A FINITE ELEMENT APPROXIMATION FOR THE STOCHASTIC MAXWELL–LANDAU–LIFSHITZ–GILBERT SYSTEM

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Abstract. The stochastic Landau–Lifshitz–Gilbert (LLG) equation coupled with the Maxwell equations (MLLG) describes creation of domain walls and vortices (fundamental objects for the novel nanostructured magnetic memories). We first reformulate the stochastic LLG equation as an equation with time-differentiable solutions. We then propose a convergent θ-linear scheme to approximate the solutions of the reformulated system. As a consequence, we prove the convergence of our numerical scheme, with no or minor conditions on time and space steps (depending on the value of θ) to a weak martingale solutions of the stochastic MLLG system. Numerical results are presented to show applicability of the method.

1. Introduction

The system of Landau–Lifshitz–Gilbert equation coupled with the Maxwell equations (MLLG) is fundamental for physics of ferromagnetic materials and applications to design and fabrication of magnetic memories. The theory and numerical methods for this system, see (1.1), (1.2) below, is an object of intense investigations, see for example [13] and the comments below.

It has been noticed a long time ago in physics community, see [27, 9] that, in order to explain the behaviour of ferromagnetic material, it is necessary to include random perturbations into the fundamental system MLLG. In particular, including random perturbations into the MLLG system is necessary for understanding the creation and stability of domain walls and vortices (fundamental objects for the novel nanostructured magnetic memories) [26]. This observation leads to a fully nonlinear system of stochastic partial differential equations. Only recently a rigorous analysis of a simplified model, in which the Maxwell equations were neglected, was initiated in [10] and the full stochastic MLLG system is studied in [23].

In this work we are concerned with the full Maxwell–Landau–Lifshitz–Gilbert system of stochastic partial differential equations. To the best of our knowledge the numerical analysis of the stochastic MLLG system given by equations (1.5)–(1.6) below is an open problem at present. In this paper we propose a convergent numerical scheme for solving this system, hence we also provide a proof of the existence of solutions to this system. This is in contrast to [23], where the existence of solutions is proved via

Date: November 12, 2019.

2000 *Mathematics Subject Classification.* Primary 35R60, 60H15, 65L60, 65L20; Secondary 82D45.

Key words and phrases. stochastic partial differential equation, martingale solutions, Landau–Lifshitz–Gilbert equation, Maxwell equation, finite element, ferromagnetism.

the Galerkin approximations that are not convenient for numerical approximations. In order to obtain a convergent numerical scheme for solving the stochastic MLLG system we will extend the novel method introduced in [15]. We note, that including the Maxwell equations into the system makes the problem of numerical simulations considerably more difficult. It becomes clear when the system is reduced to a single Landau–Lifschitz–Gilbert equation with an additional term which is non-local in space.

Let us introduce in more detail the system we will consider. We assume that a ferromagnetic material fills in a bounded domain $D \subset \mathbb{R}^3$. We also suppose that$D$ is located inside a (large) bounded cavity $\widetilde{D} \subset \mathbb{R}^3$ with perfectly conducting outer surface $\partial\widetilde{D}$), and $\widetilde{D}$ is filled in with an isotropic material with the scalar positive conductivity $\sigma : \widetilde{D} \to \mathbb{R}$.

With every point $x \in D$ we associate the value $\boldsymbol{M}(t,x)$ of the magnetisation vector at time $t \leq T$. It is known, see [9, 13], that for temperatures below the so-called Curie temperature the magnetisation vector has constant length and without loss of generality we will assume througout the paper that $|\boldsymbol{M}(t,x)| = 1$. By the theory of electromagnetics the evolving magnetisation vector gives rise to an evolving magnetic field $\boldsymbol{H}(t,x)$ defined in the full space, $x \in \mathbb{R}^3$. In this paper we consider only the so-called quasi-static approximation, when there are no electric charges in the system and the electric field is constant in time. This is the most popular model studied in the theory of ferromagnetic materials. Let $\widetilde{\boldsymbol{M}}$ denote an extension of $\boldsymbol{M}$ to the whole space that is equal to zero outside D. Then the MLLG is a system of two equations

$$\frac{\partial \boldsymbol{M}}{\partial t} = \lambda_1 \boldsymbol{M} \times \boldsymbol{H}_{\text{eff}} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}}) \quad \text{in } D, \tag{1.1}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times (\sigma \nabla \times \boldsymbol{H}) = -\mu_0 \frac{\partial \widetilde{\boldsymbol{M}}}{\partial t} \quad \text{in } \widetilde{D}, \tag{1.2}$$

where we assume that $\lambda_1 \neq 0$, $\lambda_2 > 0$, and $\mu_0 > 0$ are constants. The so-called effective magnetic field $\boldsymbol{H}_{\text{eff}}$ depends on the physical problem under consideration and its energy functional. In this paper we choose the effective field

$$\boldsymbol{H}_{\text{eff}} = \Delta \boldsymbol{M} + \boldsymbol{H}\,,$$

that contains contributions from the exchange energy and stray energy and represents the main sources of difficulties in the analysis of the MLLG system, see [9] for a thorough discussion of the physical theory. The system (1.1)–(1.2) is supplemented with the initial conditions

$$\boldsymbol{M}(0,\cdot) = \boldsymbol{M}_0 \text{ in } D \quad \text{and} \quad \boldsymbol{H}(0,\cdot) = \boldsymbol{H}_0 \text{ in } \widetilde{D}, \tag{1.3}$$

and the boundary conditions

$$\partial_{\boldsymbol{n}_D} \boldsymbol{M} = 0 \text{ on } (0,T) \times \partial D \quad \text{and} \quad (\nabla \times \boldsymbol{H}) \times \boldsymbol{n}_{\widetilde{D}} = 0 \text{ on } (0,T) \times \partial\widetilde{D}, \tag{1.4}$$

where $\boldsymbol{n}_D$ and $\boldsymbol{n}_{\widetilde{D}}$ are the unit vectors outward normal to D and $\widetilde{D}$, respectively. Here $\partial_{\boldsymbol{n}_D}$ denotes the normal derivative. We follow [6, 10] adding a noise to the effective field $\boldsymbol{H}_{\text{eff}}$ so that the stochastic version of the MLLG system takes the form

$$d\boldsymbol{M} = \big(\lambda_1 \boldsymbol{M} \times \boldsymbol{H}_{\text{eff}} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}})\big)\,dt + (\boldsymbol{M} \times \boldsymbol{g}) \circ dW(t) \ \text{ in } D_T, \tag{1.5}$$

$$\mu_0\, d\boldsymbol{H} + \nabla \times (\sigma \nabla \times \boldsymbol{H})\,dt = -\mu_0\, d\widetilde{\boldsymbol{M}} \ \text{ in } \widetilde{D}_T, \tag{1.6}$$

where $\boldsymbol{g} : D \to \mathbb{R}^3$ is a given bounded function, and W is a one-dimensional Wiener process. Here qe use the notation $D_T = (0,T) \times D$, and similar for $\widetilde{D}_T$. Here $\circ dW(t)$ stands for the Stratonovich differential. We assume without loss of generality that (see [10])

$$|\boldsymbol{g}(\boldsymbol{x})| = 1, \quad \boldsymbol{x} \in D \tag{1.7}$$

In the deterministic case, i.e. (1.1)–(1.2), the existence and uniqueness of a *local* strong solution is shown by Cimrák [12]. He also proposes [11] a finite element method to approximate this local solution and provides error estimation. Various results on the existence of global weak solutions are proved in [17, 18, 28] and a more complete list can be found in [13, 16, 20]. It should be noted that apart from [11], where a numerical scheme is suggested for a local solution, aforementioned results are non-constructive, in the sense that no computational techniques are proposed for the solution.

In [25], the stability of a semidiscrete scheme to numerically solve (1.1)–(1.2) is verified, but its convergence is not studied. Baňas, Bartels and Prohl [4] propose an implicit *nonlinear* scheme to solve the MLLG system, and succeed in proving that the finite element solution converges to a weak *global* solution of the problem. A θ-*linear* finite element scheme is proposed in [7, 21, 22] to find a weak *global* solution to the MLLG system, and convergence of the numerical solutions is proved with no condition imposed on time step and space step if $\theta \in (\frac{1}{2}, 1]$. It should be mentioned that the proofs of existence proposed in [4, 7, 21, 22] are constructive proofs, namely an approximate solution can be computed.

In the stochastic case, the Faedo–Galerkin method is used in [10] to show the existence of a weak martingale solution for the stochastic Landau–Lifshitz–Gilbert (LLG) equation (1.5). Finite element schemes for this equation are studied in [2, 6, 15] which prove that the numerical solutions converge to a weak martingale solution. It is noted that a *non-linear* scheme is proposed in [6] and *linear* schemes are proposed in [2, 15].

A full version of the stochastic Landau–Lifshitz equation coupled with the Maxwell's equations is studied firstly in [23, Section 5] where the existence of the weak martingale solution and its regularity are proved by using the Faedo-Galerkin approximation, the methods of compactness and Skorokhod's Theorem.

In this paper, we extend the θ-*linear* finite element scheme developed in [22] for the deterministic MLLG system to the stochastic case. Since this scheme seeks to approximate the time derivative of the magnetization $\boldsymbol{M}$, we adopt the technique in [15] in order to reformulate system (1.5)–(1.6) as a system of deterministic PDEs with random coefficients but with the Stratonovich differential $\circ dW(t)$ removed. Then the θ-*linear* scheme mentioned above can be applied. As a consequence, we prove that the numerical scheme converges to a weak martingale solution of the stochastic MLLG system.

The paper is organised as follows. In Section 2 we intorduce the notations to be used, and recall some technical results. In Section 3 we define weak martingale solutions to (1.5)–(1.6) and state our main result. Details of the reformulation of (1.5) are presented in Section 4. We also show in this section how a weak solution to (1.5)–(1.6) can be obtained from a weak solution of the reformulated system. In Section 5, we introduce our finite element scheme and present a proof of the convergence of finite element solutions to a weak solution of the reformulated system. Section 6 is devoted to the proof of the main theorem. Our numerical experiments are presented in Section 7.

Throughout this paper, c denotes a generic constant which may take different values at different occurences.

2. Notations and technical results

2.1. **Notations.** In this subsection, we introduce some function spaces and notations which are used in the rest of this paper.

For any open set $U \subset \mathbb{R}^3$, the *curl* operator of a vector function $\boldsymbol{u} = (u_1, u_2, u_3)$ defined on U is denoted by

$$\operatorname{curl} \boldsymbol{u} = \nabla \times \boldsymbol{u} := \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right),$$

provided the partial derivatives exist. The function spaces $\mathbb{H}^1(U)$ and $\mathbb{H}(\operatorname{curl}; U)$ are defined, respectively, by

$$\mathbb{H}^1(U) = \left\{ \boldsymbol{u} \in \mathbb{L}^2(U) : \frac{\partial \boldsymbol{u}}{\partial x_i} \in \mathbb{L}^2(U) \quad \text{for } i = 1, 2, 3 \right\},$$
$$\mathbb{H}(\operatorname{curl}; U) = \left\{ \boldsymbol{u} \in \mathbb{L}^2(U) : \nabla \times \boldsymbol{u} \in \mathbb{L}^2(U) \right\}.$$

Here, $\mathbb{L}^2(U)$ is the usual space of Lebesgue square integrable functions defined on U and taking values in $\mathbb{R}^3$. The inner product and norm in $\mathbb{L}^2(U)$ are denoted by $\langle \cdot, \cdot \rangle_U$ and $\| \cdot \|_U$, respectively.

For any vector functions $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, we denote

$$\begin{aligned}
\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} &:= \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{v}}{\partial x_i}, \\
\nabla \boldsymbol{u} \times \nabla \boldsymbol{v} &:= \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}}{\partial x_i} \times \frac{\partial \boldsymbol{v}}{\partial x_i}, \\
\boldsymbol{u} \times \nabla \boldsymbol{v} &:= \sum_{i=1}^{3} \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_i}, \\
(\boldsymbol{u} \times \nabla \boldsymbol{v}) \cdot \nabla \boldsymbol{w} &:= \sum_{i=1}^{3} \left(\boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_i} \right) \cdot \frac{\partial \boldsymbol{w}}{\partial x_i},
\end{aligned} \tag{2.1}$$

provided that the partial derivatives exist, at least in the weak sense. We also denote

$$\begin{aligned}
\mathbb{C}^\infty(\widetilde{D}) &:= \left\{\boldsymbol{u} : \widetilde{D} \to \mathbb{R}^3 \mid \boldsymbol{u} \text{ is infinitely differentiable}\right\},\\
\mathbb{C}^\infty_\times(\widetilde{D}) &:= \left\{\boldsymbol{u} \in \mathbb{C}^\infty(\widetilde{D}) \cap \mathbb{C}(\bar{D}) \mid (\nabla\times\boldsymbol{u})\times\boldsymbol{n}_D = 0 \text{ on } \partial D \text{ and } (\nabla\times\boldsymbol{u})\times\boldsymbol{n}_{\widetilde{D}} = 0 \text{ on } \partial\widetilde{D}\right\},\\
C^1_T(0,T;E) &:= \left\{\boldsymbol{u} : [0,T] \to E \mid \boldsymbol{u} \text{ is continuously differentiable and } \boldsymbol{u}(T) = 0 \text{ in } E\right\},\\
C^1_0(0,T;E) &:= \left\{\boldsymbol{u} : [0,T] \to E \mid \boldsymbol{u} \text{ is continuously differentiable and } \boldsymbol{u}(0) = \boldsymbol{u}(T) = 0 \text{ in } E\right\},
\end{aligned}$$

for any $T > 0$ and any normed vector space E.

2.2. **Technical results.** In this subsection we recall some results from [**?**]. They will be used in the next section to reformulate (3.1) to a new form.

Assume that $\boldsymbol{g} \in \mathbb{L}^\infty(D)$, and let $G : \mathbb{L}^2(D) \longrightarrow \mathbb{L}^2(D)$ be defined by

$$G\boldsymbol{u} = \boldsymbol{u}\times\boldsymbol{g} \quad \forall \boldsymbol{u} \in \mathbb{L}^2(D). \tag{2.2}$$

Then the operator G is bounded [**?**].

Lemma 2.1. *For any $s \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{L}^2(D)$ there hold*

$$e^{sG}\boldsymbol{u} = \boldsymbol{u} + (\sin s)G\boldsymbol{u} + (1-\cos s)G^2\boldsymbol{u}, \tag{2.3}$$

$$\left(e^{sG}\right)^* = e^{-sG}, \tag{2.4}$$

$$e^{sG}G\boldsymbol{u} = Ge^{sG}\boldsymbol{u}, \tag{2.5}$$

$$e^{sG}(\boldsymbol{u}\times\boldsymbol{v}) = e^{sG}\boldsymbol{u}\times e^{sG}\boldsymbol{v}. \tag{2.6}$$

In the proof of the existence of weak solutions we also need the following result for the operator e^{sG}.

Lemma 2.2. *Assume that $\boldsymbol{g} \in \mathbb{H}^2(D)$. For any $s \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{H}^1(D)$ and $\boldsymbol{v} \in \mathbb{W}^{1,\infty}_0(D)$, let*

$$\widetilde{C}(s,\boldsymbol{v}) = e^{-sG}\big((\sin s)C + (1-\cos s)(GC + CG)\big)\boldsymbol{v}$$

with C being defined by

$$C\boldsymbol{u} = \boldsymbol{u}\times\Delta\boldsymbol{g} + 2\sum_{i=1}^{3}\frac{\partial\boldsymbol{u}}{\partial x_i}\times\frac{\partial\boldsymbol{g}}{\partial x_i}.$$

There holds

$$\left\langle \widetilde{C}(s, e^{-sG}\boldsymbol{u}), \boldsymbol{v}\right\rangle_D = \left\langle \nabla\left(e^{-sG}\boldsymbol{u}\right), \nabla\boldsymbol{v}\right\rangle_D - \left\langle \nabla\boldsymbol{u}, \nabla\left(e^{sG}\boldsymbol{v}\right)\right\rangle_D,$$

From now on, we assume that $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$.

We finish this section by stating two elementary identities involving the dot and cross products of vectors in $\mathbb{R}^3$, which will be frequently used. For all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$, the following identities hold

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{2.7}$$

and

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}. \tag{2.8}$$

3. The main result

In this section we state the definition of a weak martingale solution to (1.5)–(1.6) and our main result. In what follows, our standing assumption is that the conductivity σ satisfies the condition

$$\sigma(x) = \sigma_D > 0 \text{ is constant for } x \in D \quad \text{and} \quad \inf_{x\in\widetilde{D}} \sigma(x) > 0\,.$$

Recalling that $\boldsymbol{H}_{\text{eff}} = \Delta\boldsymbol{M} + \boldsymbol{H}$, multiplying (1.5) by a test function $\boldsymbol{\psi} \in \mathbb{C}^\infty(D)$ and integrating over $(0,t)\times D$ we obtain formally

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_D - \langle \boldsymbol{M}_0, \boldsymbol{\psi}\rangle_D = \lambda_1 \int_0^t \langle \boldsymbol{M}\times\Delta\boldsymbol{M}, \boldsymbol{\psi}\rangle_D\, ds &+ \lambda_1 \int_0^t \langle \boldsymbol{M}\times\boldsymbol{H}, \boldsymbol{\psi}\rangle_D\, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}), \boldsymbol{\psi}\rangle_D\, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M}\times(\boldsymbol{M}\times\boldsymbol{H}), \boldsymbol{\psi}\rangle_D\, ds \\ &+ \int_0^t \langle \boldsymbol{M}\times\boldsymbol{g}, \boldsymbol{\psi}\rangle_D \circ dW.\end{aligned}$$

From (2.8), the Green identity and $\nabla\boldsymbol{M}\cdot(\nabla\boldsymbol{M}\times\boldsymbol{\psi}) = 0$ we define

$$\begin{aligned}\langle \boldsymbol{M}\times\Delta\boldsymbol{M}, \boldsymbol{\psi}\rangle_D &= -\langle \Delta\boldsymbol{M}, \boldsymbol{M}\times\boldsymbol{\psi}\rangle_D \\ &:= \langle \nabla\boldsymbol{M}, \nabla(\boldsymbol{M}\times\boldsymbol{\psi})\rangle_D \\ &= \langle \nabla\boldsymbol{M}, \nabla\boldsymbol{M}\times\boldsymbol{\psi}\rangle_D + \langle \nabla\boldsymbol{M}, \boldsymbol{M}\times\nabla\boldsymbol{\psi}\rangle_D \\ &= -\langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla\boldsymbol{\psi}\rangle_D\,,\end{aligned}$$

and similarly

$$\langle \boldsymbol{M}\times(\boldsymbol{M}\times\Delta\boldsymbol{M}), \boldsymbol{\psi}\rangle_D := \langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla(\boldsymbol{M}\times\boldsymbol{\psi})\rangle_D\,.$$

Therefore,

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_D - \langle \boldsymbol{M}_0, \boldsymbol{\psi}\rangle_D = -\lambda_1 \int_0^t \langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla\boldsymbol{\psi}\rangle_D\, ds &+ \lambda_1 \int_0^t \langle \boldsymbol{M}\times\boldsymbol{H}, \boldsymbol{\psi}\rangle_D\, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla(\boldsymbol{M}\times\boldsymbol{\psi})\rangle_D\, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M}\times(\boldsymbol{M}\times\boldsymbol{H}), \boldsymbol{\psi}\rangle_D\, ds + \int_0^t \langle \boldsymbol{M}\times\boldsymbol{g}, \boldsymbol{\psi}\rangle_D \circ dW(s).\end{aligned}$$

In the same manner, if we multiply (1.6) by a test function $\boldsymbol{\zeta} \in C^1_T(0,T;\mathbb{C}^\infty_\times(\widetilde{D}))$, integrate over $\widetilde{D}_T$, and note (1.3), then we obtain, formally,

$$\mu_0 \left\langle \boldsymbol{H} + \widetilde{\boldsymbol{M}}, \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0, \boldsymbol{\zeta}(0) \right\rangle_{\widetilde{D}} = \langle \nabla\times(\sigma\nabla\times\boldsymbol{H}), \boldsymbol{\zeta}\rangle_{\widetilde{D}_T}\,.$$

We remark that the time derivative is taken on $\boldsymbol{\zeta}$ because in general $\widetilde{\boldsymbol{M}}$ is not time differentiable. Since $(\nabla\times\boldsymbol{H})\times\boldsymbol{n}_{\widetilde{D}} = 0$, see (1.4), and $(\nabla\times\boldsymbol{\zeta})\times\boldsymbol{n}_{\widetilde{D}} = 0$, see the definition of $\mathbb{C}^\infty_\times(\widetilde{D})$ in Section 2, it follows from [24, Corollary 3.20] that

$$\langle \nabla \times (\sigma \nabla \times \boldsymbol{H}), \boldsymbol{\zeta} \rangle_{\widetilde{D}} = \langle \sigma \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\widetilde{D}}.$$

Hence

$$\mu_0 \left\langle \boldsymbol{H} + \widetilde{\boldsymbol{M}}, \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0, \boldsymbol{\zeta}(0) \right\rangle_{\widetilde{D}} = \langle \sigma \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\widetilde{D}_T}.$$

The above observations prompt us to define the solution of (1.5)–(1.6) as follows.

Definition 3.1. *Given $T \in (0, \infty)$, a weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, \boldsymbol{M}, \boldsymbol{H})$ to (1.5)–(1.6) on the time interval $[0,T]$, consists of*

(a) *a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with the filtration satisfying the usual conditions,*
(b) *a one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\in[0,T]}$,*
(c) *a progressively measurable process $\boldsymbol{M} : [0,T] \times \Omega \to \mathbb{L}^2(D)$,*
(d) *a progressively measurable process $\boldsymbol{H} : [0,T] \times \Omega \to \mathbb{L}^2(\widetilde{D})$*

such that there hold

(1) $\mathbb{P}\big(\boldsymbol{M} \in C([0,T]; \mathbb{H}^{-1}(D))\big) = 1$*;*
(2) $\mathbb{P}\big(\boldsymbol{H} \in L^2(0,T; \mathbb{H}(\mathrm{curl}; \widetilde{D}))\big) = 1$*;*
(3) $\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|\nabla \boldsymbol{M}(t)\|^2_D\right) < \infty$*;*
(4) *for all $t \in [0,T]$, $|\boldsymbol{M}(t,\cdot)| = 1$ a.e. in D, and $\mathbb{P}$-a.s.;*
(5) *for every $t \in [0,T]$, for all $\boldsymbol{\psi} \in \mathbb{C}^\infty(D)$, $\mathbb{P}$-a.s.:*

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi} \rangle_D - \langle \boldsymbol{M}_0, \boldsymbol{\psi} \rangle_D = &-\lambda_1 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \rangle_D \, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla (\boldsymbol{M} \times \boldsymbol{\psi}) \rangle_D \, ds \\ &+ \lambda_1 \int_0^t \langle \boldsymbol{M} \times \boldsymbol{H}, \boldsymbol{\psi} \rangle_D \, ds \\ &- \lambda_2 \int_0^t \langle \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}), \boldsymbol{\psi} \rangle_D \, ds \\ &+ \int_0^t \langle \boldsymbol{M} \times \boldsymbol{g}, \boldsymbol{\psi} \rangle_D \circ dW(s);\end{aligned} \tag{3.1}$$

(6) *for all $\boldsymbol{\zeta} \in C^1_T(0,T; \mathbb{C}^\infty_\times(\widetilde{D}))$, $\mathbb{P}$-a.s.:*

$$\mu_0 \left\langle \boldsymbol{H} + \widetilde{\boldsymbol{M}}, \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0, \boldsymbol{\zeta}(0,\cdot) \right\rangle_{\widetilde{D}} = \langle \sigma \nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\widetilde{D}_T}. \tag{3.2}$$

The main theorem of the paper is stated below.

Theorem 3.2. *Assume that $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$ satisfies (1.7) and $\big(\boldsymbol{M}_0, \boldsymbol{H}_0\big)$ satisfies*

$$\begin{aligned} &\boldsymbol{M}_0 \in \mathbb{H}^2(D), \quad |\boldsymbol{M}_0| = 1 \text{ a.e. in } D, \\ &(\boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0) \in \mathbb{H}^1(\widetilde{D}), \quad \nabla \times (\boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0) \in \mathbb{H}^1(\widetilde{D}).\end{aligned} \tag{3.3}$$

For each $T > 0$, there exists a weak martingale solution to (1.5)–(1.6).

Proof. The theorem is a direct consequence of Theorem 6.9. □

4. Equivalence of weak solutions

In this section, we use the operator G defined in Section 2 to define new variables $\boldsymbol{m}$ and $\boldsymbol{P}$ from $\boldsymbol{M}$ and $\boldsymbol{H}$.

Informally, if $\big(\boldsymbol{M},\boldsymbol{H}\big)$ is a weak solution to (3.1)–(3.2) then we can define new processes $\boldsymbol{m}$ and $\boldsymbol{P}$ (see (4.1)–(4.2) below) such that the Stratonovich differential $\circ dW(t)$ vanishes in the partial differential equation satisfied by $\boldsymbol{m}$. Moreover, it will be seen that $\boldsymbol{m}$ is differentiable with respect to t. We will make this argument more rigorous in the following lemma.

Let a filtered probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0.T]},\mathbb{P}\big)$ and a Wiener process $W(t)$ on it be given. We define a new processes $\boldsymbol{m}$ and $\boldsymbol{P}$ from processes $\boldsymbol{M}$ and $\boldsymbol{H}$

$$\boldsymbol{m}(t,\cdot) := e^{-W(t)G}\boldsymbol{M}(t,\cdot) \quad \forall t\ge 0,\ a.e.\ \text{in } D, \tag{4.1}$$

$$\boldsymbol{P}(t,\cdot) := \boldsymbol{H}(t,\cdot)+\widetilde{\boldsymbol{M}}(t,\cdot) \quad \forall t\ge 0,\ a.e.\ \text{in } \widetilde{D}, \tag{4.2}$$

$$\boldsymbol{P}_0 := \boldsymbol{H}_0+\widetilde{\boldsymbol{M}}_0 \quad a.e.\ \text{in } \widetilde{D},$$

where $\widetilde{\boldsymbol{M}}_0$ is the zero extension of $\boldsymbol{M}_0$ onto $\widetilde{D}$. Then it follows immediately from (**??**) and (2.4) that, for all $t\in[0,T]$ and almost all $x\in D$,

$$|\boldsymbol{M}(t,\cdot)| = 1 \quad \text{if and only if} \quad |\boldsymbol{m}(t,\cdot)| = 1. \tag{4.3}$$

The following lemma shows that in order to find $\boldsymbol{M}$ and $\boldsymbol{H}$, it suffices to find $\boldsymbol{m}$ and $\boldsymbol{P}$.

Lemma 4.1. *Let* $\boldsymbol{m}\in H^1\big(0,T;\mathbb{H}^1(D)\big)$ *and* $\boldsymbol{P}\in L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))$, $\mathbb{P}$-*a.s., satisfy*

$$\begin{aligned}
&\langle \boldsymbol{m}_t,\boldsymbol{\xi}\rangle_{D_T} + \lambda_1\langle \boldsymbol{m}\times\nabla\boldsymbol{m},\nabla\boldsymbol{\xi}\rangle_{D_T} + \lambda_2\langle \boldsymbol{m}\times\nabla\boldsymbol{m},\nabla(\boldsymbol{m}\times\boldsymbol{\xi})\rangle_{D_T} - \langle F(t,\boldsymbol{m}),\boldsymbol{\xi}\rangle_{D_T}\\
&\qquad - \lambda_1\big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P},\boldsymbol{\xi}\big\rangle_{D_T} + \lambda_2\big\langle \boldsymbol{m}\times(\boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}),\boldsymbol{\xi}\big\rangle_{D_T} = 0
\end{aligned} \tag{4.4}$$

and

$$\mu_0\langle \boldsymbol{P},\boldsymbol{\zeta}_t\rangle_{\widetilde{D}_T} - \mu_0\langle \boldsymbol{P}_0,\boldsymbol{\zeta}(0,\cdot)\rangle_{\widetilde{D}} = \langle \sigma\nabla\times\boldsymbol{P},\nabla\times\boldsymbol{\zeta}\rangle_{\widetilde{D}_T} - \big\langle \sigma\nabla\times(e^{W(t)G}\boldsymbol{m}),\nabla\times\boldsymbol{\zeta}\big\rangle_{D_T}, \tag{4.5}$$

where

$$F(t,\boldsymbol{m}) = \lambda_1\boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m}) - \lambda_2\boldsymbol{m}\times(\boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m})) \tag{4.6}$$

for all $\boldsymbol{\xi}\in L^2\big(0,T;\mathbb{W}^{1,\infty}(D)\big)$ *and* $\boldsymbol{\zeta}\in C^1_T(0,T;\mathbb{C}^\infty_\times(\widetilde{D}))$, *with* $\widetilde{C}$ *defined in Lemma 2.2. Then* $\boldsymbol{M}=e^{W(t)G}\boldsymbol{m}$ *and* $\boldsymbol{H}=\boldsymbol{P}-\widetilde{\boldsymbol{M}}$ *satisfy* (3.1)–(3.2) $\mathbb{P}$-*a.s.*

Proof.
<u>*Step 1:* $\boldsymbol{M}$ and $\boldsymbol{H}$ satisfy (3.1):</u>

Since $e^{W(t)G}$ is a semimartingale and $\boldsymbol{m}$ is absolutely continuous, using Itô's formula for $\boldsymbol{M} = e^{W(t)G}\boldsymbol{m}$ (see e.g. [14]), we deduce

$$\begin{aligned}\boldsymbol{M}(t) &= \boldsymbol{M}(0) + \int_0^t Ge^{W(s)G}\boldsymbol{m}\,dW(s) + \int_0^t \frac{1}{2}G^2 e^{W(s)G}\boldsymbol{m}\,ds + \int_0^t e^{W(s)G}\boldsymbol{m}_t\,ds \\ &= \boldsymbol{M}(0) + \int_0^t G\boldsymbol{M}\,dW(s) + \frac{1}{2}\int_0^t G^2\boldsymbol{M}\,ds + \int_0^t e^{W(s)G}\boldsymbol{m}_t\,ds,\end{aligned} \tag{4.7}$$

where the first integral on the right-hand side is an Itô integral and the last two are Bochner integrals. Recalling the relation between the Stratonovich and Itô differentials, namely

$$(G\boldsymbol{u})\circ dW(s) = G\boldsymbol{u}\,dW(s) + \frac{1}{2}G'(\boldsymbol{u})[G\boldsymbol{u}]\,ds, \tag{4.8}$$

and noting that $G'(\boldsymbol{u})[G\boldsymbol{u}] = G^2\boldsymbol{u}$, we rewrite (4.7) in the Stratonovich form as

$$\boldsymbol{M}(t) = \boldsymbol{M}(0) + \int_0^t G\boldsymbol{M}\circ dW(s) + \int_0^t e^{W(s)G}\boldsymbol{m}_t\,ds.$$

Multiplying both sides of the above equation by a test function $\boldsymbol{\psi} \in \mathbb{C}^\infty(D)$ and integrating over D we obtain

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_D &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_D + \int_0^t \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(s) + \int_0^t \left\langle e^{W(s)G}\boldsymbol{m}_t, \boldsymbol{\psi}\right\rangle_D\,ds \\ &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_D + \int_0^t \langle G\boldsymbol{M}, \boldsymbol{\psi}\rangle_D \circ dW(s) + \int_0^t \left\langle \boldsymbol{m}_t, e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds,\end{aligned} \tag{4.9}$$

where in the last step we used (2.4).

On the other hand, we note that $e^{-W(\cdot)G}\boldsymbol{\psi} \in L^2\big(0,t;\mathbb{W}^{1,\infty}(D)\big)$ for $t\in[0,T]$. Let the test function $\boldsymbol{\xi}$ in (4.4) be $e^{-W(\cdot)G}\boldsymbol{\psi}$, we obtain from (4.6) that

$$\begin{aligned}\int_0^t \left\langle \boldsymbol{m}_t, e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds &= -\lambda_1\int_0^t \left\langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\left(e^{-W(s)G}\boldsymbol{\psi}\right)\right\rangle_D\,ds \\ &\quad - \lambda_2\int_0^t \left\langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\left(\boldsymbol{m}\times\left(e^{-W(s)G}\boldsymbol{\psi}\right)\right)\right\rangle_D\,ds \\ &\quad + \lambda_1\int_0^t \left\langle \boldsymbol{m}\times\widetilde{C}(W(s),\boldsymbol{m}), e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds \\ &\quad - \lambda_2\int_0^t \left\langle \boldsymbol{m}\times\Big(\boldsymbol{m}\times\widetilde{C}(W(s),\boldsymbol{m})\Big), e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds \\ &\quad + \lambda_1\int_0^t \left\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds \\ &\quad - \lambda_2\int_0^t \left\langle \boldsymbol{m}\times(\boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}), e^{-W(s)G}\boldsymbol{\psi}\right\rangle_D\,ds \\ &=: \int_0^t (T_1(s)+\cdots+T_6(s))\,ds.\end{aligned}$$

Considering T_3, we use successively (2.8), Lemma 2.2, (4.1), and (2.6) to obtain

$$\begin{aligned}
T_3(s) &= \lambda_1 \left\langle \boldsymbol{m} \times \widetilde{C}(W(s), \boldsymbol{m}), e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D = -\lambda_1 \left\langle \widetilde{C}(W(s), \boldsymbol{m}), \boldsymbol{m} \times e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D \\
&= -\lambda_1 \left\langle \nabla \boldsymbol{m}, \nabla \left(\boldsymbol{m} \times e^{-W(s)G}\boldsymbol{\psi} \right) \right\rangle_D + \lambda_1 \left\langle \nabla \boldsymbol{M}, \nabla \left(\boldsymbol{M} \times \boldsymbol{\psi} \right) \right\rangle_D \\
&= -\lambda_1 \left\langle \nabla \boldsymbol{m}, \boldsymbol{m} \times \nabla \left(e^{-W(s)G}\boldsymbol{\psi} \right) \right\rangle_D + \lambda_1 \left\langle \nabla \boldsymbol{M}, \boldsymbol{M} \times \nabla \boldsymbol{\psi} \right\rangle_D \\
&= \lambda_1 \left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla \left(e^{-W(s)G}\boldsymbol{\psi} \right) \right\rangle_D - \lambda_1 \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \right\rangle_D .
\end{aligned}$$

Therefore,

$$T_1 + T_3 = -\lambda_1 \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \right\rangle_D .$$

Similarly, considering T_4 we have

$$\begin{aligned}
T_4(s) &= -\lambda_2 \left\langle \boldsymbol{m} \times \left(\boldsymbol{m} \times \widetilde{C}(W(s), \boldsymbol{m}) \right), e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D \\
&= \lambda_2 \left\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla \left(\boldsymbol{m} \times e^{-W(s)G}\boldsymbol{\psi} \right) \right\rangle_D - \lambda_2 \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla (\boldsymbol{M} \times \boldsymbol{\psi}) \right\rangle_D ,
\end{aligned}$$

so that

$$T_2 + T_4 = -\lambda_2 \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla (\boldsymbol{M} \times \boldsymbol{\psi}) \right\rangle_D .$$

On the other hand, by using (2.4), (2.6), and noting that $\boldsymbol{P} = \boldsymbol{H} + \boldsymbol{M}$ in D, we obtain

$$T_5(s) = \lambda_1 \left\langle \boldsymbol{m} \times e^{-W(s)G}\boldsymbol{P}, e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D = \lambda_1 \left\langle \boldsymbol{M} \times \boldsymbol{H}, \boldsymbol{\psi} \right\rangle_D$$

and

$$T_6(s) = -\lambda_2 \left\langle \boldsymbol{m} \times \left(\boldsymbol{m} \times e^{-W(s)G}\boldsymbol{P} \right), e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D = -\lambda_2 \left\langle \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}), \boldsymbol{\psi} \right\rangle_D .$$

Therefore,

$$\begin{aligned}
\int_0^t \left\langle \boldsymbol{m}_t, e^{-W(s)G}\boldsymbol{\psi} \right\rangle_D ds &= -\lambda_1 \int_0^t \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \right\rangle_D ds - \lambda_2 \int_0^t \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla (\boldsymbol{M} \times \boldsymbol{\psi}) \right\rangle_D ds \\
&\quad + \lambda_1 \int_0^t \left\langle \boldsymbol{M} \times \boldsymbol{H}, \boldsymbol{\psi} \right\rangle_D ds - \lambda_2 \int_0^t \left\langle \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}), \boldsymbol{\psi} \right\rangle_D ds.
\end{aligned}$$

This equation and (4.9) give

$$\begin{aligned}
\left\langle \boldsymbol{M}(t), \boldsymbol{\psi} \right\rangle_D &= \left\langle \boldsymbol{M}(0), \boldsymbol{\psi} \right\rangle_D + \int_0^t \left\langle G\boldsymbol{M}, \boldsymbol{\psi} \right\rangle_D \circ dW(s) \\
&\quad - \lambda_1 \int_0^t \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \right\rangle_D ds - \lambda_2 \int_0^t \left\langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla (\boldsymbol{M} \times \boldsymbol{\psi}) \right\rangle_D ds \\
&\quad + \lambda_1 \int_0^t \left\langle \boldsymbol{M} \times \boldsymbol{H}, \boldsymbol{\psi} \right\rangle_D ds - \lambda_2 \int_0^t \left\langle \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}), \boldsymbol{\psi} \right\rangle_D ds.
\end{aligned}$$

Hence, $\boldsymbol{M}$ and $\boldsymbol{H}$ satisfy (3.1).

Step 2: $\boldsymbol{M}$ and $\boldsymbol{H}$ satisfy (3.2):

This follows immediately from (4.5) and the fact that

$$\left\langle \sigma \nabla \times \widetilde{\boldsymbol{M}}, \nabla \times \boldsymbol{\zeta} \right\rangle_{\widetilde{D}_T} = \left\langle \sigma \nabla \times \boldsymbol{M}, \nabla \times \boldsymbol{\zeta} \right\rangle_{D_T} ,$$

completing the proof of the lemma. □

In the next lemma we provide an equivalence of equation (4.4), namely its Gilbert form.

Lemma 4.2. *Assume that* $\boldsymbol{m} \in H^1\big(0,T;\mathbb{H}^1(D)\big)$ *and* $\boldsymbol{P} \in \mathbb{L}^2(\widetilde{D}_T)$, $\mathbb{P}$-*a.s., satisfy*

$$|\boldsymbol{m}(t,\cdot)| = 1, \quad t \in (0,T), \text{ a.e. in } D, \ \mathbb{P}\text{-a.s.} \tag{4.10}$$

Assume further that $(\boldsymbol{m},\boldsymbol{P})$ *satisfies* $\mathbb{P}$-*a.s.*

$$\begin{aligned}\lambda_1 \langle \boldsymbol{m}\times\boldsymbol{m}_t, \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T} - \lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T} - \mu \langle \nabla\boldsymbol{m}, \boldsymbol{m}\times\nabla\boldsymbol{\varphi}\rangle_{D_T} \\ - \langle R(t,\boldsymbol{m}), \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T} + \mu \left\langle e^{-W(t)G}\boldsymbol{P}, \boldsymbol{m}\times\boldsymbol{\varphi}\right\rangle_{D_T} = 0,\end{aligned} \tag{4.11}$$

for all $\boldsymbol{\varphi} \in L^2\big(0,T;\mathbb{H}^1(D)\big)$, *where* $\mu = \lambda_1^2 + \lambda_2^2$ *and*

$$R(t,\boldsymbol{m}) = -\lambda_1^2 \widetilde{C}\big(W(t),\boldsymbol{m}\big) + \lambda_2^2 \boldsymbol{m}\times\big(\boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m})\big),$$

with $\widetilde{C}$ *defined in Lemma 2.2. Then* $(\boldsymbol{m},\boldsymbol{P})$ *satisfies* (4.4) $\mathbb{P}$-*a.s.*

Proof. Firstly, we observe that for each $\boldsymbol{\xi} \in L^2\big(0,T;\mathbb{W}^{1,\infty}(D)\big)$, due to Lemma 8.1, there exists $\boldsymbol{\varphi} \in L^2\big(0,T;\mathbb{H}^1(D)\big)$ satisfying

$$\boldsymbol{\xi} = \lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m}. \tag{4.12}$$

Next we derive some identities which will be used later in the proof. By using (2.7) and noting (4.10) (so that $\boldsymbol{m}\cdot\boldsymbol{m}_t = 0$), we have

$$\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{m}_t) = -\boldsymbol{m}_t. \tag{4.13}$$

Moreover,

$$\boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m}) = \boldsymbol{\varphi} - (\boldsymbol{m}\cdot\boldsymbol{\varphi})\boldsymbol{m} \quad \text{and} \quad \nabla\big(\boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m})\big) = \nabla\boldsymbol{\varphi} - \nabla\big((\boldsymbol{m}\cdot\boldsymbol{\varphi})\boldsymbol{m}\big).$$

The above identities and (2.1) imply

$$\big(\boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}\big)\cdot\big(\boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m})\big) = \big(\boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}\big)\cdot\boldsymbol{\varphi} \tag{4.14}$$

and

$$\begin{aligned}(\boldsymbol{m}\times\nabla\boldsymbol{m})\cdot\nabla\big(\boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m})\big) &= (\boldsymbol{m}\times\nabla\boldsymbol{m})\cdot\nabla\boldsymbol{\varphi} \\ &\quad - \sum_{i=1}^{3}\left(\boldsymbol{m}\times\frac{\partial\boldsymbol{m}}{\partial\, x_i}\right)\cdot\left(\frac{\partial(\boldsymbol{m}\cdot\boldsymbol{\varphi})}{\partial x_i}\boldsymbol{m} + (\boldsymbol{m}\cdot\boldsymbol{\varphi})\frac{\partial\boldsymbol{m}}{\partial x_i}\right) \\ &= (\boldsymbol{m}\times\nabla\boldsymbol{m})\cdot\nabla\boldsymbol{\varphi},\end{aligned} \tag{4.15}$$

where in the last step we used the elementary property $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a} = 0$ for all $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3$.

Now consider each term on the left-hand side of (4.4). By using (4.12)–(4.15) and noting (2.8) we obtain

$$
\begin{aligned}
\langle \boldsymbol{m}_t, \boldsymbol{\xi}\rangle_{D_T} &= \lambda_1 \langle \boldsymbol{m}_t, \boldsymbol{\varphi}\rangle_{D_T} + \lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{\varphi}\times\boldsymbol{m}\rangle_{D_T} \\
&= -\lambda_1 \langle \boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{m}_t), \boldsymbol{\varphi}\rangle_{D_T} - \lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T} \\
&= \lambda_1 \langle \boldsymbol{m}\times\boldsymbol{m}_t, \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T} - \lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T}, \\
\lambda_1 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\boldsymbol{\xi}\rangle_{D_T} &= \lambda_1^2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\boldsymbol{\varphi}\rangle_{D_T} + \lambda_1\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{D_T} \\
&= -\lambda_1^2 \langle \nabla\boldsymbol{m}, \boldsymbol{m}\times\nabla\boldsymbol{\varphi}\rangle_{D_T} + \lambda_1\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{D_T}, \\
\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{m}\times\boldsymbol{\xi})\rangle_{D_T} &= \lambda_1\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{m}\times\boldsymbol{\varphi})\rangle_{D_T} \\
&\quad + \lambda_2^2 \big\langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\big(\boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m})\big)\big\rangle_{D_T} \\
&= -\lambda_1\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{D_T} + \lambda_2^2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla\boldsymbol{\varphi}\rangle_{D_T} \\
&= -\lambda_1\lambda_2 \langle \boldsymbol{m}\times\nabla\boldsymbol{m}, \nabla(\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{D_T} - \lambda_2^2 \langle \nabla\boldsymbol{m}, \boldsymbol{m}\times\nabla\boldsymbol{\varphi}\rangle_{D_T}, \\
-\langle F(t,\boldsymbol{m}), \boldsymbol{\xi}\rangle_{D_T} &= -\lambda_1 \langle F(t,\boldsymbol{m}), \boldsymbol{\varphi}\rangle_{D_T} - \lambda_2 \langle F(t,\boldsymbol{m}), \boldsymbol{\varphi}\times\boldsymbol{m}\rangle_{D_T} \\
&= -\lambda_1^2 \Big\langle \boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m}), \boldsymbol{\varphi}\Big\rangle_{D_T} \\
&\quad + \lambda_2^2 \Big\langle \boldsymbol{m}\times\big(\boldsymbol{m}\times\widetilde{C}(W(t),\boldsymbol{m})\big), \boldsymbol{\varphi}\times\boldsymbol{m}\Big\rangle_{D_T} \\
&= -\langle R(t,\boldsymbol{m}), \boldsymbol{m}\times\boldsymbol{\varphi}\rangle_{D_T}, \\
-\lambda_1 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\xi}\big\rangle_{D_T} &= \lambda_1^2 \big\langle e^{-W(t)G}\boldsymbol{P}, \boldsymbol{m}\times\boldsymbol{\varphi}\big\rangle_{D_T} \\
&\quad - \lambda_1\lambda_2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\varphi}\times\boldsymbol{m}\big\rangle_{D_T}, \\
\lambda_2 \big\langle \boldsymbol{m}\times\big(\boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}\big), \boldsymbol{\xi}\big\rangle_{D_T} &= \lambda_1\lambda_2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\varphi}\times\boldsymbol{m}\big\rangle_{D_T} \\
&\quad - \lambda_2^2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{m}\times(\boldsymbol{\varphi}\times\boldsymbol{m})\big\rangle_{D_T} \\
&= \lambda_1\lambda_2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\varphi}\times\boldsymbol{m}\big\rangle_{D_T} \\
&\quad - \lambda_2^2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\varphi}\big\rangle_{D_T} \\
&= \lambda_1\lambda_2 \big\langle \boldsymbol{m}\times e^{-W(t)G}\boldsymbol{P}, \boldsymbol{\varphi}\times\boldsymbol{m}\big\rangle_{D_T} \\
&\quad + \lambda_2^2 \big\langle e^{-W(t)G}\boldsymbol{P}, \boldsymbol{m}\times\boldsymbol{\varphi}\big\rangle_{D_T}.
\end{aligned}
$$

Adding the above equations side by side we deduce that the left-hand side of (4.4) equals that of (4.11). Thus (4.4) holds if (4.11) holds. The lemma is proved. □

Thanks to Lemma 4.1 and Lemma 4.2, in order to solve (1.5)–(1.6), we solve (4.11) and (4.5). It is therefore necessary to define the weak martingale solutions for these two latter equations.

Definition 4.3. *Given $T \in (0,\infty)$, a weak martingale solution to* (4.11) *and* (4.5) *on the time interval* $[0,T]$*, denoted by* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P},W,\boldsymbol{m},\boldsymbol{P})$*, consists of*

(a) *a filtered probability space* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ *with the filtration satisfying the usual conditions,*

(b) *a one-dimensional* $(\mathcal{F}_t)$*-adapted Wiener process* $W = (W_t)_{t\in[0,T]}$,
(c) *a progressively measurable process* $\boldsymbol{m} : [0,T]\times\Omega \to \mathbb{L}^2(D)$,
(d) *a progressively measurable process* $\boldsymbol{P} : [0,T]\times\Omega \to \mathbb{L}^2(\widetilde{D})$,

such that there hold

(1) $\boldsymbol{m} \in \mathbb{H}^1(D_T)$, $\mathbb{P}$*-a.s.;*
(2) $\boldsymbol{P} \in L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))$, $\mathbb{P}$*-a.s.;*
(3) $\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|\nabla \boldsymbol{m}(t)\|_D^2\right) < \infty$*;*
(4) $|\boldsymbol{m}(t,\cdot)| = 1$ *for all* $t \in [0,T]$*, a.e. in* D*, and* $\mathbb{P}$*-a.s.;*
(5) $(\boldsymbol{m}, \boldsymbol{P})$ *satisfies* (4.11) *and* (4.5) $\mathbb{P}$*-a.s.*

We state the following lemma which is a direct consequence of Lemma 4.1, Lemma 4.2, and statement (4.3).

Lemma 4.4. *If* $(\boldsymbol{m}, \boldsymbol{P})$ *is a weak martingale solution of* (4.11) *and* (4.5) *in the sense of Definition 4.3, then* $(\boldsymbol{M}, \boldsymbol{H})$ *is a weak martingale solution of* (1.5) *and* (1.6) *in the sense of Definition 3.1.*

In the next section, we present a finite element scheme to approximate the solutions of (4.11) and (4.5).

5. The finite element scheme

In this section we introduce the θ-linear finite element scheme which approximates a weak solution $(\boldsymbol{m}, \boldsymbol{P})$ defined in Definition 4.3.

Let $\mathbb{T}_h$ be a regular tetrahedrization of the domain $\widetilde{D}$ into tetrahedra of maximal mesh-size h. Let $\mathbb{T}_h|_D$ be its restriction to $D \subset \widetilde{D}$. We denote by $\mathcal{N}_h := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ the set of vertices in $\mathbb{T}_h|_D$ and by $\mathcal{M}_h := \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_M\}$ the set of edges in $\mathbb{T}_h$.

To discretize the equation (4.11), we introduce the finite element space $\mathbb{V}_h \subset \mathbb{H}^1(D)$ defined by

$$\mathbb{V}_h := \left\{ \boldsymbol{u} \in \mathbb{H}^1(D) : \boldsymbol{u}|_K \in \left(P_1|_K\right)^3 \quad \forall K \in \mathbb{T}_K \right\},$$

where P_1 is the set of polynomials of maximum total degree 1 in x_1, x_2, x_3. A basis for $\mathbb{V}_h$ can be chosen to be $\{\phi_n\boldsymbol{\xi}_1, \phi_n\boldsymbol{\xi}_2, \phi_n\boldsymbol{\xi}_3\}_{1\le n\le N}$, where ϕ_n is a continuous piecewise linear function on $\mathbb{T}_h$ satisfying $\phi_n(\boldsymbol{x}_m) = \delta_{n,m}$ (the Kronecker delta) and $\{\boldsymbol{\xi}_j\}_{j=1,\cdots,3}$ is the canonical basis for $\mathbb{R}^3$. The interpolation operator from $\mathbb{C}^0(D)$ onto $\mathbb{V}_h$ is defined by

$$I_{\mathbb{V}_h}(\boldsymbol{v}) = \sum_{n=1}^{N} \boldsymbol{v}(\boldsymbol{x}_n)\phi_n(\boldsymbol{x}) \quad \forall \boldsymbol{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

To discretize (4.5), we introduce the lowest order edge elements of Nédélec's first family (see [24]) defined by

$$\mathbb{Y}_h := \left\{ \boldsymbol{u} \in \mathbb{H}(\mathrm{curl};\widetilde{D}) : \boldsymbol{u}|_K \in \mathcal{D}_K \quad \forall K \in \mathbb{T}_h \right\},$$

where

$$\mathcal{D}_K := \left\{ \boldsymbol{v} : K \to \mathbb{R}^3 \,:\, \exists \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \text{ such that } \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} \quad \forall \boldsymbol{x} \in K \right\}.$$

A basis $\{\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_M\}$ of $\mathbb{Y}_h$ can be defined by

$$\int_{\boldsymbol{e}_p} \boldsymbol{\psi}_q \cdot \boldsymbol{\tau}_p \, ds = \delta_{q,p} \,,$$

where $\boldsymbol{\tau}_p$ is the unit vector in the direction of edge $\boldsymbol{e}_p$. For any $\delta > 0$ and $p > 2$, the interpolation operator $I_{\mathbb{Y}_h}$ from $\mathbb{H}^{1/2+\delta}(\widetilde{D}) \cap \mathbb{W}^{1,p}(\widetilde{D})$ onto $\mathbb{Y}_h$ is defined by

$$I_{\mathbb{Y}_h}(\boldsymbol{u}) = \sum_{q=1}^{M} u_q \boldsymbol{\psi}_q \quad \forall \boldsymbol{u} \in \mathbb{H}^{1/2+\delta}(\widetilde{D}) \cap \mathbb{W}^{1,p}(\widetilde{D}),$$

where $u_q = \int_{\boldsymbol{e}_q} \boldsymbol{u} \cdot \boldsymbol{\tau}_q \, ds$.

Before introducing our approximation scheme, we state the following result, proved in [5], which will be used in the analysis.

Lemma 5.1. *If there holds*

$$\int_D \nabla\phi_i \cdot \nabla\phi_j \, d\boldsymbol{x} \le 0 \quad \textit{for all} \quad i, j \in \{1, 2, \cdots, N\} \textit{ and } i \ne j, \tag{5.1}$$

then for all $\boldsymbol{u} \in \mathbb{V}_h$ *satisfying* $|\boldsymbol{u}(\boldsymbol{x}_l)| \ge 1$, $l = 1, 2, \cdots, N$, *there holds*

$$\int_D \left| \nabla I_{\mathbb{V}_h} \left(\frac{\boldsymbol{u}}{|\boldsymbol{u}|} \right) \right|^2 d\boldsymbol{x} \le \int_D |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}. \tag{5.2}$$

When $d = 2$, condition (5.1) holds for Delaunay triangulations. When $d = 3$, it holds if all dihedral angles of the tetrahedra in $\mathbb{T}_h|_D$ are less than or equal to $\pi/2$; see [5]. In the sequel we assume that (5.1) holds.

With the finite element spaces defined as above, we are ready to define our approximation scheme. Fixing a positive integer J, we choose the time step k to be $k = T/J$ and define $t_j = jk$, $j = 0, \cdots, J$. For $j = 1, 2, \dots, J$, the functions $\boldsymbol{m}(t_j, \cdot)$ and $\boldsymbol{P}(t_j, \cdot)$ are approximated by $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$ and $\boldsymbol{P}_h^{(j)} \in \mathbb{Y}_h$, respectively. If $\boldsymbol{v}_h^{(j)}$ is an approximation of $\boldsymbol{m}_t(t_j, \cdot)$, then since

$$\boldsymbol{m}_t(t_j, \cdot) \approx \frac{\boldsymbol{m}(t_{j+1}, \cdot) - \boldsymbol{m}(t_j, \cdot)}{k} \approx \frac{\boldsymbol{m}_h^{(j+1)} - \boldsymbol{m}_h^{(j)}}{k},$$

we can define $\boldsymbol{m}_h^{(j+1)}$ from $\boldsymbol{m}_h^{(j)}$ by

$$\boldsymbol{m}_h^{(j+1)} := \boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}, \tag{5.3}$$

To maintain the condition $|\boldsymbol{m}_h^{(j+1)}| = 1$, we normalise the right-hand side of (5.3) and therefore define $\boldsymbol{m}_h^{(j+1)}$ belonging to $\mathbb{V}_h$ by

$$\boldsymbol{m}_h^{(j+1)} = I_{\mathbb{V}_h} \left(\frac{\boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}}{|\boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}|} \right) = \sum_{n=1}^{N} \frac{\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)}{\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|} \phi_n,$$

which ensures that $|\boldsymbol{m}_h^{(j+1)}| = 1$ at vertices. Hence it suffices to propose a scheme to compute $\boldsymbol{v}_h^{(j)}$.

We first rewrite (4.11) as

$$\begin{aligned}
&\lambda_2 \langle \boldsymbol{m}_t, \boldsymbol{w}\rangle_{D_T} - \lambda_1 \langle \boldsymbol{m}\times\boldsymbol{m}_t, \boldsymbol{w}\rangle_{D_T} + \mu \langle \nabla\boldsymbol{m}, \nabla\boldsymbol{w}\rangle_{D_T} \\
&\qquad = -\langle R(t,\boldsymbol{m}), \boldsymbol{w}\rangle_{D_T} + \mu \left\langle e^{-W(t)G}\boldsymbol{P}, \boldsymbol{w}\right\rangle_{D_T} \qquad (5.4)
\end{aligned}$$

where $\boldsymbol{w} = \boldsymbol{m}\times\boldsymbol{\varphi}$. Then, noting that $\boldsymbol{m}_t\cdot\boldsymbol{m} = 0$ (which follows from $|\boldsymbol{m}| = 1$) and $\boldsymbol{w}\cdot\boldsymbol{m} = 0$, we can design a Galerkin method in which the unknown $\boldsymbol{v}_h^{(j)}$ and the test function $\boldsymbol{w}_h$ reflect the above property. Hence we follow [1, 3] to define

$$\mathbb{W}_h^{(j)} := \left\{\boldsymbol{w}\in\mathbb{V}_h \mid \boldsymbol{w}(\boldsymbol{x}_n)\cdot\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) = 0,\ n = 1,\dots,N\right\},$$

and we will seek $\boldsymbol{v}_h^{(j)}$ in this space. It remains to approximate the other terms in (5.4).

Considering the piecewise constant approximation $W_k(t)$ of $W(t)$, namely,

$$W_k(t) = W(t_j), \quad t\in[t_j, t_{j+1}), \qquad (5.5)$$

we define

$$\begin{aligned}
\boldsymbol{g}_h &:= I_{\mathbb{V}_h}(\boldsymbol{g}), \\
G_h\boldsymbol{u} &:= \boldsymbol{u}\times\boldsymbol{g}_h \quad \forall\boldsymbol{u}\in\mathbb{V}_h\cup\mathbb{Y}_h, \\
e^{W_k(t)G_h}\boldsymbol{u} &:= \boldsymbol{u} + (\sin W_k(t))G_h\boldsymbol{u} + (1-\cos W_k(t))G_h^2\boldsymbol{u} \quad \forall\boldsymbol{u}\in\mathbb{V}_h\cup\mathbb{Y}_h, \qquad (5.6)\\
C_h(\boldsymbol{u}) &:= \boldsymbol{u}\times I_{\mathbb{V}_h}(\Delta\boldsymbol{g}) + 2\nabla\boldsymbol{u}\times I_{\mathbb{V}_h}(\nabla\boldsymbol{g}) \quad \forall\boldsymbol{u}\in\mathbb{V}_h,
\end{aligned}$$

$$D_{h,k}(t,\boldsymbol{u}) = \Big(\big(\sin W_k(t)\big)C_h + \big(1-\cos W_k(t)\big)(G_hC_h + C_hG_h)\Big)\boldsymbol{u} \qquad (5.7)$$

$$\widetilde{C}_{h,k}(t,\boldsymbol{u}) = \Big(I - \sin W_k(t)G_h + (1-\cos W_k(t))G_h^2\Big)D_{h,k}(t,\boldsymbol{u}), \qquad (5.8)$$

$$R_{h,k}(t,\boldsymbol{u}) = \lambda_2^2\boldsymbol{u}\times(\boldsymbol{u}\times\widetilde{C}_{h,k}(t,\boldsymbol{u})) - \lambda_1^2\widetilde{C}_{h,k}(t,\boldsymbol{u}). \qquad (5.9)$$

We can now discretise (5.4) as: For some $\theta\in[0,1]$, find $\boldsymbol{v}_h^{(j)}\in\mathbb{W}_h^{(j)}$ satisfying

$$\begin{aligned}
&\lambda_2\left\langle\boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)}\right\rangle_D - \lambda_1\left\langle\boldsymbol{m}_h^{(j)}\times\boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)}\right\rangle_D + \mu\left\langle\nabla(\boldsymbol{m}_h^{(j)} + k\theta\boldsymbol{v}_h^{(j)}), \nabla\boldsymbol{w}_h^{(j)}\right\rangle_D \\
&\qquad = -\left\langle R_{h,k}(t_j,\boldsymbol{m}_h^j), \boldsymbol{w}_h\right\rangle_D + \mu\left\langle e^{-W_k(t_j)G_h}\boldsymbol{P}_h^{(j)}, \boldsymbol{w}_h^{(j)}\right\rangle_D \quad \forall\boldsymbol{w}_h^{(j)}\in\mathbb{W}_h^{(j)}. \qquad (5.10)
\end{aligned}$$

To discretise (4.5), even though $\boldsymbol{P}$ is not time differentiable we formally use integration by parts to bring the time derivative to $\boldsymbol{P}$, and thus with $d_t\boldsymbol{P}_h^{(j+1)}$ defined by

$$d_t\boldsymbol{P}_h^{(j+1)} := k^{-1}\big(\boldsymbol{P}_h^{(j+1)} - \boldsymbol{P}_h^{(j)}\big),$$

the discretisation of (4.5) reads: Compute $\boldsymbol{P}_h^{(j+1)}\in\mathbb{Y}_h$ by solving

(5.11)

$$\mu_0\left\langle d_t\boldsymbol{P}_h^{(j+1)}, \boldsymbol{\zeta}_h\right\rangle_{\widetilde{D}} + \left\langle\sigma\nabla\times\boldsymbol{P}_h^{(j+1)}, \nabla\times\boldsymbol{\zeta}_h\right\rangle_{\widetilde{D}} = \sigma_D\left\langle\nabla\times\big(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)}\big), \nabla\times\boldsymbol{\zeta}_h\right\rangle_D \quad \forall\boldsymbol{\zeta}_h\in\mathbb{Y}_h.$$

We summarise the above procedure in the following algorithm.

Algorithm 5.1.

Step 1: *Set $j = 0$. Choose $\boldsymbol{m}_h^{(0)} = I_{\mathbb{V}_h}\boldsymbol{m}_0$ and $\boldsymbol{P}_h^{(0)} = I_{\mathbb{Y}_h}\boldsymbol{P}_0$.*

Step 2: *Solve* (5.10) *and* (5.11) *to find* $(\boldsymbol{v}_h^{(j)}, \boldsymbol{P}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$.
Step 3: *Define*

$$\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}) := \sum_{n=1}^{N} \frac{\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)}{\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|}\phi_n(\boldsymbol{x}).$$

Step 4: *Set* $j = j+1$ *and and return to Step* 2 *if* $j < J$. *Stop if* $j = J$.

By the Lax–Milgram theorem, for each $j > 0$ there exists a unique solution $(\boldsymbol{v}_h^{(j)}, \boldsymbol{P}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$ of equations (5.10)–(5.11). Since $\left|\boldsymbol{m}_h^{(0)}(\boldsymbol{x}_n)\right| = 1$ and $\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n) \cdot \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) = 0$ for all $n = 1, \ldots, N$ and $j = 0, \ldots, J$, there hold (by induction)

$$\left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j+1)}(\boldsymbol{x}_n)\right| \geq 1 \quad \text{and} \quad \left|\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n)\right| = 1, \quad j = 0, \ldots, J. \tag{5.12}$$

In particular, the above inequality shows that Step 3 of the algorithm is well defined.

We finish this section by proving the following lemmas concerning boundedness of $\boldsymbol{m}_h^{(j)}$, $\boldsymbol{P}_h^{(j)}$ and $R_{h,k}$.

Lemma 5.2. *For any* $j = 0, \ldots, J$ *there hold*

$$\|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1 \quad \textit{and} \quad \|\boldsymbol{m}_h^{(j)}\|_D \leq |D|,$$

where $|D|$ *denotes the measure of* D.

Proof. The first inequality follows from (5.12) and the second can be obtained by integrating over D. ☐

Lemma 5.3. *Assume that* $\boldsymbol{g}$ *satisfies* (1.7) *and* $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$. *There exists a deterministic constant* c *depending only on* $\boldsymbol{g}$ *such that, for any* $j = 0, \cdots, J$, *there holds* $\mathbb{P}$-*a.s.,*

$$\left\|R_{h,k}(t_j, \boldsymbol{m}_h^{(j)})\right\|_D^2 \leq c + c\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_D^2, \tag{5.13}$$

$$\|e^{-W_k(t_j)G_h}\boldsymbol{u}\|_D^2 \leq \|\boldsymbol{u}\|_D^2 \quad \forall \boldsymbol{u} \in \mathbb{L}^2(D), \tag{5.14}$$

$$\|\nabla \times \left(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)}\right)\|_D^2 \leq c + c\|\nabla \boldsymbol{m}_h^{(j)}\|_D^2. \tag{5.15}$$

Proof. The proof of (5.13) is similar to that of [**?**, Lemma 5.3]. To prove (5.14) we first note that the definition of $e^{-W_k(t_j)G_h}\boldsymbol{u}$ gives

$$\begin{aligned}
\left|e^{-W_k(t_j)G_h}\boldsymbol{u}\right|^2 &= \left|\boldsymbol{u} - \big(\sin W_k(t_j)\big)\boldsymbol{u} \times \boldsymbol{g}_h + \big(1 - \cos W_k(t_j)\big)(\boldsymbol{u} \times \boldsymbol{g}_h) \times \boldsymbol{g}_h\right|^2 \\
&= |\boldsymbol{u}|^2 + (1 - \cos W_k(t_j))^2 \left(|(\boldsymbol{u} \times \boldsymbol{g}_h) \times \boldsymbol{g}_h|^2 - |\boldsymbol{u} \times \boldsymbol{g}_h|^2\right) \\
&= |\boldsymbol{u}|^2 + \big(1 - \cos W_k(t_j)\big)^2 (|\boldsymbol{g}_h|^2 - 1)\, |\boldsymbol{u} \times \boldsymbol{g}_h|^2\,,
\end{aligned}$$

where in the last step we used $|(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}|^2 = |\boldsymbol{a}\times\boldsymbol{b}|^2|\boldsymbol{b}|^2$ for all $\boldsymbol{a}$, $\boldsymbol{b}\in\mathbb{R}^3$. Since $|\boldsymbol{g}(\boldsymbol{x}_i)| = 1$ and $\sum_{i=1}^N \phi_i(\boldsymbol{x}) = 1$ for all $\boldsymbol{x}\in D$, we have

$$|\boldsymbol{g}_h(\boldsymbol{x})|^2 = \left|\sum_{i=1}^N \boldsymbol{g}(\boldsymbol{x}_i)\phi_i(\boldsymbol{x})\right|^2 \le 1.$$

Therefore,

$$\left|e^{-W_k(t_j)G_h}\boldsymbol{u}\right|^2 \le |\boldsymbol{u}|^2 \quad a.e. \text{ in } D,$$

proving (5.14).

Finally, in order to prove (5.15) we use the inequality

$$\|\nabla\times\boldsymbol{u}\|_D^2 \le c\|\nabla\boldsymbol{u}\|_D^2 \quad \forall\boldsymbol{u}\in\mathbb{H}^1(D)$$

to obtain

$$\|\nabla\times(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)})\|_D^2 \le c\|\nabla\big(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)}\big)\|_D^2.$$

On the other hand from the definition of $e^{W_k(t_j)G_h}$, we deduce

$$\begin{aligned}\nabla\big(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)}\big) &= e^{W_k(t_j)G_h}\nabla\boldsymbol{m}_h^{(j)} + \big(\sin W_k(t_j)\big)\boldsymbol{m}_h^{(j)}\times\nabla\boldsymbol{g}_h \\ &\quad + (1-\cos W_k(t_j))\Big((\boldsymbol{m}_h^{(j)}\times\nabla\boldsymbol{g}_h)\times\boldsymbol{g}_h + (\boldsymbol{m}_h^{(j)}\times\boldsymbol{g}_h)\times\nabla\boldsymbol{g}_h\Big).\end{aligned}$$

Since $\boldsymbol{g}\in\mathbb{W}^{2,\infty}(D)$, by using Lemma 5.2 and (5.14) we obtain from the above equality

$$\left|\nabla(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)})\right|^2 \le c + \left|e^{W_k(t_j)G_h}(\nabla\boldsymbol{m}_h^{(j)})\right|^2 \le c + c\left|\nabla\boldsymbol{m}_h^{(j)}\right|^2.$$

This completes the proof. $\square$

Lemma 5.4. *The sequence* $\Big\{\big(\boldsymbol{m}_h^{(j)},\boldsymbol{v}_h^{(j)},\boldsymbol{P}_h^{(j)}\big)\Big\}_{j=0,1,\cdots,J}$ *produced by Algorithm 5.1 satisfies* $\mathbb{P}$*-a.s.*,

$$\begin{aligned}&\|\nabla\boldsymbol{m}_h^{(j)}\|_D^2 + k\sum_{i=0}^{j-1}\|\boldsymbol{v}_h^{(i)}\|_D^2 + k^2(2\theta-1)\sum_{i=0}^{j-1}\|\nabla\boldsymbol{v}_h^{(i)}\|_D^2 + \|\boldsymbol{P}_h^{(j)}\|_{\widetilde{D}}^2 \\ &\quad + \sum_{i=0}^{j-1}\|\boldsymbol{P}_h^{(i+1)} - \boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2 + k\sum_{i=0}^{j-1}\|\nabla\times\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2 \le c.\end{aligned} \tag{5.16}$$

Proof. Choosing $\boldsymbol{w}_h^{(j)} = \boldsymbol{v}_h^{(j)}$ in (5.10), we obtain

$$\begin{aligned}\lambda_2\|\boldsymbol{v}_h^{(j)}\|_D^2 + \mu k\theta\|\nabla\boldsymbol{v}_h^{(j)}\|_D^2 &= -\mu\left\langle\nabla\boldsymbol{m}_h^{(j)},\nabla\boldsymbol{v}_h^{(j)}\right\rangle_D - \left\langle R_{h,k}(t_j,\boldsymbol{m}_h^j),\boldsymbol{v}_h^{(j)}\right\rangle_D \\ &\quad + \mu\left\langle e^{-W_k(t_j)G_h}\boldsymbol{P}_h^{(j)},\boldsymbol{v}_h^{(j)}\right\rangle_D,\end{aligned}$$

or equivalently

$$\begin{aligned}\left\langle\nabla\boldsymbol{m}_h^{(j)},\nabla\boldsymbol{v}_h^{(j)}\right\rangle_D &= -\lambda_2\mu^{-1}\|\boldsymbol{v}_h^{(j)}\|_D^2 - k\theta\|\nabla\boldsymbol{v}_h^{(j)}\|_D^2 - \mu^{-1}\left\langle R_{h,k}(t_j,\boldsymbol{m}_h^j),\boldsymbol{v}_h^{(j)}\right\rangle_D \\ &\quad + \left\langle e^{-W_k(t_j)G_h}\boldsymbol{P}_h^{(j)},\boldsymbol{v}_h^{(j)}\right\rangle_D.\end{aligned}$$

Lemma 5.1 and the above equation yield

$$\begin{aligned}\|\nabla \boldsymbol{m}_h^{(j+1)}\|_D^2 &\le \|\nabla(\boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)})\|_D^2 \\ &= \|\nabla \boldsymbol{m}_h^{(j)}\|_D^2 + k^2(1-2\theta)\|\nabla \boldsymbol{v}_h^{(j)}\|_D^2 - 2k\lambda_2\mu^{-1}\|\boldsymbol{v}_h^{(j)}\|_D^2 \\ &\quad - 2k\mu^{-1}\left\langle R_{h,k}(t_j, \boldsymbol{m}_h^j), \boldsymbol{v}_h^{(j)}\right\rangle_D + 2k\left\langle e^{-W_k(t_j)G_h}\boldsymbol{P}_h^{(j)}, \boldsymbol{v}_h^{(j)}\right\rangle_D.\end{aligned}$$

By using the elementary inequality

$$2ab \le \alpha^{-1}a^2 + \alpha b^2 \quad \forall \alpha > 0, \forall a, b \in \mathbb{R}, \tag{5.17}$$

for the last two terms on the right hand side, we deduce

$$\begin{aligned}&\|\nabla \boldsymbol{m}_h^{(j+1)}\|_D^2 + 2k\lambda_2\mu^{-1}\|\boldsymbol{v}_h^{(j)}\|_D^2 + k^2(2\theta-1)\|\nabla \boldsymbol{v}_h^{(j)}\|_D^2 \\ &\le \|\nabla \boldsymbol{m}_h^{(j)}\|_D^2 + k\lambda_2\mu^{-1}\|\boldsymbol{v}_h^{(j)}\|_D^2 + 2k\lambda_2^{-1}\mu^{-1}\|R_{h,k}(t_j, \boldsymbol{m}_h^j)\|_D^2 + 2k\lambda_2^{-1}\mu\|e^{-W_k(t_j)G_h}\boldsymbol{P}_h^{(j)}\|_D^2.\end{aligned}$$

By rearranging the above inequality and using (5.13)–(5.14) we obtain

$$\begin{aligned}&\|\nabla \boldsymbol{m}_h^{(j+1)}\|_D^2 + k\lambda_2\mu^{-1}\|\boldsymbol{v}_h^{(j)}\|_D^2 + k^2(2\theta-1)\|\nabla \boldsymbol{v}_h^{(j)}\|_D^2 \\ &\qquad \le \|\nabla \boldsymbol{m}_h^{(j)}\|_D^2 + 2k\lambda_2^{-1}\mu\|\boldsymbol{P}_h^{(j)}\|_D^2 + k\lambda_2^{-1}\mu^{-1}c\|\nabla \boldsymbol{m}_h^j\|_D^2 + k\lambda_2^{-1}\mu^{-1}c.\end{aligned}$$

Replacing j by i in the above inequality and summing for i from 0 to $j-1$ yields

$$\begin{aligned}&\|\nabla \boldsymbol{m}_h^{(j)}\|_D^2 + \lambda_2\mu^{-1}k\sum_{i=0}^{j-1}\|\boldsymbol{v}_h^{(i+1)}\|_D^2 + (2\theta-1)k^2\sum_{i=0}^{j-1}\|\nabla \boldsymbol{v}_h^{(i+1)}\|_D^2 \\ &\qquad \le \|\nabla \boldsymbol{m}_h^{(0)}\|_D^2 + ck\sum_{i=0}^{j-1}\|\boldsymbol{P}_h^{(i)}\|_D^2 + ck\sum_{i=0}^{j-1}\|\nabla \boldsymbol{m}_h^i\|_D^2 + c.\end{aligned}$$

Since $\boldsymbol{m}_0 \in \mathbb{H}^2(D)$ it can be shown that there exists a deterministic constant c depending only on $\boldsymbol{m}_0$ such that

$$\|\nabla \boldsymbol{m}_h^{(0)}\|_D^2 \le c. \tag{5.18}$$

By using (5.18) we deduce

$$\begin{aligned}&\|\nabla \boldsymbol{m}_h^{(j)}\|_D^2 + k\sum_{i=0}^{j-1}\|\boldsymbol{v}_h^{(i+1)}\|_D^2 + k^2(2\theta-1)\sum_{i=0}^{j-1}\|\nabla \boldsymbol{v}_h^{(i+1)}\|_D^2 \\ &\qquad \le c + c\sum_{i=0}^{j-1}k\|\boldsymbol{P}_h^{(i)}\|_D^2 + c\sum_{i=0}^{j-1}k\|\nabla \boldsymbol{m}_h^i\|_D^2.\end{aligned} \tag{5.19}$$

In order to estimate the two sums on the right-hand side, we take $\boldsymbol{\zeta}_h = \boldsymbol{P}_h^{(j+1)}$ in (5.11) to obtain the following identity

$$\mu_0\left\langle d_t\boldsymbol{P}_h^{(j+1)}, \boldsymbol{P}_h^{(j+1)}\right\rangle_{\widetilde{D}} + \left\langle \sigma\nabla\times\boldsymbol{P}_h^{(j+1)}, \nabla\times\boldsymbol{P}_h^{(j+1)}\right\rangle_{\widetilde{D}} = \sigma_D\left\langle \nabla\times(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)}), \nabla\times\boldsymbol{P}_h^{(j+1)}\right\rangle_D.$$

Let σ_0 is the lower bound of σ on $\widetilde{D}$. By using successively (5.17) and (5.15) we deduce from the above equality

$$\begin{aligned}\mu_0\left\langle \boldsymbol{P}_h^{(j+1)}-\boldsymbol{P}_h^{(j)},\boldsymbol{P}_h^{(j+1)}\right\rangle_{\widetilde{D}}+k\sigma_0\|\nabla\times\boldsymbol{P}_h^{(j+1)}\|_{\widetilde{D}}^2 &\le +k\frac{\sigma_D^2}{2\sigma_0}\|\nabla\times(e^{W_k(t_j)G_h}\boldsymbol{m}_h^{(j)})\|_D^2\\ &\quad+\frac{1}{2}k\sigma_0\|\nabla\times\boldsymbol{P}_h^{(j+1)}\|_D^2\\ &\le \frac{1}{2}k\sigma_0\|\nabla\times\boldsymbol{P}_h^{(j+1)}\|_{\widetilde{D}}^2\\ &\quad+ck\|\nabla\boldsymbol{m}_h^{(j)}\|_D^2+ck,\end{aligned}$$

or equivalently

$$\mu_0\left\langle \boldsymbol{P}_h^{(j+1)}-\boldsymbol{P}_h^{(j)},\boldsymbol{P}_h^{(j+1)}\right\rangle_{\widetilde{D}}+\frac{1}{2}k\sigma_0\|\nabla\times\boldsymbol{P}_h^{(j+1)}\|_{\widetilde{D}}^2\le ck\|\nabla\boldsymbol{m}_h^{(j)}\|_D^2+ck.$$

Replacing j by i in the above inequality and summing over i from 0 to $j-1$ and using the following Abel summation

$$\sum_{i=0}^{j-1}(\boldsymbol{a}_{i+1}-\boldsymbol{a}_i)\cdot\boldsymbol{a}_{i+1}=\frac{1}{2}|\boldsymbol{a}_j|^2-\frac{1}{2}|\boldsymbol{a}_0|^2+\frac{1}{2}\sum_{i=0}^{j-1}|\boldsymbol{a}_{i+1}-\boldsymbol{a}_i|^2,\quad \boldsymbol{a}_i\in\mathbb{R}^3,$$

we obtain

$$\begin{aligned}\|\boldsymbol{P}_h^{(j)}\|_{\widetilde{D}}^2+\sum_{i=0}^{j-1}\|\boldsymbol{P}_h^{(i+1)}-\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2+\sigma_0\mu_0^{-1}\sum_{i=0}^{j-1}k\|\nabla\times\boldsymbol{P}_h^{(i+1)}\|_{\widetilde{D}}^2&\\ \le\|\boldsymbol{P}_h^{(0)}\|_{\widetilde{D}}^2+c\sum_{i=0}^{j-1}k\|\nabla\boldsymbol{m}_h^{(i)}\|_D^2+cT\sigma.&\end{aligned}$$

By using (3.3) and the error estimate for the interpolant $\boldsymbol{P}_h^{(0)}=I_{\mathbb{Y}_h}\boldsymbol{P}_0$, it can be shown that there exists a constant c depending only on $\boldsymbol{P}_0$ such that

$$\|\boldsymbol{P}_h^{(0)}\|_{\widetilde{D}}^2+\|\nabla\times\boldsymbol{P}_h^{(0)}\|_{\widetilde{D}}^2\le c. \tag{5.20}$$

By using (5.20) we deduce

(5.21)

$$\|\boldsymbol{P}_h^{(j)}\|_{\widetilde{D}}^2+\sum_{i=0}^{j-1}\|\boldsymbol{P}_h^{(i+1)}-\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2+k\sum_{i=0}^{j-1}\|\nabla\times\boldsymbol{P}_h^{(i+1)}\|_{\widetilde{D}}^2\quad\le c+ck\sum_{i=0}^{j-1}\|\nabla\boldsymbol{m}_h^{(i)}\|_D^2.$$

From (5.19) and (5.21) we obtain

$$\|\nabla\boldsymbol{m}_h^{(j)}\|_D^2+\|\boldsymbol{P}_h^{(j)}\|_{\widetilde{D}}^2\le c+ck\sum_{i=0}^{j-1}\|\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2+ck\sum_{i=0}^{j-1}\|\nabla\boldsymbol{m}_h^{(i)}\|_{\widetilde{D}}^2.$$

By using induction and (5.18)-(5.20) we can show that

$$\|\nabla\boldsymbol{m}_h^{(i)}\|_D^2+\|\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2\le c(1+ck)^i.$$

Summing over i from 0 to $j-1$ and using $1+x \le e^x$ we obtain

$$k\sum_{i=0}^{j-1} \|\nabla \boldsymbol{m}_h^{(i)}\|_D^2 + k\sum_{i=0}^{j-1} \|\boldsymbol{P}_h^{(i)}\|_{\widetilde{D}}^2 \le ck\frac{(1+ck)^j-1}{ck} \le e^{ckJ} = c. \tag{5.22}$$

The required result (5.16) now follows from (5.19), (5.21) and (5.22). ☐

6. Proof of the main theorem

The discrete solutions $\boldsymbol{m}_h^{(j)}$, $\boldsymbol{v}_h^{(j)}$ and $\boldsymbol{P}_h^{(j)}$ constructed via Algorithm 5.1 are interpolated in time in the following definition.

Definition 6.1. *For all $x \in D$ and all $t \in [0,T]$, let $j \in \{0,...,J-1\}$ be such that $t \in [t_j, t_{j+1})$. We then define*

$$\begin{aligned}
\boldsymbol{m}_{h,k}(t,\boldsymbol{x}) &:= \frac{t-t_j}{k}\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}) + \frac{t_{j+1}-t}{k}\boldsymbol{m}_h^{(j)}(\boldsymbol{x}),\\
\boldsymbol{m}_{h,k}^-(t,\boldsymbol{x}) &:= \boldsymbol{m}_h^{(j)}(\boldsymbol{x}),\\
\boldsymbol{v}_{h,k}(t,\boldsymbol{x}) &:= \boldsymbol{v}_h^{(j)}(\boldsymbol{x}),\\
\boldsymbol{P}_{h,k}(t,\boldsymbol{x}) &:= \frac{t-t_j}{k}\boldsymbol{P}_h^{(j+1)}(\boldsymbol{x}) + \frac{t_{j+1}-t}{k}\boldsymbol{P}_h^{(j)}(\boldsymbol{x}),\\
\boldsymbol{P}_{h,k}^-(t,\boldsymbol{x}) &:= \boldsymbol{P}_h^{(j)}(\boldsymbol{x}),\\
\boldsymbol{P}_{h,k}^+(t,\boldsymbol{x}) &:= \boldsymbol{P}_h^{(j+1)}(\boldsymbol{x}).
\end{aligned}$$

The above sequences have the following obvious bounds.

Lemma 6.2. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\boldsymbol{P}_0$, $\boldsymbol{g}$, μ, σ and T such that for all $\theta \in [0,1]$ there holds $\mathbb{P}$-a.s.*

$$\|\boldsymbol{m}_{h,k}^*\|_{D_T}^2 + \left\|\nabla \boldsymbol{m}_{h,k}^*\right\|_{D_T}^2 + \|\boldsymbol{v}_{h,k}\|_{D_T}^2 + k(2\theta-1)\,\|\nabla \boldsymbol{v}_{h,k}\|_{D_T}^2 \le c,$$

where $\boldsymbol{m}_{h,k}^* = \boldsymbol{m}_{h,k}$ or $\boldsymbol{m}_{h,k}^-$. In particular, when $\theta \in [0,\frac{1}{2})$, there holds $\mathbb{P}$-a.s.

$$\|\boldsymbol{m}_{h,k}^*\|_{D_T}^2 + \left\|\nabla \boldsymbol{m}_{h,k}^*\right\|_{D_T}^2 + \left(1+(2\theta-1)kh^{-2}\right)\|\boldsymbol{v}_{h,k}\|_{D_T}^2 \le c.$$

Proof. Both inequalities are direct consequences of Definition 6.1, Lemmas 5.2 and 5.4, noting that the second inequality requires the use of the inverse estimate (see e.g. [19])

$$\|\nabla \boldsymbol{v}_h^{(i)}\|_D^2 \le ch^{-2}\|\boldsymbol{v}_h^{(i)}\|_D^2.$$

☐

Lemma 6.3. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\boldsymbol{P}_0$, $\boldsymbol{g}$, μ, σ and T such that for all $\theta \in [0,1]$ there holds $\mathbb{P}$-a.s.*

$$\|\boldsymbol{P}_{h,k}\|_{\widetilde{D}_T}^2 + \|\boldsymbol{P}_{h,k}^+\|_{\widetilde{D}_T}^2 + \|\nabla\times\boldsymbol{P}_{h,k}^+\|_{\widetilde{D}_T}^2 \le c, \tag{6.1}$$

$$\|\boldsymbol{P}_{h,k} - \boldsymbol{P}_{h,k}^*\|_{\widetilde{D}_T}^2 \le kc, \tag{6.2}$$

where $\boldsymbol{P}_{h,k}^* = \boldsymbol{P}_{h,k}^+$ or $\boldsymbol{P}_{h,k}^-$.

Proof. It is easy to prove (6.1) by using Lemma 5.4 and Definition 6.1. Inequality (6.2) can be deduced from Lemma 5.4 by noting that for $t \in [t_j, t_{j+1})$ there holds

$$\left|\boldsymbol{P}_{h,k}(t,\boldsymbol{x}) - \boldsymbol{P}^+_{h,k}(t,\boldsymbol{x})\right| = \left|\frac{t-t_{j+1}}{k}\big(\boldsymbol{P}^{(j+1)}_h(\boldsymbol{x}) - \boldsymbol{P}^{(j)}_h(\boldsymbol{x})\big)\right| \le \left|\boldsymbol{P}^{(j+1)}_h(\boldsymbol{x}) - \boldsymbol{P}^{(j)}_h(\boldsymbol{x})\right|,$$

completing the proof of the lemma. □

The next lemma provides a bound of $\boldsymbol{m}_{h,k}$ in the $\mathbb{H}^1$-norm and relationships between $\boldsymbol{m}^-_{h,k}$, $\boldsymbol{m}_{h,k}$ and $\boldsymbol{v}_{h,k}$.

Lemma 6.4. *Assume that h and k go to 0 with a further condition $k = o(h^2)$ when $\theta \in [0, \frac{1}{2})$ and no condition otherwise. The sequences $\{\boldsymbol{m}_{h,k}\}$, $\{\boldsymbol{m}^-_{h,k}\}$, and $\{\boldsymbol{v}_{h,k}\}$ defined in Definition 6.1 satisfy the following properties $\mathbb{P}$-a.s.*

$$\|\boldsymbol{m}_{h,k}\|_{\mathbb{H}^1(D_T)} \le c, \tag{6.3}$$

$$\|\boldsymbol{m}_{h,k} - \boldsymbol{m}^-_{h,k}\|_{D_T} \le ck, \tag{6.4}$$

$$\|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \le ck, \tag{6.5}$$

$$\||\boldsymbol{m}_{h,k}| - 1\|_{D_T} \le chk. \tag{6.6}$$

Proof. The proof of this lemma is similar to that of [**?**, Lemma 6.3] □

The following two Lemmas 6.5 show that $\boldsymbol{m}_{h,k}$ and $\boldsymbol{P}_{h,k}$, respectively, satisfy discrete forms of (4.11) and (4.5).

Lemma 6.5. *Assume that h and k go to 0 with the following conditions*

$$\begin{cases} k = o(h^2) & \text{when } 0 \le \theta < 1/2, \\ k = o(h) & \text{when } \theta = 1/2, \\ \text{no condition} & \text{when } 1/2 < \theta \le 1. \end{cases} \tag{6.7}$$

Then for any $\boldsymbol{\varphi} \in C\big(0,T;\mathbb{C}^\infty(D)\big)$ and $\boldsymbol{\zeta} \in C^1_T(0,T;\mathbb{C}^\infty_\times(\widetilde{D}))$, there holds $\mathbb{P}$-a.s.

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}^-_{h,k} \times \boldsymbol{v}_{h,k}, \boldsymbol{m}^-_{h,k} \times \boldsymbol{\varphi}\right\rangle_{D_T} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}^-_{h,k} \times \boldsymbol{\varphi}\right\rangle_{D_T} \\ &+ \mu \left\langle \nabla(\boldsymbol{m}^-_{h,k} + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}^-_{h,k} \times \boldsymbol{\varphi})\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot, \boldsymbol{m}^-_{h,k}), \boldsymbol{m}^-_{h,k} \times \boldsymbol{\varphi}\right\rangle_{D_T} \\ &- \mu \left\langle e^{W_k G_h} \boldsymbol{P}^-_{h,k}, \boldsymbol{m}^-_{h,k} \times \boldsymbol{\varphi}\right\rangle_{D_T} = O(h+k) \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \mu_0 \left\langle \boldsymbol{P}_{h,k}, \boldsymbol{\zeta}_t\right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{P}^{(0)}_h, \boldsymbol{\zeta}(0,\cdot)\right\rangle_{\widetilde{D}} - \left\langle \sigma \nabla \times \boldsymbol{P}^+_{h,k}, \nabla \times \boldsymbol{\zeta}\right\rangle_{\widetilde{D}_T} \\ + \sigma_D \left\langle e^{W_k G_h} \boldsymbol{m}^-_{h,k}, \nabla \times (\nabla \times \boldsymbol{\zeta})\right\rangle_{D_T} = O(h+k). \end{aligned} \tag{6.9}$$

Proof.

Proof of (6.8): For $t \in [t_j, t_{j+1})$, we use (5.10) with $\boldsymbol{w}_h^{(j)} = I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big) \in \mathbb{W}_h^{(j)}$ to have

$$
\begin{aligned}
-\lambda_1 &\left\langle \boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big)\right\rangle_D \\
&+ \lambda_2 \left\langle \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big)\right\rangle_D \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^-(t,\cdot) + k\theta \boldsymbol{v}_{h,k}(t,\cdot)), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big)\right\rangle_D \\
&+ \left\langle R_{h,k}(t_j, \boldsymbol{m}_{h,k}^-(t,\cdot)), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big)\right\rangle_D \\
&- \mu \left\langle e^{W_k(t)G_h} \boldsymbol{P}_{h,k}^-(t), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)\big)\right\rangle_D = 0.
\end{aligned}
$$

Integrating both sides of the above equation over (t_j, t_{j+1}) and summing over $j = 0, \cdots, J-1$ we deduce

$$
\begin{aligned}
&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\big)\right\rangle_{D_T} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\big)\right\rangle_{D_T} \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\big)\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}^-), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\big)\right\rangle_{D_T} \\
&- \mu \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^-, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\big)\right\rangle_{D_T} = 0.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\right\rangle_{D_T} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\right\rangle_{D_T} \\
&+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot, \boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\right\rangle_{D_T} \\
&- \mu \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}\right\rangle_{D_T} = I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \left\langle -\lambda_1 \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k} + \lambda_2 \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\right\rangle_{D_T}, \\
I_2 &= \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\big)\right\rangle_{D_T}, \\
I_3 &= \left\langle R_{h,k}(., \boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\right\rangle_{D_T}, \\
I_4 &= - \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}))\right\rangle_{D_T}.
\end{aligned}
$$

Hence it suffices to prove that $I_i = O(h+k)$ for $i = 1, \cdots, 4$. Firstly, by using Lemma 5.2 we obtain

$$
\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \le \sup_{0 \le j \le J} \|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \le 1.
$$

This inequality, Lemma 6.2 and Lemma 8.2 yield

$$
\begin{aligned}
|I_1| &\le c\left(\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} + 1\right) \|\boldsymbol{v}_{h,k}\|_{D_T} \|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\|_{D_T} \\
&\le c\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi})\|_{D_T} \le ch.
\end{aligned}
$$

The bounds for I_2, I_3 and I_4 can be obtained similarly by using Lemma 6.2 and Lemma 5.3, respectively, noting that when $\theta \in [0, \frac{1}{2}]$, a bound of $\|\nabla \boldsymbol{v}_{h,k}\|_{D_T}$ can be deduced from the inverse estimate $\|\nabla \boldsymbol{v}_{h,k}\|_{D_T} \le ch^{-1} \|\boldsymbol{v}_{h,k}\|_{D_T}$. This completes the proof (6.8).

Proof of (6.9): For $t \in [t_j, t_{j+1})$, we use (5.11) with $\boldsymbol{\zeta}_h(t,\cdot) = I_{\mathbb{Y}_h}\boldsymbol{\zeta}(t,\cdot)$ to have

$$\begin{aligned}\mu_0 \left\langle \partial_t \boldsymbol{P}_{h,k}(t,\cdot), I_{\mathbb{Y}_h}\boldsymbol{\zeta}(t,\cdot)\right\rangle_{\widetilde{D}} = &- \left\langle \sigma\nabla\times \boldsymbol{P}^+_{h,k}(t,\cdot), \nabla\times I_{\mathbb{Y}_h}\boldsymbol{\zeta}(t,\cdot)\right\rangle_{\widetilde{D}} \\ &+ \sigma_D \left\langle \nabla\times e^{W_k(t)G_h}\boldsymbol{m}^-_{h,k}(t,\cdot), \nabla\times I_{\mathbb{Y}_h}\boldsymbol{\zeta}(t,\cdot)\right\rangle_D .\end{aligned}$$

Integrating both sides of the above equation over (t_j, t_{j+1}) andsumming over $j = 0,\cdots,J-1$, and using integration by parts (noting that $\boldsymbol{\zeta}_h(T,\cdot) = 0$) we deduce

$$\begin{aligned}\mu_0 \left\langle \boldsymbol{P}_{h,k}, \partial_t\boldsymbol{\zeta}_h\right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{P}^{(0)}_h, \boldsymbol{\zeta}_h(0,\cdot)\right\rangle_{\widetilde{D}} = &\left\langle \sigma\nabla\times\boldsymbol{P}^+_{h,k}, \nabla\times\boldsymbol{\zeta}_h\right\rangle_{\widetilde{D}_T} \\ &- \sigma_D\left\langle \nabla\times e^{W_kG_h}\boldsymbol{m}^-_{h,k}, \nabla\times\boldsymbol{\zeta}_h\right\rangle_{D_T} .\end{aligned}$$

By using Lemma 6.3 and the following error estimate, see e.g. [24],

$$\|\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}_h(t)\|_{\widetilde{D}} + h\|\nabla\times(\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}_h(t))\|_{\widetilde{D}} \le Ch^2\|\nabla^2\boldsymbol{\zeta}\|_{\widetilde{D}},$$

we deduce

$$\begin{aligned}\mu_0 \left\langle \boldsymbol{P}_{h,k}, \boldsymbol{\zeta}_t)\right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{P}^{(0)}_h, \boldsymbol{\zeta}(0,\cdot)\right\rangle_{\widetilde{D}} &- \left\langle \sigma\nabla\times\boldsymbol{P}^+_{h,k}, \nabla\times\boldsymbol{\zeta}\right\rangle_{\widetilde{D}_T} \\ &+ \sigma_D\left\langle \nabla\times e^{W_kG_h}\boldsymbol{m}^-_{h,k}, \nabla\times\boldsymbol{\zeta}\right\rangle_{D_T} = O(h).\end{aligned}$$

Using Green's identity (see [24, Corollary 3.20]) we obtain (6.9), completing the proof of the lemma. ∎

In the next lemma we show that $\boldsymbol{v}_{h,k}$ can be replaced by $\partial_t\boldsymbol{m}_{h,k}$, as indeed the latter approximates $\boldsymbol{m}_t$.

Lemma 6.6. *Assume that h and k go to 0 satisfying* (6.7). *Then for any $\boldsymbol{\varphi} \in C^1_0(0,T;C^\infty(D))$ and $\boldsymbol{\zeta} \in C^1_T(0,T;\mathbb{C}^\infty_\times(\widetilde{D}))$, there holds $\mathbb{P}$-a.s.*

$$\begin{aligned}&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}\times\partial_t\boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} + \lambda_2\left\langle \partial_t\boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} \\ &+ \mu\left\langle \nabla\boldsymbol{m}_{h,k}, \nabla(\boldsymbol{m}_{h,k}\times\boldsymbol{\varphi})\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot,\boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} \\ &- \mu\left\langle e^{W_kG_h}\boldsymbol{P}^+_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} = O(h+k);\end{aligned}\tag{6.10}$$

and

$$\begin{aligned}\mu_0 \left\langle \boldsymbol{P}^+_{h,k}, \boldsymbol{\zeta}_t\right\rangle_{\widetilde{D}_T} - \mu_0\left\langle \boldsymbol{P}^{(0)}_h, \boldsymbol{\zeta}(0,\cdot)\right\rangle_{\widetilde{D}} &- \left\langle \sigma\nabla\times\boldsymbol{P}^+_{h,k}, \nabla\times\boldsymbol{\zeta}\right\rangle_{\widetilde{D}_T} \\ &+ \sigma_D\left\langle e^{W_kG_h}\boldsymbol{m}_{h,k}, \nabla\times(\nabla\times\boldsymbol{\zeta})\right\rangle_{D_T} = O(h+k).\end{aligned}\tag{6.11}$$

Proof.
Proof of (6.10): From (6.8) it follows that

$$\begin{aligned}&- \lambda_1 \left\langle \boldsymbol{m}_{h,k}\times\partial_t\boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} + \lambda_2\left\langle \partial_t\boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} \\ &+ \mu\left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k}\times\boldsymbol{\varphi})\right\rangle_{D_T} + \left\langle R_{h,k}(\cdot,\boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} \\ &- \mu\left\langle e^{W_kG_h}\boldsymbol{P}^+_{h,k}, \boldsymbol{m}_{h,k}\times\boldsymbol{\varphi}\right\rangle_{D_T} = O(h+k) + I_1 + \cdots + I_5,\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} \right\rangle_{D_T} - \lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T}, \\
I_2 &= -\lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} \right\rangle_{D_T} + \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T}, \\
I_3 &= -\mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi}) \right\rangle_{D_T} + \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\varphi}) \right\rangle_{D_T}, \\
I_4 &= -\left\langle R_{h,k}(.,\boldsymbol{m}_{h,k}^-), \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} \right\rangle_{D_T} + \left\langle R_{h,k}(\cdot,\boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T}, \\
I_5 &= -\mu \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^+, \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T} + \mu \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} \right\rangle_{D_T}.
\end{aligned}
$$

Hence it suffices to prove that $I_i = O(k)$ for $i = 1, \cdots, 5$.

First, by using the triangle inequality we obtain

$$
\begin{aligned}
\lambda_1^{-1}|I_1| &\le \left| \left\langle (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\varphi} \right\rangle_{D_T} \right| + \left| \left\langle \boldsymbol{m}_{h,k} \times \boldsymbol{v}_{h,k}, (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{\varphi} \right\rangle_{D_T} \right| \\
&\quad + \left| \left\langle \boldsymbol{m}_{h,k} \times (\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T} \right| \\
&\le 2\|\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}\|_{D_T} \|\boldsymbol{v}_{h,k}\|_{D_T} \|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\quad + \|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)}.
\end{aligned}
$$

Therefore, the bound of I_1 can be obtained by using Lemmas 6.2 and 6.4. The bounds for I_2, I_3 and I_4 can be obtained similarly.

Finally, using (5.14), Lemmas 6.2 and 6.3 we obtain

$$
\begin{aligned}
\mu^{-1}|I_5| &\le \left| \left\langle e^{W_k G_h} (\boldsymbol{P}_{h,k}^+ - \boldsymbol{P}_{h,k}^-), \boldsymbol{m}_{h,k} \times \boldsymbol{\varphi} \right\rangle_{D_T} \right| + \left| \left\langle e^{W_k G_h} \boldsymbol{P}_{h,k}^-, (\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-) \times \boldsymbol{\varphi} \right\rangle_{D_T} \right| \\
&\le \|e^{W_k G_h} (\boldsymbol{P}_{h,k}^+ - \boldsymbol{P}_{h,k}^-)\|_{D_T} \|\boldsymbol{m}_{h,k}\|_{D_T} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\quad + \|e^{W_k G_h} \boldsymbol{P}_{h,k}^-\|_{D_T} \|\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-\|_{D_T} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\le c\|\boldsymbol{P}_{h,k}^+ - \boldsymbol{P}_{h,k}^-\|_{D_T} + c\|\boldsymbol{P}_{h,k}^-\|_{D_T} \|\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-\|_{D_T} \le ck.
\end{aligned}
$$

This completes the proof of (6.10).

Proof of (6.11): It follows from (6.9) that

$$
\begin{aligned}
\mu_0 \left\langle \boldsymbol{P}_{h,k}^+, \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{P}_h^{(0)}, \boldsymbol{\zeta}(0,\cdot) \right\rangle_{\widetilde{D}} - \left\langle \sigma \nabla \times \boldsymbol{P}_{h,k}^+, \nabla \times \boldsymbol{\zeta} \right\rangle_{\widetilde{D}_T} \\
+ \sigma_D \left\langle e^{W_k G_h} \boldsymbol{m}_{h,k}, \nabla \times (\nabla \times \boldsymbol{\zeta}) \right\rangle_{D_T} = O(h+k) + I_6 + I_7,
\end{aligned}
$$

where

$$
\begin{aligned}
I_6 &= \mu_0 \left\langle \boldsymbol{P}_{h,k}^+ - \boldsymbol{P}_{h,k}, \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T}, \\
I_7 &= \sigma \left\langle e^{W_k G_h} (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}), \nabla \times (\nabla \times \boldsymbol{\zeta}) \right\rangle_{D_T}.
\end{aligned}
$$

By using (6.4) and (6.2) we obtain that $I_i = O(k)$ for $i = 6, 7$. This completes the proof of (6.11). $\square$

In order to prove the $\mathbb{P}$-a.s. convergence of random variables $\boldsymbol{m}_{h,k}$ and $\boldsymbol{P}_{h,k}^+$, we first show that the family $\mathcal{L}(\boldsymbol{m}_{h,k})$ and $\mathcal{L}(\boldsymbol{P}_{h,k}^+)$ are tight.

Lemma 6.7. *Assume that h and k go to 0 satisfying* (6.7). *Then the set of laws* $\{\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k)\}$ *on the space* $C\big(0,T;\mathbb{H}^{-1}(D)\big) \times H^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$ *is tight. Here,* $\mathbb{D}(0,T)$ *is the Skorokhod space; see e.g.* [8].

Proof. Firstly, from Definition 5.5, the approximation W_k of the Wiener process W belongs to $\mathbb{D}(0,T)$. The tightness of $\{\mathcal{L}(W_k)\}$ in $\mathbb{D}(0,T)$ is proved in [8, Theorem 2.5.6]. The tightness of $\{\mathcal{L}(\boldsymbol{m}_{h,k})\}$ on $C\big(0,T;\mathbb{H}^{-1}(D)\big)$ and of $\{\mathcal{L}(\boldsymbol{P}_{h,k}^{+})\}$ on $H^{-1}(\widetilde{D}_T)$ can be obtained as in the proof of [**?**, Lemma 6.6] and is therefore omitted. □

The following proposition is a consequence of the tightness of $\{\mathcal{L}(\boldsymbol{m}_{h,k})\}$, $\{\mathcal{L}(\boldsymbol{P}_{h,k}^{+})\}$ and $\{\mathcal{L}(W_k)\}$.

Proposition 6.8. *Assume that h and k go to 0 satisfying* (6.7). *Then there exist*

(a) *a probability space* $(\Omega', \mathcal{F}', \mathbb{P}')$,
(b) *a sequence* $\{(\boldsymbol{m}'_{h,k}, \boldsymbol{P}'_{h,k}, W'_k)\}$ *of random variables defined on* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and taking values in the space* $C\big(0,T;\mathbb{H}^{-1}(D)\big) \times \mathbb{H}^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$,
(c) *a random variable* $(\boldsymbol{m}', \boldsymbol{P}', W')$ *defined on* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and taking values in* $C\big([0,T];\mathbb{H}^{-1}(D)\big) \times \mathbb{H}^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$,

satisfying

(1) $\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k) = \mathcal{L}(\boldsymbol{m}'_{h,k}, \boldsymbol{P}'_{h,k}, W'_k)$,
(2) $\boldsymbol{m}'_{h,k} \to \boldsymbol{m}'$ *in* $C\big(0,T;\mathbb{H}^{-1}(D)\big)$ *strongly,* $\mathbb{P}'$*-a.s.,*
(3) $\boldsymbol{P}'_{h,k} \to \boldsymbol{P}'$ *in* $\mathbb{H}^{-1}(\widetilde{D}_T)$ *strongly,* $\mathbb{P}'$*-a.s.,*
(4) $W'_k \to W'$ *in* $\mathbb{D}(0,T)$ $\mathbb{P}'$*-a.s.*

Moreover, the sequence $\{\boldsymbol{m}'_{h,k}\}$ *and* $\{\boldsymbol{P}'_{h,k}\}$ *satisfy* $\mathbb{P}'$*-a.s.*

$$\|\boldsymbol{m}'_{h,k}(\omega')\|_{\mathbb{H}^1(D_T)} \le c, \tag{6.12}$$

$$\|\boldsymbol{m}'_{h,k}(\omega')\|_{\mathbb{L}^\infty(D_T)} \le c, \tag{6.13}$$

$$\||\boldsymbol{m}'_{h,k}(\omega')| - 1\|_{\mathbb{L}^2(D_T)} \le c(h+k), \tag{6.14}$$

$$\text{and} \quad \|\boldsymbol{P}'_{h,k}(\omega)\|_{L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))} \le c. \tag{6.15}$$

Proof. By Lemma 6.7 and the Donsker theorem [8, Theorem 8.2], the family of probability measures $\{\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k)\}$ is tight on $C\big(0,T;\mathbb{H}^{-1}(D)\big) \times \mathbb{H}^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$. Then by Theorem 5.1 in [8] the family of measures $\{\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k)\}$ is relatively compact on $C\big(0,T;\mathbb{H}^{-1}(D)\big) \times \mathbb{H}^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$, that is there exists a subsequence, still denoted by $\{\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k)\}$, such that $\{\mathcal{L}(\boldsymbol{m}_{h,k}, \boldsymbol{P}_{h,k}^{+}, W_k)\}$ converges weakly. Hence, the existence of (a)–(c) satisfying (1)–(4) follows immediately from the Skorokhod Theorem [8, Theorem 6.7] since $C\big(0,T;\mathbb{H}^{-1}(D)\big) \times \mathbb{H}^{-1}(\widetilde{D}_T) \times \mathbb{D}(0,T)$ is a separable metric space.

We note that from the Kuratowski theorem, the Borel subsets of $\mathbb{H}^1(D_T)$ or $\mathbb{H}^1(D_T) \cap \mathbb{L}^\infty(D_T)$ are Borel subsets of $C\big(0,T;\mathbb{H}^{-1}(D)\big)$ and the Borel subsets of $L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))$ are Borel subsets of $\mathbb{H}^{-1}(\widetilde{D}_T)$. The estimates (6.12)–(6.15) are direct consequences of Lemmas 6.3–6.4 and the equality of laws stated in part (1). □

We now ready to prove the main result of this paper.

Theorem 6.9. *Assume that $T > 0$, $\boldsymbol{M}_0 \in \mathbb{H}^2(D)$ and $\boldsymbol{g} \in \mathbb{W}^{2,\infty}(D)$ satisfy (??) and (1.7), respectively. Then $\boldsymbol{m}'$, $\boldsymbol{P}'$, W', the sequences $\{\boldsymbol{m}'_{h,k}\}$, $\{\boldsymbol{P}'_{h,k}\}$ and the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ given by Proposition 6.8 satisfy*

(1) *the sequence $\{\boldsymbol{m}'_{h,k}\}$ converges to $\boldsymbol{m}'$ weakly in $\mathbb{H}^1(D_T)$, $\mathbb{P}'$-a.s.*
(2) *the sequence $\{\boldsymbol{P}'_{h,k}\}$ converges to $\boldsymbol{P}'$ weakly in $L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D})$, $\mathbb{P}'$-a.s.*
(3) *$\big(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\in[0,T]}, \mathbb{P}', W', \boldsymbol{M}', \boldsymbol{P}'\big)$ is a weak martingale solution of (1.5), where*

$$\boldsymbol{M}'(t) := e^{W'(t)G}\boldsymbol{m}'(t) \quad \forall t \in [0,T], \ \text{a.e. } \boldsymbol{x} \in D.$$

Proof. By Proposition 6.8 there exists a set $V \subset \Omega'$ such that $\mathbb{P}'(V) = 1$,

$$\boldsymbol{m}'_{h,k}(\omega') \to \boldsymbol{m}'(\omega') \quad \text{strongly in} \quad C\big(0,T;\mathbb{H}^{-1}(D)\big),$$

$$\boldsymbol{P}'_{h,k}(\omega') \to \boldsymbol{P}'(\omega') \quad \text{strongly in} \quad \mathbb{H}^{-1}(\widetilde{D}_T),$$

and (6.12), (6.15) hold for every $\omega' \in V$. In what follows, we work with a fixed $\omega' \in V$.

The convergences of sequences $\{\boldsymbol{m}'_{h,k}(\omega')\}$ and $\{\boldsymbol{P}'_{h,k}(\omega')\}$ are obtained by using the same arguments as in [15, Theorem 6.8].

In order to prove (3), by noting Lemma 4.4 we need to prove that $\boldsymbol{m}', \boldsymbol{P}'$ and W' satisfy (4.10), (4.11) and (4.5).

Prove that $\boldsymbol{m}'$ satisfies (4.10): Since $\mathbb{H}^1(D_T)$ is compactly embedded in $\mathbb{L}^2(D_T)$, there exists a subsequence of $\{\boldsymbol{m}'_{h,k}(\omega')\}$ (still denoted by $\{\boldsymbol{m}'_{h,k}(\omega')\}$) such that

$$\boldsymbol{m}'_{h,k}(\omega') \to \boldsymbol{m}'(\omega') \quad \text{strongly in} \quad \mathbb{L}^2(D_T). \tag{6.16}$$

Therefore (4.10) follows from (6.16) and (6.14).

Prove that $\boldsymbol{m}', \boldsymbol{P}'$ satisfy (4.11) and (4.5): From Lemma 6.6 , $\big(\boldsymbol{m}_{h,k}, \boldsymbol{P}^+_{h,k}, W_k\big)$ satisfies (6.10)–(6.11) $\mathbb{P}$-a.s.. Therefore, it follows from the equality of laws in Proposition 6.8 that $\big(\boldsymbol{m}'_{h,k}, \boldsymbol{P}'_{h,k}, W'_k\big)$ satisfies the following equations for all $\boldsymbol{\psi} \in C_0^\infty\big((0,T);\mathbb{C}^\infty(D)\big)$ and $\boldsymbol{\zeta} \in C_c^\infty([0,T), \mathbb{C}^\infty_\times(\widetilde{D}))$, $\mathbb{P}'$-a.s.

$$\begin{aligned}
&- \lambda_1 \left\langle \boldsymbol{m}'_{h,k}(\omega') \times \partial_t \boldsymbol{m}'_{h,k}(\omega'), \boldsymbol{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} + \lambda_2 \left\langle \partial_t \boldsymbol{m}'_{h,k}(\omega'), \boldsymbol{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \\
&+ \mu \left\langle \nabla(\boldsymbol{m}'_{h,k}(\omega')), \nabla(\boldsymbol{m}'_{h,k}(\omega') \times \boldsymbol{\psi}) \right\rangle_{D_T} + \left\langle R_{h,k}(\cdot, \boldsymbol{m}'_{h,k}(\omega')), \boldsymbol{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \\
&\quad - \mu \left\langle e^{W'_k G_h} \boldsymbol{P}'_{h,k}(\omega'), \boldsymbol{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} = O(h+k),
\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}
&\mu_0 \left\langle \boldsymbol{P}'_{h,k}(\omega'), \boldsymbol{\zeta}_t \right\rangle_{\widetilde{D}_T} - \mu_0 \left\langle \boldsymbol{P}^{(0)}_h, \boldsymbol{\zeta}(0,\cdot) \right\rangle_{\widetilde{D}_T} + \left\langle \sigma \nabla \times \boldsymbol{P}'_{h,k}(\omega'), \nabla \times \boldsymbol{\zeta} \right\rangle_{\widetilde{D}_T} \\
&\qquad - \sigma_D \left\langle e^{W'_k G_h} \boldsymbol{m}'_{h,k}(\omega'), \nabla \times (\nabla \times \boldsymbol{\zeta}) \right\rangle_{D_T} = O(h+k).
\end{aligned} \tag{6.18}$$

It suffices now to use the same arguments as in [15, Theorem 6.8] to pass the limit in (6.17) and (6.18). Indeed, from [15, Theorem 6.8] there hold

$$
\begin{aligned}
\langle \boldsymbol{m}'(\omega')\times\partial_t\boldsymbol{m}'(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\rangle_{D_T} &\to \langle \boldsymbol{m}'(\omega')\times\partial_t\boldsymbol{m}'(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\rangle_{D_T}\\
\left\langle \partial_t\boldsymbol{m}'_{h,k}(\omega'),\boldsymbol{m}'_{h,k}(\omega')\times\boldsymbol{\psi}\right\rangle_{D_T} &\to \langle \partial_t\boldsymbol{m}'(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\rangle_{D_T}\\
\left\langle \nabla(\boldsymbol{m}'_{h,k}(\omega')),\nabla(\boldsymbol{m}'_{h,k}(\omega')\times\boldsymbol{\psi})\right\rangle_{D_T} &\to \langle \nabla(\boldsymbol{m}'(\omega')),\nabla(\boldsymbol{m}'(\omega')\times\boldsymbol{\psi})\rangle_{D_T}\\
\left\langle R_{h,k}(\cdot,\boldsymbol{m}'_{h,k}(\omega')),\boldsymbol{m}'_{h,k}(\omega')\times\boldsymbol{\psi}\right\rangle_{D_T} &\to \langle R(\cdot,\boldsymbol{m}'(\omega')),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\rangle_{D_T}.
\end{aligned}
$$

To prove the convergence of the last term in (6.17), we use the triangle inequality, Hölder inequaliy, (5.14), (5.6) and (2.3) to obtain

$$
\begin{aligned}
\mathcal{I} := \Big|\Big\langle e^{W'_kG_h}\boldsymbol{P}'_{h,k}(\omega'),\boldsymbol{m}'_{h,k}(\omega')\times\boldsymbol{\psi}\Big\rangle_{D_T} &- \Big\langle e^{W'G}\boldsymbol{P}'(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\Big\rangle_{D_T}\Big|\\
\le\quad & \Big|\Big\langle e^{W'_kG_h}\boldsymbol{P}'_{h,k}(\omega'),\big(\boldsymbol{m}'_{h,k}(\omega')-\boldsymbol{m}'(\omega')\big)\times\boldsymbol{\psi}\Big\rangle_{D_T}\Big|\\
&+\Big|\Big\langle (e^{W'_kG_h}-e^{W'_kG})\boldsymbol{P}'_{h,k}(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\Big\rangle_{D_T}\Big|\\
&+\Big|\Big\langle (e^{W'_kG}-e^{W'G})\boldsymbol{P}'_{h,k}(\omega'),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\Big\rangle_{D_T}\Big|\\
&+\Big|\Big\langle e^{W'G}\big(\boldsymbol{P}'_{h,k}(\omega')-\boldsymbol{P}'(\omega')\big),\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\Big\rangle_{D_T}\Big|\\
\le\quad & \|\boldsymbol{P}'_{h,k}(\omega')\|_{D_T}\|\boldsymbol{m}'_{h,k}(\omega')-\boldsymbol{m}'(\omega')\|_{D_T}\|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)}\\
&+c\|I_{\mathbb{V}_h}(\boldsymbol{g})-\boldsymbol{g}\|_D\|\boldsymbol{P}'_{h,k}(\omega')\|_{D_T}\|\boldsymbol{m}'(\omega')\|_{\mathbb{L}^\infty(D_T)}\|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)}\\
&+c\|W_k(\omega')-W'(\omega')\|_{\mathbb{L}^\infty([0,T])}\|\boldsymbol{P}'_{h,k}(\omega')\|_{D_T}\|\boldsymbol{m}'(\omega')\|_{\mathbb{L}^\infty(D_T)}\|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)}\\
&+\Big|\Big\langle \boldsymbol{P}'_{h,k}(\omega')-\boldsymbol{P}'(\omega'),e^{-W'G}\big(\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\big)\Big\rangle_{D_T}\Big|\\
\le c\|\boldsymbol{m}'_{h,k}(\omega')-\boldsymbol{m}'(\omega')\|_{D_T} &+ c\|I_{\mathbb{V}_h}(\boldsymbol{g})-\boldsymbol{g}\|_D+c\|W_k(\omega')-W'(\omega')\|_{\mathbb{L}^\infty([0,T])}\\
&+\Big|\Big\langle \boldsymbol{P}'_{h,k}(\omega')-\boldsymbol{P}'(\omega'),e^{-W'G}\big(\boldsymbol{m}'(\omega')\times\boldsymbol{\psi}\big)\Big\rangle_{D_T}\Big|,
\end{aligned}
$$

here the last inequality is obtained by using (6.15) and $|\boldsymbol{m}'(\omega')|=1$ a.e..

Hence, it follows from (6.16), part (4) in Proposition 6.8 and the weak convergence of $\{\boldsymbol{P}'_{h,k}(\omega')\}$ in $L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))$ that

$$
\mathcal{I}\to 0 \quad \text{as } h,k\to 0.
$$

This implies that $\boldsymbol{m}',\boldsymbol{P}'$ satisfy (4.11).

The convergence of (6.18) can be proved in the same manner by noting that $\{\boldsymbol{P}'_{h,k}(\omega')\}$ converges weakly in $L^2(0,T;\mathbb{H}(\mathrm{curl};\widetilde{D}))$, completing the proof of the theorem. □

7. Numerical experiment

In order to carry out physically relevant experiments (see [16]), the initial fields $\boldsymbol{M}_0$, $\boldsymbol{H}_0$ must satisfy the following conditions

$$\operatorname{div}(\boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0) = 0 \text{ in } \widetilde{D} \quad \text{and} \quad (\boldsymbol{H}_0 + \widetilde{\boldsymbol{M}}_0) \cdot \boldsymbol{n} = 0 \text{ on } \partial\widetilde{D}.$$

This can be achieved by taking

$$\boldsymbol{H}_0 = \boldsymbol{H}_0^* - \chi_D \boldsymbol{M}_0,$$

where $\operatorname{div} \boldsymbol{H}_0^* = 0$ in $\widetilde{D}$. In our experiment, for simplicity, we choose $\boldsymbol{H}_0^*$ to be a constant. We solve an academic example with $D = \widetilde{D} = (0,1)^3$ and

$$\begin{aligned} \boldsymbol{M}_0(\boldsymbol{x}) &= \begin{cases} (0,0,-1), & |\boldsymbol{x}^*| \geq \frac{1}{2}, \\ (2\boldsymbol{x}^* A, A^2 - |\boldsymbol{x}^*|^2)/(A^2 + |\boldsymbol{x}^*|^2), & |\boldsymbol{x}^*| \leq \frac{1}{2}, \end{cases} \\ \boldsymbol{H}_0^*(\boldsymbol{x}) &= (0,0,H_s), \quad \boldsymbol{x} \in \widetilde{D}, \end{aligned}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{x}^* = (x_1 - 0.5, x_2 - 0.5, 0)$ and $A = (1 - 2|\boldsymbol{x}^*|)^4/4$. The constant H_s represents the strength of $\boldsymbol{H}_0$ in the x_3-direction. We carried out the experiments for $H_s = 30$. We set the values for the other parameters in (3.1) and (3.2) as $\lambda_1 = \lambda_2 = \mu_0 = \sigma = 1$.

For each time step k, we generate a discrete Brownian path by:

$$W_k(t_{j+1}) - W_k(t_j) \sim \mathcal{N}(0,k) \quad \text{for all } j = 0, \cdots, J-1.$$

An approximation of any expected value is computed as the average of L discrete Brownian paths. In our experiments, we choose $L = 400$.

At each iteration we solve two linear systems of sizes $2N \times 2N$ and $M \times M$, recalling that N is the number of vertices and M is the number of edges in the triangulation. The code is written in Fortran90. The parameter θ in Algorithm 5.1 is chosen to be 0.7.

In the first set of experiments, to observe convergence of the method, we solve with $T = 1$, $h = 1/n$ where $n = 2, \dots, 7$, and different time steps $k = h$, $k = h/2$, and $k = h/4$. For each value of h, the domain D is partitioned into uniform cubes of size h. Each cube is then partitioned into six tetrahedra. Noting that

$$E_{h,k}^2 := \int_{D_T} \left|1 - |\boldsymbol{m}_{h,k}^-|\right|^2 \, d\boldsymbol{x}\, dt = \||\boldsymbol{m}| - |\boldsymbol{m}_{h,k}^-|\|_{D_T}^2 \leq \|\boldsymbol{m} - \boldsymbol{m}_{h,k}^-\|_{D_T}^2,$$

we compute and plotte in Figure 1 the error $\mathbb{E}[E_{h,k}^2]$ for different values of h and k.

In the second set of experiments to observe boundedness of discrete energies, we solve the problem with fixed values of $h = 1/7$ and $k = 1/20$. We plot $t \mapsto \|\nabla \boldsymbol{m}_{h,k}(t)\|_D^2$ in Figure 2 and $t \mapsto \|\boldsymbol{P}_{h,k}(t)\|_{\widetilde{D}}^2$ in Figure 3 for three individual paths and the expectations which seems to suggest that these energies are bounded when $t \to \infty$. Figure 4 shows that the total energy $\mathcal{E}(t) := \|\nabla \boldsymbol{m}_{h,k}(t)\|_D^2 + \|\boldsymbol{P}_{h,k}(t)\|_{\widetilde{D}}$ is bounded as in Lemma 5.4.

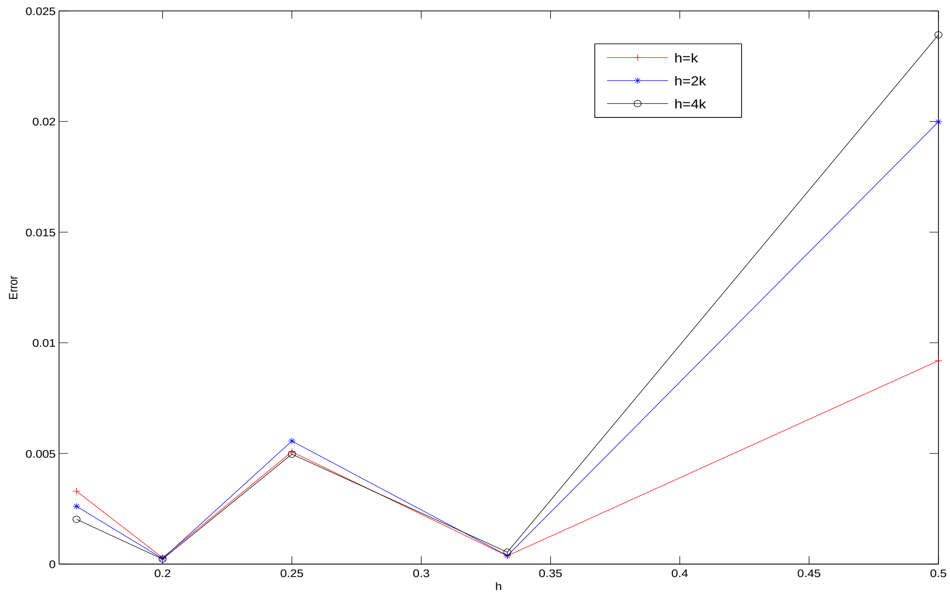

FIGURE 1. Plot of error $\mathbb{E}[E_{h,k}^2]$

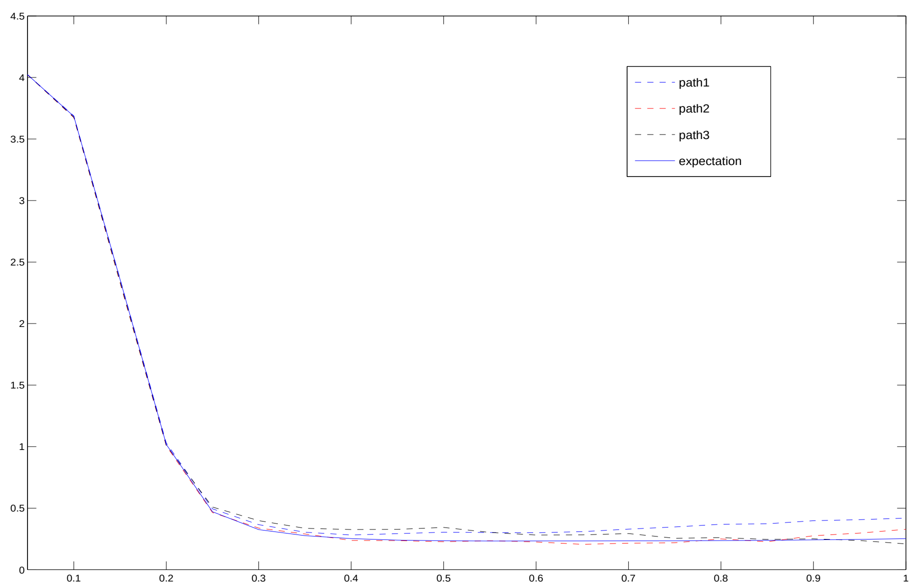

FIGURE 2. Plot of $t \mapsto \|\nabla \boldsymbol{M}_{h,k}(t)\|_D$, expectation and three individual paths

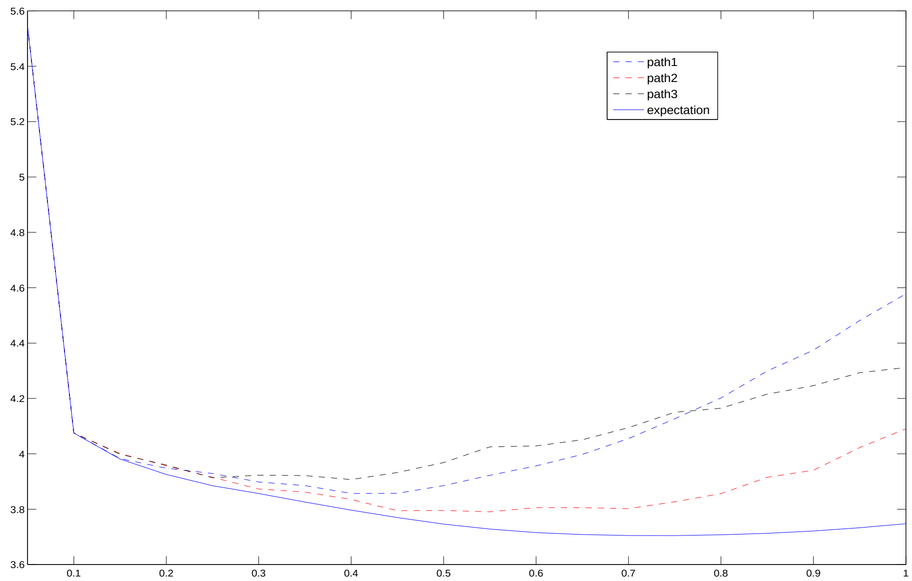

FIGURE 3. Plot of $t \mapsto \|\boldsymbol{P}_{h,k}(t)\|_{\widetilde{D}}$, expectation and three individual paths

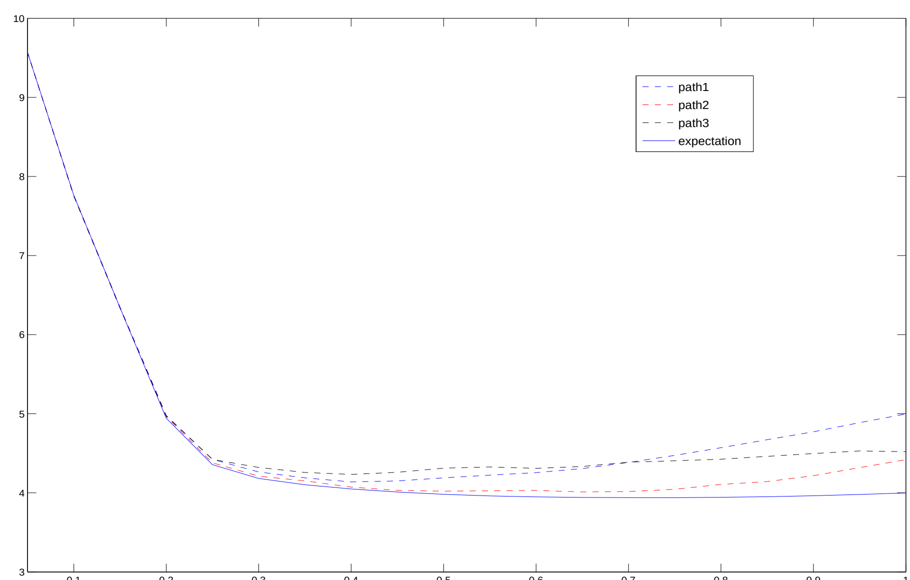

FIGURE 4. Plot of $t \mapsto \mathcal{E}(t)$, expectation and three individual paths

8. Appendix

For the reader's convenience we will recall the following lemmas proved in [15].

Lemma 8.1. *For any real constants λ_1 and λ_2 with $\lambda_1 \neq 0$, if $\boldsymbol{\psi}, \boldsymbol{\zeta} \in \mathbb{R}^3$ satisfy $|\boldsymbol{\zeta}| = 1$, then there exists $\boldsymbol{\varphi} \in \mathbb{R}^3$ satisfying*

$$\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{\zeta} = \boldsymbol{\psi}. \tag{8.1}$$

As a consequence, if $\boldsymbol{\zeta} \in H^1\big((0,T); \mathbb{H}^1(D)\big)$ with $|\boldsymbol{\zeta}(t,x)| = 1$ a.e. in D_T and $\boldsymbol{\psi} \in L^2\big((0,T); W^{1,\infty}(D)\big)$, then $\boldsymbol{\varphi} \in L^2\big((0,T); \mathbb{H}^1(D)\big)$.

Lemma 8.2. *For any $\boldsymbol{v} \in \mathbb{C}(D)$, $\boldsymbol{v}_h \in \mathbb{V}_h$ and $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$ there hold*

$$\|I_{\mathbb{V}_h} \boldsymbol{v}\|_{\mathbb{L}^\infty(D)} \leq \|\boldsymbol{v}\|_{\mathbb{L}^\infty(D)},$$

$$\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|^2_{\mathbb{L}([0,T],\mathbb{H}^1(D))} \leq ch^2 \|\boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}([0,T],\mathbb{H}^1(D))} \|\boldsymbol{\psi}\|^2_{\mathbb{W}^{2,\infty}(D_T)},$$

where $\boldsymbol{m}_{h,k}^-$ is defined in Defintion 6.1

The next lemma defines a discrete $\mathbb{L}^p$-norm in $\mathbb{V}_h$ which is equivalent to the usual $\mathbb{L}^p$-norm.

Lemma 8.3. *There exist h-independent positive constants C_1 and C_2 such that for all $p \in [1, \infty)$ and $\boldsymbol{u} \in \mathbb{V}_h$ there holds*

$$C_1 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)} \leq h^d \sum_{n=1}^{N} |\boldsymbol{u}(\boldsymbol{x}_n)|^p \leq C_2 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)},$$

where $\Omega \subset \mathbb{R}^d$, d=1,2,3.

Acknowledgements

The authors acknowledge financial support through the ARC projects DP160101755 and DP120101886.

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