate

$$C_2 \mathrm{e}^{-\frac{\rho^2}{C_3 t}} t^{\frac{d-1}{2}} \le I(t,x).$$

Our proof of (8.4) is elementary. In Lemma 8.9 from Section 8.3 we will establish estimate (8.4) for an arbitrary bounded region in $\mathbb{R}^d$. In the proof of Lemma 8.9 we use the Laplace method. To see (8.4), note that we may assume that $x = (x_1, 0, 0, \ldots, 0) = (1-\rho(x), 0, \ldots, 0)$. Note that for $\rho, r \in [0,1]$ we have

$$1 + \sqrt{1-r^2} - \rho \ge \left|1 - \sqrt{1-r^2} - \rho\right|.$$

Therefore

$$\begin{aligned}
I(t,x) &= \int_{\mathbb{S}^{d-1}} \exp\left\{-\frac{|1-\rho(x)-y_1|^2 + \sum_{j=2}^d y_j^2}{ct}\right\}\mathrm{d}s(y) \\
&\le 2\int_{\mathbb{B}^{d-1}} \exp\left\{-\frac{\left|1-\rho(x)-\sqrt{1-|z|^2}\right|^2 + |z|^2}{ct}\right\}\left(1 + \sum_{j=1}^{d-1}\frac{z_j^2}{1-|z|^2}\right)^{1/2}\mathrm{d}z \\
&\le c_3 \int_0^1 \exp\left\{-\frac{\left|1-\rho(x)-\sqrt{1-r^2}\right|^2 + r^2}{ct}\right\}\left(1-r^2\right)^{-1/2} r^{d-2}\mathrm{d}r.
\end{aligned}$$

Note that that for all $r\in[0,1]$ we have

$$\frac{1}{2}r^2 \le 1 - \sqrt{1-r^2} \le r^2.$$

For, let $f(r)=(1-\sqrt{1-r})/r$. Then $f(0)=1/2$, $f(1)=1$ and

$$f'(r)=\frac{(\sqrt{1-r}-1)^2}{2r^2\sqrt{1-r}}\geq 0.$$

Therefore there is a $c_4>0$ such that for all $r,\rho\in[0,1]$ we have

$$\left|1-\rho-\sqrt{1-r^2}\right|^2+r^2\geq c_4(\rho^2+r^2).$$

For, we have

$$\begin{aligned}\left|1-\rho-\sqrt{1-r^2}\right|^2+r^2&=\rho^2+(1-\sqrt{1-r^2})^2-2\rho(1-\sqrt{1-r^2})+r^2\\&\geq\rho^2+\frac{r^4}{4}-2\rho r^2+r^2\\&\geq\rho^2+\frac{r^4}{4}-\rho^2\kappa-\frac{r^4}{\kappa}+r^2=\rho^2(1-\kappa)+r^2\left(1-\frac{r^2}{\kappa}+\frac{r^2}{4}\right).\end{aligned}$$

Let $\kappa\in(4/5,1)$. Then

$$\left|1-\rho-\sqrt{1-r^2}\right|^2+r^2\geq(1-\kappa)\rho^2+r^2\left(\frac{5}{4}-\frac{1}{\kappa}\right),$$

which gives the desired estimate.

Summing up we have

$$I(t,x)\leq c_3\int_0^1\exp\left\{-\frac{c_4(\rho^2(x)+r^2)}{ct}\right\}\left(1-r^2\right)^{-1/2}r^{d-2}\mathrm{d}r\leq \mathrm{e}^{-\frac{c_4\rho^2(x)}{ct}}C(t),$$

where

$$\begin{aligned}C(t)&=c_3\int_0^1\exp\left\{-\frac{c_4r^2}{ct}\right\}\left(1-r^2\right)^{-1/2}r^{d-2}\mathrm{d}r\\&\leq c_3\mathrm{e}^{-\frac{c_4}{4ct}}\int_{1/2}^1\left(1-r^2\right)^{-1/2}\mathrm{d}r+c_3\frac{2}{\sqrt{5}}\int_0^{1/2}\exp\left\{-\frac{c_4r^2}{ct}\right\}r^{d-2}\mathrm{d}r\\&\leq C_1t^{\frac{d-1}{2}},\end{aligned}$$

which gives (8.4).

Combining (8.3) with (8.4) we obtain

$$\begin{aligned}\mathcal{J}&\leq c_2\int_{\mathbb{B}^d}\left[\int_0^T t^{-1-\alpha-d}\left(\int_{\mathbb{S}^{d-1}}\mathrm{e}^{-\frac{|x-y|^2}{ct}}\mathrm{d}s(y)\right)^2\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_5\int_{\mathbb{B}^d}\left[\int_0^T t^{-2-\alpha}\mathrm{e}^{-\frac{2\rho^2(x)}{C_1t}}\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_6\int_{\mathbb{B}^d}\rho^{-p-\alpha p+\theta}(x)\mathrm{d}x\leq c_7\int_0^1 r^{-p+\theta-\alpha p}\mathrm{d}r.\end{aligned}$$

□

*Remark* 8.5. Assume that $A$ is equal to the Laplace operator $\Delta$. Since the semigroup is exponentially stable on $L^p$, $1<p<+\infty$, then by Theorem 5.2, it is exponentially stable on $L^p_{\theta,\delta}$. Therefore the Markov family defined by the boundary problem (1.1) on $L^p_{\theta,0}$ for $p-1<\theta<2p-1$, $p\in(1,+\infty)$ has a unique invariant measure.

Assumption (8.2) ensures that $W$ is a random field on $[0,+\infty)\times\mathbb{S}^{d-1}$. Below we present a natural example of the process satisfying the above assumptions.

**Example 8.6.** Let $(W_k)$ and $(\tilde{W}_k)$ be sequences of independent Wiener processes. Let $(a_k)$ and $(b_k)$ be sequences of real numbers such that $\sum_k a_k^2 < +\infty$. Then

$$W(t,y) = \sum_k a_k \left(\cos\langle y, b_k\rangle W_k(t) + \sin\langle y, b_k\rangle \tilde{W}_k(t)\right)$$

can obviously be written in the form (8.1). Moreover, condition (8.2) is satisfied.

For each $t\geq 0$, $W(t,\cdot)$ is rotational invariant random field on $\mathbb{S}^{d-1}$. Indeed we have

$$\begin{aligned}\mathbb{E}\, W(t,y)W(t,z) &= \sum_k a_k^2 t\left(\cos\langle y,b_k\rangle\cos\langle z,b_k\rangle + \sin\langle y,b_k\rangle\sin\langle z,b_k\rangle\right)\\ &= \sum_k a_k^2\cos\langle b_k, y-z\rangle t.\end{aligned}$$

In the case of the so-called *white noise on* $\mathbb{S}^{d-1}$, $W$ is formally defined by (8.1) with $\{e_k\}$ being an orthonormal basis of $L^2(\mathbb{S}^{d-1},\mathrm{d}s)$.

**Proposition 8.7.** *Assume that $W$ is a white noise on $\mathbb{S}^1$. Let $p>1$ and $\theta\in\left(\frac{3p}{2}-1, 2p-1\right)$. Then the boundary problem (1.1) defines a Markov family with continuous trajectories in $L^p_{\theta,0}$. If $A$ is equal to Laplace operator, then the Markov family defined by the boundary problem (1.1) on $L^p_{\theta,0}$ for $p\in(1,+\infty)$ and $\theta\in\left(\frac{3p}{2}-1,2p-1\right)$ has a unique invariant measure.*

*Proof.* Assume that $d>1$. Let $\mathcal{J} := \mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,0)$. Using (7.1) and then Corollary 6.5, we obtain

$$\begin{aligned}\mathcal{J} &= \int_{\mathbb{B}^d}\left[\sum_k\int_0^T t^{-\alpha}\left(-\int_{\mathbb{S}^{d-1}}\frac{\partial}{\partial\mathbf{n}^a(y)}G(t,x,y)e_k(y)\mathrm{d}s(y)\right)^2\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x\\ &= \int_{\mathbb{B}^d}\left[\int_0^T t^{-\alpha}\int_{\mathbb{S}^{d-1}}\left|\frac{\partial}{\partial\mathbf{n}^a(y)}G(t,x,y)\right|^2\mathrm{d}s(y)\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x\\ &\leq c_1\int_{\mathbb{B}^d}\left[\int_0^T t^{-1-\alpha}\int_{\mathbb{S}^{d-1}} g^2_{ct}(x-y)\mathrm{d}s(y)\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x\\ &\leq c_2\int_{\mathbb{B}^d}\left[\int_0^T t^{-1-d-\alpha}\int_{\mathbb{S}^{d-1}} \mathrm{e}^{-\frac{2|x-y|^2}{ct}}\mathrm{d}s(y)\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x.\end{aligned}$$

By (8.4) there is a constant $C>0$ such that

$$\begin{aligned}\mathcal{J} &\leq C\int_{\mathbb{B}^d}\left[\int_0^T t^{-1-\alpha-d+\frac{d-1}{2}}\mathrm{e}^{-\frac{\rho^2(x)}{Ct}}\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x\\ &\leq C\int_{\mathbb{B}^d}\left[\int_0^T t^{-\frac{d+3}{2}-\alpha}\mathrm{e}^{-\frac{\rho^2(x)}{Ct}}\mathrm{d}t\right]^{p/2} w_{\theta,0}(x)\mathrm{d}x\\ &\leq C_1\int_{\mathbb{B}^d}\rho^{\frac{p}{2}(2-(d+3))-p\alpha}(x)w_{\theta,0}(x)\mathrm{d}x \leq C_2\int_0^1 r^{-\frac{p}{2}(d+1)+\theta-p\alpha}\mathrm{d}r.\end{aligned}$$

Therefore $\mathcal{J}<+\infty$ if $-1<-\frac{p}{2}(d+1)+\theta-p\alpha$. In general case we need $\theta<2p-1$ and $p>1$. This requires

$$-1+\frac{p}{2}(d+1)+p\alpha<\theta<2p-1 \quad \text{and} \quad p>1.$$

Consequently, it is required that $d=1$ or $d=2$. However the case $d=1$ has been already excluded. $\square$

*Remark* 8.8. Let us drop the assumption that $e_k \in L^2(\partial\mathcal{O}, \mathrm{d}s(y))$. Namely, assume that $W(t,x) = B(t)\delta_{\widehat{y}}$, where $B$ is a real valued Wiener process and $\delta_{\widehat{y}}$ is the Dirac delta function at $\widehat{y} \in \mathbb{S}^1$. Then, using (7.1) and then Corollary 6.5, we obtain

$$\begin{aligned}
\mathcal{J}_{T,\alpha}(\delta_{\widehat{y}}, p, \theta, 0) &= \int_{\mathbb{B}^2} \left[ \sum_k \int_0^T t^{-\alpha} \left( \frac{\partial}{\partial \mathbf{n}^a(y)} G(t,x,\widehat{y}) \right)^2 \mathrm{d}t \right]^{p/2} w_{\theta,0}(x)\mathrm{d}x \\
&\leq c_1 \int_{\mathbb{B}^2} \left[ \int_0^T t^{-1-2-\alpha} \mathrm{e}^{-\frac{2|x-\widehat{y}|^2}{ct}} \, \mathrm{d}t \right]^{p/2} w_{\theta,0}(x)\mathrm{d}x \\
&\leq c_2 \int_{\mathbb{B}^2} |x-\widehat{y}|^{-2p-\alpha p} \rho^{\theta}(x)\mathrm{d}x \leq c_3 \int_0^1 r^{-2p+\theta-\alpha p}\mathrm{d}r.
\end{aligned}$$

We need $-2p+\theta-\alpha p > -1$, which is in contradiction with $\theta < 2p-1$. Therefore, we cannot treat this case.

8.3. **The case of a bounded region in $\mathbb{R}^d$.** Let $\mathcal{O}$ be a bounded $C^{1,\alpha}$ region in $\mathbb{R}^d$, $d \geq 2$. Set

$$I(t,x) := \int_{\partial\mathcal{O}} \mathrm{e}^{-\frac{|x-y|^2}{ct}} \mathrm{d}s(y), \qquad t \in (0,T], \; x \in \mathcal{O}.$$

Recall that $\rho(x) := \operatorname{dist}(x, \partial\mathcal{O})$. We have the following generalization of (8.4) established for $\mathcal{O} = \mathbb{B}^d$.

**Lemma 8.9.** *There exist constants $C_1, C_2 > 0$ such that*

$$I(t,x) \leq C_1 t^{\frac{d-1}{2}} \mathrm{e}^{-\frac{\rho^2(x)}{C_2 t}}, \qquad \forall\, t \in (0,T], \; x \in \mathcal{O}.$$

*Proof.* Fix $\varepsilon > 0$ and $x \in \mathcal{O}$. Let $\varepsilon > 0$ be fixed. Since $\mathcal{O}$ is a bounded $C^{1,\alpha}$-domain, there exist open sets $\mathcal{O}_i \subset \mathbb{R}^d$, $i = 1, 2, \ldots, n$, such that for every $i$:

(1) for every $i$ there exists, up to a shift and a rotation, a $C^{1,\alpha}$ function $a_i$ such that $x \in \mathcal{O} \cap \mathcal{O}_i$ if and only if $x = (\bar{x}, x_d - a_i(\bar{x}))$ with $x_d - a_i(\bar{x}) > 0$.
(2) $\partial\mathcal{O} \subset \bigcup_i \mathcal{O}_i$,
(3) if $x \in \mathcal{O} \setminus \bigcup_i \mathcal{O}_i$ then $\rho(x) > \varepsilon$,
(4) for each $i \leq n$ there exists a $C^{1,\alpha}$-diffeomorphism

$$g^i \colon \mathcal{O}_i \to C^d := \left\{ z \in \mathbb{R}^d \colon \; -1 < z_k < 1, \; k = 1, \ldots d \right\}$$

such that $h^i := (g^i)^{-1} : C^d \to \mathcal{O}_i$ is of class $C^{1,\alpha}$ as well and

$$\begin{aligned}
g^i(\mathcal{O}_i \cap \partial\mathcal{O}) = C^{d-1} &= \left\{ z \in C^d \colon z_d = 0 \right\}, \\
\mathcal{O}_i \cap \mathcal{O} &= \left\{ z \in C^d \colon z_d > 0 \right\}, \\
\mathcal{O}_i \cap \overline{\mathcal{O}}^c &= \left\{ z \in C^d \colon z_d < 0 \right\}.
\end{aligned}$$

Assume that $x \in \mathcal{O}_i \cap \mathcal{O}$ for certain $i \leq n$. Then for $y = (\bar{y}, 0) \in \mathcal{O}_i \cap \partial\mathcal{O}$

$$|x-y|^2 = |(\bar{x}, x_d - a_i(\bar{x})) - (\bar{y}, 0)|^2 \geq |\bar{x} - \bar{y}|^2 + \rho^2(x),$$

hence

$$\begin{aligned}
I(t,x) &\leq C t^{\frac{d-1}{2}} \mathrm{e}^{-\frac{\rho^2(x)}{ct}} \int_{\mathcal{O}_i \cap \partial\mathcal{O}} g_{ct}(\bar{x} - \bar{y}) \, \mathrm{d}s(y) \\
&= C t^{\frac{d-1}{2}} \mathrm{e}^{-\frac{\rho^2(x)}{ct}} \int_{C^{d-1}} g_{ct}\left(h^i(u) - h^i(v)\right) J\left(h^i\right)(v)\mathrm{d}v.
\end{aligned}$$

Since $h^i\colon C^d\to\mathcal{O}_i$, there exists a constant $c_1>0$ such that

$$\left|h^i(u)-h^i(v)\right|\geq c_1|u-v|,\quad u,v\in C^d\,.$$

Therefore,

$$I(t,x)\leq Ct^{\frac{d-1}{2}}\mathrm{e}^{-\frac{\rho^2(x)}{ct}}\int_{C^{d-1}}g_{ct}(u-v)\mathrm{d}v\leq Ct^{\frac{d-1}{2}}\mathrm{e}^{-\frac{\rho^2(x)}{ct}}\,,$$

and the lemma follows for $x\in\mathcal{O}\cap\mathcal{O}_i$ for every $i\leq n$.

If $x\in\mathcal{O}\setminus\bigcup_i\mathcal{O}_i$, then $\rho(x)\geq\varepsilon$ and the lemma trivially follows. □

Recall that

$$W(t,y)=\sum_k e_k(y)W_k(t),\qquad t\geq 0,\ y\in\partial\mathcal{O},$$

where $(e_k)$ is a sequence of functions on $\partial\mathcal{O}$ and $(W_k)$ are independent real-valued Wiener processes.

**Proposition 8.10.** *(i) Assume that $\sum_k\sup_{y\in\partial\mathcal{O}}e_k^2(y)<+\infty$. Then for any $1<p$ and $\theta\in(p-1,2p-1)$, the boundary problem (1.1) defines a Markov family with continuous trajectories in $L^p_{\theta,0}$.*
*(ii) Assume that $d=2$ and $W$ is a white-noise on $\partial\mathcal{O}$. Then for $1<p$ and $\theta\in\left(\frac{3p}{2}-1,2p-1\right)$ the boundary problem (1.1) defines a Markov family with continuous trajectories in $L^p_{\theta,0}$.*

*Proof of (i).* Using the calculations from the proof of Proposition 8.4, and then our Lemma 8.9 we obtain

$$\begin{aligned}\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,0)&\leq c_1\int_{\mathcal{O}}\left[\int_0^T t^{-1-d-\alpha}\left(I(t,x)\right)^2\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_2\int_{\mathcal{O}}\left[\int_0^T t^{-1-d+d-1-\alpha}\mathrm{e}^{-2\frac{\rho^2(x)}{C_1t}}\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_3\int_{\mathcal{O}}\rho(x)^{-p+\theta-p\alpha}(x)\mathrm{d}x.\end{aligned}$$

It is easy to show, see the proof of Lemma 4.5, that the integral is finite if and only if $-p+\theta-p\alpha>-1$.

□

*Proof of (ii).* Using the calculations from the proof of Proposition 8.7, and then our Lemma 8.9 we obtain

$$\begin{aligned}\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,0)&\leq c_1\int_{\mathcal{O}}\left[\int_0^T t^{-1-d-\alpha}\int_{\partial\mathcal{O}}\mathrm{e}^{-\frac{2|x-y|^2}{ct}}\mathrm{d}s(y)\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_2\int_{\mathcal{O}}\left[\int_0^T t^{-1-d+\frac{d-1}{2}-\alpha}\mathrm{e}^{-2\frac{\rho^2(x)}{C_1t}}\mathrm{d}t\right]^{p/2}w_{\theta,0}(x)\mathrm{d}x\\&\leq c_3\int_{\mathcal{O}}\rho^{-\frac{p}{2}(d+1)+\theta-\alpha p}(x)\mathrm{d}x.\end{aligned}$$

□

8.4. **Half-space with spatially homogeneous Wiener process.** In this section $\mathcal{O} = (0,+\infty)\times\mathbb{R}^m$, and $W$ is the so-called spatially homogeneous Wiener process on $\mathbb{R}^m \equiv \{0\}\times\mathbb{R}^m = \partial\mathcal{O}$. We adopt the notation $x = (x_0, x_1, \ldots, x_n) = (x_0, \mathbf{x})$.

**Definition 8.11.** A process $W$ taking values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^m)$ is called a *spatially homogeneous Wiener process* if and only if:

- $(i)$ It is Gaussian process with continuous trajectories in $\mathcal{S}'(\mathbb{R}^m)$.
- $(ii)$ For each $\psi \in \mathcal{S}(\mathbb{R}^m)$, $t \mapsto (W(t), \psi)$ is a one dimensional Wiener process.
- $(iii)$ For each fixed $t \geq 0$ the law of $W(t)$ is invariant with respect to all translations $\tau'_h\colon \mathcal{S}'(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^m)$, $h \in \mathbb{R}^m$, where $\tau_h\colon \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$, $\tau_h\psi(\cdot) = \psi(\cdot + h)$ for $\psi \in \mathcal{S}(\mathbb{R}^m)$.

The law of a spatially homogeneous Wiener process $W$ on $\mathbb{R}^m$ is characterized by its *spectral measure* $\mu$ on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$. Recall, see [33] that $\mu$ is a positive symmetric Radon tempered measure on $\mathbb{R}^m$, and for any test functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^m)$,

$$\mathbb{E}\langle W(t), \psi\rangle\langle W(s), \phi\rangle = t \wedge s \int_{\mathbb{R}^m} \mathcal{F}\psi(\mathbf{x})\overline{\mathcal{F}\phi(\mathbf{x})}\mu(\mathrm{d}\mathbf{x}),$$

where $\mathcal{F}$ denotes the Fourier transform. Note, see e.g. [33] that if the spectral measure is finite, then $W$ is a random field, such that for any $\mathbf{x}$, $W(\cdot, \mathbf{x})$ is a one dimensional Wiener process. Moreover, for fixed $t$, the field $W(t, \mathbf{x})$ is stationary in $\mathbf{x}$.

Let

$$L^2_{(s)}(\mu) := \left\{\psi \in L^2(\mathbb{R}^d \mapsto \mathbb{C}, \mathcal{B}(\mathbb{R}^m), \mu)\colon \psi(-\mathbf{x}) = \overline{\psi(\mathbf{x})}\right\}.$$

Then, see [33], the Reproducing Kernel Hilbert Space $H_W$ of $W$ is given by

$$H_W = \left\{\mathcal{F}\psi\colon \psi \in L^2_{(s)}(\mu)\right\}$$

and

$$\langle \mathcal{F}\psi, \mathcal{F}\phi\rangle_{H_W} = \int_{\mathbb{R}^d} \psi(\mathbf{x})\overline{\phi(\mathbf{x})}\mu(\mathrm{d}\mathbf{x}).$$

Thus any orthonormal basis $\{e_k\}$ of $H_W$ is of the form $e_k = \mathcal{F}(f_k\mu)$, where $\{f_k\}$ is an orthonormal basis of $L^2_{(s)}(\mu)$. We will identify $\mathbb{R}^m$ with $\partial(0,+\infty)\times\mathbb{R}^m = \{0\}\times\mathbb{R}^m$.

We have

$$\begin{aligned}
\mathcal{J}_{T,\alpha}(\{e_k\}, p, \theta, \delta) &:= \int_{\mathcal{O}} \left[t^{-\alpha}\int_0^T \sum_k \left(\int_{\mathbb{R}^m} \frac{\partial G}{\partial \mathbf{n}_\mathbf{y}}(t, x, (0, \mathbf{y}))e_k(\mathbf{y})\mathrm{d}\mathbf{y}\right)^2 \mathrm{d}t\right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x \\
&= \int_{\mathcal{O}} \left[\int_0^T t^{-\alpha} \sum_k \left(\int_{\mathbb{R}^m} \frac{\partial G}{\partial \mathbf{n}_\mathbf{y}}(t, x, (0, \mathbf{y}))\mathcal{F}(f_k\mu)(\mathbf{y})\mathrm{d}\mathbf{y}\right)^2 \mathrm{d}t\right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x \\
&\leq \int_{\mathcal{O}} \left[\int_0^T t^{-\alpha} \int_{\mathbb{R}^m} \left|\mathcal{F}^{-1}_\mathbf{y} \frac{\partial G}{\partial \mathbf{n}_\mathbf{y}}(t, x, (0, \mathbf{y}))\right|^2 \mu(\mathrm{d}\mathbf{y})\mathrm{d}t\right]^{p/2} w_{\theta,\delta}(x)\mathrm{d}x. \qquad (8.5)
\end{aligned}$$

**Proposition 8.12.** *Assume that: the spectral measure of $W$ is finite, $\delta > (m+1)/2$, $p \in (1,+\infty)$ and $\theta \in (p-1, 2p-1)$. Then boundary problem (1.1) defines Markov family with continuous trajectories in $L^p_{\theta,\delta}$.*

*Proof.* If the measure $\mu$ is finite, then

$$\begin{aligned}\int_{\mathbb{R}^m} \left|\mathcal{F}_{\mathbf{y}}^{-1}\frac{\partial G}{\partial \mathbf{n}_{\mathbf{y}}}(t,x,(0,\mathbf{y}))\right|^2 \mu(\mathrm{d}\mathbf{y}) &\le \mu(\mathbb{R}^m)\sup_{\mathbf{y}\in\mathbb{R}^m}\left|\mathcal{F}_{\mathbf{y}}^{-1}\frac{\partial G}{\partial \mathbf{n}_{\mathbf{y}}}(t,x,(0,\mathbf{y}))\right|^2\\ &\le \mu(\mathbb{R}^m)\left[\int_{\mathbb{R}^m}\left|\frac{\partial G}{\partial \mathbf{n}_{\mathbf{y}}}(t,x,(0,\mathbf{y}))\right|\mathrm{d}\mathbf{y}\right]^2\\ &\le ct^{-2}\mathrm{e}^{-\frac{x_0^2}{ct}},\end{aligned}$$

where in the last estimate we use (7.1). We have

$$\int_0^T t^{-2-\alpha}\mathrm{e}^{-\frac{x_0^2}{ct}}\mathrm{d}t \le \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{2s}}\mathrm{d}s x_0^{-2-2\alpha} \le c_1 x_0^{-2}.$$

Therefore, by (8.5), we have

$$\begin{aligned}\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) &\le c_1\int_0^{+\infty}\int_{\mathbb{R}^m}\min\{x_0,1\}^{\theta}x_0^{-p-\alpha p}\left(1+|x_0|^2+|\mathbf{x}|^2\right)^{-\delta}\mathrm{d}x_0\mathrm{d}\mathbf{x}\\ &\le c_2 + c_2\int_0^1 x_0^{\theta-p-\alpha p}\mathrm{d}x_0.\end{aligned}$$

□

*Remark* 8.13. If $\mathcal{A}=\Delta$ then $G$ is given by (4.6). Then, with $\overline{x}$ defined by (4.7),

$$\frac{\partial G}{\partial \mathbf{n}_y}(t,x,y) = -\frac{\partial G}{\partial y_0}(t,x,y) = \frac{y_0-x_0}{2t}g_{2t}(x-y) + \frac{-y_0-x_0}{2t}g_{2t}(\overline{x}-y).$$

*Remark* 8.14. (7.1) gives estimates for $\frac{\partial G}{\partial \mathbf{n}_y}(t,x,(0,\mathbf{y}))$. Unfortunately, we are not able to use them to compare the Fourier transforms of $\frac{\partial G}{\partial \mathbf{n}_y}(t,x,(0,\mathbf{y}))$ and $\tilde{g}_{ct}$. This problem can be solved under a certain technical assumption on the spectral measure $\mu$, see the lemma below. For a measure for which this assumption is violated see [32].

Write

$$K_\alpha(r) := \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}-r^2 s}\mathrm{d}s, \qquad r\ge 0,$$

**Lemma 8.15.** *Assume that either $A=\Delta$ or there is a finite symmetric measure $\mu_0$ on $\mathbb{R}^m$ such that $\mathcal{F}\mu+\mathcal{F}\mu_0$ is a non-negative measure. Let $\delta>(m+1)/2$, $p>1$ and $\theta\in(p-1,2p-1)$. Then for any $T>0$, there is a constant $C>0$ such that*

$$\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) \le C + C\int_{\mathbb{R}^m}\int_0^{+\infty}\left[\int_{\mathbb{R}^m}K_\alpha\left(\frac{x_0|\mathbf{y}|}{C}\right)\mu(\mathrm{d}\mathbf{y})\right]^{p/2} x_0^{-p-\alpha p}w_{\theta,\delta}((x_0,\mathbf{y}))\mathrm{d}x_0\mathrm{d}\mathbf{y}.$$

*Proof.* Taking into account (8.5) and Proposition 8.12 we can assume that $\mathcal{F}\mu$ is a non-negative measure. Then

$$\begin{aligned}&\int_{\mathbb{R}^m}\left|\mathcal{F}_{\mathbf{y}}^{-1}\frac{\partial G}{\partial \mathbf{n}_{\mathbf{y}}}(t,x,(0,\mathbf{y}))\right|^2\mu(\mathrm{d}\mathbf{y})\\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\partial G}{\partial \mathbf{n}_{\mathbf{u}}}(t,x,(0,\mathbf{u}))\frac{\partial G}{\partial \mathbf{n}_{\mathbf{v}}}(t,x,(0,\mathbf{v}))\mathcal{F}\mu(\mathbf{u}-\mathbf{v})\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}\\ &\le c_1^2 t^{-1}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} g_{ct}(x-(0,\mathbf{u}))g_{ct}(x-(0,\mathbf{v}))\mathcal{F}\mu(\mathbf{u}-\mathbf{v})\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}\\ &\le c_2 t^{-2}\mathrm{e}^{-\frac{x_0^2}{c_2 t}}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{t|\mathbf{z}|^2}{c_2}}\mu(\mathrm{d}\mathbf{z}).\end{aligned}$$

Therefore, by (8.5),

$$\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta) \le \int_{\mathbb{R}^m}\int_0^{+\infty}\left[\int_{\mathbb{R}^m}\int_0^T c_2 t^{-2-\alpha}\mathrm{e}^{-\frac{x_0^2}{c_2 t}}\mathrm{e}^{-\frac{t|\mathbf{z}|^2}{c_2}}\mathrm{d}t\mu(\mathrm{d}\mathbf{z})\right]^{p/2} w_{\theta,\delta}((x_0,\mathbf{y}))\mathrm{d}x_0\mathrm{d}\mathbf{y}.$$

Since

$$\int_0^T c_2 t^{-2-\alpha}\mathrm{e}^{-\frac{x_0^2}{c_2 t}}\mathrm{e}^{-\frac{t|\mathbf{z}|^2}{c_2}}\mathrm{d}t \le x_0^{-2-2\alpha}\int_0^{+\infty} s^{-2}\mathrm{e}^{-\frac{1}{s}-\frac{x_0^2|\mathbf{z}|^2 s}{c_2^2}}\mathrm{d}s = x_0^{-2-\alpha}K_\alpha\left(\frac{x_0|\mathbf{z}|}{c_2}\right),$$

the desire conclusion follows. □

If $\mu(\mathrm{d}\mathbf{y}) = \mathrm{d}\mathbf{y}$, then $W$ is the co-called *cylindrical Wiener process on* $L^2(\mathbb{R}^m)$ or equivalently $\frac{\partial W}{\partial t}(t,\mathbf{y})$, $t\ge 0$, $\mathbf{y}\in\mathbb{R}^m$, is the *space-time white noise*. Then $\mathcal{F}\mu$ is the Dirac delta measure. We have the following consequence of Lemma 8.15 and our general Theorem 2.10. Note that our results on heat semigroups on weighted spaces do not allow $m>1$.

**Proposition 8.16.** *Let $m=1$, $\delta>1$, and let $W$ be a cylindrical Wiener process on $L^2(\mathbb{R})$. Then the boundary problem (1.1) defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for $p>1$ and $\theta\in\left(\frac{3p}{2}-1,2p-1\right)$.*

*Proof.* We have

$$\begin{aligned}\int_{\mathbb{R}^m} K_\alpha\left(\frac{x_0|\mathbf{z}|}{C}\right)\mu(\mathrm{d}\mathbf{z}) &= \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{x_0^2|\mathbf{z}|^2}{C^2}s}\mathrm{d}\mathbf{z}\mathrm{d}s\\ &= \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}}\left(2\pi\frac{C^2}{2x_0^2 s}\right)^{m/2}\mathrm{d}s = \tilde{C}x_0^{-m}.\end{aligned}$$

Then, by Lemma 8.15, $\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta)$ is finite if

$$\int_0^1 r^{-\frac{mp}{2}-p+\theta-\alpha p}\mathrm{d}r < +\infty.$$

This requires $\theta > -1+\frac{mp}{2}+p$. Since we need $2p-1>\theta$ we arrive at the condition $p>\frac{mp}{2}$. Since $p\ge 1$ we are able to deal with the boundary problem only in the case of $m=1$. □

In many interesting cases, see e.g. [10], the spectral measure $\mu$ of $W$ is absolutely continuous and its density is the so-called *Bessel potential*. Namely for a parameter $\kappa>0$ let

$$\mu_\kappa(\mathrm{d}\mathbf{y}) := \left(1+|\mathbf{y}|^2\right)^{-\kappa/2}\mathrm{d}\mathbf{y}.$$

Then the space correlation $\Gamma_\kappa(\mathbf{y}) := \mathcal{F}\mu_\kappa(\mathbf{y})$ is a non-negative continuous function on $\mathbb{R}^m\setminus\{0\}$. Moreover, asymptotically as $|\mathbf{y}|\to 0$,

$$\Gamma_\kappa(\mathbf{y}) \approx \begin{cases} C_{m,\kappa}|\mathbf{y}|^{\kappa-m} & \text{for } 0<\kappa<m,\\ C_{m,\kappa}\log\frac{1}{|\mathbf{y}|} & \text{for } \kappa=m,\\ C_{m,\kappa} & \text{for } \kappa>m.\end{cases} \tag{8.6}$$

Finally

$$\Gamma_\kappa(\mathbf{y}) \approx C_{m,\kappa}\mathrm{e}^{-|\mathbf{y}|}, \qquad \text{as } |\mathbf{y}|\to+\infty. \tag{8.7}$$

Note that in the limit $\kappa\downarrow 0$ we obtain Lebesgue measure corresponding to the cylindrical Wiener process on $L^2(\mathbb{R}^m)$ treated in Proposition 8.16.

**Proposition 8.17.** *Let $W$ be a Wiener process on $\mathbb{R}^m$ with the spectral measure $\mu_\kappa$, $0<\kappa\le m$. (i) If $\kappa \ge m$, then boundary problem (1.1) defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for $p>1$ and $\theta\in(p-1,2p-1)$.*
*(ii) If $m-2<\kappa<m$, then boundary problem (1.1) defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for*

$$1<p<+\infty, \qquad p+\frac{p}{2}(m-\kappa)-1<\theta<2p-1.$$

*Proof.* Note that if $\kappa>m$ then $\mu_\kappa$ is finite and we may apply our Proposition 8.12. Therefore we restrict our attention to the case of $0<\kappa\le m$. We have

$$\begin{aligned}\int_{\mathbb{R}^m} K_\alpha\left(\frac{x_0|\mathbf{z}|}{C}\right)\mu_\kappa(\mathrm{d}\mathbf{z}) &= \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{x_0^2|\mathbf{z}|^2}{C}s}\left(1+|\mathbf{z}|^2\right)^{-\kappa/2}\mathrm{d}\mathbf{z}\,\mathrm{d}s\\ &= \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}}\int_{\mathbb{R}^m}\left[\mathcal{F}^{-1}_{\mathbf{z}}\mathrm{e}^{-\frac{x_0^2 s}{C}|\mathbf{z}|^2}\right]\Gamma_\kappa(\mathbf{z})\mathrm{d}\mathbf{z}\,\mathrm{d}s\\ &= \int_0^{+\infty} s^{-2-\alpha}\mathrm{e}^{-\frac{1}{s}}\int_{\mathbb{R}^m}\left(2\pi\sigma^2\right)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_\kappa(\mathbf{z})\mathrm{d}\mathbf{z}\,\mathrm{d}s,\end{aligned}$$

where $\sigma^2:=\frac{x_0^2 s}{2C}$. Then by (8.7),

$$\int_{\{|\mathbf{z}|\ge 1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_\kappa(\mathbf{z})\mathrm{d}\mathbf{z}\le C\int_{\{|\mathbf{z}|\ge 1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}-\frac{|\mathbf{z}|}{C}}\mathrm{d}\mathbf{z}\le C_1<+\infty,$$

where $C_1<+\infty$ does not depend on $\sigma$.

Next, by (8.6), if $\kappa=m$, then

$$\begin{aligned}\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_\kappa(\mathbf{z})\mathrm{d}\mathbf{z}&\le C\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\log\frac{1}{C|\mathbf{z}|}\mathrm{d}\mathbf{z}\\ &\le C_1\int_0^1(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}\left|\log Cr\right|r^{m-1}\mathrm{d}r.\end{aligned}$$

If $m>1$, then $r\mapsto\left|\log Cr\right|r^{m-1}$ is a continuous function on a closed interval $[0,1]$. Therefore

$$\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_1(\mathbf{z})\mathrm{d}\mathbf{z}\le\tilde{C}_1,$$

where again $\tilde{C}_1<+\infty$ does not depend on $\sigma$.

If $\kappa=m=1$ then $r\mapsto|\log Cr|$ in integrable on $[0,1]$ with any power $>1$. Therefore, using the Hölder inequality we obtain that for any $q>1$ there is an independent of $\sigma$ constant $C(q)$ such that

$$\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-1/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_1(\mathbf{z})\mathrm{d}\mathbf{z}\le C(q)\left(\int_0^1(2\pi\sigma^2)^{-q/2}\mathrm{e}^{-q\frac{r^2}{2\sigma^2}}\mathrm{d}r\right)^{1/q}\le C(q)\sigma^{\frac{1-q}{2q}}.$$

Taking $q>1$ small enough we conclude that for any $\varepsilon>0$ there is an independent of $\sigma$ constant $C_\varepsilon$ such that

$$\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-1/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_1(\mathbf{z})\mathrm{d}\mathbf{z}\le C_\varepsilon\sigma^{-\varepsilon}.$$

Finally if $0<\kappa<m$, then

$$\begin{aligned}\int_{\{|\mathbf{z}|<1\}}(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{|\mathbf{z}|^2}{2\sigma^2}}\Gamma_\kappa(\mathbf{z})\mathrm{d}\mathbf{z} &\leq C_2\int_0^1(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}r^{\kappa-m+m-1}\mathrm{d}r\\ &\leq C_2\int_0^1(2\pi\sigma^2)^{-m/2}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}r^{\kappa-1}\mathrm{d}r\\ &\leq C_3\sigma^{-m+1+\kappa-1}\int_0^{+\infty}\mathrm{e}^{-\frac{u^2}{2}}u^{\kappa-1}\mathrm{d}r\\ &\leq C_4\sigma^{\kappa-m}.\end{aligned}$$

Summing up, we see that there is an independent of $x_0$ constant $c_1$ such that

$$\int_{\mathbb{R}^m}K_\alpha\left(\frac{x_0|\mathbf{z}|}{C}\right)\mu_\kappa(\mathrm{d}\mathbf{z})\leq c_1,\qquad \text{if } \kappa=m>1,$$

and

$$\int_{\mathbb{R}^m}K_\alpha\left(\frac{x_0|\mathbf{z}|}{C}\right)\mu_\kappa(\mathrm{d}\mathbf{z})\leq c_1+c_1x_0^{\kappa-m},\qquad \text{if } \kappa<m.$$

If $\kappa=m=1$ then for any $\varepsilon>0$ then there is an independent of $x_0$ constant $c(\varepsilon)$ such that

$$\int_{\mathbb{R}^m}K_\alpha\left(\frac{x_0|\mathbf{z}|}{C}\right)\mu_\kappa(\mathrm{d}\mathbf{z})\leq c_1+c(\varepsilon)x_0^{-\varepsilon}.$$

Thus, by Lemma 8.15, $\mathcal{J}_{T,\alpha}(\{e_k\},p,\theta,\delta)$ is finite if

$$\int_0^1 x_0^{-p+\theta-p\alpha}dx_0<+\infty,\qquad \text{if } \kappa=m\geq 1$$

and

$$\int_0^1 x_0^{-p+\frac{p}{2}(\kappa-m)+\theta-p\alpha}\mathrm{d}x_0<+\infty,\qquad \text{if } \kappa<m.$$

□

*Remark* 8.18. Note that, we are not able to treat the case of $m>2$ and $\kappa\leq m-2$.

## Appendix A. Proof of Lemma 4.5

Assume that $\mathcal{O}$ is a half space. Without any loss of generality we may assume that $\mathcal{O}=\{x=(x_1,\mathbf{x})\in\mathbb{R}^d\colon x_1>0\}$. We also assume that $d>1$. Then

$$\begin{aligned}\int_{\mathcal{O}}g_{ct}(x-y)\rho^\alpha(y)\,\mathrm{d}y &= \int_0^{+\infty}\int_{\mathbb{R}^{d-1}}g_{ct}(x-y)y^\alpha\,\mathrm{d}y\\ &= \int_0^{+\infty}(2\pi ct)^{-1/2}\,\mathrm{e}^{-\frac{(x_1-y_1)^2}{2ct}}y_1^\alpha\,\mathrm{d}y_1\\ &\leq t^{\frac{\alpha}{2}}\int_{\mathbb{R}}(2\pi c)^{-1/2}\,\mathrm{e}^{-\frac{(t^{-\frac12}x_1-z)^2}{2c}}|z|^\alpha\mathrm{d}z.\end{aligned}$$

Since

$$\begin{aligned}&\sup_{r\in\mathbb{R}}\int_{\mathbb{R}}(2\pi c)^{-1/2}\mathrm{e}^{-\frac{(r-z)^2}{2c}}|z|^\alpha\mathrm{d}z\\ &\leq\int_{|z|\leq 1}(2\pi c)^{-1/2}|z|^\alpha\mathrm{d}z+\sup_{r\in\mathbb{R}}\int_{\mathbb{R}}(2\pi c)^{-1/2}\mathrm{e}^{-\frac{(z-r)^2}{2c}}\mathrm{d}z<+\infty,\end{aligned}$$

the desired conclusion follows. □

Assume that $\mathcal{O}$ is a bounded $C^{1,\alpha}$-domain. For $y \in \mathbb{R}^d$ we will write $y = (y_1, \mathbf{y}) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $g_{ct}(y) = g_{ct}^{(1)}(y_1)\, g_{ct}^{(d-1)}(\mathbf{y})$. Since $\mathcal{O}$ is a bounded $C^{1,\alpha}$-domain, its boundary $\partial\mathcal{O}$ can be covered with a finite number of open sets $\mathcal{O}_i$, such that for every $i$ there exists a $C^{1,\alpha}$ function $h^i$ such that (up to a shift and rotation of the domain)

$$\mathcal{O} \cap \mathcal{O}_i = \left\{ y \in \mathbb{R}^d \colon y_1 > h^i(\mathbf{y}) \right\} .$$

Moreover, for $t$ small enough we have

$$\mathcal{O}_t \subset \bigcup_i \mathcal{O}_i .$$

For each $i$ we can define a $C^1$-diffeomorphism

$$g^i \colon \mathcal{O}_i \to g^i(\mathcal{O}_i), \quad g^i(y_1, \mathbf{y}) = \left( y_1 - h^i(\mathbf{y}), \mathbf{y} \right)$$

Clearly, the Jacobian $J^i$ of $g^i$ satisfies the condition $|J^i(x)| = 1$. Therefore, for $z = (z_1, \mathbf{z}) \in g^i(\mathcal{O}_i)$ we have

$$z_1 = y_1 - h^i(\mathbf{y}) = \inf_{v \in \mathcal{O}_i \cap \partial\mathcal{O}} \left| g^i(y) - g^i(v) \right| .$$

Since

$$c_1 |y - v| \leq \left| g^i(y) - g^i(v) \right| \leq c_2 |y - v|, \quad y, v \in \mathcal{O}_i \cap \mathcal{O}$$

we find that

$$c_1 z_d \leq \rho(y) = \rho(\bar{y}, y_d) \leq c_2 z_d, \quad y \in \mathcal{O}_t .$$

We have

$$\int_{\mathcal{O}_t} \rho^\alpha(y) g_{ct}(x-y) \mathrm{d}y = \sum_i \int_{\mathcal{O}_t \cap \mathcal{O}_i} \rho^\alpha(y) g_{ct}(x-y) \mathrm{d}y =: \sum_i J_i(t,x) ,$$

and it is enough to show that for every $i$

$$\sup_{t \leq 1} \sup_{x \in \mathcal{O}} t^{-\frac{\alpha}{2}} J_i(t,x) < +\infty .$$

Changing variables we obtain

$$\begin{aligned} J_i(t,x) &\leq C \int_{\mathbb{R}^d} |z_1|^\alpha g_{ct}(z-x) \mathrm{d}z = C t^{\frac{\alpha}{2}} \int_{\mathbb{R}} |y|^\alpha g_c^{(1)}\left( x_d t^{-1/2} - y \right) \mathrm{d}y \\ &\leq C t^{\frac{\alpha}{2}} (2\pi c)^{-\frac{1}{2}} \int_{-1}^{1} |y|^\alpha \mathrm{d}y + C t^{\frac{\alpha}{2}} \int_{\{|y| \geq 1\}} g_c^{(1)}\left( x_d t^{-1/2} - y \right) \mathrm{d}y \\ &\leq C t^{\frac{\alpha}{2}} (2\pi c)^{-\frac{1}{2}} \int_{-1}^{1} |y|^\alpha \mathrm{d}y + C t^{\frac{\alpha}{2}} . \qquad \square \end{aligned}$$

## Appendix B. $C_0$-property without Assumption 4.4

We are showing that for any $p \in [1, +\infty)$ and $\theta \in [0, p)$ there exists a constant $M_{p,\theta}$ such that

$$|S(t)\psi|_{L^p_\theta} \leq M_{p,\theta} |\psi|_{L^p_\theta}, \qquad \forall\, t \in (0,1],\ \forall\, \psi \in L^p. \tag{B.1}$$

By (4.3) and the Jensen inequality we have

$$|S(t)\psi|^p_{L^p_\theta} \leq C^p \int_{\mathcal{O}} \rho^\theta(x) \int_{\mathcal{O}} m_t^p(y) g_{ct}(x-y) |\psi(y)|^p \,\mathrm{d}y \,\mathrm{d}x.$$

Changing variables we obtain

$$
\text{(B.2)}\qquad
\begin{aligned}
&|S(t)\psi|^p_{L^p_\theta}\\
&\quad\le C^p t^{d/2}\int_{\mathcal{O}/\sqrt{t}}\rho^\theta(x\sqrt{t})\int_{\mathcal{O}/\sqrt{t}} m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p\,\mathrm{d}y\,\mathrm{d}x\\
&\quad\le C^p t^{d/2}\int_{\mathcal{O}/\sqrt{t}}\rho^\theta(x\sqrt{t})\int_{\mathcal{O}/\sqrt{t}} m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Recall that $\rho(x)=\mathrm{dist}\,(x,\partial\mathcal{O})$. Define

$$
(\mathcal{O}/\sqrt{t})_1 := \left\{x\in\mathcal{O}/\sqrt{t}\colon m_t(x\sqrt{t})=\rho(x\sqrt{t})/\sqrt{t}<1\right\}
$$

and

$$
(\mathcal{O}/\sqrt{t})_1^c := \left\{x\in\mathcal{O}/\sqrt{t}\colon m_t(x\sqrt{t})=1\right\}.
$$

Then we have

$$
|S(t)\psi|^p_{L^p_\theta}\le C^p t^{d/2}\left(I_1+I_2+I_3+I_4\right),
$$

where

$$
\begin{aligned}
I_1 &:= \int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}x\,\rho^\theta(x\sqrt{t})\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\, m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p,\\
I_2 &:= \int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}x\,\rho^\theta(x\sqrt{t})\int_{(\mathcal{O}/\sqrt{t})_1^c}\mathrm{d}y\, m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p,\\
I_3 &:= \int_{(\mathcal{O}/\sqrt{t})_1^c}\mathrm{d}x\,\rho^\theta(x\sqrt{t})\int_{(\mathcal{O}/\sqrt{t})_1^c}\mathrm{d}y\, m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p,\\
I_4 &:= \int_{(\mathcal{O}/\sqrt{t})_1^c}\mathrm{d}x\,\rho^\theta(x\sqrt{t})\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\, m_t^p(y\sqrt{t})g_c(x-y)|\psi(y\sqrt{t})|^p.
\end{aligned}
$$

Set $\phi(y)=\psi(y\sqrt{t})$. Taking into account that $\rho(z\sqrt{t})/\sqrt{t}\le 1$ for $z\in(\mathcal{O}/\sqrt{t})_1$, and the fact that since $\theta<p$

$$
\frac{\rho^p(y\sqrt{t})}{t^{p/2}}\le\frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}},\qquad y\in(\mathcal{O}/\sqrt{t})_1,
$$

we have

$$
\begin{aligned}
I_1 &= \int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}x\,\rho^\theta(x\sqrt{t})\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\,\frac{\rho^p(y\sqrt{t})}{t^{p/2}}g_c(x-y)|\phi(y)|^p\\
&\le \int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}x\,\frac{\rho^\theta(x\sqrt{t})}{t^{\theta/2}}\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\,\rho^\theta(y\sqrt{t})g_c(x-y)|\phi(y)|^p\\
&\le \int_{(\mathcal{O}/\sqrt{t})_1} g_c(x-y)\mathrm{d}x\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\,\rho^\theta(y\sqrt{t})|\phi(y)|^p\\
&\le \int_{\mathbb{R}^d} g_c(x-y)\mathrm{d}x\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\,\rho^\theta(y\sqrt{t})|\phi(y)|^p\\
&\le C_1\int_{(\mathcal{O}/\sqrt{t})_1}\mathrm{d}y\,\rho^\theta(y\sqrt{t})|\phi(y)|^p\le C_1|\phi|^p_{L^p_\theta(\mathcal{O}/\sqrt{t})}.
\end{aligned}
$$

Taking into account that $(\mathcal{O}/\sqrt{t})_1^c$, $1 \le \rho(z\sqrt{t})/\sqrt{t}$ we find that

$$\begin{aligned}
I_2 &= \int_{(\mathcal{O}/\sqrt{t})_1} \mathrm{d}x\, \rho^\theta(x\sqrt{t}) \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}y\, g_c(x-y)|\phi(y)|^p \\
&\le \int_{\mathbb{R}^d} g_c(x-y)\mathrm{d}x t^{\theta/2} \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}y|\phi(y)|^p \\
&\le C_1 \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}y \frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}} t^{\theta/2}|\phi(y)|^p \\
&\le C_1 |\phi|^p_{L^p_\theta(\mathcal{O}/\sqrt{t})}
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \rho^\theta(x\sqrt{t}) \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}y\, g_c(x-y)|\phi(y)|^p \\
&= \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \frac{\rho^\theta(x\sqrt{t})}{\rho^\theta(y\sqrt{t})} g_c(x-y) \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}y\, \rho^\theta(y\sqrt{t})|\phi(y)|^p \\
&\le \sup_{y\in(\mathcal{O}/\sqrt{t})_1^c} \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \frac{\rho^\theta(x\sqrt{t})}{\rho^\theta(y\sqrt{t})} g_c(x-y)|\phi|^p_{L^p_\theta(\mathcal{O}/\sqrt{t})},
\end{aligned}$$

and finally, as

$$\frac{\rho^p(y\sqrt{t})}{t^{p/2}} \le \frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}}, \qquad y \in (\mathcal{O}/\sqrt{t})_1,$$

we have

$$\begin{aligned}
I_4 &= \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \rho^\theta(x\sqrt{t}) \int_{(\mathcal{O}/\sqrt{t})_1} \mathrm{d}y\, \frac{\rho^p(y\sqrt{t})}{t^{p/2}} g_c(x-y)|\phi(y)|^p \\
&\le \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \rho^\theta(x\sqrt{t}) \int_{(\mathcal{O}/\sqrt{t})_1} \mathrm{d}y\, \frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}} g_c(x-y)|\phi(y)|^p \\
&\le \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}} g_c(x-y) \int_{(\mathcal{O}/\sqrt{t})_1} \mathrm{d}y\, \rho^\theta(x\sqrt{t})|\phi(y)|^p \\
&\le \sup_{y\in(\mathcal{O}/\sqrt{t})_1} \int_{(\mathcal{O}/\sqrt{t})_1^c} \mathrm{d}x\, \frac{\rho^\theta(y\sqrt{t})}{t^{\theta/2}} g_c(x-y)|\phi|^p_{L^p_\theta(\mathcal{O}/\sqrt{t})}.
\end{aligned}$$

Note that

$$|\phi|^p_{L^p_\theta(\mathcal{O}/\sqrt{t})} = t^{-d/2}|\psi|^p_{L^p_\theta}.$$

Therefore the proof will be completed as soon as we show that

$$A_1 := \sup_{t\in(0,1]} \sup_{y\in(\mathcal{O}/\sqrt{t})_1^c} \int_{(\mathcal{O}/\sqrt{t})_1^c} \frac{\rho^\theta(x\sqrt{t})}{\rho^\theta(y\sqrt{t})} g_c(x-y)\mathrm{d}x < +\infty$$

and

$$A_2 := \sup_{t\in(0,1]} \sup_{y\in(\mathcal{O}/\sqrt{t})_1} \int_{(\mathcal{O}/\sqrt{t})_1^c} \frac{\rho^\theta(x\sqrt{t})}{t^{\theta/2}} g_c(x-y)\mathrm{d}x < +\infty.$$

To do this note that

$$\rho(z\sqrt{t}) = \operatorname{dist}\left(z\sqrt{t}, \partial\mathcal{O}\right) = \sqrt{t}\,\operatorname{dist}\left(z, \partial\mathcal{O}/\sqrt{t}\right).$$

Therefore

$$A_1 = \sup_{t\in(0,1]} \sup_{y\in(\mathcal{O}/\sqrt{t})_1^c} \int_{(\mathcal{O}/\sqrt{t})_1^c} \left( \frac{\operatorname{dist}\left(x, \partial\mathcal{O}/\sqrt{t}\right)}{\operatorname{dist}\left(y, \partial\mathcal{O}/\sqrt{t}\right)} \right)^{\theta} g_c(x-y)\mathrm{d}x$$

and

$$A_2 = \sup_{t\in(0,1]} \sup_{y\in(\mathcal{O}/\sqrt{t})_1} \int_{(\mathcal{O}/\sqrt{t})_1^c} \operatorname{dist}\left(x, \partial\mathcal{O}/\sqrt{t}\right)^{\theta} g_c(x-y)\mathrm{d}x.$$

Given a domain $\mathcal{D} \subset \mathbb{R}^d$, $\mathcal{D} \neq \mathbb{R}^d$, set

$$\begin{aligned}
\mathcal{D}_1 &= \{x \in \mathcal{D} \colon \operatorname{dist}(x, \partial\mathcal{D}) < 1\}, \\
\mathcal{D}_1^c &= \{x \in \mathcal{D} \colon \operatorname{dist}(x, \partial\mathcal{D}) \geq 1\}, \\
A_1(\mathcal{D}) &= \sup_{y\in\mathcal{D}_1^c} \int_{\mathcal{D}_1^c} \left( \frac{\operatorname{dist}(x, \partial\mathcal{D})}{\operatorname{dist}(y, \partial\mathcal{D})} \right)^{\theta} \exp\left\{ -\frac{|x-y|^2}{c} \right\} \mathrm{d}x, \\
A_2(\mathcal{D}) &= \sup_{y\in\mathcal{D}_1} \int_{\mathcal{D}_1^c} \operatorname{dist}(x, \partial\mathcal{D})^{\theta} \exp\left\{ -\frac{|x-y|^2}{c} \right\} \mathrm{d}x.
\end{aligned}$$

We have to show that there is a constant $N$ (independent of $\mathcal{D}$ but it can depend on $d$, $\theta$ and $c$) such that

$$A_1(\mathcal{D}) + A_2(\mathcal{D}) \leq N. \tag{B.3}$$

We note first that for any $x, y \in \mathbb{R}^d$

$$|\operatorname{dist}(x, \partial\mathcal{D}) - \operatorname{dist}(y, \partial\mathcal{D})| \leq |x-y|.$$

We will consider $A_1(\mathcal{D})$ first. For every $y \in \mathcal{D}_1^c$ we obtain

$$\begin{aligned}
\left( \frac{\operatorname{dist}(x, \partial D)}{\operatorname{dist}(y, \partial\mathcal{D})} \right)^{\theta} &= \left( \frac{\operatorname{dist}(x, \partial\mathcal{D}) - \operatorname{dist}(y, \partial\mathcal{D})}{\operatorname{dist}(y, \partial\mathcal{D})} + 1 \right)^{\theta} \\
&\leq (|x-y| + 1)^{\theta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_1(\mathcal{D}) &\leq \sup_{y\in\mathcal{D}_1^c} \int_{\mathcal{D}_1^c} (|x-y|+1)^{\theta} \exp\left\{ -\frac{|x-y|^2}{c} \right\} \mathrm{d}x \\
&\leq \sup_{y\in\mathcal{D}_1^c} \int_{\mathbb{R}^d} (|x-y|+1)^{\theta} \exp\left\{ -\frac{|x-y|^2}{c} \right\} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} (1+|z|)^{\theta} \exp\left\{ -\frac{|z|^2}{c} \right\} \mathrm{d}x < +\infty.
\end{aligned}$$

Consider now $A_2(\mathcal{D})$. Then, by similar arguments for every $y \in \mathcal{D}_1$ we obtain

$$\begin{aligned}
\operatorname{dist}(x, \partial\mathcal{D})^{\theta} &\leq (\operatorname{dist}(x, \partial\mathcal{D}) - \operatorname{dist}(y, \partial\mathcal{D}) + \operatorname{dist}(y, \partial\mathcal{D}))^{\theta} \\
&\leq (|x-y|+1)^{\theta}
\end{aligned}$$

and again

$$A_2(\mathcal{D}) \leq \int_{\mathbb{R}^d} (1+|z|)^{\theta} \exp\left\{ -\frac{|z|^2}{c} \right\} \mathrm{d}z < +\infty.$$

Combining the two estimates above we obtain (B.3) with

$$N = 2\int_{\mathbb{R}^d} (1+|z|)^{\theta} \exp\left\{ -\frac{|z|^2}{c} \right\} \mathrm{d}z < +\infty. \qquad \square$$

We are showing now the the gradient estimates. Using Assumption 4.2 ((4.2) and (4.3)) and the Jensen inequality we obtain

$$\begin{aligned}
\left|\frac{\partial S(t)\psi}{\partial x_j}\right|^p_{L^p_\theta} &= \int_{\mathcal{O}} \rho^\theta(x) \left|\int_{\mathcal{O}} \frac{\partial}{\partial x_j} G(t,x,y)\psi(y)dy\right|^p \mathrm{d}x \\
&\leq \frac{C^p}{t^{p/2}} \int_{\mathcal{O}} \rho^\theta(x) \int_{\mathcal{O}} m_t^p(y) g_{ct}(x-y)|\psi(y)|^p \mathrm{d}y\mathrm{d}x \\
&= \frac{C^p t^{d/2}}{t^{p/2}} \int_{\mathcal{O}/\sqrt{t}} \rho^\theta(x\sqrt{t}) \int_{\mathcal{O}/\sqrt{t}} m_t^p(y\sqrt{t}) g_c(x-y)|\psi(y\sqrt{t})|^p \mathrm{d}y\mathrm{d}x \\
&= \frac{C^p t^{d/2}}{t^{p/2}} \left(I_1 + I_2 + I_3 + I_4\right) ,
\end{aligned}$$

where $I_i$, $i = 1,2,3,4$ are defined in the previous section. Therefore we can use the estimates for $I_i$ and the desired conclusion follows. □


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