Center at the critical level for centralizers in type A

A. I. Molev

Abstract

We consider the affine vertex algebra at the critical level associated with the centralizer of a nilpotent element in the Lie algebra $\mathfrak{gl}_N$. Due to a recent result of Arakawa and Premet, the center of this vertex algebra is an algebra of polynomials. We construct a family of free generators of the center in an explicit form. As a corollary, we obtain generators of the corresponding quantum shift of argument subalgebras and recover free generators of the center of the universal enveloping algebra of the centralizer produced earlier by Brown and Brundan.

1 Introduction

For any finite-dimensional Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ equipped with an invariant symmetric bilinear form let $\widehat{\mathfrak{a}}$ denote the corresponding affine Kac–Moody algebra. The vacuum module over $\widehat{\mathfrak{a}}$ is a vertex algebra. The center of this vertex algebra is a commutative associative algebra. In the case of a simple Lie algebra $\mathfrak{a}$ the structure of the center $\mathfrak{z}(\widehat{\mathfrak{a}})$ at the critical level is described by a celebrated theorem of Feigin and Frenkel [5] (see also [6]), which states that the center is an algebra of polynomials in infinitely many variables. This theorem was extended in a recent work by Arakawa and Premet [1] to the case where $\mathfrak{a}$ is the centralizer of a certain nilpotent element e in a simple Lie algebra. As a consequence, they showed the existence of the regular quantum shift of argument subalgebras and proved that they are free polynomial algebras. Moreover, explicit formulas for generators of $\mathfrak{z}(\widehat{\mathfrak{a}})$ were produced in [1] in the case where $\mathfrak{a} = \mathfrak{g}^e$ is the centralizer of a minimal nilpotent e in $\mathfrak{g} = \mathfrak{gl}_N$.

Our goal in this note is to extend these formulas to the case of an arbitrary nilpotent element $e \in \mathfrak{g}$. As a corollary, we get explicit generators of the quantum shift of argument subalgebras. Furthermore, by applying an evaluation homomorphism we produce free generators of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ found earlier by Brown and Brundan [2]. In the particular case $e = 0$ our formulas coincide with those in [3] and [4]; see also [11] for more details and extension to the other classical types.

2 Segal–Sugawara vectors

Using the notation of [2], suppose that $e \in \mathfrak{g} = \mathfrak{gl}_N$ is a nilpotent matrix with Jordan blocks of sizes $\lambda_1, \dots, \lambda_n$, where $\lambda_1 \leqslant \dots \leqslant \lambda_n$ and $\lambda_1 + \dots + \lambda_n = N$. Consider the corresponding *pyramid* which is a left-justified array of rows of unit boxes such that the top row contains λ_1 boxes, the next row contains λ_2 boxes, etc. The *row-tableau* is obtained by writing the numbers $1,\dots,N$ into the boxes of the pyramid consecutively by rows from left to right. For instance, the row-tableau

1	2		
3	4	5	
6	7	8	9

corresponds to the Jordan blocks of sizes $2,3,4$ and $N=9$. We will use the notation $\mathrm{row}(a)$ and $\mathrm{col}(a)$ for the row and column number of the box containing the entry a.

Denote by e_{ab} with $a,b=1,\dots,N$ the standard basis elements of $\mathfrak{g}$. For any $1\leqslant i,j\leqslant n$ and $\lambda_j-\min(\lambda_i,\lambda_j)\leqslant r<\lambda_j$ set

$$E_{ij}^{(r)}=\sum_{\substack{\mathrm{row}(a)=i,\ \mathrm{row}(b)=j\\ \mathrm{col}(b)-\mathrm{col}(a)=r}} e_{ab},$$

summed over $a,b\in\{1,\dots,N\}$. The elements $E_{ij}^{(r)}$ form a basis of the Lie algebra $\mathfrak{a}=\mathfrak{g}^e$.

Following [1, Sec. 5], equip the Lie algebra $\mathfrak{a}$ with the invariant symmetric bilinear form $\langle\ ,\ \rangle$ defined by the formulas

$$\big\langle E_{ii}^{(0)},E_{jj}^{(0)}\big\rangle=\min(\lambda_i,\lambda_j)-\delta_{ij}\big(\lambda_1+\dots+\lambda_{i-1}+(n-i+1)\lambda_i\big),$$

and if $\lambda_i=\lambda_j$ for some $i\neq j$ then

$$\big\langle E_{ij}^{(0)},E_{ji}^{(0)}\big\rangle=-\big(\lambda_1+\dots+\lambda_{i-1}+(n-i+1)\lambda_i\big),$$

whereas all remaining values of the form on the basis vectors are zero. Note that the sum in the brackets equals the number of boxes in the first i columns of the pyramid.

The corresponding affine Kac–Moody algebra $\widehat{\mathfrak{a}}$ is the central extension

$$\widehat{\mathfrak{a}}=\mathfrak{a}[t,t^{-1}]\oplus\mathbb{C}\mathbf{1},$$

where $\mathfrak{a}[t,t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in $\mathfrak{a}$. For any $r\in\mathbb{Z}$ and $X\in\mathfrak{g}$ we will write $X[r]=X\,t^r$. The commutation relations of the Lie algebra $\widehat{\mathfrak{a}}$ have the form

$$\big[X[r],Y[s]\big]=[X,Y][r+s]+r\,\delta_{r,-s}\langle X,Y\rangle\,\mathbf{1},\qquad X,Y\in\mathfrak{a},$$

and the element $\mathbf{1}$ is central in $\widehat{\mathfrak{a}}$. The *vacuum module at the critical level* over $\widehat{\mathfrak{a}}$ is the quotient

$$V(\mathfrak{a})=\mathrm{U}(\widehat{\mathfrak{a}})/\mathrm{I},$$

where I is the left ideal of $\mathrm{U}(\widehat{\mathfrak{a}})$ generated by $\mathfrak{a}[t]$ and the element $\mathbf{1}-1$. By the Poincaré–Birkhoff–Witt theorem, the vacuum module is isomorphic to the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$, as a vector space. This vector space is equipped with a vertex algebra structure; see [7], [8]. Denote by $\mathfrak{z}(\widehat{\mathfrak{a}})$ the *center* of this vertex algebra which is defined as the subspace

$$\mathfrak{z}(\widehat{\mathfrak{a}}) = \{v \in V(\mathfrak{a}) \mid \mathfrak{a}[t]\, v = 0\}.$$

It follows from the axioms of vertex algebra that $\mathfrak{z}(\widehat{\mathfrak{a}})$ is a unital commutative associative algebra. It can be regarded as a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$. This subalgebra is invariant with respect to the *translation operator* T which is the derivation of the algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$ whose action on the generators is given by

$$T : X[r] \mapsto -r\, X[r-1], \qquad X \in \mathfrak{a}, \quad r < 0.$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{a}})$ is called a *Segal–Sugawara vector*. By [1, Thm 1.4], there exists a *complete set of Segal–Sugawara vectors* $S_1, \dots, S_N$, which means that all translations $T^r S_l$ with $r \geqslant 0$ and $l = 1, \dots, N$ are algebraically independent and any element of $\mathfrak{z}(\widehat{\mathfrak{a}})$ can be written as a polynomial in the shifted vectors; that is,

$$\mathfrak{z}(\widehat{\mathfrak{a}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, N, \quad r \geqslant 0].$$

In the case $e = 0$ this reduces to the Feigin–Frenkel theorem in type A [5, 6].

To produce a complete set of Segal–Sugawara vectors, we will use the extended Lie algebra $\widehat{\mathfrak{a}} \oplus \mathbb{C}\tau$ where the additional element τ satisfies the commutation relations

$$\big[\tau, X[r]\big] = -r\, X[r-1], \qquad \big[\tau, \mathbf{1}\big] = 0. \tag{2.1}$$

We will identify the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}] \oplus \mathbb{C}\tau\big)$ with the tensor product space $V(\mathfrak{a}) \otimes \mathbb{C}[\tau]$. For all $i, j \in \{1, \dots, n\}$ introduce its elements $\mathcal{E}_{ij}$ by

$$\mathcal{E}_{ij} = \begin{cases} \delta_{ij}\tau^{\lambda_j} + E_{ij}^{(0)}[-1]\,\tau^{\lambda_j - 1} + \dots + E_{ij}^{(\lambda_j - 1)}[-1] & \text{if} \quad i \geqslant j, \\ E_{ij}^{(\lambda_j - \lambda_i)}[-1]\,\tau^{\lambda_i - 1} + \dots + E_{ij}^{(\lambda_j - 1)}[-1] & \text{if} \quad i < j. \end{cases}$$

Define elements $S_1^\circ, \dots, S_N^\circ \in V(\mathfrak{a})$ by expanding the column-determinant of the matrix $\mathcal{E} = [\mathcal{E}_{ij}]$,

$$\operatorname{cdet} \mathcal{E} = \tau^N + S_1^\circ\, \tau^{N-1} + \dots + S_N^\circ.$$

We will say that the element $E_{ij}^{(k)}[r]$ of the Lie algebra $t^{-1}\mathfrak{a}[t^{-1}]$ has *weight* k. By extending the weight function to the universal enveloping algebra we get a grading on $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$. Denote by S_i the homogeneous component of maximal weight of the coefficient S_i°.

Theorem 2.1. *The elements $S_1, \dots, S_N$ belong to the center $\mathfrak{z}(\widehat{\mathfrak{a}})$ of the vertex algebra $V(\mathfrak{a})$. Moreover, they form a complete set of Segal–Sugawara vectors for the Lie algebra $\mathfrak{a}$.*

As with the minimal nilpotent case considered in [1], the proof follows the same approach as in the paper [3] which deals with the case $e = 0$. We outline some necessary additional details in Section 3.

Examples 2.2. In the principal nilpotent case with $n = 1$ and $e = e_{12} + \dots + e_{N-1\,N}$ we have

$$S_k = E_{11}^{(k-1)}[-1] = e_{1\,k}[-1] + \dots + e_{N-k+1\,N}[-1], \qquad k = 1, \dots, N.$$

For $n = 2$ write the column-determinant $\operatorname{cdet} \mathcal{E}$ as

$$\begin{vmatrix} \tau^{\lambda_1} + E_{11}^{(0)}[-1]\tau^{\lambda_1-1} + \dots + E_{11}^{(\lambda_1-1)}[-1] & E_{12}^{(\lambda_2-\lambda_1)}[-1]\tau^{\lambda_1-1} + \dots + E_{12}^{(\lambda_2-1)}[-1] \\ E_{21}^{(0)}[-1]\tau^{\lambda_1-1} + \dots + E_{21}^{(\lambda_1-1)}[-1] & \tau^{\lambda_2} + E_{22}^{(0)}[-1]\tau^{\lambda_2-1} + \dots + E_{22}^{(\lambda_2-1)}[-1] \end{vmatrix}.$$

We have

$$\begin{aligned} S_1 &= E_{11}^{(0)}[-1] + E_{22}^{(0)}[-1], \ \dots, \ S_{\lambda_1} = E_{11}^{(\lambda_1-1)}[-1] + E_{22}^{(\lambda_1-1)}[-1], \\ S_{\lambda_1+1} &= E_{22}^{(\lambda_1)}[-1], \ \dots, \ S_{\lambda_2} = E_{22}^{(\lambda_2-1)}[-1], \end{aligned}$$

whereas

$$S_{\lambda_2+1} = \begin{vmatrix} E_{11}^{(0)}[-1] & E_{12}^{(\lambda_2-1)}[-1] \\ E_{21}^{(0)}[-1] & E_{22}^{(\lambda_2-1)}[-1] \end{vmatrix} + \dots + \begin{vmatrix} E_{11}^{(\lambda_1-1)}[-1] & E_{12}^{(\lambda_2-\lambda_1)}[-1] \\ E_{21}^{(\lambda_1-1)}[-1] & E_{22}^{(\lambda_2-\lambda_1)}[-1] \end{vmatrix} + \lambda_1 \, E_{22}^{(\lambda_2-1)}[-2]$$

and

$$S_{\lambda_2+a+1} = \begin{vmatrix} E_{11}^{(a)}[-1] & E_{12}^{(\lambda_2-1)}[-1] \\ E_{21}^{(a)}[-1] & E_{22}^{(\lambda_2-1)}[-1] \end{vmatrix} + \dots + \begin{vmatrix} E_{11}^{(\lambda_1-1)}[-1] & E_{12}^{(\lambda_2-\lambda_1+a)}[-1] \\ E_{21}^{(\lambda_1-1)}[-1] & E_{22}^{(\lambda_2-\lambda_1+a)}[-1] \end{vmatrix}$$

for $a = 1, \dots, \lambda_1 - 1$.

The minimal nilpotent case $e = e_{n\,n+1} \in \mathfrak{gl}_{n+1}$ corresponds to the pyramid with the n rows $1, \dots, 1, 2$. The column-determinant takes the form

$$\begin{vmatrix} \tau + E_{11}^{(0)}[-1] & \dots & E_{1\,n-1}^{(0)}[-1] & E_{1\,n}^{(1)}[-1] \\ E_{21}^{(0)}[-1] & \dots & E_{2\,n-1}^{(0)}[-1] & E_{2\,n}^{(1)}[-1] \\ \vdots & \vdots & \vdots & \vdots \\ E_{n1}^{(0)}[-1] & \dots & E_{n\,n-1}^{(0)}[-1] & \tau^2 + E_{nn}^{(0)}[-1]\,\tau + E_{nn}^{(1)}[-1] \end{vmatrix} = \tau^{n+1} + S_1^{\circ}\,\tau^n + \dots + S_{n+1}^{\circ}.$$

By taking the maximal weight components S_i of S_i° we find, in particular, that

$$\begin{aligned} S_1 &= E_{11}^{(0)}[-1] + \dots + E_{nn}^{(0)}[-1] = e_{11}[-1] + \dots + e_{n+1\,n+1}[-1], \\ S_2 &= E_{nn}^{(1)}[-1] = e_{n\,n+1}[-1], \end{aligned}$$

cf. [1, Sec. 5]. □

By adapting Rybnikov's construction [14] to the case of Lie algebra $\widehat{\mathfrak{a}}$ as in [1], for any element $\chi \in \mathfrak{a}^*$ and a variable z consider the homomorphism

$$\varrho_{\chi} : \mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{a}) \otimes \mathbb{C}[z^{-1}], \qquad X[r] \mapsto X\,z^r + \delta_{r,-1}\,\chi(X), \tag{2.2}$$

for any $X \in \mathfrak{a}$ and $r < 0$. If $S \in \mathfrak{z}(\widehat{\mathfrak{a}})$ is a homogeneous element of degree d with respect to the grading defined by

$$\deg X[r] = -r, \qquad r < 0, \tag{2.3}$$

define the elements $S_{(k)} \in \mathrm{U}(\mathfrak{a})$ (depending on χ) by the expansion

$$\varrho_\chi(S) = S_{(0)}\, z^{-d} + \dots + S_{(d-1)}\, z^{-1} + S_{(d)}. \tag{2.4}$$

If the variable z takes a particular nonzero value in $\mathbb{C}$, then the formula (2.2) defines a homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{a}), \qquad X[r] \mapsto X\, z^r + \delta_{r,-1}\, \chi(X), \tag{2.5}$$

for any $X \in \mathfrak{a}$ and $r < 0$. Since $\mathfrak{z}(\widehat{\mathfrak{a}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$, its image under the homomorphism (2.5) is a commutative subalgebra of $\mathrm{U}(\mathfrak{a})$ which we denote by $\mathcal{A}_\chi$. This subalgebra does not depend on the value of z.

As we will see in Sec. 3, the respective degrees $d_1, \dots, d_N$ of the Segal–Sugawara vectors $S_1, \dots, S_N$ constructed in Theorem 2.1 coincide with the degrees of the basic invariants of the symmetric algebra $\mathrm{S}(\mathfrak{a})$ given by

$$\underbrace{1, \dots, 1}_{\lambda_n}, \underbrace{2, \dots, 2}_{\lambda_{n-1}}, \dots, \underbrace{n, \dots, n}_{\lambda_1},$$

as found in [13]; see also [2]. Introduce the corresponding polynomials (2.4) by

$$\varrho_\chi(S_k) = S_{k\,(0)}\, z^{-d_k} + \dots + S_{k\,(d_k-1)}\, z^{-1} + S_{k\,(d_k)}.$$

By applying the results of [1] we come to the following.

Corollary 2.3. *Suppose the element $\chi \in \mathfrak{a}^*$ is regular. Then the elements*

$$S_{k\,(i)}, \qquad k = 1, \dots, N, \qquad i = 0, 1, \dots, d_k - 1,$$

are algebraically independent generators of the algebra $\mathcal{A}_\chi$. Moreover, $\mathcal{A}_\chi$ is a quantization of the shift of argument subalgebra $\overline{\mathcal{A}}_\chi \subset \mathrm{S}(\mathfrak{a})$ so that $\mathrm{gr}\, \mathcal{A}_\chi = \overline{\mathcal{A}}_\chi$. □

The subalgebra $\overline{\mathcal{A}}_\chi \subset \mathrm{S}(\mathfrak{a})$ is known to be Poisson-commutative and it is also called the *Mishchenko–Fomenko subalgebra*; see [9]. Corollary 2.3 provides an explicit solution of *Vinberg's quantization problem* [15] for centralizers. We conjectured in [12, Conjecture 5.8] that the subalgebra $\mathcal{A}_\chi$ can also be obtained via a symmetrization map.

As another corollary of Theorem 2.1, we recover the algebraically independent generators of the center of $\mathrm{U}(\mathfrak{a})$ constructed in [2] with the use of the shifted Yangians; see also [10] for the particular case of rectangular pyramids. For all $i, j \in \{1, \dots, n\}$ introduce polynomials $\mathcal{E}_{ij}(u)$ in a variable u with coefficients in $\mathrm{U}(\mathfrak{a})$ by

$$\mathcal{E}_{ij}(u) = \begin{cases} \delta_{ij} u^{\lambda_j} + \widetilde{E}_{ij}^{(0)}\, u^{\lambda_j - 1} + \dots + \widetilde{E}_{ij}^{(\lambda_j - 1)} & \text{if} \quad i \geqslant j, \\ \widetilde{E}_{ij}^{(\lambda_j - \lambda_i)}\, u^{\lambda_i - 1} + \dots + \widetilde{E}_{ij}^{(\lambda_j - 1)} & \text{if} \quad i < j, \end{cases}$$

where $\widetilde{E}^{(r)}_{ij} = E^{(r)}_{ij} + \delta_{r\,0}\,\delta_{ij}(n-i)\,\lambda_i$. Expand the column-determinant of the matrix $\mathcal{E}(u) = \big[\mathcal{E}_{ij}(u)\big]$ as a polynomial in u:

$$\mathrm{cdet}\,\mathcal{E}(u) = u^N + Z_1\,u^{N-1} + \dots + Z_N.$$

Corollary 2.4. *The coefficients $Z_1,\dots,Z_N$ are algebraically independent generators of the center of the algebra* $\mathrm{U}(\mathfrak{a})$. $\square$

The elements Z_r are slightly different from the corresponding elements z_r given by [2, Eq. (1.3)]. Relations (2.1) are preserved by the shift $\tau \mapsto \tau + c$ for any given constant c. Therefore, this constant can be used as a parameter in Theorem 2.1 and Corollary 2.4. By taking $c = -n+1$ we get the correspondence $Z_r \mapsto z_r$.

3 Proof of Theorem 2.1

To prove the first part of the theorem, it will be sufficient to verify that the coefficients $S_k \in V(\mathfrak{a})$ are annihilated by a family of elements which generate $\mathfrak{a}[t]$ as a Lie algebra. The commutation relations in $\mathfrak{a}$ have the form

$$\big[E^{(r)}_{ij}, E^{(s)}_{kl}\big] = \delta_{kj}\,E^{(r+s)}_{il} - \delta_{il}\,E^{(r+s)}_{kj},$$

assuming that $E^{(r)}_{ij} = 0$ for $r \geqslant \lambda_j$. Hence, for a family of generators we can take $E^{(0)}_{i+1\,i}[0]$, $E^{(\lambda_{i+1}-\lambda_i)}_{i\,i+1}[0]$ for $i = 1,\dots,n-1$, and $E^{(r)}_{i\,i}[p]$ for $r = 0,\dots,\lambda_i - 1$ and $p = 0,1,\dots$.

We will relabel the elements $S_1,\dots,S_N$ by indicating their weights as superscripts, while the subscripts will coincide with their degrees with respect to the grading (2.3) so that

$$\tau^N + S_1\,\tau^{N-1} + \dots + S_N = \tau^N + \sum_{d=1}^{n}\ \sum_{\lambda_1+\dots+\lambda_{n-d}\leqslant a<\lambda_1+\dots+\lambda_{n-d+1}} S^{(N-a-d)}_{(d)}\,\tau^a$$

and $S_{N-a} = S^{(N-a-d)}_{(d)}$. Note that the subscript d of a coefficient $S^{(k)}_{(d)}$ coincides with its filtration degree in the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$.

Begin with the generators $E^{(0)}_{i+1\,i}[0]$ and show that for all coefficients A_k of the powers of τ in the expansion

$$E^{(0)}_{i+1\,i}[0]\,\mathrm{cdet}\,\mathcal{E} = A_1\,\tau^{N-1} + \dots + A_N \tag{3.6}$$

we have the property that the weight of A_{N-a} is less than the weight of S_{N-a} given by $N-a-d$. Our calculations will use some simple properties of column-determinants described in [3, Lemmas 4.1 and 4.2] which allow us to write the left hand side of (3.6) as the difference of two column-determinants

$$\begin{vmatrix} \mathcal{E}_{1\,1} & \dots & \mathcal{E}_{1\,i} & \mathcal{E}_{1\,i+1} & \dots & \mathcal{E}_{1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \widetilde{\mathcal{E}}_{i+1\,1} & \dots & \widetilde{\mathcal{E}}_{i+1\,i} & \widetilde{\mathcal{E}}_{i+1\,i+1} & \dots & \widetilde{\mathcal{E}}_{i+1\,n} \\ \mathcal{E}_{i+1\,1} & \dots & \mathcal{E}_{i+1\,i} & \mathcal{E}_{i+1\,i+1} & \dots & \mathcal{E}_{i+1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n\,1} & \dots & \mathcal{E}_{n\,i} & \mathcal{E}_{n\,i+1} & \dots & \mathcal{E}_{n\,n} \end{vmatrix} - \begin{vmatrix} \mathcal{E}_{1\,1} & \dots & \mathcal{E}_{1\,i} & \widetilde{\mathcal{E}}_{1\,i} & \dots & \mathcal{E}_{1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{i\,1} & \dots & \mathcal{E}_{i\,i} & \widetilde{\mathcal{E}}_{i\,i} & \dots & \mathcal{E}_{i\,n} \\ \mathcal{E}_{i+1\,1} & \dots & \mathcal{E}_{i+1\,i} & \widetilde{\mathcal{E}}_{i+1\,i} & \dots & \mathcal{E}_{i+1\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n\,1} & \dots & \mathcal{E}_{n\,i} & \widetilde{\mathcal{E}}_{n\,i} & \dots & \mathcal{E}_{n\,n} \end{vmatrix}.$$

Here we set $\widetilde{\mathcal{E}}_{i+1\,j} = \mathcal{E}_{i+1\,j}$ for $j \leqslant i$,

$$\widetilde{\mathcal{E}}_{i+1\,j} = \delta_{i+1\,j}\tau^{\lambda_{i+1}} + E_{i+1\,j}^{(\lambda_j-\lambda_i)}[-1]\,\tau^{\lambda_i-1} + \dots + E_{i+1\,j}^{(\lambda_j-1)}[-1]$$

for $j \geqslant i+1$, and

$$\widetilde{\mathcal{E}}_{k\,i} = \begin{cases} E_{ki}^{(0)}[-1]\,\tau^{\lambda_{i+1}-1} + \dots + E_{ki}^{(\lambda_i-1)}[-1]\,\tau^{\lambda_{i+1}-\lambda_i} & \text{if} \quad k \geqslant i+1, \\ \delta_{ki}\,\tau^{\lambda_{i+1}} + E_{ij}^{(\lambda_{i+1}-\lambda_k)}[-1]\,\tau^{\lambda_k-1} + \dots + E_{ki}^{(\lambda_i-1)}[-1]\,\tau^{\lambda_{i+1}-\lambda_i} & \text{if} \quad k \leqslant i. \end{cases}$$

In the first determinant subtract row $i+1$ from row i and expand the resulting column-determinant along the row i. Since $\widetilde{\mathcal{E}}_{i+1\,j} - \mathcal{E}_{i+1\,j} = 0$ for $j \leqslant i$, and

$$\mathcal{E}_{i+1\,j} - \widetilde{\mathcal{E}}_{i+1\,j} = E_{i+1\,j}^{(\lambda_j-\lambda_{i+1})}[-1]\,\tau^{\lambda_{i+1}-1} + \dots + E_{i+1\,j}^{(\lambda_j-\lambda_i-1)}[-1]\,\tau^{\lambda_i}$$

for $j \geqslant i+1$, we can see that the weight of the coefficient of τ^a in the expansion of the first determinant is less than $N-a-d$ if $\lambda_1 + \dots + \lambda_{n-d} \leqslant a < \lambda_1 + \dots + \lambda_{n-d+1}$. The same conclusion is reached for the second determinant by using its simultaneous expansion along the columns i and $i+1$.

The arguments for the generators $E_{i\,i+1}^{(\lambda_{i+1}-\lambda_i)}[0]$ are quite similar; we only need to take into account the property that the action of such a generator increases the weights by $\lambda_{i+1}-\lambda_i$. The relation $E_{i\,i}^{(0)}[0]\,\mathrm{cdet}\,\mathcal{E} = 0$ follows by even a simpler calculation.

Now consider the action of $E_{i\,i}^{(0)}[1]$. We have the commutation relations

$$\big[E_{ii}^{(0)}[1], \mathcal{E}_{kl}\big] = \delta_{k\,i}\,\mathcal{E}_{i\,l}[0] - \delta_{i\,l}\,\mathcal{E}_{k\,i}[0] + \mathcal{E}'_{kl}\,E_{ii}^{(0)}[0] + \delta_{k\,l}\,\varkappa_{i\,k}\,\tau^{\lambda_k-1},$$

where we set $\varkappa_{i\,k} = \big\langle E_{ii}^{(0)}, E_{kk}^{(0)}\big\rangle$ and use the notation

$$\mathcal{E}_{ij}[0] = \begin{cases} E_{ij}^{(0)}[0]\,\tau^{\lambda_j-1} + \dots + E_{ij}^{(\lambda_j-1)}[0] & \text{if} \quad i \geqslant j, \\ E_{ij}^{(\lambda_j-\lambda_i)}[0]\,\tau^{\lambda_i-1} + \dots + E_{ij}^{(\lambda_j-1)}[0] & \text{if} \quad i < j, \end{cases}$$

while $\mathcal{E}'_{kl}$ stands for the derivative of $\mathcal{E}_{kl}$ over τ. By [3, Lemma 4.1], we get

$$\big[E_{ii}^{(0)}[1], \mathrm{cdet}\,\mathcal{E}\big] = \sum_{j=1}^{n} \begin{vmatrix} \mathcal{E}_{11} & \dots & \big[E_{ii}^{(0)}[1], \mathcal{E}_{1\,j}\big] & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & \big[E_{ii}^{(0)}[1], \mathcal{E}_{n\,j}\big] & \dots & \mathcal{E}_{nn} \end{vmatrix}. \tag{3.7}$$

Similar to the expansion (3.6), we will look at the components of maximal weight of the coefficients of the resulting polynomial in τ. One verifies that nonzero contributions to these components could only come from the entries (i,j) and (j,j) of the j-th summand in (3.7) with $j \neq i$, as well as from the entries (k,i) of the i-th summand in (3.7) for $k = 1,\dots,n$. Moreover, modulo terms of lower weight, column j with $j \neq i$ in the j-th summand can be replaced with the column whose j-th component is $\tau^{\lambda_j - 1}\big(\varkappa_{ij} + \lambda_j\, E^{(0)}_{ii}[0]\big)$ and the i-th component is $\mathcal{E}_{ij}[0]$, while the remaining components are zero. Similarly, column i of the i-th summand in (3.7) can be replaced with the column whose (k,i) entry is $-\mathcal{E}_{ki}[0]$ for $k \neq i$, while the i-th entry is $\tau^{\lambda_i - 1}\big(\varkappa_{ii} + \lambda_i\, E^{(0)}_{ii}[0]\big)$. In other words, we find that modulo terms of lower weight, the commutator $\big[E^{(0)}_{ii}[1], \operatorname{cdet}\mathcal{E}\big]$ equals

$$\sum_{j=1}^{n} \varkappa_{ij}\, \mathcal{E}^{\widehat{\jmath}}_{\widehat{\jmath}}\, \tau^{\lambda_j - 1} + \sum_{j=1}^{n} \begin{vmatrix} \mathcal{E}_{11} & \dots & 0 & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \lambda_j\, \tau^{\lambda_j - 1} E^{(0)}_{ii}[0] & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & 0 & \dots & \mathcal{E}_{nn} \end{vmatrix}$$

plus the difference of two determinants

$$\begin{vmatrix} \mathcal{E}_{11} & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots \\ \mathcal{E}_{i1}[0] & \dots & \mathcal{E}_{in}[0] \\ \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & \mathcal{E}_{nn} \end{vmatrix} - \begin{vmatrix} \mathcal{E}_{11} & \dots & \mathcal{E}_{1i}[0] & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & \mathcal{E}_{ni}[0] & \dots & \mathcal{E}_{nn} \end{vmatrix},$$

where $\mathcal{E}^{\widehat{\jmath}}_{\widehat{\jmath}}$ denotes the column-determinant of the matrix obtained from $\mathcal{E}$ by deleting row and column j. Now we proceed as in [3] relying on Lemma 4.2 therein to evaluate the action of the elements of the form $E^{(0)}_{ii}[0]$, $\mathcal{E}_{ij}[0]$ and $\mathcal{E}_{ki}[0]$. Suppose first that $j > i$. For the generator $E^{(0)}_{ii}[0]$ occurring as the (j,j) entry we have

$$\begin{vmatrix} \mathcal{E}_{11} & \dots & 0 & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & E^{(0)}_{ii}[0] & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & 0 & \dots & \mathcal{E}_{nn} \end{vmatrix} = \sum_{m=j+1}^{n} \begin{vmatrix} \mathcal{E}_{11} & \dots & 0 & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \mathcal{E}_{im} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & 0 & \dots & \mathcal{E}_{nn} \end{vmatrix},$$

where row and column j are deleted in the column-determinants in the sum, and $\mathcal{E}_{im}$ occurs in row i and column m. In the case $j < i$ the same expansion holds plus the additional sum $-\sum_{j=1}^{i-1} \mathcal{E}^{\widehat{\jmath}}_{\widehat{\jmath}}$. Similarly, for the expression $\mathcal{E}_{ij}[0]$ occurring as the (i,j) entry, we can write

$$\begin{vmatrix} \mathcal{E}_{11} & \dots & 0 & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \mathcal{E}_{ij}[0] & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & 0 & \dots & \mathcal{E}_{nn} \end{vmatrix} = (-1)^{i+j} \sum_{m=j+1}^{n} \begin{vmatrix} \mathcal{E}_{11} & \dots & \big[\mathcal{E}_{ij}[0], \mathcal{E}_{1m}\big] & \dots & \mathcal{E}_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{n1} & \dots & \big[\mathcal{E}_{ij}[0], \mathcal{E}_{nm}\big] & \dots & \mathcal{E}_{nn} \end{vmatrix},$$

where row i and column j are deleted in the column-determinants in the sum. Furthermore, observe that the commutators $\big[\mathcal{E}_{ij}[0], \mathcal{E}_{lm}\big]$ with $l \neq j$ and $m \neq i$ do not contribute to the maximal weight components and so can be replaced by 0. The argument is completed as in [3] by analyzing the maximal weights of the coefficients of the powers of τ which occur in thus obtained resulting expression for $E_{ii}^{(0)}[1]\,\mathrm{cdet}\,\mathcal{E}$ in the vacuum module. Although $\mathcal{E}$ is no longer a Manin matrix in general, a version of [3, Lemma 4.3] is replaced by the property that swapping neighbouring columns results in a changed sign modulo elements of lower weight. Relations $E_{ii}^{(r)}[p]\,\mathrm{cdet}\,\mathcal{E} = 0$ modulo lower weight terms, obviously hold for $p > n$ by the degree observation. The remaining values of p and r are considered in a way similar to the above case with $r = 0$ and $p = 1$.

The proof of the second part of the theorem relies on [1, Thm 3.2]. It reduces the task to the verification that the symbols of the elements $S_1, \dots, S_N$ in the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{a}[t^{-1}]\big)$ coincide with the respective images of certain algebraically independent generators of the algebra of $\mathfrak{a}$-invariants $\mathrm{S}(\mathfrak{a})^{\mathfrak{a}}$ under the embedding $X \mapsto X[-1]$ for $X \in \mathfrak{a}$. The existence of such generators was established in [13], which proved *Premet's conjecture* in type A, in particular. On the other hand, by the remark following Corollary 2.4, we can see that the desired property holds for the generators of the algebra $\mathrm{S}(\mathfrak{a})^{\mathfrak{a}}$ produced explicitly in [2, Thm 4.1].

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School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au