ASPHERICAL 4-MANIFOLDS WITH ELEMENTARY AMENABLE FUNDAMENTAL GROUP

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Abstract. We shall complement and strengthen a result of Freedman and Quinn by showing that if the fundamental group of an aspherical compact 4-manifold with boundary is elementary amenable then it is either polycyclic or is a solvable Baumslag-Solitar group. Moreover, two such manifolds are homeomorphic if and only if their peripheral group systems are equivalent, and each such manifold is the boundary connected sum of an aspherical 4-manifold with prime boundary and a contractible 4-manifold.

Let M be an aspherical topological compact 4-manifold with boundary, and let $\pi = \pi_1(M)$. In order to be able to use currently available surgery techniques to classify such manifolds we must restrict attention to elementary amenable groups π. The group is then in fact solvable, as we shall see. In [10, Theorem 11.5] it is shown that if π is polycyclic then a homotopy equivalence of such manifolds which restricts to a homeomorphism on the boundaries is homotopic to a homeomorphism. We shall extend this result to all solvable groups, and reformulate it in purely group-theoretic terms.

We shall show first that if π is elementary amenable then either it is polycyclic or it is a Baumslag-Solitar group $BS(1,m)$, with $|m| > 1$. If $\pi = 1$ or $\mathbb{Z}$ or if $\operatorname{cd}\pi = 3$ then we can characterize the possible boundary components, but when $\operatorname{cd}\pi = 2$ our results are fragmentary. We then show that aspherical 4-manifolds with elementary amenable fundamental groups are homeomorphic if and only if their peripheral group systems (the fundamental group with orientation character and the homomorphisms on π_1 induced by the inclusions of the boundary components) are equivalent. As a consequence, it follows that each such manifold is the boundary connected sum of an aspherical 4-manifold with prime boundary and a contractible 4-manifold.

Although our use of 4-dimensional surgery requires that the group π be "good", many of our arguments can easily by made in greater

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generality, and we shall do so where possible, without further comment. When M is closed, $\chi(M) = 0$ and π is polycyclic of Hirsch length 4 thn M is smoothable. Beyond this, we have no general results on the existence of smooth structures on the aspherical 4-manifolds considered here. Even the case when M is contractible and ∂M is an integral homology 3-sphere is notoriously difficult in the smooth context.

1. General remarks on the fundamental group systems of aspherical manifolds

Basic invariants of a manifold with boundary include its fundamental group, the number of boundary components, and the fundamental groups of the boundary components. We formalize this by introducing the notion of a peripheral system and then study this applied to aspherical manifolds.

Note that we will always assume our manifolds with boundary are connected, although, of course, we will not assume the boundary is connected.

Let M be a compact n-manifold with boundary $\partial M = \sqcup_{i=1}^{k} \partial_i M$. Let $\mathrm{inc}_i : \partial_i M \to M$ be the inclusion of the i-th boundary component. Choosing base points $x_0 \in M$ and $x_i \in \partial_i M$ and paths γ_i from x_0 to x_i, define the *fundamental group system of* M to be the tuple $(\pi_1 M, \pi_1 \partial_1 M, \ldots, \pi_1 \partial_k M, \pi_1(\mathrm{inc}_1), \ldots, \pi_1(\mathrm{inc}_k))$. The *orientation character* of M is the homomorphism $w = w_1(TM) : \pi_1 M \to \mathbb{Z}^\times = \{\pm 1\}$. The *peripheral system of* M is the fundamental group system together with the orientation character.

An isomorphism $(G, G_1, \ldots, G_k, j_1 : G_1 \to G, \ldots, j_k : G_k \to G, w) \to (G', G'_1, \ldots, G'_k, j'_1 : G'_1 \to G', \ldots, j'_k : G'_k \to G', w')$ of peripheral systems is a permutation $\sigma \in S_k$, group isomorphisms $\theta : G \to G'$, $\theta_i : G_i \to G'_{\sigma(i)}$, and elements $g_i \in G_i$ so that $w = w' \circ \theta$ and $\theta_i \circ j_i \circ c_{g_i} = j'_i \circ \theta_{\sigma(i)}$ where c_{g_i} is conjugation by g_i. The isomorphism class of the peripheral system is an invariant of M, independent of the choice of base points and paths.

The *cohomological dimension* of a group π is

$$\mathrm{cd}\, \pi = \sup\{n \mid H^n(\pi; A) \neq 0 \text{ for some } \mathbb{Z}\pi\text{-module } A.\}$$

A group π is a PD_n-group (a *Poincaré duality group* of dimension n) if

$$H^k(\pi; \mathbb{Z}\pi) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

and if π has type FP, that is, $\mathbb{Z}$ has a finite length resolution by finitely generated projective $\mathbb{Z}\pi$-modules. A group π is a D_n-group (a *duality* *group* of dimension n) if $H^k(\pi;\mathbb{Z}\pi)$ is zero for $k \neq n$ and if π has type FP.

If $w : \pi \to \mathbb{Z}^\times$ is a homomorphism then let $\mathbb{Z}^w$ be the $\mathbb{Z}\pi$-module which is additively the infinite cyclic group but with $(\sum a_g g)n = \sum a_g w(g)n$. If A is a $\mathbb{Z}\pi$-module, let A^w denote the $\mathbb{Z}\pi$-module $A \otimes_{\mathbb{Z}} \mathbb{Z}^w$, with the diagonal π-action.

A space is *aspherical* if its universal cover is contractible. The fundamental group of a closed aspherical n-manifold is a PD_n-group.

Lemma 1. *Let M be a compact aspherical n-manifold with fundamental group π.*

(1) $\operatorname{cd}\pi = n \iff \partial M$ is empty.
(2) $\operatorname{cd}\pi \leq n-2 \implies \partial M$ has one component.
(3) π is $PD_{n-1} \implies \partial M$ has one or two components.

Proof. (1) If ∂M is empty, then $H^n(\pi;\mathbb{F}_2) = H^n(M;\mathbb{F}_2) = H_0(M;\mathbb{F}_2) = \mathbb{F}_2 \neq 0$. If ∂M is nonempty, then

$$H^n(\pi;\mathbb{Z}\pi) = H^n(M;\mathbb{Z}\pi) = H_0(M,\partial M;\mathbb{Z}\pi) = H_0(\widetilde{M},\partial\widetilde{M};\mathbb{Z}) = 0,$$

and so $\operatorname{cd}\pi < n$.

(2) and (3) follow from the exact sequence

$$H_1(M,\partial M;\mathbb{F}_2) \to H_0(\partial M;\mathbb{F}_2) \to H_0(M;\mathbb{F}_2) \to H_0(M,\partial M;\mathbb{F}_2)$$

which is isomorphic to

$$H^{n-1}(\pi;\mathbb{F}_2) \to H_0(\partial M;\mathbb{F}_2) \to \mathbb{F}_2 \to H^n(\pi;\mathbb{F}_2)$$

by Poincaré-Lefschetz duality. □

An example to keep in mind is $X^l \times D^{n-l}$. Another example is a compact contractible manifold with boundary a homology sphere. Note that the boundary components of an aspherical manifold need not be aspherical. A compact aspherical manifold can have many boundary components, for example the cartesian product of torus with a 2-sphere with many open disks removed.

Now we move from the discussion of the number of boundary components to their fundamental groups.

Theorem 2. *Let M be a compact aspherical n-manifold with fundamental group π.*

(1) If $\operatorname{cd}\pi < n$ and $\pi \neq 1$, then $\pi_1(\mathrm{inc}_i)$ is nontrivial for all i.
(2) If $\operatorname{cd}\pi \leq n-2$, then $\pi_1(\mathrm{inc})$ is an epimorphism.
(3) If $\operatorname{cd}\pi \leq n-3$, then $\ker\pi_1(\mathrm{inc})$ is a perfect group.
(4) If π is a PD_{n-2}-group, then $\ker\pi_1(\mathrm{inc})$ has abelianization $\mathbb{Z}$.

(5) If π is a PD_{n-1}-group, then each $\ker \pi_1(\mathrm{inc}_i)$ is a perfect group and either

(a) ∂M has two components and each $\pi_1(\mathrm{inc}_i)$ is an epimorphism, or

(b) ∂M has one component, the image of $\pi_1(\mathrm{inc})$ has index two and

$$w_1 M = (w_\pi)(w_\partial) : \pi \to \mathbb{Z}^\times$$

where w_π is the orientation character of the PD_{n-1}-group π and $w_\partial : \pi \to \mathbb{Z}^\times$ is the homomorphism whose kernel is the image of $\pi_1(\mathrm{inc})$.

Proof. For all k,

$$\overline{H^k(\pi; \mathbb{Z}\pi)} \cong \widetilde{H}_{n-k-1}(\partial\widetilde{M}). \tag{1}$$

This follows from Poincaré-Lefschetz duality:

$$\overline{H^k(\pi; \mathbb{Z}\pi)} \cong H_{n-k}(M, \partial M; \mathbb{Z}\pi),$$

the identification $H_{n-k}(M, \partial M; \mathbb{Z}\pi) = H_{n-k}(\widetilde{M}, \partial\widetilde{M})$, and the long exact sequence of the pair $(\widetilde{M}, \partial\widetilde{M})$. The isomorphism is an isomorphism of left $\mathbb{Z}\pi$-modules, where the overbar indicates the conjugate module, defined in terms of the involution $g \mapsto w(g)g^{-1}$.

(1) Take $k = 0$. Note that since $M = K(\pi, 1)$ is finite-dimensional, π is torsionfree. If $\pi \neq 1$, then π is infinite so $H^0(\pi; \mathbb{Z}\pi) = (\mathbb{Z}\pi)^\pi = 0$. Thus $H_{n-1}(\partial\widetilde{M}) = 0$. Hence $\partial\widetilde{M}$ has no closed components, as we can see by either replacing $\mathbb{Z}$ by $\mathbb{F}_2$ or passing to the orientation double cover of M.

(2) Take $k = n - 1$. If $\mathrm{cd}\,\pi < n - 1$, then $\widetilde{H}_0(\partial\widetilde{M}) = 0$, and hence $\pi_1(\mathrm{inc})$ is an epimorphism.

(3),(4) Take $k = n - 2$. If $\mathrm{cd}\,\pi \leq n - 3$, then $H_1(\partial\widetilde{M}) = 0$ and if π is a PD_{n-2}-group, $\mathbb{Z} \cong H_1(\partial\widetilde{M})$. In both cases, $H_1(\partial\widetilde{M})$ is the abelianization of $\ker(\pi_1(\mathrm{inc}))$.

(5) Take $k = n-1, n-2$. Taking $k = n-1$ we see $\widetilde{H}_0(\partial\widetilde{M}) = \mathbb{Z}$, and thereby we see that either ∂M has two components with $\pi_1(\mathrm{inc}_i)$ an epimorphism or that ∂M has one component and the image of $\pi_1(\mathrm{inc})$ has index 2. In either case, taking $k = n-2$, we see that $H_1(\partial\widetilde{M}) = 0$, so that maps $\pi_1(\mathrm{inc}_i)$ have perfect kernels.

Finally, assume M is a compact, aspherical n-manifold with fundamental group π, which is a PD_{n-1}-group, and that ∂M has one component with $\rho := \mathrm{image}(\pi_1(\mathrm{inc}))$ having index 2 in π. Consider the exact sequence of $\mathbb{Z}\pi$-modules

$$0 \to H_1(M, \partial M; \mathbb{Z}\pi) \to H_0(\partial M; \mathbb{Z}\pi) \xrightarrow{\beta} H_0(M; \mathbb{Z}\pi).$$

The map β is isomorphic to the quotient map $\mathbb{Z}\pi\otimes_{\mathbb{Z}\rho}\mathbb{Z} \to \mathbb{Z}\pi\otimes_{\mathbb{Z}\pi}\mathbb{Z}$ and so thus $H_1(M, \partial M; \mathbb{Z}\pi) \cong \ker\beta \cong \mathbb{Z}^{w_\partial}$. By Poincaré-Lefschetz duality, $H_1(M, \partial M; \mathbb{Z}\pi)^{w_1 M} \cong H^{n-1}(M; \mathbb{Z}\pi)$. But $H^{n-1}(M; \mathbb{Z}\pi) \cong \mathbb{Z}^{w_\pi}$ by the definition of w_π. Thus $(\mathbb{Z}^{w_\pi})^{w_1 M} \cong \mathbb{Z}^{w_\partial}$. Hence $w_1 M = (w_\pi)(w_\partial)$. □

In particular, if $\operatorname{cd}\pi \le n-2$ or if π is a PD_{n-1}-group and ∂M has two components then ∂M is orientable if and only if M is orientable. If $n \ge 4$ there are examples of each type considered in the theorem. In case (5)(b) we may take a closed aspherical $(n-1)$-manifold X with a double cover $\widehat{X} \to X$ and set $M = \widehat{X} \times_{\mathbb{Z}/2} [-1, 1]$. Note that in this case M and X must have different orientation characters.

2. Fundamental group systems of aspherical 4-manifolds

The work of Waldhausen on P^2-irreducible, sufficiently large 3-manifolds and the Geometrization Theorem of Perelman together imply that aspherical 3-manifolds are determined up to homeomorphism by their peripheral systems. In Theorem 14 we shall show that a similar result holds for 4-manifolds, provided that $\pi_1(M)$ is elementary amenable.

Lemma 3. *Let N be a closed 3-manifold such that $\nu = \pi_1(N)$ has an infinite perfect normal subgroup κ. If $G = \nu/\kappa$ has one end, then G is a PD_3-group. Furthermore $H_1(\kappa)$ and $H_2(\kappa)$ both vanish.*

Proof. Let $N_\kappa \to N$ be the cover with $\pi_1(N_\kappa) = \kappa$, and let $C_*(N; \mathbb{Z}G) = C_*(N_\kappa)$ be the cellular chain complex for N_κ, considered as a complex of finitely generated free left $\mathbb{Z}G$-modules. We first claim that N_κ is acyclic, i.e. has the homology of a point. Note that $H_1(N_\kappa) = \kappa/\kappa' = 0$. Note that $H_2(N_\kappa) = H_2(N; \mathbb{Z}G) \cong \overline{H^1(N; \mathbb{Z}G)} = \overline{H^1(G; \mathbb{Z}G)} = 0$ since G has one end (see section 3 of [23]). Hence also $H_2(\kappa) = 0$. Next note that $H_i(N_\kappa) = 0$ for $i \ge 3$, since N_κ is a connected, noncompact 3-manifold.

Hence $C_*(N; \mathbb{Z}G)$ is a finite free resolution for $H_0(N; \mathbb{Z}G) = \mathbb{Z}$ as a $\mathbb{Z}G$-module. Since $\overline{H^3(G; \mathbb{Z}G)} \cong \overline{H^3(N; \mathbb{Z}G)} \cong H_0(N; \mathbb{Z}G) = \mathbb{Z}$, we see that G is a PD_3-group. □

Note that solvable Poincaré duality groups are polycyclic [3].

The Baumslag-Solitar group $BS(1, m)$ is the group with presentation $\langle a, t \mid tat^{-1} = a^m\rangle$. (We shall assume $m \ne 0$.) These groups are semidirect products $\mathbb{Z}[1/m] \rtimes \mathbb{Z}$ and include the fundamental group of the torus $\mathbb{Z}^2 = BS(1, 1)$ and of the Klein bottle $BS(1, -1)$. For $m \ne \pm 1$, they are examples of solvable groups which are not polycyclic.

Recall that the class of *elementary amenable groups* is the smallest class of groups containing all finite and all abelian groups and which is closed under subgroups, quotients, extensions, and directed unions.

Theorem 4. *Let M be an compact aspherical 4-manifold such that $\pi = \pi_1(M)$ is elementary amenable. Then one of the following conditions holds*

(1) $\pi = 1$ *and* ∂M *is an homology* 3*-sphere;*
(2) $\pi \cong \mathbb{Z}$ *and* $\pi_1(\mathrm{inc}) : \pi_1(\partial M) \to \pi_1 M$ *is an epimorphism with perfect kernel;*
(3) $\pi \cong BS(1, m)$ *for some* $m \neq 0$ *and* $\pi_1(\mathrm{inc})$ *is an epimorphism, with kernel* κ *such that* $H_1(\kappa) \cong \overline{H^2(\pi; \mathbb{Z}\pi)}$*;*
(4) π *is a polycyclic* PD_3*-group and either* ∂M *has two components and* $\pi_1(\mathrm{inc}_i)$ *is an epimorphism for* $i = 1, 2$*, or* ∂M *is connected and* $[\pi : \mathrm{Im}(\pi_1(\mathrm{inc}))] = 2$*, and in each case the kernels are perfect;*
(5) π *is a polycyclic* PD_4*-group and* M *is closed.*

Proof. Since M is an aspherical 4-manifold, $\mathrm{cd}\,\pi \leq 4$, with equality if and only if M is closed. Since M is compact, aspherical, π has type FP. Since π is elementary amenable and $\mathrm{cd}\,\pi < \infty$, it is virtually solvable [16]. A virtually solvable group which is FP is constructible, and is a duality group, with Hirsch length $h(\pi) = \mathrm{cd}\,\pi$ [18].

The trivial group is the only group with cohomological dimension 0.

If $\mathrm{cd}\,\pi = 1$ then π is a nontrivial free group, and since π is elementary amenable, $\pi \cong \mathbb{Z}$.

The only finitely generated solvable groups with cohomological dimension 2 are the Baumslag-Solitar groups $BS(1, m)$ with $m \neq 0$ [13]. If π is virtually solvable and $\mathrm{cd}\,\pi = 2$ then π has a normal subgroup K of finite index which is a Baumslag Solitar group. Hence $\chi(\pi) = [\pi : K]^{-1}\chi(K) = 0$, and so $Hom(\pi, \mathbb{Z}) = H^1(\pi; \mathbb{Z}) \neq 0$. The kernel of an epimorphism from π to $\mathbb{Z}$ is torsion-free, and virtually abelian of rank 1. Hence it is abelian, and so π is solvable, and thus is also a solvable Baumslag-Solitar group.

If $\mathrm{cd}\,\pi = 3$ then $h(\pi) = 3$ and $H^k(\pi; \mathbb{Z}\pi) = 0$ for $k \leq 2$ since π is a duality group. Hence, by (1), $H_1(\partial\widetilde{M})$, $H_2(\partial\widetilde{M})$, and $H_3(\partial\widetilde{M})$ all vanish. Since $H_1(\partial\widetilde{M}) = 0$, each $\ker(\pi_1(\mathrm{inc}_i))$ is perfect. The vanishing of H_3 implies that each $G_i = \mathrm{image}(\pi_1(\mathrm{inc}_i))$ is infinite (in the nonorientable case either replace $\mathbb{Z}$ by $\mathbb{F}_2$ or pass to the 2-fold orientation double cover).

An LHS spectral sequence argument shows that $H^1(\pi_i; \mathbb{Z}[\pi]) \cong H^1(G_i; \mathbb{Z}[\pi])$, since $\kappa_i = \ker(\pi_1(\mathrm{inc}_i))$ is perfect and so $Hom(\kappa_i; \mathbb{Z}[\pi]) = 0$. Hence if $G_i \cong \mathbb{Z}$ then $H^1(\pi_i;\mathbb{Z}[\pi]) \cong \mathbb{Z}[G_i\backslash\pi] \neq 0$, and so $H_2(\partial_i M;\mathbb{Z}\pi_i) = H^1(\partial_i M;\mathbb{Z}\pi_i) = H^1(\pi_i;\mathbb{Z}\pi_i) \neq 0$. Therefore the images G_i are non-cyclic finitely generated torsion-free virtually solvable groups, and so each have one end. Hence they are PD_3-groups, by Lemma 3. Solvable Poincaré duality groups are polycyclic [3], and virtually polycyclic PD_3-groups are polycyclic.

A constructible solvable group is a group which can be built up from the trivial group by a finite sequence of finite extensions and ascending HNN extensions [4]. It follows by an induction on the Hirsch length that if $K \leq H$ are finitely generated solvable groups such that K is polycyclic, H is constructible and $h(K) = h(H)$ then $[H : K] < \infty$. Therefore $[\pi : G_i] < \infty$, so π is polycyclic also. Hence π is a 3-manifold group [7].

The final case is when M is closed and $\operatorname{cd}\pi = 4$, and then π is a virtually solvable Poincaré duality group. Hence it is virtually polycyclic. Such PD_4-groups are essentially known, in so far as the classification can be largely reduced to questions of conjugacy in $GL(3,\mathbb{Z})$ (and related groups) [15, Chapter 8]. Inspection of the possible groups shows that all are solvable, and thus polycyclic. (This case is well-known, and is included here only for completeness.)

The assertions about the boundary components and the homomorphisms $\pi_1(\mathrm{inc}_i)$ follow from the results of the previous section. □

The following list of simple examples of aspherical 4-manifolds with non-empty boundary includes examples with polycyclic groups of each cohomological dimension ≤ 3. (It also includes examples whose groups have non-cyclic free subgroups, and thus for which the DET has not been proven.)

(1) $\pi = 1$: let $M = D^4$, with $\partial M = S^3$;
(2) $\pi = F(r)$: let $M = \natural^r S^1 \times D^3$, with $\partial M = \sharp^r S^1 \times S^2$;
(3) $\pi = \pi_1(T_g)$, T_g a closed surface of genus $g \geq 1$: let M be the total space of a D^2-bundle over T_g, with ∂M the associated S^1-bundle;
(4) $\pi = \pi_1(N)$, N an aspherical closed 3-manifold: let $M = M(\eta)$ be the total space of the I-bundle over N induced by $\eta \in H^1(\pi;\mathbb{Z}/2\mathbb{Z})$. Then $\partial M = N \times \{0,1\}$ if the bundle is trivial and ∂M a connected 2-fold covering space of N otherwise.

The examples of type (4) include all the possibilities with π a PD_3-group and all the $\pi_1(\mathrm{inc}_i)$ injective. For then $\pi_1(\partial_i M)$ is also a PD_3-group, since it has finite index in π, by Theorem 2. Hence $\partial_i M$ is aspherical. It then follows from Mostow Rigidity and the Geometrization Theorem (via [20, 24]) that π is also a 3-manifold group.

3. CRITERIA FOR ASPHERICITY

It is easy to that a closed 4-manifold M is aspherical if and only if π has one end and $\pi_2(M) = 0$. In the bounded case the condition on ends must be modified, as the boundary connected sum of aspherical manifolds with boundary is again aspherical.

Theorem 5. *Let M be a compact 4-manifold with $\partial M = \sqcup_{i=1}^{k} \partial_i M$. Assume that $\pi = \pi_1(M) \neq 1$, and let $\Gamma = \mathbb{Z}[\pi]$. Then M is aspherical if and only if $\nu_i = \pi_1(\partial_i M)$ has infinite image in π, for $i \leq k$, the inclusions induce an isomorphism $H^1(\pi;\Gamma) \cong \oplus H^1(\nu_i;\Gamma)$, and $\pi_2(M) = 0$. If $\operatorname{cd}\pi \leq 2$ then M is aspherical if and only if $\partial M \neq \emptyset$, $(M, \partial M)$ is 1-connected and $\chi(M) = \chi(\pi)$.*

Proof. If M is aspherical then $H_i(M;\Gamma) = 0$ for all $i \geq 1$. In particular, $\pi_2(M) = 0$. If $\pi \neq 1$ then π is infinite, so $H^0(M;\Gamma) = 0$. Hence $H_3(\partial M;\Gamma) = 0$, and so each ν_i has infinite image in π. Since $H^1(M, \partial M;\Gamma) \cong H_3(M;\Gamma) = 0$ and $H^2(M, \partial M;\Gamma) \cong H_2(M;\Gamma) = 0$, the inclusion of the boundary induces an isomorphism $H^1(\pi;\Gamma) \to \oplus H^1(\nu_i;\Gamma)$.

Conversely, the first two of these conditions imply that $H^1(M, \partial M;\Gamma) = 0$ and $H_3(\partial M;\Gamma) = 0$, and so $H_3(M;\Gamma) = 0$. Since π is infinite $H_4(M;\Gamma) = 0$. Thus if also $\pi_2(M) = 0$ then M is aspherical.

Suppose now that $\operatorname{cd}\pi \leq 2$. If M is aspherical then $\partial M \neq \emptyset$, $\chi(M) = \chi(\pi)$, and $(M, \partial M)$ is 1-connected, by Theorem [REF]. Conversely, if these conditions hold then $H^3(M;\Gamma) = H_1(M, \partial M;\Gamma) = 0$, by Poincaré duality and the fact that $(M, \partial M)$ is 1-connected. Since $\partial M \neq \emptyset$ the cellular chain complex $C_*(M;\Gamma)$ is chain homotopy equivalent to a finite projective chain complex of length ≤ 3. Two applications of Schanuel's Lemma show that

(1) Z_1 is projective and $C_2 \cong Z_1 \oplus Z_2$, where $Z_i \leq C_i$ is the submodule of i-cycles; and

(2) $H_3(C_*) = H_3(M;\Gamma)$ is projective.

Now $Hom_\Gamma(H_3(M;\Gamma), \Gamma) = 0$, since it is a quotient of $H^3(M;\Gamma) = 0$, by the Universal Coefficient spectral sequence. Since $H_3(M;\Gamma)$ is projective, it must then be 0. Hence $H_3(C_*) = H^3(C_*) = 0$, and so we may assume that C_* has length ≤ 2. Hence $C_* \cong D_* \oplus \Pi$ where D_* is a projective resolution of $\mathbb{Z}$ and $\Pi = Z_2$ is a projective module concentrated in degree 2. On tensoring with $\mathbb{Z}$ and using the condition $\chi(M) = \chi(\pi)$, we see that $\mathbb{Z} \otimes_\Gamma \Pi = 0$. Since $\operatorname{cd}\pi \leq 2$, the weak Bass conjecture holds for π [6], and so $\Pi = 0$. Hence M is aspherical. □

Every finite 2-complex is homotopy equivalent to a compact 4-manifold with boundary, and so every group G with a finite 2-dimensional $K(G,1)$

complex is realized in this way. However, it remains an open question whether every finitely presentable group G with $\mathrm{cd}\, G = 2$ has such a $K(G, 1)$-complex.

4. Realization of boundaries: PD_4-pairs

Each of the polycyclic groups π allowed by Theorem 4 is the fundamental group of an aspherical 4-manifold with boundary, and we shall see in §6 below that the other Baumslag-Solitar groups are also realizable. We would like to show that every 3-manifold N and homomorphism from $\pi_1(N)$ (or pair of such data) compatible with Theorem 4 can be realized as the boundary of an aspherical 4-manifold. We shall show here that this is so on the homotopy level.

The case when $\pi = 1$ is well understood. The manifold M must be contractible, and taking the boundary gives a bijective correspondence between compact contractible TOP 4-manifolds and homology 3-spheres [10]. More generally, if M is a 4-manifold with non-empty boundary then taking boundary connected sum with a contractible 4-manifold does not change π (or the homotopy type of M) but changes ∂M by connected sum with an homology 3-sphere. (We could insist that ∂M be prime. See §6 below.)

Lemma 6. *Let π be a finitely presentable group, N a 3-manifold, and $w : \pi \to \mathbb{Z}^\times$ and $p : \nu = \pi_1(N) \to \pi$ be homomorphisms such that $w_1(N) = wp$. Let X be the mapping cylinder of the map from N to $K(\pi, 1)$ corresponding to p, and let $\kappa = \mathrm{Ker}(p)$. Suppose that one of the following conditions holds.*

(1) $\mathrm{cd}\, \pi = 1$, p is an epimorphism and κ is perfect;
(2) $\mathrm{cd}\, \pi = 2$, π has one end, p is an epimorphism and $\kappa/\kappa' \cong H^2(\pi; \mathbb{Z}[\pi])$; or
(3) π is a PD_3-group, $[\pi : p(\nu)] = 2$ and κ is perfect.

Then the pair (X, N) is a PD_4-pair.

Proof. Since $H_4(X; \mathbb{Z}^w) = H_3(X; \mathbb{Z}^w) = 0$, the connecting homomorphism from $H_4(X, N; \mathbb{Z}^w)$ to $H_3(N; \mathbb{Z}^w)$ is an isomorphism. Cap product with a generator $[X] \in H_4(X, N; \mathbb{Z}^w)$ and its image $[N]$ in $H_3(N; \mathbb{Z}^w)$ gives rise to a commuting diagram with rows the exact sequences of cohomology and homology for (X, N), with coefficients $\mathbb{Z}[\pi]$.

Since N is non-empty , $H_0(X, N; \mathbb{Z}[\pi]) = 0$, and since X is aspherical, $H_i(X, N; \mathbb{Z}[\pi]) = H_{i-1}(N; \mathbb{Z}[\pi])$, for $i \geq 2$. If p is an epimorphism then we also have $H_1(X, N; \mathbb{Z}[\pi]) = 0$, while if $[\pi : p(\nu)] = 2$ then $H_1(X, N; \mathbb{Z}[\pi]) \cong \mathbb{Z}$. We may then use homological algebra (and Poincaré duality for N), to determine the cohomology groups.

In each case, many of the terms are 0, so the diagram reduces largely to a collection of commuting squares in which the horizontal maps are isomorphisms, and the vertical maps are

$$-\cap [X] : \overline{H^j(X;\mathbb{Z}[\pi])} \to H_{4-j}(X,N;\mathbb{Z}[\pi])$$

and

$$-\cap [N] : \overline{H^j(N;\mathbb{Z}[\pi])} \to H_{3-j}(N;\mathbb{Z}[\pi]).$$

(When p is not an epimorphism the situation is slightly more complicated at one end of the diagram.)

We shall give further details only for case (2), when $\mathrm{cd}\,\pi = 2$ and π has one end. (The other cases are similar but easier.) Then $H^k(X;\mathbb{Z}[\pi]) = 0$ for $k > 2$. The augmentation $\mathbb{Z}[\pi]$-module $\mathbb{Z}$ has a finitely generated projective resolution of length two. Dualizing this gives a resolution for $H^2(\pi;\mathbb{Z}[\pi])$, and on dualizing again we see that

$$Hom_{\mathbb{Z}[\pi]}(\overline{H^2(\pi;\mathbb{Z}[\pi])}, \mathbb{Z}[\pi]) = 0$$

and

$$Ext^1_{\mathbb{Z}[\pi]}(\overline{H^2(\pi;\mathbb{Z}[\pi])}, \mathbb{Z}[\pi]) = 0.$$

Hence $H_3(X,N;\mathbb{Z}[\pi]) = H_2(N;\mathbb{Z}[\pi]) = 0$ and a homological argument then shows that $H^i(X,N;\mathbb{Z}[\pi]) = 0$, for $i \leq 3$.

Since N satisfies Poincaré duality with fundamental class $[N]$, the result follows. □

The hypothesis in (2) that π has one end is probably not necessary, but holds if $\pi \cong BS(1,m)$, and simplifies the exposition.

A similar argument applies if π is a PD_3-group, and we have two 3-manifolds N_i and epimorphisms $p_i : \nu_i = \pi_1(N_i) \to \pi$ with perfect kernels and such that $w_1(N_i) = wp_i$ for $i = 1,2$.

It remains for us to show that there are degree-1 normal maps $(M,N) \to (X,N)$ with M a manifold and trivial surgery invariant.

5. Realization of boundaries: $\pi = \mathbb{Z}$

In this section we shall show that if $\pi \cong \mathbb{Z}$ then any closed 3-manifold satisfying the conditions of Theorem 4 bounds such a 4-manifold.

We may separate the underlying question into two parts.

(1) Given a 3-manifold N and a homomorphism $p : \nu = \pi_1(N) \to \pi$, compatible with the conditions of Theorem 4, is N a boundary component of some 4-manifold M with $\pi_1(M) \cong \pi$?
(2) If so, can we choose M to be aspherical?

We shall use low-dimensional surgery to tackle these questions. This is straightforward when $\pi = \mathbb{Z}$ or $\mathrm{cd}\,\pi = 3$ (as in Theorems 7 and 12), but some questions remain when $\mathrm{cd}\,\pi = 2$.

In dimension 3 we use the modification of the surgery exact sequence for low dimensions given in [17], in which elements of the structure set $\overline{\mathcal{S}}^{TOP}(N)$ for a 3-manifold N with $\pi_1(N) = \pi$ are represented by $\mathbb{Z}[\pi]$-homology equivalences with trivial Whitehead torsion, and two such are equivalent if there is a normal cobordism between them with surgery obstruction 0 in $L_4(\pi, w)$.

When $\pi \cong \mathbb{Z}$ then $H_1(\partial M; \mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(\partial M)'$ is perfect. (More generally, if $\mathrm{cd}\,\pi = 1$ then π is free, $\pi_1(\mathrm{inc})$ is an epimorphism with perfect kernel, and $\pi_1(\partial M) \cong P \rtimes \pi$ with P perfect.)

Theorem 7. *Let N be a 3-manifold such that $\nu = \pi_1(N) \cong \kappa \rtimes \mathbb{Z}$, where $\kappa = \nu'$ is perfect. Then there is a 4-manifold $M \simeq S^1$ with $\partial M \cong N$. The homotopy class of* inc *is essentially unique.*

Proof. Suppose first that N is orientable, and let $f : N \to S^1$ and $g : N \to S^2$ be maps corresponding to the generators of $H^1(N; \mathbb{Z})$ and $H^2(N; \mathbb{Z})$, respectively. Then $h = (f, g) : N \to S^1 \times S^2$ has degree 1, and so is a Λ-homology equivalence, where $\Lambda = \mathbb{Z}[\mathbb{Z}]$.

If N is non-orientable, the infinite cyclic covering N' is homotopy equivalent to a 2-complex with the homology of S^2, and so there is a map $u : N' \to S^2$ which induces isomorphisms on cohomology. Let t generate $Aut(N'/N) \cong \mathbb{Z}$. Then ut is homotopic to Au, where A is the antipodal map, since N is non-orientable. Hence u gives rise to a map h from the mapping torus $M(t) \simeq N$ to $M(A) \simeq S^1 \tilde{\times} S^2$, which is easily seen to be a Λ-homology equivalence.

If N is orientable then N and $S^1 \times S^2$ are parallelizable. If N is non-orientable then $w_1(N) = h^* w_1(S^1 \tilde{\times} S^2)$ (and hence also $w_2(N) = h^* w_2(S^1 \tilde{\times} S^2)$). The difference of stable normal bundles $\nu_N \oplus h^* \tau_{S^1 \tilde{\times} S^2}$ is a stable *Spin*-bundle over a 3-manifold, and so is trivial. Thus in each case h extends to a normal map. The surgery invariant is 0 since it is the obstruction to surgering h to a Λ-homology equivalence. Hence h is normally bordant to $id_{S^1 \times S^2}$ or $id_{S^1 \tilde{\times} S^2}$. (See also [10, page 207].)

If $H : W \to S^1 \times S^2 \times [0, 1]$ (or $S^1 \tilde{\times} S^2 \times [0, 1]$) is a normal bordism then we may modify it by taking connected sums with a standard normal map from the E_8 manifold to S^4 to obtain a normal cobordism with surgery obstruction 0 in $L_4(\mathbb{Z}) = L_4(1) = \mathbb{Z}$ (or $L_4(\mathbb{Z}^-) = \mathbb{Z}/2\mathbb{Z}$). We may then perform surgery *rel* ∂ to obtain a Λ-homology-cobordism $\widehat{W}$ with $\pi_1(\widehat{W}) \cong \mathbb{Z}$. Let $M = \widehat{W} \cup S^1 \times D^3$ (or $\widehat{W} \cup S^1 \tilde{\times} D^3$). Then $M \simeq S^1$ and $\partial M \cong N$. Since $H_1(\mathrm{inc})$ is an isomorphism, the homotopy class of inc is a generator of $[\partial M, M] \cong H^1(\partial M; \mathbb{Z}) \cong \mathbb{Z}$. $\square$

Example. Let K be a knot which bounds a smooth slice disc Δ such that $\pi_1(D^4 \setminus \Delta) \cong \mathbb{Z}$. Let $X(\Delta)$ be the complement of an open regular neighbourhood of Δ in D^4. Then $X(\Delta)$ is aspherical, by Theorem 5, and so $X(\Delta) \simeq S^1$, while $\partial X(\Delta) \cong M(K, 0)$, the 3-manifold obtained by 0-framed surgery on K. If K is non-trivial (e.g., if K is the Kinoshita-Terasaka 11-crossing knot – see [14, page 23]) then $M(K; 0)$ is also aspherical [11].

6. Realization of boundaries: $\mathrm{cd}\,\pi = 2$

The case $\mathrm{cd}\,\pi = 2$ presents the greatest difficulties for us, and we do not yet have a clear picture of the possible boundaries, even in the polycyclic case $\pi = \mathbb{Z}^2$ or $\pi_1(Kb)$. (The problem may be related to the fact that $\mathrm{Ker}(\pi_1(\mathrm{inc}))$ is not perfect.)

Every group π with a finite 2-dimensional $K(\pi, 1)$-complex is the fundamental group of an aspherical 4-manifold, for the cellular structure of the 2-complex provides a model for constructing a 4-dimensional handlebody M of the same homotopy type. If $w : \pi \to \mathbb{Z}^\times$ is a homomorphism, we may assume that the 1-handles corresponding to generators g are orientable if and only if $w(g) = 1$, and then $w_1(M) = w$. The boundary of such a 4-manifold is a closed 3-manifold N such that $\nu = \pi_1(N)$ maps onto π with kernel κ such that $\kappa/\kappa' \cong \overline{H^2(\pi; \mathbb{Z}\pi)}$ as left $\mathbb{Z}\pi$-modules. (Conjugation in ν determines an action $\alpha : \pi \to Out(\kappa)$ and hence a module structure on the abelianization.)

In particular, $\pi = BS(1, m)$ is the fundamental group of an aspherical 4-manifold with boundary. The simplest examples of bounded 4-manifolds with such groups have Kirby calculus diagrams based on the presentations given in §1 above, with a 3-component link having two dotted components representing the generators, and the third component with integral framing. Since the 2-complexes corresponding to these presentations are aspherical, so are the resulting 4-manifolds.

When $m = 1$ (i.e., $\pi = \mathbb{Z}^2$) we may take the link to be the Borromean rings. Varying the framing on the third component gives the different total spaces $E_n(\mathbb{Z}^2, 1)$ of orientable D^2-bundles over the torus.

When $m = 2$ and the framing is 0 the boundary is an homology $S^2 \times S^1$. There is an alternative construction of such an example.

Example. The stevedore's knot 6_1 bounds an "obvious" ribbon disc in S^3. (Give the standard ribbon disc with boundary the reef knot a half-twist, between the two throughcuts.) Let $\Delta \subset D^4$ be the slice disc obtained from this ribbon disc, and let $X(\Delta)$ be the exterior of a regular neighbourhood of Δ. Then $X(\Delta)$ has a handlebody structure with two 1-handles and one 2-handle and $\pi_1(X(\Delta))$ is the "ribbon group" associated to the ribbon disc, as defined in [14, Chapter 1]. Calculation shows that $\pi_1(X(\Delta)) \cong BS(1,2)$, and so $X(\Delta)$ is aspherical, by Theorem 5. In this case $\partial X(\Delta)$ is the 3-manifold obtained by 0-framed surgery on the knot 6_1.

Suppose that $\mathrm{cd}\,\pi = 2$, N is a 3-manifold and $p : \nu = \pi_1(N) \to \pi$ is an epimorphism such that $w_1(N) = wp$ and $\kappa = \mathrm{Ker}(p)$ satisfy the condition $\kappa/\kappa' \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$ of Theorem 4. (If π is solvable we may use Lemma 11 to reduce to the case when N is aspherical.) Let (X,N) be the PD_4-pair constructed in Lemma 6. If (X,N) is reducible, there is a degree-1 normal map $(M,N) \to (X,N)$ with normal invariant in

$$[X/\partial, G/TOP] \cong H_2(\pi;\mathbb{Z}/2\mathbb{Z}) \oplus H_0(\pi;\mathbb{Z}^w).$$

We may modify M by connected sum with copies of the E_8-manifold so that the image of the normal invariant in the second component is 0. If $\pi \cong BS(1,m)$ with m odd then $H_2(\pi;\mathbb{Z}/2\mathbb{Z})$ has order 2, and maps nontrivially to $L_4(\pi,w)$. If m is even this group is 0, and so N bounds an aspherical 4-manifold with fundamental group $\pi \cong BS(1,m)$.

The condition $\kappa/\kappa' \cong \overline{H^2(\pi;\mathbb{Z}\pi)}$ is not transparent, but may be testable.

Theorem 8. *If π has trivial centre there is at most one extension by κ corresponding to a given homomorphism $\alpha : \pi \to Out(\kappa)$.*

Proof. The action α induces a $\mathbb{Z}[\pi]$-module structure on $\zeta\kappa$, and the extensions of π by κ with action α are classified by $H^2(\pi;\zeta\kappa^\alpha)$.

If the centre $\zeta\kappa$ is non-trivial then $\nu = \pi_1(N)$ has a non-trivial abelian normal subgroup A. Since $\mathrm{cd}\,\pi = 2$ and π has trivial centre, π is not polycyclic. Hence ν is not polycyclic either, and so N is Seifert fibred. We may assume that M is orientable. Since $m \neq 1$, $H^2(\pi;\mathbb{Z})$ is finite, and so $\beta_1(N) = \beta_1(M) = 1$. It then follows from standard properties of Seifert 3-manifolds that A is central in π, and $A \cap \pi' = 1$. Hence the image of A in π/π' is infinite. On the other hand, the image of A in π is central, since $\pi_1(\mathrm{inc})$ is onto, and so is trivial. But this is a contradiction, and so $\zeta\kappa = 1$. The theorem follows. ∎

Theorem 8 applies if $\pi \cong BS(1,m)$ for some $m \neq \pm 1$. By contrast, when $m = \pm 1$, so π is a PD_2-group and $\kappa = \mathbb{Z}$, then $\alpha = w_1(M) - w_1(\pi)$, and the possible extensions are parametrized by $H^2(\pi;\mathbb{Z}^\alpha)$.

Lemma 9. *Let N be a 3-manifold such that $\nu = \pi_1(N)$ is an extension of a finitely presentable group π with $\mathrm{cd}\,\pi = 2$ by a normal subgroup κ such that $H_1(\kappa;\mathbb{Z}) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$. Then the following are equivalent*

(1) $\kappa \cong \mathbb{Z}$;

(2) κ *is finitely generated;*
(3) κ *is solvable.*

Proof. We note first that $H^2(\pi;\mathbb{Z}[\pi])$ is nontrivial and torsion-free as an abelian group, since π is finitely presentable and $\mathrm{cd}\,\pi = 2$, and so κ is infinite. Clearly (1) $\Rightarrow$ (2) and (3).

If κ is finitely generated then so is $H^2(\pi;\mathbb{Z}[\pi])$, as an abelian group. Hence π is a PD_2-group [8]. Since κ must be infinite, ν has one end, and so ν is a PD_3-group. Hence $\kappa \cong \mathbb{Z}$ [15, Theorem 1.19].

If κ is solvable then ν again has one end, and so is a PD_3-group. Since κ is a nontrivial solvable normal subgroup, ν must be either the fundamental group of a Seifert fibred 3-manifold or polycyclic; in either case, κ is finitely generated. □

If π is a PD_2-group then $\kappa/\kappa' \cong \mathbb{Z}$. Hence $\nu/\kappa' \cong \pi_1(N_n)$, where N_n is the total space of an S^1-bundle over $F = K(\pi,1)$, with Euler invariant $n \in H^2(\pi;\mathbb{Z}^u)$, where $u : \pi \to \mathbb{Z}^\times$ is the action of π on κ/κ'. (Note that $H^2(\pi;\mathbb{Z}^u) \cong \mathbb{Z}$ if $u = w_1(\pi)$, and otherwise has order 2.) Let $E_n(\pi,u)$ be the total space of the associated D^2-bundle over F. Then $\partial E_n(\pi,u) = N_n$.

Suppose now that N and π are orientable. Since $\mathrm{cd}\,\pi = 2$ and the oriented bordism groups Ω_i are 0 for $1 \leq i \leq 3$, it follows from the Atiyah-Hirzebruch spectral sequence that $\Omega_3(K(\pi,1)) = 0$. Thus there is an orientable 4-manifold M with boundary N and a map $g : M \to E_n(\pi,u)$ such that $\pi_1(g \circ \mathrm{inc}) = p$. After elementary surgery on $\mathrm{Ker}(\pi_1(g))$ we may arrange that $\pi_1(M) \cong \pi$. Since $g \circ \mathrm{inc}$ factors through N_n, we get a map of pairs $h : (M,N) \to (E_n(\pi,u), N_n)$ which is 2-connected and is easily seen to be a $\mathbb{Z}[\pi]$-homology equivalence of the boundaries.

If the restriction $h|_N : N \to N_n$ is a $\mathbb{Z}[\pi_1(N_n)]$-homology equivalence then the argument of Lemma 10 (below) applies. In general we cannot expect the strategy used in Theorem 7 (above) for $\pi = \mathbb{Z}$ and in Lemma 10 for π a polycyclic PD_3-group should apply when $\mathrm{cd}\,\pi = 2$. For example, let K be a knot with non-trivial Alexander polynomial, and let L be the 3-component link obtained by tying K into one of the components of the Borromean rings. Let M be the 4-manifold with Kirby calculus presentation given by the link L with the two trivial components dotted and the knotted component having framing 0. Then $M \simeq T_1 = K(\mathbb{Z}^2,1)$ and $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}^3$. Hence there is an esssentially unique degree-1 map $h : \partial M \to T^3 = K(\mathbb{Z}^3,1)$. However, h is not a $\mathbb{Z}[\mathbb{Z}^3]$-homology equivalence.

7. REALIZATION OF BOUNDARIES: $\operatorname{cd}\pi = 3$

If π is the group of an aspherical 3-manifold N_0 then the inclusion of ∂M into M factors through some $\partial M(\eta)$, up to homotopy, and so there is a map

$$F : (M, \partial M) \to (M(\eta), \partial M(\eta)).$$

such that $F : M \to M(\eta)$ is a homotopy equivalence and $F|_{\partial M}$ is a $\mathbb{Z}\pi$-homology equivalence. In this case we may again use the strategy of Theorem 7.

In the following lemma we use the modification of the surgery exact sequence for low dimensions given in [17], in which elements of the structure set $\overline{\mathcal{S}}^{TOP}(N)$ for a 3-manifold N with $\pi_1(N) = \pi$ are represented by $\mathbb{Z}\pi$-homology equivalences with trivial Whitehead torsion, and two such are equivalent if there is a normal cobordism between them with surgery obstruction 0 in $L_4(\pi, w)$.

Lemma 10. *Let N_0 be an aspherical 3-manifold with $\pi_1(N_0) = \pi$, and let $h : N \to N_0$ be a $\mathbb{Z}[\pi]$-homology equivalence. Then h extends to a normal map (h, b) which is normally bordant to id_{N_0}, via a normal bordism with trivial surgery obstruction in $L_4(\pi, w)$.*

Proof. It follows from the Geometrization Theorem that $Wh(\pi) = 0$. If N_0 is orientable then source and target are parallelizable, and so h extends to a normal map (h, b). This remains true if N_0 is non-orientable, by the argument of Theorem 7 above. In [21] (as corrected in [2]) it is shown that the surgery obstruction homomorphisms σ_{3+i} are bijective, if $3 + i \geq 5$. Periodicity of these homomorphisms implies that $\overline{\mathcal{S}}^{TOP}(N_0)$ has just one member, so there is a normal cobordism with trivial surgery invariant. □

Lemma 11. *Let ν be a finitely generated group which is an extension of a solvable group π by a perfect group κ. If $\nu \cong G * H$ then G or H is perfect.*

Proof. Clearly ν maps onto $\rho = G^{ab} * H^{ab}$, and hence the subquotients of the derived series for ν map onto the corresponding terms for ρ. If neither G nor H is perfect then the derived series of ρ is infinite, contrary to the hypothesis on ν. □

Theorem 12. *Let π be a polycyclic PD_3-group and let N_1 and N_2 be 3-manifolds such that $\nu_i = \pi_1(N_i)$ is an extension of π by a perfect group κ_i, for $i = 1, 2$. Then there is an aspherical 4-manifold M such that $\pi_1(M) \cong \pi$ and $\partial M \cong N_1 \sqcup N_2$.*

Proof. Let N_0 be a 3-manifold with $\pi_1(N_0) = \pi$, and let $h_i : N_i \to N_0$ be maps corresponding to the surjections $\nu_i \to \nu_i/\kappa_i \cong \pi$.

If N_i is aspherical then $\mathrm{cd}\,\kappa_i \leq 2$, since $[\nu_i : \kappa_i]$ is infinite. Hence κ_i is acyclic, by Lemma 3, and so h_i is a $\mathbb{Z}[\pi]$-homology equivalence. There is a normal cobordisms W_i from h_i to id_{N_0}, with trivial surgery obstruction in $L_4(\pi, w)$, by Lemma 10.

Thus if both N_1 and N_2 are aspherical we may perform surgery *rel* ∂ to obtain normal cobordisms $(\widehat{H}_i, \hat{b}_i) : \widehat{W}_i \to N_0 \times [0,1]$ such that $\widehat{H}_i$ is a homotopy equivalence, for $i = 1, 2$. Let $M = \widehat{W_1} \cup N_0 \times [0,1] \cup \widehat{W_2}$ (identified along copies of N_0). Then $M \simeq N_0$ and $\partial M \cong N_1 \sqcup N_2$.

In general, we have $\nu_i \cong \rho_i * \sigma_i$, where ρ_i is indecomposable and σ_i is perfect, by Lemma 11. Hence $N_i \cong P_i \sharp \Sigma_i$, where P_i is aspherical and Σ_i is an integral homology 3-sphere. Let M_{asp} be the 4-manifold with boundary $P_1 \sqcup P_2$ constructed as in the above two paragraphs, and let C_i be contractible 4-manifolds with $\partial C_i = \Sigma_i$, for $i = 1, 2$. Then the boundary connected sum $M = C_1 \natural M_{asp} \natural C_2$ has the required properties. $\square$

There is a similar result for the case when π is a polycyclic PD_3-group and the image of ν in π has index 2. Let $\eta \in H^1(\pi; \mathbb{Z}/2\mathbb{Z})$ be the corresponding epimorphism. Then N bounds $\widehat{W} \cup M(\eta)$, where $\widehat{W}$ is a $\mathbb{Z}[\nu]$-homology cobordism from N to N_0 with $\pi_1(\widehat{W}) \cong \nu$.

8. The role of the peripheral system

If M is an aspherical 4-manifold with $\pi = \pi_1(M)$ polycyclic then the surgery obstruction maps

$$[\Sigma^j(M/\partial), G/TOP] \to L_{4+j}(\pi, w)$$

are bijective for all $i > 0$ [9, Theorem 4.1]. A similar result holds if $\pi \cong BS(1, m)$ for some $m \neq 0$, since such groups are fundamental groups of (very small) graphs of groups with all vertex and edge groups infinite cyclic, and this case is settled in [22]. In either case, M is determined up to homeomorphism by ∂M and π [10, Chapter 11.§5].

We shall strengthen this result to show that the peripheral system is a complete invariant for the homeomorphism type of an aspherical 4-manifold with elementary amenable fundamental group.

Lemma 13. *Each boundary component N of an aspherical 4-manifold M with elementary amenable fundamental group is a connected sum $N_0 \sharp \Sigma_N$, where either N_0 is aspherical or $\pi_1(N_0) \cong \mathbb{Z}$ or $N_0 = S^3$, and $\pi_1(\Sigma_N)$ is perfect. In particular, Σ_N is a connected sum of aspherical homology 3-spheres with copies of S^3/I^*.*

Proof. Since $w_1(N) = w_1(M)|_N$ and $\pi = \pi_1(M)$ is torsion-free, N has no 2-sided projective planes. Therefore N is a connected sum of aspherical 3-manifolds, copies of $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$ and summands with finite fundamental group.

If $\pi = 1$ then $\partial M = N$, and N must be an homology 3-sphere, so we may set $N_0 = S^3$ and $\Sigma_N = N$.

If $\pi \cong \mathbb{Z}$ then $\partial M = N$ and $\mathrm{Ker}(\pi_1(\mathrm{inc}))$ is perfect. It follows from Lemma 11 that $N \cong N_0 \sharp \Sigma_N$, where N_0 is irreducible and $H_1(N_0; \mathbb{Z}) \cong \mathbb{Z}$, while $\pi_1(\Sigma_N)$ is perfect.

A similar argument applies if π is a solvable PD_3-group.

When $\mathrm{cd}\,\pi = 2$ then we must extend the strategy of Lemma 11. If $\pi = BS(1, m)$ then ν is an extension of π by a group κ such that $\kappa/\kappa' \cong \overline{H^2(\pi; \mathbb{Z}\pi)}$. The cohomology group $H^2(\pi; \mathbb{Z}\pi)$ is torsion free [12, Chapter 13], and so ν/κ' has a composition series with three torsion-free factors. Any homomorphism from a group with finite abelianization to ν must have image in $\kappa' \leq \nu'$. It follows that if $\nu \cong G * H$ and $\beta_1(H) = 0$ then H is perfect.

Suppose first that M is orientable. If $m \neq 1$ then $H_2(\pi; \mathbb{Z}) = 0$ and $H^2(\pi; \mathbb{Z}) \cong \mathbb{Z}/(m-1)\mathbb{Z}$, so $\beta_1(\nu) = 1$, by the long exact sequence for (M, N). Thus if $\nu \cong G * H$ with $\beta_1(G) \geq \beta_1(H)$ then $\beta_1(H) = 0$, and so H is perfect. If $m = 1$, so $\pi \cong \mathbb{Z}^2$, then $\beta_1(\nu) = 2$ or 3, and $H^2(\pi; \mathbb{Z}[\pi]) \cong \mathbb{Z}$. If $\nu \cong G * H$ with $\beta_1(G) = \beta_1(H) = 1$ then the epimorphism from ν to π factors through an epimorphism to $F(2)$. But then $H_1(\nu; \mathbb{Z}[\pi])$ maps onto $H_1(F(2); \mathbb{Z}[\pi])$, which has infinite rank as an abelian group. Thus if $\beta_1(\nu) = 2$ and $\beta_1(G) \geq \beta_1(H)$ then H must be perfect. A similar argument applies if $\beta_1(\nu) = 3$.

If M is non-orientable and $\nu = \pi_1(N) \cong G * H$ with $\beta_1(G), \beta_1(H) > 0$ then the orientable double cover M^+ has $\nu^+ = \pi_1(\partial M^+) \cong G_1 * H_1$ with $\beta_1(G_1), \beta_1(H_1) > 0$. Thus we may assume that M is orientable.

Since S^3/I^* is the only homology 3-sphere with finite fundamental group, the final assertion is clear. □

Theorem 14. *Let M and $\widehat{M}$ be aspherical 4-manifolds with boundary, and with elementary amenable fundamental groups. Then M and $\widehat{M}$ are homeomorphic if and only if their peripheral systems are equivalent.*

Proof. The condition is clearly necessary. Suppose that it holds.

If M and $\widehat{M}$ are closed then $\pi = \pi_1(M)$ is polycyclic, by Theorem 4, and the result follows from [9, Corollary A] and [10].

Suppose next that $N = \partial M$ and $\widehat{N} = \partial\widehat{M}$ are connected, and let θ and $\varphi : \pi_1(N) \to \pi_1(\widehat{N})$ be the isomorphisms provided by the hypothesis. We may assume that $\theta\pi_1(\mathrm{inc}^M) = \pi_1(\mathrm{inc}^{\widehat{M}})\varphi$, by absorbing any conjugacy into the isomorphism θ, if necessary. Although φ itself may not be induced by a homeomorphism (see [19]), we may modify the equivalence to achieve this. We may write $N = N_0\sharp\Sigma_N$ and $\widehat{N} = \widehat{N}_0\sharp\Sigma_{\widehat{N}}$, where N_0 and $\widehat{N_0}$ are aspherical and Σ_N and $\Sigma_{\widehat{N}}$ are homology 3-spheres, by Lemma 13. Since $\pi_1(N) \cong \pi_1(\widehat{N})$, there is a bijective correspondance between the indecomposable factors of these groups, and hence between the irreducible summands of the 3-manifolds. Since none of these summands are lens spaces or contain two-sided projective planes, they are determined up to homeomorphism by their fundamental groups.

If π is perfect then $\pi = 1$, and N and $\widehat{N}$ are homology 3-spheres, so we may assume that $N_0 = \widehat{N}_0 = S^3$.

If $\pi \cong \mathbb{Z}$ then either $\pi_1(N_0) \cong \pi_1(\widehat{N}_0) \cong \mathbb{Z}$ or N_0 and $\widehat{N}_0$ are both aspherical. Since $w_1(M)$ and $w_1(\widehat{M})$ are trivial on the homology sphere summands and $w_1(M) = w_1(\widehat{M})\theta$, we may again assume that $N_0 \cong \widehat{N}_0$, and both are either $S^1 \times S^2$, $S^1\tilde{\times}S^2$ or are aspherical.

If $\mathrm{cd}\,\pi > 1$ then N_0 and $\widetilde{N}_0$ are aspherical, and are homeomorphic.

Thus in each case there is a homeomorphism $h_0 : N_0 \to \widehat{N}_0$ which induces the isomorphism given by the composite of the inclusion of $\pi_1(N_0)$ into $\pi_1(N)$ as a factor, φ and the epimorphism $\pi_1(\widehat{N}) \to \pi_1(\widehat{N}_0)$ induced by collapsing $\Sigma_{\widehat{N}}$. Let $h_\Sigma : \Sigma_N \to \Sigma_{\widehat{N}}$ be any homeomorphism of the homology 3-sphere summands. Then $\breve{h} = h_0\sharp h_\Sigma$ is a homeomorphism such that $\theta\pi_1(\mathrm{inc}^M) = \pi_1(\mathrm{inc}^{\widehat{M}}\breve{h})$, and we may now apply the result of [10, Chapter 11.§5] (which relies on [9]), that M is determined up to homeomorphism by ∂M and π. If $\pi = BS(1, m)$, we may invoke [22] instead of [9], to obtain a similar result.

A similar argument applies to each component if π is a PD_3-group and ∂M and $\partial\widehat{M}$ each have two components. In this case the homomorphisms $\pi_1(\mathrm{inc}_i)$ are epimorphisms, and so conjugacies may be absorbed into the isomorphisms φ_i. □

Lemma 15. *Let M be an aspherical 4-manifold such that $\partial M \cong N\sharp\Sigma$, where Σ is an homology 3-sphere. If $\pi_1(\Sigma)$ has trivial image in $\pi = \pi_1(M)$ then there is an aspherical 4-manifold M_1 such that $M_1 \simeq M$ and $\partial M_1 \cong N$.*

Proof. We may write $\partial M = N_o \cup \Sigma_o$, where N_o and Σ_o are the complements of open 3-discs in N and Σ, respectively. The homology 3-sphere Σ bounds a contractible 4-manifold C. Then $\partial C = \Sigma_o \cup D^3$. Let $M_1 = M \cup_{\Sigma_o} C$. Then $\pi_1(M_1) \cong \pi$ and $H_*(M_1; \mathbb{Z}[\pi]) \cong H_*(M; \mathbb{Z}[\pi])$, and so the inclusion $M \to M_1$ is a homnotopy equivalence. Clearly $\partial M_1 = N_o \cup D^3 \cong N$. □

Corollary 16. *The* 4*-manifold* M *is homeomorphic to a boundary connected sum* $M_0 \natural \Delta$*, where* ∂M_0 *is prime and* Δ *is contractible.*

Proof. This follows from Theorems 7 and 12 together with Lemmas 13 and 15, and Theorem 14. □

9. Other good groups

Here we interpret “good” in a very broad sense; having no noncyclic free subgroups.

It seems unlikely that there are any other good groups π with $\mathrm{cd}\,\pi = 2$. For assume π finitely generated, $\mathrm{cd}\,\pi = 2$ and π has no non-cyclic free subgroup. Then the following are equivalent [15, Corollary 2.6.1]:

(1) $\pi \cong BS(1, m) = \mathbb{Z}*_m$ for some integer $m \neq 0$;
(2) π is elementary amenable;
(3) π is almost coherent and virtually indicable;
(4) π is almost coherent and amenable;
(5) π is almost coherent and $\mathrm{def}(\pi) = 1$.

Here a group is almost coherent if every finitely generated subgroup is FP_2.

Question. Is it enough to assume that π is almost coherent and residually finite (in addition to the conditions before the list)?

If π is an almost coherent, residually finite PD_3-group which has no non-cyclic free subgroup then it is polycyclic [5]. In particular, “good” 3-manifold groups are polycyclic, since they are coherent and residually finite.

10. Some remarks on the closed case

Grigorchuk (1998) has an example of a finitely presentable group which is not elementary amenable, but does have sub-exponential growth, and so the DEC holds over this group, by Freedman-Teichner (1995). (We do not know whether this group could realized by an aspherical 4-manifold, and have no reason to expect so. In fact no known aspherical finite complex has such a fundamental group.)

At the other extreme, the fundamental groups of (necessarily aspherical) closed 4-manifolds with one of the geometries $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^2\times\mathbb{H}^2$, $\mathbb{H}^3\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^2$ or $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$ contain noncyclic free subgroups.

If $f : M \to M_0$ is a homotopy equivalence of closed 4-manifolds then the difference of the stable normal bundles $\xi = \nu_M \oplus f^*\tau_{M_0}$ is a stable *Spin*-bundle, since the Stiefel-Whitney classes are homotopy invariants. If M and M_0 are orientable then the Pontrjagin class $p_1(\xi)$ is 0 (since M and M_0 have the same signature). Since the base M is 4-dimensional, ξ is trivial, and so f extends to a normal map (f, b). Does this remain true if M and M_0 are not orientable?

If M and M_0 are aspherical and the FJCs hold for $\pi = \pi_1(M)$ then $f \times id_{S^1}$ is homotopic to a homeomorphism, by 5-dimensional surgery, and $Wh(\pi) = 0$. Hence $M \times \mathbb{R} \cong M_0 \times \mathbb{R}$. These products contain an h-cobordism between M and M_0, which is an s-cobordism since $Wh(\pi) = 0$, by assumption.

Question. Does the argument of the last paragraph go through for bounded manifolds *rel* ∂, provided the Borel conjecture holds?

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