

SEPARATION-TYPE COMBINATORIAL INVARIANTS FOR TRIANGULATIONS OF MANIFOLDS

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ABSTRACT. We propose and study a variation of Bagchi and Datta's σ -vector of a simplicial complex C , whose entries are defined as weighted sums of Betti numbers of induced subcomplexes of C . We prove that these invariants satisfy an Alexander-Dehn-Sommerville type identity, and behave nicely under natural operations on triangulated manifolds and spheres such as connected sums and bistellar flips.

In the language of commutative algebra, the invariants are weighted sums of graded Betti numbers of the Stanley-Reisner ring of C . This interpretation implies, by a result of Migliore and Nagel, that the Billera-Lee sphere maximizes these invariants among polytopal spheres with a given f -vector. We provide theoretical and experimental evidence that this bound also holds for non-polytopal spheres, and establish a slightly weaker upper bound for arbitrary pure and strongly connected simplicial complexes. Concerning lower bounds, our experimental evidence shows that these depend on whether only polytopal or arbitrary triangulated spheres are considered.

As an application of the upper bound, we show how this can be used to obtain lower bounds on the f -vector of a triangulated 4-manifold with transitive symmetry and prescribed vector of Betti numbers.

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1. INTRODUCTION

In this article we investigate a combinatorial invariant of simplicial complexes that we call the τ -vector, defined as follows: for a simplicial complex C with ground set V , and for each $i = \{-1, 0, 1, 2, \dots\}$,

$$\tau_i(C) = \frac{1}{|V|+1} \sum_{W \subset V} \frac{\tilde{\beta}_i(C[W])}{\binom{|V|}{|W|}}.$$

Here $C[W]$ denotes the subcomplex induced by a set $W \subset V$, and $\tilde{\beta}_i$ is the reduced i -th Betti number with respect to a certain field \mathbb{F} . The τ -vector depends on the choice of \mathbb{F} but our results are independent of \mathbb{F} .

Put differently, τ_i is the weighted average of the i -th reduced Betti number of all induced subcomplexes $C[W]$, with respect to weights that are uniform on subsets $W \subset V$ of equal size and add up to $\frac{1}{|V|+1}$ for each size $j \in \{0, \dots, |V|\}$.

The τ -vector is a variation of the σ -vector introduced by Bagchi and Datta [6] and studied in [5, 14]. Its original motivation was the study of *tight* triangulations of manifolds, that is, triangulations with the property that all the homomorphisms induced

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in homology by inclusions of induced subcomplexes are injective (see Section 6.2, in particular Definition 6.2 for more details):

Theorem 1.1 ([5, Theorem 1.7(c), Corollary 1.8]; [6, Theorems 2.6(b) and 2.10] for the 2-neighborly case). *For a simplicial d -manifold M with vertex set V and for every $i \in \{0, \dots, d\}$, let*

$$\mu_i := \sum_{v \in V} \tau_{i-1}(\text{lk}_M(v)).$$

Then $\tilde{\beta}_i(C) \leq \mu_i(C)$, with equality occurring for all i if and only if C is \mathbb{F} -tight.

Moreover, the τ -vector is also interesting from a more combinatorial viewpoint. For instance, it produces the following characterization of stacked 2-dimensional spheres (the only dimension where stackedness is not characterized by the f -vector).

Theorem 1.2 (Burton, Datta, Singh and Spreer [14]). *Let S be a triangulated 2-sphere with n -vertices. Then*

$$\tau_0(S) \leq \frac{1}{10(n+1)} \binom{n-3}{2} = \frac{n-8}{20} + \frac{1}{n+1},$$

with equality if and only if S is stacked.

The only differences between the τ - and the original σ -vector from [5, 6] are that we introduce a factor of $|V| + 1$ in the denominator and that we treat the case $W = \emptyset$ differently than in [5, 6]. We believe that these two (minor) differences make the definition more natural. For instance, they simplify Theorem 1.1 (compare our definition of μ in Theorem 1.1 to [5, Definition 1.5] and [6, Definition 2.1], where a distinction needs to be made for the cases $i = 0, 1$).

More importantly, however, our definition is “independent of the ground set” in the following sense. Suppose that we have a complex C with vertex set V_0 but which can also be considered as a complex on a bigger ground set $V \supsetneq V_0$. This happens naturally for each link $\text{lk}_M(v)$ appearing in Theorem 1.1, where V_0 is the set of vertices adjacent to v in the 1-skeleton of M . Then it makes sense to calculate the τ - or σ -vector of this complex both with respect to V and with respect to V_0 . Our first result (see Theorem 2.7) is that the τ -vector is independent of this choice, which is not the case for the original σ -vector. The underlying reason for this nice property is that our normalization causes the weights used on 2^V to define an “exchangeable probability measure” (see Remark 2.8).

This independence of the ground set simplifies computations in several places. For example, in the context of connected sums it has the consequence that one can apply Mayer-Vietoris type sequences to all induced subcomplexes simultaneously, to easily obtain the following statement Section 4.2.

Theorem 1.3 (Theorem 4.6). *Let M_1 and M_2 be simplicial d -manifolds, $d \geq 2$. Then*

$$\begin{aligned} \tau_i(M_1 \# M_2) &= \tau_i(M_1) + \tau_i(M_2), & i \in \{1, \dots, d-2\} & \quad \text{and} \\ \tau_j(M_1 \# M_2) &= \tau_j(M_1) + \tau_j(M_2) + c(d, n_1, n_2), & j \in \{0, d-1\}, \end{aligned}$$

where $n_i = f_0(M_i)$ is the number of vertices in M_i and

$$c(d, n_1, n_2) := \frac{1}{d+2} - \frac{1}{n_1+1} - \frac{1}{n_2+1} + \frac{1}{n_1+n_2-d-1}.$$

In the special case where M_2 is the boundary of a $(d+1)$ -simplex, the change from M_1 to M is a so-called *bistellar flip of type $(1, d+1)$* . In particular, Theorem 1.3 yields a formula for how the τ -vector changes under such flips (Corollary 4.7) and for the τ -vector of stacked spheres, which are the simplicial spheres that can be obtained from the boundary of a simplex via such flips. For the more general case of (i, j) -flips in manifolds there is no closed formula on how the τ -vector changes, but some partial results can be stated, see Section 4.1.

For neighborly and stacked manifolds some components of the τ -vector become zero. More precisely, for a k -neighborly complex all τ_i with $i \leq k - 2$ are zero (see Proposition 4.9) and for a k -stacked d -sphere all τ_i with $k \leq i \leq d - k - 1$ are zero (see Theorem 5.7). These two results can already be found in [6, Lemma 3.9 and Theorem 3.5a] due to Bagchi and Datta, but their proof of Theorem 5.7 needs a shellability condition that our proof does not require (see Remark 5.9). Going further, we look at manifolds that are almost 2-neighborly or almost 1-stacked and prove bounds for the entries of their τ -vectors (Theorems 4.10, 5.12 and 5.13). In particular, Section 5.3 completely characterizes the possible τ -vectors of spheres with $g_2 = 1$, following a classification of such spheres by Nevo and Novinsky [41].

The τ -vector also has a commutative algebra interpretation. As customary, to a simplicial complex C we associate its Stanley-Reisner ideal $I(C)$. The minimal resolution of $I(C)$ gives rise to a triangular array of *graded Betti numbers* $r_{i,j}$ for $-1 \leq i < j \leq |V|$. Hochster's formula says that each $r_{i,j}$ equals the sum over all subsets $W \subset V$ of size j of $\tilde{\beta}_{j-i-2}(C[W])$ (see Equation (2)).

In particular, the entries of the τ -vector are non-negative linear combinations of the graded Betti numbers, which implies that known upper and lower bounds for the latter apply to the former. For example, it was shown by Migliore and Nagel [38, 40] that, among all the *polytopal* spheres with a given f -vector, the Billera-Lee spheres [11] maximize every $r_{i,j}$ over any field of characteristic zero. In particular:

Theorem 1.4 (Corollary of [38, 40]). *Let S be a simplicial polytopal d -sphere and let T be the Billera-Lee d -sphere with the same f -vector as S . Then $\tau_i(S) \leq \tau_i(T)$ for all i , and with respect to any field of characteristic zero.*

We prove a similar statement for $i = 0$ in the much wider class of pure complexes: among all pure strongly connected complexes with a given dimension $d \geq 3$ and numbers of vertices and edges (f_0, f_1) the Billera-Lee balls with those parameters (which exist since a strongly connected d -manifold has $f_1 \in [df_0 - \binom{d+1}{2}, \binom{f_0}{2}]$) maximize τ_0 (Corollary 3.9 and Theorem 3.10). The proof uses the fact that the τ_0 value, an invariant of the 1-skeleton, can be upper bounded in terms of what is called the in-degree sequence associated to an ordering of the vertices, with equality in the upper bound if and only if the ordering is a *partial elimination order* (Lemma 3.2).

We conjecture that Theorem 1.4 holds also for non-polytopal spheres and for arbitrary fields. That is:

Conjecture 1.5. *Let S be a d -sphere with the f -vector of some $(d+1)$ -polytope and let T be the Billera-Lee d -sphere with f -vector $f(S)$. Then $\tau_i(S) \leq \tau_i(T)$ for all i .*

Observe that it is not known whether all triangulated d -spheres have the f -vector of some $(d+1)$ -polytope; this is the so-called g -conjecture.

Partial evidence for Conjecture 1.5 is that: (a) it holds for nearly neighborly and nearly stacked spheres. See Sections 4.3 and 5.3, where we compute exact values and upper bounds for the τ -vectors of (nearly) stacked (Section 5.3) and (nearly) neighborly (see Section 4.3) spheres and manifolds; (b) it is closed under connected sums (Lemma 3.12); and (c) exhaustive computations show that it holds for all 3-spheres up to 10 vertices (Section 6.1).

Incidentally, these exhaustive computations also suggest that *lower bounds* for the entries of the τ -vector of polytopal spheres are not satisfied by non-polytopal spheres.

The topological Alexander duality implies that the τ -vector of a d -sphere S is symmetric, i.e., that we have $\tau_i(S) = \tau_{d-i-1}(S)$. Together with the Dehn-Sommerville relations this yields a formula relating the first half of the τ - and h -vector to each other (see Corollary 5.5). For example, for 3-spheres Alexander duality implies any of τ_0, τ_1, τ_2 (together with the f -vector or, equivalently, the h -vector) gives the other two, via the

following relations (Corollary 5.6):

$$(1) \quad \tau_0 = \tau_2, \quad \tau_0 - \tau_1 + \tau_2 = \frac{h_1(h_1 + 1)}{10(h_1 + 5)} - \frac{h_2}{30}.$$

Moving towards more topological applications of the τ -vector, we use the conjectured and proven upper bounds for τ in dimension three to construct lower bounds for triangulations of 4-manifolds with vertex transitive symmetry and prescribed vector of Betti numbers (see Section 6.3). These lower bounds are interesting because, when attained, they not only provide a minimal symmetric triangulation, but also one that is tight.

2. DEFINITION AND FIRST PROPERTIES

2.1. Notation and conventions. In this section we briefly introduce most of the combinatorial/topological concepts used all throughout the paper and set notation and conventions, although some of the details are deferred to the sections where they are used.

Simplicial complexes. An (abstract) *simplicial complex* C on a ground set V (typically $V = [n]$ for an $n \in \mathbb{N}$) is a family of subsets of V closed under taking subsets. The elements of C are called *faces*, and the *dimension* of a face is its size minus one. Maximal faces are called *facets* and faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. Observe that the set of vertices can be properly contained in V , that is, not every element of V is necessarily a vertex of C . The *empty complex* has only one face, the empty set. The k -skeleton of C is the subcomplex consisting of faces of dimension at most k . The dimension of C is the maximum dimension of a face, and a complex of dimension d is sometimes called a d -complex.

The following properties that a simplicial complex may have are each more restrictive than the previous one:

- C is *pure* of dimension d if all facets have dimension d . In this case the faces of size d are called *ridges* and the *adjacency graph* of C , sometimes also referred to as the *dual graph* of C , is defined as having as vertices the facets of C and as edges the pairs of facets that share a common ridge. This is different from the 1-skeleton of C , which is also a graph and sometimes called the *graph* of C .
- A pure complex C is called *strongly connected* if its adjacency graph is connected.
- A *(closed) pseudo-manifold* is a strongly-connected pure complex such that every ridge is contained in exactly two facets. In this case there exists a bijective correspondence between ridges of C and edges in the adjacency graph. A *pseudo-manifold with boundary* is a strongly-connected pure complex such that every ridge is contained in at most two facets. The ridges contained in only one facet, together with all their faces, form the *boundary* of C , which is a pure $(d - 1)$ -complex.
- A *(closed) triangulated manifold*, or simply a *(closed) manifold*, is a simplicial complex whose topological realization is a (closed) manifold. We say the manifold is a sphere, a ball, etc if its topological realization is.
- A *combinatorial manifold* is a simplicial complex in which the *link* of every vertex v , i.e., the boundary of the complex consisting of all facets of C containing v and their faces, is a triangulated sphere PL-homeomorphic to the standard sphere.

f -, h -, and g -vectors. The f -vector of a d -complex is defined as $f(C) = (f_{-1}, \dots, f_d)$ where f_i is the number of faces of size $i + 1$.

For a pure complex C with f -vector $(f_{-1}, f_0, \dots, f_d)$ one defines the h -vector $h(C) = (h_0, \dots, h_{d+1})$ by

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1}.$$

The h -vector contains the same information as the f -vector since the above formula can be reversed (Proposition 5.3). But it has nicer properties; for example, the h -vector of a closed manifold satisfies the Klee-Dehn-Sommerville equations [27]:

$$h_{d+1-i} - h_i = (-1)^i \binom{d+1}{i} (\chi(C) - \chi(S^d)) \quad \text{for all } i.$$

From the h -vector one can, in turn, deduce the g -vector $g(C) = (g_0, \dots, g_{d+2})$ as $g_i = h_i - h_{i-1}$. See [24, Chapter 17] for more details.

Betti numbers. For any given base field \mathbb{F} the *reduced chain complex* of C is the naturally defined sequence of linear maps

$$\mathbb{F}f_d \xrightarrow{\partial_d} \mathbb{F}f_{d-1} \rightarrow \dots \rightarrow \mathbb{F}f_1 \xrightarrow{\partial_1} \mathbb{F}f_0 \xrightarrow{\partial_0} \mathbb{F}f_{-1} = \mathbb{F} \rightarrow 0.$$

The *reduced Betti numbers* $\tilde{\beta}_d, \dots, \tilde{\beta}_0, \tilde{\beta}_{-1}$ of C are the dimensions of the corresponding homology (or cohomology) groups. Equivalently,

$$\tilde{\beta}_i := f_i - \text{rank } \partial_i - \text{rank } \partial_{i+1}.$$

Thus, the alternating sum of Betti numbers coincides with the alternating sum of f -vector entries; the *Euler characteristic* of C equals that sum plus one:

$$\chi(C) := \sum_{i=-1}^d (-1)^i \tilde{\beta}_i + 1 = \sum_{i=-1}^i (-1)^i f_i + 1.$$

Remark 2.1. Observe that every complex has $f_{-1} = 1$, for the empty face; in particular $\chi(C) = \sum_{i=0}^i (-1)^i f_i$. Every complex other than the empty complex has $\tilde{\beta}_{-1} = 0$, and thus $\chi(C) = \sum_{i=0}^d (-1)^i \tilde{\beta}_i + 1$.

(Induced) subcomplexes, Alexander duality, Hochster's formula. Let C be a simplicial complex on a ground set V . A subcomplex of C is a subset of the faces of C that is itself a simplicial complex. For any $W \subset V$, the subcomplex of C *induced by* W is

$$C[W] := \{F \in C : F \subset W\}$$

and the *deletion of* W *in* C is the induced subcomplex $C[V \setminus W]$. When C is a d -sphere, the Betti numbers of these two subcomplexes are related by *Alexander duality*:

$$\tilde{\beta}_i C[W] = \tilde{\beta}_{d-1-i} C[V \setminus W].$$

Remark 2.2. In this paper we always mean Alexander duality in the above, topological, sense. This is related but not to be confused with the combinatorial version of Alexander duality as, for instance, stated in [12].

The *Stanley-Reisner ideal* I_C of C is the ideal in $R := \mathbb{F}[x_1, \dots, x_n]$ generated by all monomials whose support is not a face in C . It has a minimal resolution which is unique modulo isomorphism

$$R^{r_n} \rightarrow R^{r_{n-1}} \rightarrow \dots \rightarrow R^{r_0} \rightarrow I_C \rightarrow 0.$$

The ranks r_i appearing in the resolution can be refined as follows: The usual grading in I_C induces a grading in all the R^{r_i} 's, so that we can write

$$R^{r_i} = \bigoplus_{j=i+1}^n R^{r_{i,j}},$$

where $R^{r_{i,j}}$ is the part of R^{r_i} of degree j . By convention we take $r_{-1,0} = 1$ and $r_{-1,j} = 0$ for every other j . (The convention for $i = -1$ is justified by looking at the resolution of R/I rather than I . That resolution ends in $\rightarrow R \rightarrow R/I \rightarrow 0$, where the generator in $R = R^{r_{-1}}$ has degree zero.) The numbers $r_{i,j} = r_{i,j}(I_C)$ for $-1 \leq i < j \leq n$ are called the *graded Betti numbers* of I_C .

Hochster's formula [39, Corollary 5.12] implies that

$$(2) \quad r_{i,j}(I_C) = \sum_{\substack{W \subset V \\ |W|=j}} \tilde{\beta}_{j-i-2}(C[W]).$$

In particular, $r_{i,n}(I_C) = \tilde{\beta}_{n-i-2}(C)$ and $r_{0,j}(I_C)$ equals the number of minimal non-faces of size j in C , since these correspond to the generators of degree j in I_C .

2.2. The τ -vector. In [6], Bagchi and Datta introduce the σ -vector of a simplicial complex C . It is defined as follows:

Definition 2.3 (σ -vector, Definition 2.1 in [6]). Let C be a simplicial complex of dimension d with vertex set V . For each $i = -1, \dots, d$ we define

$$\sigma_i(C) = \sum_{W \subset V} \frac{\tilde{\beta}_i(C[W])}{\binom{|V|}{|W|}}.$$

Here, $\tilde{\beta}_i$ denotes the i -th reduced Betti number with respect to a certain field \mathbb{F} , that we omit.

In this article, we slightly adapt the definition of the σ -vector:

Definition 2.4. Let C be a simplicial complex of dimension d on a ground set V of size $|V| = n$. For each integer $i \geq -1$ we define

$$\tau_i(C) = \frac{1}{n+1} \sum_{W \subset V} \frac{\tilde{\beta}_i(C[W])}{\binom{n}{|W|}}.$$

Remark 2.5. In [5], [6] and [14], $\tilde{\beta}_0$ is defined to be equal to $\beta_0 - 1$. This coincides with our convention, see Section 2.1, except for the empty complex where their convention gives $\tilde{\beta}_0 = -1$ and ours gives $\tilde{\beta}_0 = 0$. We believe our convention is more commonly used, see for instance [25]. It also behaves more nicely, e.g., regarding Alexander duality. All statements from the above papers that we cite in this article have been adapted to our convention.

Hochster's Equation (2) gives the following interpretation of the τ -vector:

Proposition 2.6. Let C be a simplicial complex on a vertex set V of size n and let $\{r_{i,j}\}_{0 \leq i \leq j \leq n}$ be the graded Betti numbers of the Stanley-Reisner ideal I_C . Then

$$\tau_i = \frac{1}{n+1} \sum_{j=i+1}^n \frac{r_{j-i-2,j}}{\binom{n}{j}}.$$

The value τ_i is essentially the same as $\sigma_i/(n+1)$, except for the fact that we do not need to assume all elements of V to be used as vertices in C . That is, we may have $f_0(C) < |V|$. As the primary example, observe that the link $\text{lk}_C(v)$ of a vertex v of the simplicial complex C can be considered as a complex having as ground set the set of all vertices of C or having only those vertices joined to v . The value of the σ -vector as defined in [6] depends on the choice of ground set, but the τ -vector does not:

Theorem 2.7. Let C be a simplicial complex on a ground set V but assume that it only uses as vertices a subset $V_0 \subsetneq V$. Then, $\tau_i(C)$ is independent of whether we compute it using V or V_0 as a ground set.

Proof. Assume that $V \setminus V_0$ is a single vertex v , from which the general case follows by induction. Then, using V as a ground set we have that each subset $W_0 \subset V_0$ appears

twice in the sum for τ_i , once truly as W_0 and once as $W_0 \cup \{v\}$ but with the same induced subcomplex $C[W_0 \cup \{v\}] = C[W_0]$. Hence, we can write

$$\begin{aligned} \tau_i(C) &= \frac{1}{|V|+1} \sum_{W_0 \subset V_0} \tilde{\beta}_i(C[W_0]) \left(\frac{1}{\binom{|V|}{|W_0|}} + \frac{1}{\binom{|V|}{|W_0|+1}} \right) \\ &= \frac{1}{|V_0|+2} \sum_{W_0 \subset V_0} \tilde{\beta}_i(C[W_0]) \left(\frac{1}{\binom{|V_0|+1}{|W_0|}} + \frac{1}{\binom{|V_0|+1}{|W_0|+1}} \right) \\ &= \frac{1}{|V_0|+2} \sum_{W_0 \subset V_0} \frac{\tilde{\beta}_i(C[W_0])}{\binom{|V_0|}{|W_0|}} \left(\frac{|V_0|+1-|W_0|}{|V_0|+1} + \frac{|W_0|+1}{|V_0|+1} \right) \\ &= \frac{1}{|V_0|+1} \sum_{W_0 \subset V_0} \frac{\tilde{\beta}_i(C[W_0])}{\binom{|V_0|}{|W_0|}}, \end{aligned}$$

which coincides with the definition when taking V_0 itself as the ground set. \square

Another motivation for preferring τ over σ is that it admits a probabilistic interpretation. Indeed, the coefficients

$$P_V(W) := \frac{1}{(|V|+1)\binom{|V|}{|W|}}$$

add up to 1 when we sum over all subsets W of V , hence they are a probability distribution in 2^V . Our $\tau_i(C)$ is nothing but the expected reduced Betti number $\tilde{\beta}_i$ over all induced subcomplexes of C , with respect to this probability distribution.

In this interpretation P_V can be thought as the joint distribution of $|V|$ binary random variables (the indicators of the subsets $W \subset V$). Then, Theorem 2.7 follows from (and is in fact equivalent to) the fact that for every subset $V_0 \subset V$ of variables we have that P_{V_0} is the restriction of P_V to that subset.

Remark 2.8. The probability distribution P_V on $2^{|V|}$ is equivalent to the *Pólya urn model with one initial ball of each color* [36]. Suppose that we have an urn, initially containing one ball of color 0 and one ball of color 1. We repeat the following procedure n times: take a ball uniformly at random from the urn, then place it back in the urn together with an additional ball of the same color. It can easily be checked that the probability of obtaining a certain sequence $S \in \{0,1\}^n$ after n trials equals $\frac{1}{(n+1)\binom{n}{k}}$ where k is the number of 1's in S .

One may ask what other probability distributions over 2^V can be used to define invariants with similar properties as the τ -vector. There are two nice properties of P_V that are implicitly used throughout this paper, and which a viable alternative probability distribution should satisfy as well:

- (1) P_V causes the $|V|$ individual binary variables to be *exchangeable*. That is, the probability of a subset W only depends on $|W|$ and not on the particular elements it contains. This property makes the τ -vector manageable from the combinatorial point of view, and is also needed in order to have (a statement analogue to) Proposition 2.6.
- (2) P_V is symmetric under *complementation*. Equivalently, the individual probability of each element is $1/2$. We implicitly use this property often, especially in connection to Alexander duality in Section 5.1.

Assuming both exchangeability and symmetry under complementation, a very natural alternative choice of probability distribution is the uniform distribution, giving probability $(1/2)^{|V|}$ to every subset. Our reason to prefer P_V is that, this way, the τ -vector allows to determine whether a simplicial manifold is tight or not, see Theorem 1.1 and its applications in Section 6.

3. τ_0 OF (CHORDAL) GRAPHS AND STRONGLY CONNECTED PURE COMPLEXES

In this section, we look at the first non-trivial entry of the τ -vector, the 0-dimensional one. By definition, $\tau_i(C)$ only depends on the $(i+1)$ -skeleton of C , so in particular τ_0 only depends on the 1-skeleton and can be studied for arbitrary graphs. Computing τ_0 exactly for a given graph is likely to be a computationally hard problem in general, since it is closely related to hard network reliability problems [13]. Here we show that computing it for chordal graphs is straightforward, and discuss implications for the τ_0 of arbitrary manifolds (or, more generally, for strongly connected pure complexes).

3.1. τ_0 via perfect elimination orderings and in-degree sequences. Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and consider the vertices given in a particular order v_1, \dots, v_n . An *in-degree sequence* of G with respect to that ordering is a sequence $(\delta_1, \dots, \delta_n)$ where δ_i is the number of neighbors of v_i among $\{v_1, \dots, v_{i-1}\}$. Equivalently, it is the in-degree sequence of the digraph obtained from G by directing all edges from the smaller to the larger vertex (according to the given ordering). Let $G_i = G[v_1, \dots, v_i]$ be the subgraph induced by the first i vertices.

An ordering of the vertices of G is called a *perfect elimination ordering* (or p.e.o., for short) [19, 23] if for every $i = 2, \dots, n$ we have that $G[\{v_j : j < i \text{ and } v_i v_j \in E\}]$ is a clique in G . Equivalently, if G_i is obtained from G_{i-1} by joining the new vertex v_i to a clique.

Remark 3.1. Dirac's Theorem says that the existence of a p.e.o in a graph G is equivalent to G being *chordal*; that is, no cycle in G of length greater than three is induced.

Several generalizations of chordality to higher dimension have been proposed, the closest to our work being the recent homological one by Adiprasito, Nevo and Samper [2]. More precisely, Adiprasito et al. call a simplicial complex C *resolution k -chordal* if $\tau_k(C) = 0$ and *decomposition k -chordal* if the k -th homology of every induced subcomplex of C is generated by k -cycles isomorphic to the boundary of a $(k+1)$ -simplex. Among other results, they prove that all complexes with a certain k -Dirac property are decomposition k -chordal [2, Proposition 6.3] and explore the converse implication.

Lemma 3.2. *In the above notation, we have*

$$(3) \quad \tau_0(G_i) - \tau_0(G_{i-1}) \leq \frac{1}{(\delta_i + 1)(\delta_i + 2)} - \frac{1}{i(i+1)},$$

with equality if and only if $G[\{v_j : j < i \text{ and } v_i v_j \in E\}]$ is a clique. In particular,

$$\tau_0(G) \leq \frac{1}{n+1} + \sum_{i=1}^n \frac{1}{(\delta_i + 1)(\delta_i + 2)} - 1,$$

with equality if and only if the ordering v_1, \dots, v_n is a perfect elimination ordering.

Proof. The second part of the lemma easily follows from the first one by induction on i (for the base case $i = 1$ observe that $\delta_1 = 0$).

For the first part, denote $V_i = \{v_1, \dots, v_i\}$ for each i and rewrite $\tau_0(G_i)$ as

$$\tau_0(G_i) = \frac{1}{i+1} \sum_{W \subset V_{i-1}} \frac{\tilde{\beta}_0(G_i[W])}{\binom{i}{|W|}} + \frac{1}{i+1} \sum_{W \subset V_{i-1}} \frac{\tilde{\beta}_0(G_i[W \cup \{i\}])}{\binom{i}{|W|+1}}.$$

By Theorem 2.7, we can consider G_{i-1} as a graph on the ground set V_i and write

$$\tau_0(G_{i-1}) = \frac{1}{i+1} \sum_{W \subset V_{i-1}} \frac{\tilde{\beta}_0(G_i[W])}{\binom{i}{|W|}} + \frac{1}{i+1} \sum_{W \subset V_{i-1}} \frac{\tilde{\beta}_0(G_i[W])}{\binom{i}{|W|+1}}.$$

The difference of these two expressions gives

$$\tau_0(G_i) - \tau_0(G_{i-1}) = \frac{1}{i+1} \sum_{W \subset V_{i-1}} \frac{\tilde{\beta}_0(G_i[W \cup \{i\}]) - \tilde{\beta}_0(G_i[W])}{\binom{i}{|W|+1}}.$$

We thus need to look at the difference $\tilde{\beta}_0(G_i[W \cup \{i\}]) - \tilde{\beta}_0(G_i[W])$. For each $W \subset V_{i-1}$ we distinguish depending on whether W contains some neighbor of v_i or not:

- If W contains no neighbor of v_i then $\tilde{\beta}_0(G_i[W \cup \{i\}]) - \tilde{\beta}_0(G_i[W]) = 1$ except for the case $W = \emptyset$, where it equals zero. The contribution of all such subsets W to $\tau_0(G_i) - \tau_0(G_{i-1})$ can thus be computed exactly.

$$\frac{1}{i+1} \sum_{k=1}^{i-1-\delta_i} \frac{\binom{i-1-\delta_i}{k}}{\binom{i}{k+1}} = \frac{1}{i+1} \sum_{k=0}^{i-1-\delta_i} \frac{\binom{i-1-\delta_i}{k}}{\binom{i}{k+1}} - \frac{1}{i(i+1)} = \frac{1}{(\delta_i+1)(\delta_i+2)} - \frac{1}{i(i+1)}.$$

The last equality is a consequence of Lemma 3.3 below.

- If W contains neighbors of v_i then $\tilde{\beta}_0(G_i[W \cup \{i\}]) - \tilde{\beta}_0(G_i[W]) \leq 0$, and equality holds for every W if and only if the neighbors form a clique. This finishes the proof. \square

The following combinatorial identity used in the proof of Lemma 3.2 appears several times in the paper.

Lemma 3.3. *For any non-negative integers n and $a \leq b$ one has*

$$\sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+b}{k+a}} = \frac{n+b+1}{(b+1)\binom{b}{a}}.$$

Proof.

$$\begin{aligned} \binom{b}{a} \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+b}{k+a}} &= \frac{b!}{a!(b-a)!} \sum_{k=0}^n \frac{n!(n+b-k-a)!(k+a)!}{(n+b)!(n-k)!k!} \\ &= \frac{n!b!}{(n+b)!} \sum_{k=0}^n \frac{(n+b-k-a)!(k+a)!}{(n-k)!k!a!(b-a)!} \\ &= \frac{n!b!}{(n+b)!} \sum_{k=0}^n \binom{n+b-k-a}{b-a} \binom{k+a}{a} \\ &= \frac{n!b!}{(n+b)!} \binom{n+b+1}{b+1} = \frac{n+b+1}{b+1}. \end{aligned}$$

The second to last equality

$$\sum_{k=0}^n \binom{n+b-k-a}{b-a} \binom{k+a}{a} = \binom{n+b+1}{b+1}$$

is a form of the Chu-Vandermonde identity, and follows from the fact that the left-hand side enumerates subsets of $[n+b+1]$ of size $b+1$: each summand counts the possibilities for the $(a+1)$ -th element in the subset to be $k+a+1$. \square

3.2. Billera-Lee spheres and balls. The Billera-Lee spheres are polytopal $(d-1)$ -spheres realizing all possible f -vectors allowed by McMullen's conditions, see [11]. They are constructed as the boundaries of certain d -balls that we call *Billera-Lee balls*. We here introduce this construction, motivated by the fact (proven in Section 3.3) that Billera-Lee d -balls maximize τ_0 among all strongly connected d -complexes with a fixed number of vertices and edges.

Consider the cyclic $(d+1)$ -polytope with n vertices, $C_{d+1}(n)$. In its standard embedding along the moment curve, we define its *lower facets* to be the facets whose exterior normal has a negative last coordinate. By Gale's evenness criterion (see, for instance, [18, Corollary 6.1.9]), the lower facets of $C_{d+1}(n)$ are

- $(i_1, i_1 + 1, \dots, i_k, i_k + 1)$, for $1 \leq i_1 < \dots < i_k \leq n - 1$ non-consecutive, if $d + 1 = 2k$ is even; and
- $(1, i_1, i_1 + 1, \dots, i_k, i_k + 1)$, for $2 \leq i_1 < \dots < i_k \leq n - 1$ non-consecutive, if $d + 1 = 2k + 1$ is odd.

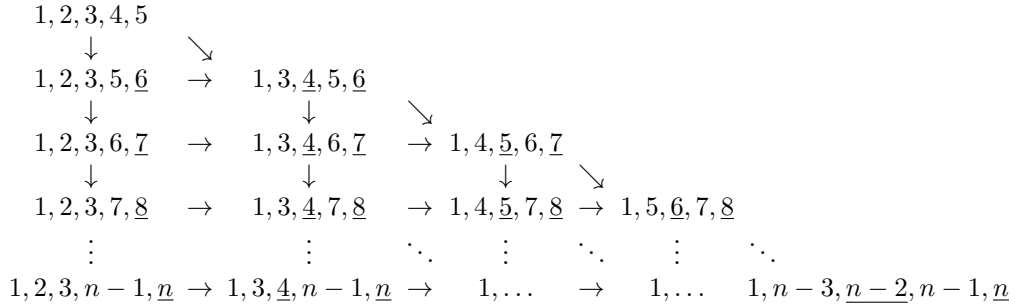


FIGURE 1. Directed dual graph, or shelling order, of the 4-ball that is the lower envelope of the cyclic 5-polytope with n vertices.

As a complex, these lower facets form a d -ball that we denote $L_d(n)$. See Figure 1 for the list of facets of $L_d(n)$. As shown in the figure, we consider the adjacency graph of $L_{d+1}(n)$ with its edges directed towards the lexicographically larger facet incident to the ridge they represent. This orientation is obviously acyclic. It corresponds to a shelling of $L_{d+1}(n)$ in the sense that any linear ordering compatible with this orientation is a shelling order. Since the in-degree of each facet in this directed graph equals the number of ridges it has in common with the complex it is glued to in the shelling process, the h -vector of $L_d(n)$ has as h_i the number of facets of in-degree i . An easy calculation shows that

$$h_i = \binom{n-d-2+i}{i}, \quad \forall i = 0, \dots, \lfloor d/2 \rfloor,$$

and $h_i = 0$ for $i = \lfloor d/2 \rfloor + 1, \dots, d + 1$.

Let B be any ideal in the partial order. That is, let B be a subset of facets of $L_d(n)$ such that $F \in B$ and $F' \rightarrow F$ is a directed edge implies $F' \in B$. Then B is an initial segment of a shelling of $L_d(n)$ and, in particular, B is a shellable d -ball whose h_i equals the number of facets F in B that have in-degree equal to i . The *Billera-Lee balls* are given by some particular ideals of this type.

Theorem 3.4 (Billera and Lee [11]). *Let $k = (k_0, \dots, k_{\lfloor d/2 \rfloor})$ be any vector satisfying McMullen's conditions for the g -vector of a d -polytope and let $n \geq k_0 + d + 1$. Let B_k^d be the subset of facets of $L_d(n)$ given by*

$$B_k^d := \bigcup_{i=0}^{\lfloor d/2 \rfloor} \{\text{first } k_i \text{ facets of } L_d(n) \text{ with in-degree } i, \text{ in reverse lexicographical order}\}.$$

Then, B_k^d is an ideal in the partial order of facets of $L_d(n)$. In particular, it is a shellable ball with h -vector equal to k . Moreover, ∂B_k^d is polytopal.

We call B_k^d constructed in the theorem the *Billera-Lee ball* with h -vector equal to the given k , and its boundary $S_k^{d-1} := \partial B_k^d$ the *Billera-Lee sphere* with g -vector

$$(k_0, \dots, k_{\lfloor d/2 \rfloor - 1}, k_{\lfloor d/2 \rfloor}, -k_{\lfloor d/2 \rfloor}, -k_{\lfloor d/2 \rfloor - 1}, \dots, -1).$$

Here $k_{\lfloor d/2 \rfloor}$ is considered to be zero when d is odd. That this is indeed the g -vector of S_k^{d-1} follows from the fact that the h -vector of a ball and the g -vector of its boundary are related by $g(\partial B) = h(B) - \bar{h}(B)$, where $\bar{h}(B)$ denotes the h -vector written in reverse. (See [18, Theorem 2.6.11] or [24, Chapter 17, Theorem 7.3.6].)

Example 3.5. For $d = 4$, assume that we want to construct a 3-sphere with prescribed g -vector (g_1, g_2) . In Figure 1 take the first $g_1 + 1$ facets of the first column (the first facet contributes to h_0 , the others contribute to h_1), and the first g_2 facets of the other columns in the order they are read (13456, 13467, 14567, 13478, ...). Observe that each row, considered as a sequence of flips in the boundary of the previously constructed ball, connects the vertex i , inserted by a $(1, 4)$ -flip by the first facet in the row, to all other vertices of the sphere. In particular, the last row corresponds to a sequence of flips turning $C_4(n - 1)$ into $C_4(n)$. In Figure 1, the (minimal) faces introduced by the respective flip are underlined.

For dimension $d \geq 3$, number of vertices $f_0 \geq d + 1$ and number of edges $f_1 \in [df_0 - \binom{d+1}{2}, \binom{f_0}{2}]$, the graph of the Billera-Lee d -ball with f_0 vertices and f_1 edges consists of a) a clique of size $k \in [d + 1, f_0]$, b) if $k < f_0$, a vertex attached to $j \in [d, k - 1]$ vertices from that clique, and c) $f_0 - k - 1$ additional vertices each attached to d vertices forming a d -clique in the previous list. The parameters j and k can be deduced from f_0, f_1 and d in the following way: k is the largest integer such that $\binom{k}{2} + (f_0 - k)d \leq f_1$, and $j = f_1 - \binom{k}{2} - (f_0 - k - 1)d$.

Definition 3.6. We call a graph obtained in this way a *Billera-Lee graph* with parameters (f_0, f_1, d) .

Observe that the graph of the Billera-Lee $(d - 1)$ -sphere with f_0 vertices and f_1 edges is isomorphic to some Billera-Lee graph with parameters (f_0, f_1, d) .

Since the given ordering of vertices is a p.e.o., we can deduce τ_0 of a Billera-Lee graph from Lemma 3.2.

Corollary 3.7. *With the above notation, the τ_0 -value of a Billera-Lee graph G with parameters (f_0, f_1, d) equals*

$$\tau_0(G) = \frac{1}{f_0 + 1} - \frac{1}{k + 1} + \frac{1}{(j + 1)(j + 2)} + \frac{f_0 - k - 1}{(d + 1)(d + 2)}.$$

3.3. In-degree sequences of pure complexes and an upper bound for τ_0 .

Proposition 3.8. *Every pure d -complex with connected adjacency graph (that is, every strongly connected pure simplicial complex) has an ordering of its vertices with in-degree sequence $(0, 1, 2, 3, \dots, d, k_1, \dots, k_{n-d-1})$ with $k_i \geq d$ for all $1 \leq i \leq n - d - 1$. We call sequences with this property d -dimensional.*

Note that every d -dimensional in-degree sequence is also $(d - 1)$ -dimensional. The d only indicates a lower bound for the entries in the sequence.

Proof. In order to construct a d -dimensional in-degree sequence, build up the complex step by step adding one vertex at the time and forming the respective induced subcomplex. Start with the vertices of a facet to obtain the first $d + 1$ entries of the sequence. By the connectedness assumption, in any further induced subcomplex we must have some facet F for which some adjacent facet F' is still not in the subcomplex. Since the subcomplex is induced, the unique vertex of $F' \setminus F$ is not in it either. Since F' and F

are adjacent, this vertex must connect to at least d vertices of the current subcomplex. Add this vertex to construct the next induced subcomplex. \square

Corollary 3.9. *Let C be a strongly connected d -complex, and let G be the Billera-Lee graph with parameters $(f_0(C), f_1(C), d)$. Then $\tau_0(C) \leq \tau_0(G)$.*

Proof. Let $\delta = (0, 1, 2, 3, \dots, d, k_1, \dots, k_{n-d-1})$ be a d -dimensional in-degree sequence realizing C . Such a sequence exists due to Proposition 3.8. Then $\tau_0(C)$ satisfies the upper bound specified in Lemma 3.2. The proof is completed by the observation that this upper bound increases when δ is modified into a d -dimensional in-degree sequence of Billera-Lee type keeping the sum of degrees constant. \square

Combining all these observations we have the following application of Corollary 3.9.

Theorem 3.10. *Given (f_0, f_1) and $d \geq 3$, the maximum τ_0 among all strongly connected d -complexes with f_0 vertices and f_1 edges lies between the τ_0 of the Billera-Lee graph with parameters $(f_0, f_1, d+1)$ (realized by a Billera-Lee d -sphere) and that of the Billera-Lee graph with parameters (f_0, f_1, d) . The quotient between these two values is smaller than $(d+3)/(d+1)$.*

Proof. The lower bound follows from the existence of the Billera-Lee d -spheres (which have as graph the Billera-Lee graph with parameter $(f_0, f_1, d+1)$). The upper bound is implied by Corollary 3.9. \square

3.4. A conjectured upper bound for τ of spheres.

Conjecture 3.11. *Let S be a triangulated d -sphere and let T be the Billera-Lee d -sphere with f -vector $f(S)$. Then $\tau_i(S) \leq \tau_i(T)$, $0 \leq i \leq d$.*

Conjecture 3.11 holds for polytopal spheres due to [38, Theorem 9.6] (see also [40, Theorem 2.1]), and for τ_0 and $\binom{n-4}{2} - g_2 \leq n - d - 2$ (see Theorem 4.10). Moreover, for $g_2 \leq 1$ it is true for separate reasons (see Theorems 5.7, 5.12 and 5.13). For the general case, we have the following theoretical evidence for Conjecture 3.11.

Lemma 3.12. *Let S_i , $i = 1, 2$ denote two triangulated d -spheres. If Conjecture 3.11 holds for S_1 and S_2 , then it holds for $S_1 \# S_2$.*

Proof. Let $f(S)$ and $\tau(S)$ denote the f -vector and τ -vector of any given sphere S . Let B_1 , B_2 and B be the Billera-Lee spheres with f -vectors $f(S_1)$, $f(S_2)$ and $f(S_1 \# S_2)$. Our hypothesis is that $\tau(S_1)$ and $\tau(S_2)$ are bounded above by $\tau(B_1)$ and $\tau(B_2)$. By Theorem 1.3 we have that

$$\tau(S_1 \# S_2) = \tau(S_1) + \tau(S_2) + c(d, n_1, n_2) \leq \tau(B_1) + \tau(B_2) + c(d, n_1, n_2) = \tau(B_1 \# B_2),$$

where $n_i = f_0(S_i)$. But the connected sum of polytopal spheres is polytopal, so Theorem 1.4 gives $\tau(B_1 \# B_2) \leq \tau(B)$. \square

4. THE τ -VECTOR UNDER MANIFOLD OPERATIONS

This section is dedicated to a detailed analysis of how the τ -vector behaves under certain standard operations on simplicial manifolds.

4.1. Bistellar flips. *Bistellar flips, bistellar moves*, or just *flips* are local modifications that change one triangulation of a manifold into another without changing the PL-topological type of the manifold.

They are sometimes called *Pachner moves* due to the following result of Pachner [43, 44]: two triangulated manifolds C_1 and C_2 are PL-homeomorphic if and only if there is a sequence of flips transforming one in the other.

For the precise definition we need the following setup: let F be a ground set of size $d+2$, and for each $\emptyset \neq F_1 \subsetneq F$, let B_{F, F_1} be defined as the simplicial complex with vertex

set F and unique minimal non-face F_1 . More explicitly, B_{F,F_1} is the pure d -complex with $|F_1|$ facets, given by the subsets $\{F \setminus \{i\} : i \in F_1\}$.

It is a fact that B_{F,F_1} is always a simplicial ball and that its boundary equals the boundary of B_{F,F_2} , where $F_2 = F \setminus F_1$. It is also obvious that the isomorphism type of B_{F,F_i} only depends on $|F|$ and $|F_i|$.

Definition 4.1 (Bistellar flip). Let M_1 be a triangulated d -manifold on a ground set V and let $F \subset V$ be of size $d + 2$. Suppose that $M_1[F] = B_{F,F_1}$ for some $\emptyset \neq F_1 \subsetneq F$ and let $F_2 = F \setminus F_1$. Assume F_2 is not a face in M_1 . Then, the *bistellar flip* of F_1 (or of F) in M_1 is the triangulated manifold M_2 obtained by removing the subcomplex B_{F,F_1} from M_1 and inserting B_{F,F_2} in its place. We say that the flip is of *type* $(|F_1|, |F_2|)$ and we call F_1 and F_2 the *face removed* and *face inserted* by the flip, respectively. (More precisely, the flip inserts/removes all faces containing F_1 and F_2 , respectively; F_1 and F_2 are the unique minimal inserted/removed faces).

A flip of type (i, j) , $i + j = d + 2$, replaces i facets in a triangulation by j of them. Flips of types (i, j) and (j, i) are inverse operations to one another. More precisely, for all choices of F and F_1 , the simplicial complex $B_{F,F_1} \cup B_{F,F_2}$ is isomorphic to the boundary of the $(d-1)$ -simplex. Flips of type $(1, d+1)$ are also called *stellar subdivisions* or *stacking operations*. Spheres obtained from the boundary of a simplex by stacking operations are called *stacked* (see Section 5.2).

Remark 4.2. It is straightforward to describe how a bistellar flip changes the f -, h -, and g -vector of a d -manifold. For the latter, this takes the following very simple form. If M_2 is obtained from M_1 by a flip of type (i, j) then

$$g_i(M_2) = g_i(M_1) + 1, \quad g_j(M_2) = g_j(M_1) - 1, \quad g_k(M_2) = g_k(M_1) \text{ for all } k \notin \{i, j\}.$$

(Assuming $i \neq j$. Flips with $i = j$ do not change the f -, h -, or g -vector).

The homotopy type of most induced subcomplexes are not affected by the flip, which allows us to give a qualitative statement on how the τ -vector of the manifold changes under the operation.

Lemma 4.3. *Let M_1 and M_2 be d -manifolds, M_2 obtained from M_1 by a flip with removed face F_1 and inserted face F_2 . Let $F = F_1 \cup F_2$, $i = |F_1|$ and $j = |F_2|$. In particular, the flip is of type (i, j) . Let $W \subset V$ be a subset of the ground set.*

- If $W \cap F \notin \{F_1, F_2\}$ then $M_1[W] \simeq M_2[W]$.
- If $W \cap F = F_1$ then $\tilde{\beta}_k(M_2[W]) = \tilde{\beta}_k(M_1[W])$, $\forall k \notin \{i-1, i-2, j-1, j-2\}$.
If, moreover, $i \neq j$ then

$$\begin{aligned} \tilde{\beta}_{i-2}(M_2[W]) &\leq \tilde{\beta}_{i-2}(M_1[W]), & \tilde{\beta}_{j-2}(M_2[W]) &\geq \tilde{\beta}_{j-2}(M_1[W]), \\ \tilde{\beta}_{i-1}(M_2[W]) &\geq \tilde{\beta}_{i-1}(M_1[W]), & \tilde{\beta}_{j-1}(M_2[W]) &\leq \tilde{\beta}_{j-1}(M_1[W]). \end{aligned}$$

- If $W \cap F = F_2$ then the same happens, with inequalities in the opposite direction.

Proof. For the first part, just observe that if $W \cap F \notin \{F_1, F_2\}$ then $M_1[W \cap F]$ and $M_2[W \cap F]$ deformation retract to $M_1[W \cap F] \cap M_2[W \cap F]$.

For the second part, assume $F = F_1$ (the case $F = F_2$ is similar, since it is the reverse flip). Removing the $(i-1)$ -face F_1 can only decrease $\tilde{\beta}_{i-1}$ and increase $\tilde{\beta}_{i-2}$, and inserting the $(j-1)$ -face F_2 can only increase $\tilde{\beta}_{j-1}$ and decrease $\tilde{\beta}_{j-2}$. \square

Corollary 4.4. *Let M_1 and M_2 be two d -manifolds, with M_2 obtained from M_1 by an (i, j) -flip. Then $\tau_k(M_2) = \tau_k(M_1)$, $\forall k \notin \{i-2, i-1, j-2, j-1\}$. If, moreover, $i \neq j$ then*

$$\begin{aligned} \tau_{i-2}(M_2) &< \tau_{i-2}(M_1), & \tau_{j-2}(M_2) &> \tau_{j-2}(M_1), \\ \tau_{i-1}(M_2) &> \tau_{i-1}(M_1), & \tau_{j-1}(M_2) &< \tau_{j-1}(M_1). \end{aligned}$$

Proof. All statements, except for the strictness of the inequalities, follow directly from Lemma 4.3. Strictness follows from considering the cases $W = F_1$ and $W = F_2$. \square

4.2. Connected sum. Building simplicial connected sums or, conversely, decomposing a manifold into its connected summands, is sometimes applied as a step to organise proofs and/or to provide an argument with additional combinatorial structure. For instance, decomposing a triangulated 2-sphere along all of its induced 3-cycles yields a collection of flag 2-spheres (plus possibly some boundaries of the tetrahedron). More generally, the 1-skeleton of a d -manifold, which is always at least $(d+1)$ -connected, is at least $(d+2)$ -connected if and only if it is not a simplicial connected sum.

Definition 4.5 (Simplicial connected sum). Given two d -manifolds triangulated M_1 and M_2 , their *simplicial connected sum*, written $M_1 \# M_2$ is obtained by the following procedure. Remove a facet from each M_1 and M_2 , and glue the resulting boundaries (both isomorphic to $\partial\Delta^d$).

The combinatorics of $C_1 \# C_2$ depends on the choice of F_1 and F_2 , but its topology only depends on (the parity of) the bijection between F_1 and F_2 (in case both C_1 and C_2 are chiral, i.e., orientable and do not admit orientation-reversing automorphisms). Our next result implies that the τ -vector does not depend on the choice either.

Theorem 4.6. *Let M_1, M_2 be n_j -vertex d -manifolds for $d \geq 2$. Let n_i be the number of vertices of M_i . Then*

$$\begin{aligned} \tau_i(M_1 \# M_2) &= \tau_j(M_1) + \tau_j(M_2) + c(d, n_1, n_2), & i \in \{0, d-1\}, \\ \tau_i(M_1 \# M_2) &= \tau_i(M_1) + \tau_i(M_2), & i \in \{1, \dots, d-2\}. \end{aligned}$$

Here,

$$\begin{aligned} c(d, n_1, n_2) &:= \frac{1}{d+2} - \frac{1}{n_1+1} - \frac{1}{n_2+1} + \frac{1}{n_1+n_2-d-1} \\ &= \frac{1}{d+2} - \frac{1}{n_1+1} - \frac{1}{n_2+1} + \frac{1}{n+1}, \end{aligned}$$

where $n = n_1 + n_2 - d - 1$ denotes the number of vertices of $M_1 \# M_2$.

Proof. Let $M = M_1 \# M_2$. Denote by V_k the set of vertices of M_k , $k = 1, 2$, and let $D = V_1 \cap V_2$ be the common facet along which we take the connected sum. Let $\tilde{M}_k = M_k \setminus \{D\}$, so that $M_1 \# M_2 = \tilde{M}_1 \cup \tilde{M}_2$. By Theorem 2.7 we regard all these complexes as having the same ground set $V = V_1 \cup V_2$. That is,

$$\tau_i(M) = \sum_{W \subset V} \frac{\tilde{\beta}_i(M[W])}{(|V|+1)\binom{|V|}{|W|}}, \quad \tau_i(M_k) = \sum_{W \subset V} \frac{\tilde{\beta}_i(M_k[W])}{(|V|+1)\binom{|V|}{|W|}}.$$

For $i \in \{1, \dots, d-2\}$ we have that

$$(4) \quad \beta_i(M[W]) = \beta_i(\tilde{M}_1[W]) + \beta_i(\tilde{M}_2[W]) = \beta_i(M_1[W]) + \beta_i(M_2[W])$$

for every $W \subset V$. The first equality comes from Mayer-Vietoris, thanks to the fact that $\tilde{M}_1[W] \cap \tilde{M}_2[W]$ is either empty, contractible, or a $(d-1)$ -sphere. (In particular, its i -th and $(i-1)$ -th Betti numbers vanish). The second equality comes from the fact that removing a d -face only affects homology in dimensions d and $d-1$. This implies $\tau_i(M_1 \# M_2) = \tau_i(M_1) + \tau_i(M_2)$ for $i \in \{1, \dots, d-2\}$.

For $\tilde{\beta}_0$ the second equality in Equation (4) still holds but the first one needs a correction term whenever $W \cap D = \emptyset$ but both $W \cap V_k \neq \emptyset$. More precisely, denoting $W_D = W \cap D$, $W_1 = W \setminus V_2$ and $W_2 = W \setminus V_1$, we have

$$(5) \quad \tilde{\beta}_0(M[W]) = \tilde{\beta}_0(M_1[W]) + \tilde{\beta}_0(M_2[W]) + \begin{cases} 0 & \text{if } W_D \neq \emptyset, W_1 = \emptyset \text{ or } W_2 = \emptyset, \\ 1 & \text{if } W_D = \emptyset \text{ and } W_1 \neq \emptyset \neq W_2. \end{cases}$$

For $i = d-1$ we have the opposite. If $D \not\subset W$ (that is, if $W_D \neq D$) Equation (4) still holds, and the same happens if $V_1 \subset W$ or $V_2 \subset W$ (in this last case both inequalities

in (4) may fail, but their failures cancel out since the $(d-1)$ -cycle ∂D is trivial in all of $M_1[W]$, $M_2[W]$ and $M[W]$. However, if $D \subset W$ but W contains none of V_1 or V_2 then ∂D adds one to $\beta_{d-1}(M[W])$. That is,

$$(6) \quad \beta_{d-1}(M[W]) = \beta_{d-1}(M_1[W]) + \beta_{d-1}(M_2[W]) + \begin{cases} 0 & \text{if } W_D \neq D, W_1 = V_1, \text{ or } W_2 = V_2 \\ 1 & \text{if } W_D = \emptyset \text{ and } W_1 \neq \emptyset \neq W_2. \end{cases}$$

Observe that a W contributes to the correction term in Equation (5) if and only if its complement contributes in Equation (6). Thus, the global correction is identical and, due to Equation (5), equal to

$$\begin{aligned} & \sum_{\substack{\emptyset \neq W_1 \subset V_1 \setminus D, \\ \emptyset \neq W_2 \subset V_2 \setminus D}} \frac{1}{(n+1) \binom{n}{|W_1 \cup W_2|}} = \\ &= \frac{1}{n+1} \sum_{i=1}^{n_1-d-1} \sum_{j=1}^{n_2-d-1} \frac{\binom{n_1-d-1}{i} \binom{n_2-d-1}{j}}{\binom{n}{i+j}} \\ &= \frac{1}{n+1} \left(\sum_{i=0}^{n_1-d-1} \sum_{j=0}^{n_2-d-1} \frac{\binom{n_1-d-1}{i} \binom{n_2-d-1}{j}}{\binom{n}{i+j}} - \sum_{j=0}^{n_2-d-1} \frac{\binom{n_2-d-1}{j}}{\binom{n}{j}} - \sum_{i=0}^{n_1-d-1} \frac{\binom{n_1-d-1}{i}}{\binom{n}{i}} + 1 \right) \\ &= \frac{1}{n+1} \left(\sum_{k=0}^{n-d-1} \frac{\binom{n-d-1}{k}}{\binom{n}{k}} - \sum_{j=0}^{n_2-d-1} \frac{\binom{n_2-d-1}{j}}{\binom{n}{j}} - \sum_{i=0}^{n_1-d-1} \frac{\binom{n_1-d-1}{i}}{\binom{n}{i}} + 1 \right) \\ &= \frac{1}{d+2} - \frac{1}{n_1+1} - \frac{1}{n_2+1} + \frac{1}{n+1}. \end{aligned}$$

Here the last equality is implied by a special case of Lemma 3.3, namely

$$\sum_{k=0}^{n-t} \frac{\binom{n-t}{k}}{\binom{n}{k}} = \frac{n+1}{t+1}.$$

□

As a first application of Theorem 4.6, we compute the change of the τ -vector under arbitrarily many stacking operations (i.e. bistellar $(1, d+1)$ -moves). Note that stacking is the same as performing a simplicial connected sum of a complex with the boundary of a $(d+1)$ -simplex, which has $d+2$ vertices and whose τ -vector vanishes except for $\tau_{-1} = \tau_d = 1$.

Corollary 4.7. *Let M be an n -vertex combinatorial d -manifold, $d \geq 2$. Then for every combinatorial d -manifold N obtained from M by k bistellar $(1, d+1)$ -moves (or stacking operations) we have*

$$(7) \quad \begin{aligned} \tau_j(N) &= \tau_j(M) + \frac{k}{(d+2)(d+3)} - \frac{1}{n+1} + \frac{1}{n+k+1}, & j \in \{0, d-1\}, \\ \tau_i(N) &= \tau_i(M), & i \in \{1, \dots, d-2\}. \end{aligned}$$

Proof. We prove the result for τ_0 and by induction on k . All other cases follow immediately using the same arguments. For the base case $k=1$ we apply Theorem 4.6 to obtain

$$(8) \quad \tau_0(N) = \tau_0(M) + \frac{1}{(d+2)(d+3)} - \frac{1}{n+1} + \frac{1}{n+2}.$$

Let N_k be obtained from M by stacking k times, and N_{k+1} by stacking once in N_k . Applying Theorem 4.6 to N_k (that is, using Equation (8)) and using the inductive hypothesis we obtain

$$\begin{aligned}\tau_0(N_{k+1}) &= \tau_0(N_k) + \frac{1}{(d+2)(d+3)} - \frac{1}{n+k+1} + \frac{1}{n+k+2} \\ &= \tau_0(M) + \frac{k+1}{(d+2)(d+3)} - \frac{1}{n+1} + \frac{1}{n+k+2}.\end{aligned}$$

□

Remark 4.8. The proof of Theorem 4.6 still works for $d = 1$, except for the fact that, since $0 = d - 1$, the contributions for the cases $i = 0$ and $i = d - 1$ are added yielding

$$\tau_0(M_1 \# M_2) = \tau_0(M_1) + \tau_0(M_2) + \frac{2}{3} - \frac{2}{n_1+1} - \frac{2}{n_2+1} + \frac{2}{n_1+n_2-2}.$$

In particular, for a cycle C_m with m vertices we have that

$$\tau_0(C_m) = \frac{m}{6} + \frac{2}{m+1} - 1 = \frac{(m-2)(m-3)}{6(m+1)}.$$

4.3. (Nearly)-neighborly manifolds. A d -complex is called k -neighborly if its $(k-1)$ -skeleton is complete. This is equivalent to any of the following two equalities:

$$f_{k-1} = \binom{f_0}{k} \Leftrightarrow h_k = \binom{h_1 + k - 1}{k}$$

Every complex is 1-neighborly. The only $\lfloor d/2 \rfloor + 1$ -neighborly sphere is the boundary of a $d+1$ -simplex, and $\lfloor d/2 \rfloor$ -neighborly manifolds are simply called *neighborly*.

Observe that any k -neighborly combinatorial d -manifold M has $\tilde{\beta}_j(M) = 0$ for all $j \leq k - 2$, because $\tilde{\beta}_j$ only depends on the $(j+1)$ -skeleton, and $(j+2)$ -neighborly means that the $(j+1)$ -skeleton coincides with that of a simplex, which is contractible. Therefore the following statement necessarily holds.

Proposition 4.9 (Bagchi, Datta, Lemma 3.9 of [6]). *Let M be a k -neighborly combinatorial d -manifold, then $\tau_i(M) = 0$ for all $i \in \{0, \dots, k-2\}$.*

Conversely, if $\tau_i(M) = 0$ then M is (at least) $(i+2)$ -neighborly.

More generally, assume that M is close to $(i+2)$ -neighborly with only very few $(i+1)$ -faces missing. Then $\tau_i(M)$ can only take few distinct values corresponding to the few possibilities for the isomorphism type of the $(i+1)$ -skeleton. Here we explore the case $i = 0$.

Assume only one edge is missing from the 1-skeleton of M , then $\tau_0(M) = \frac{1}{(n+1)\binom{n}{2}}$. The only non-zero contribution comes from the induced subcomplex on the two vertices that do not share an edge. If two edges are missing, these can either be disjoint or they meet in an edge. In the former case we have $\tau_0(M) = \frac{2}{(n+1)\binom{n}{2}}$ in the latter we have $\tau_0(M) = \frac{1}{n+1} (2/\binom{n}{2} + 1/\binom{n}{3})$.

Going further, we can prove the following special case of Conjecture 1.5 for “nearly 2-neighborly” manifolds.

Theorem 4.10. *Let M be an n -vertex combinatorial d -manifold with $f_1 \geq \binom{n}{2} - n + d + 2 = \binom{n-1}{2} + d + 1$ edges, and let S be an n -vertex Billera-Lee d -sphere with f_1 edges from Section 3.2. Then*

$$\tau_0(M) \leq \tau_0(S).$$

Proof. Following Corollary 3.9, it suffices to show that the graph of M has an in-degree sequence of type $(0, 1, 2, \dots, d, k_1, \dots, k_{n-d-1})$ with $k_i \geq d+1$, $1 \leq i \leq n-d-1$.

Recall that at most $n-d-2$ edges are missing from the graph of M . Since M is d -dimensional, there exists an ordering of the vertices of M such that its corresponding in-degree sequence starts with $(0, 1, 2, \dots, d)$. Now assume that there exists an in-degree sequence of the graph of M with the first ℓ entries as required, $\ell \geq d+1$. For the $(\ell+1)$ -st entry to be necessarily strictly less than $d+1$, at least $(\ell-d)(n-\ell)$ edges must be missing from the graph of M . Since $(\ell-d)(n-\ell) > n-d-2$ for all $d+1 \leq \ell \leq n-d-1$, this implies the statement of the theorem. \square

5. THE τ -VECTOR OF SPHERES

In this section we refine results stated in previous sections in the case that the simplicial complexes in question are triangulations of d -spheres.

5.1. Euler, Dehn-Sommerville and Alexander relations. Let C be a simplicial complex. The Euler-Poincaré formula for C

$$\chi(C) := \sum_{k=0}^{\infty} (-1)^k f_k(C) = \sum_{k=0}^{\infty} (-1)^k \beta_k(C)$$

or its reduced version

$$\tilde{\chi}(C) := \sum_{k=-1}^{\infty} (-1)^k f_k(C) = \sum_{k=-1}^{\infty} (-1)^k \tilde{\beta}_k(C)$$

translates into an expression for the alternating sum of entries in the τ -vector in terms of the f -vector.

Lemma 5.1. *Let C be a simplicial complex of dimension d with f -vector $f(C) = (f_{-1}, f_0, f_1, \dots, f_d)$, vertex set V , and τ -vector $\tau(C) = (\tau_0, \tau_1, \dots, \tau_d)$. Then*

$$(9) \quad \sum_{i=-1}^d (-1)^i \tau_i = \sum_{k=-1}^d \frac{(-1)^k}{k+2} f_k.$$

Proof. By Euler's formula

$$\sum_{i=-1}^d (-1)^i \tau_i = \sum_{W \subset V} \frac{\sum_{i=-1}^d (-1)^i \tilde{\beta}_i(C[W])}{(f_0+1) \binom{f_0}{|W|}} = \sum_{W \subset V} \frac{\sum_{k=-1}^d (-1)^k f_k(C[W])}{(f_0+1) \binom{f_0}{|W|}}.$$

Since each k -face contributes to the $f_k(C[W])$ of exactly $\binom{f_0-k-1}{j-k-1}$ induced subcomplexes of size j , the latter equals

$$\begin{aligned} &= \sum_{j=0}^{f_0} \sum_{k=-1}^d \frac{\binom{f_0-k-1}{j-k-1}}{(f_0+1) \binom{f_0}{j}} (-1)^k f_k \\ &= \sum_{j=0}^{f_0} \sum_{k=-1}^d \frac{(f_0-k-1)! j!}{(f_0+1)! (j-k-1)!} (-1)^k f_k \\ &= \sum_{k=-1}^d \frac{\sum_{j=0}^{f_0} j \dots (j-k)}{(f_0+1) f_0 \dots (f_0-k)} (-1)^k f_k \\ &\stackrel{*}{=} \sum_{k=-1}^d \frac{(-1)^k}{k+2} f_k. \end{aligned}$$

Here, we use the convention that an empty product (the case $k = -1$) is always equal to 1. The equality “ $\stackrel{*}{=}$ ” is implied by the following identity

$$\sum_{i=0}^n i \cdots (i-k) = \frac{(n+1) \cdot n \cdots (n-k)}{k+2},$$

which can be verified using induction, starting with $k = -1$. \square

If C is pure of dimension d , we can rewrite Lemma 5.1 in terms of the h -vector (see Section 2.1 for a definition and some background).

Corollary 5.2. *Let C be a d -dimensional simplicial complex. Then*

$$\sum_{i=-1}^d (-1)^i \tau_i(C) = \sum_{k=0}^{d+1} \frac{(-1)^{k+1}}{(d+2) \binom{d+1}{k}} h_k.$$

Corollary 5.2 is a consequence of the following identity relating the f - and the h -vector of a pure d -dimensional simplicial complex (which can also be used as an alternative definition of the h -vector).

Proposition 5.3 (Stanley). *Let C be a simplicial complex with facets of dimension d , f -vector $f(C) = (f_{-1}, f_0, f_1, \dots, f_d)$, and h -vector $h(C) = (h_0, h_1, \dots, h_{d+1})$. Then*

$$f_j = \sum_{k=0}^{j+1} \binom{d+1-k}{d-j} h_k.$$

Proof of Corollary 5.2. Substituting Proposition 5.3 into Lemma 5.1 we obtain

$$\sum_{i=-1}^d (-1)^i \tau_i(C) = \sum_{j=-1}^d \frac{(-1)^j}{j+2} f_j = \sum_{j=-1}^d \frac{(-1)^j}{j+2} \left(\sum_{k=0}^{j+1} \binom{d+1-k}{d-j} h_k \right).$$

Swapping the summation orders, the coefficient of h_k in the right hand side equals

$$\begin{aligned} \sum_{j=k-1}^d \frac{(-1)^j}{j+2} \binom{d+1-k}{d-j} &= (-1)^{k-1} \sum_{j=0}^{d+1-k} \frac{(-1)^j}{j+k+1} \binom{d+1-k}{j} \\ &\stackrel{*}{=} (-1)^{k-1} \frac{1}{(k+1) \binom{d+2}{d+1-k}} = \frac{(-1)^{k-1}}{(d+2) \binom{d+1}{k}}. \end{aligned}$$

In the equality “ $\stackrel{*}{=}$ ” we use the following combinatorial identity (see [37] or Chapter 7 in [45]):

$$\sum_{j=0}^n \frac{(-1)^j}{j+r} \binom{n}{j} = \frac{1}{r \binom{n+r}{n}}.$$

\square

Recall that when C is topologically a d -sphere the h -vector entries satisfy the Dehn-Sommerville equations $h_k = h_{d+1-k}$. For even dimensional spheres, Corollary 5.2 simply states $\sum_{i=-1}^d (-1)^i \tau_i(C) = 0$. This can also be deduced from Alexander duality, or, more precisely, is implied by the following statement.

Lemma 5.4 (Lemma 2.2 in [6]). *Let S be a triangulated d -sphere, then*

$$\tau_i = \tau_{d-i-1}, \quad \text{for all } -1 \leq i \leq d.$$

Proof. Let V be the vertex set of S . By Alexander duality, the contribution of each W to τ_i equals the contribution of $V \setminus W$ to τ_{d-i-1} . \square

For odd-dimensional spheres we can combine the Dehn-Sommerville relations and Alexander duality in order to obtain yet another special case of Lemma 5.1.

Corollary 5.5. *Let S be a triangulation of the d -sphere, d odd. Then*

$$(10) \quad \sum_{i=-1}^d (-1)^i \tau_i(C) = (-1)^{\frac{d-1}{2}} \tau_{\frac{d-1}{2}} + 2 \sum_{i=-1}^{\frac{d-3}{2}} (-1)^i \tau_i(C)$$

$$(11) \quad = \frac{1}{d+2} \left(2 \sum_{k=0}^{\frac{d-1}{2}} \frac{(-1)^{k+1} h_k}{\binom{d+1}{k}} + \frac{(-1)^{\frac{d-1}{2}} h_{\frac{d+1}{2}}}{\binom{d+1}{(d+1)/2}} \right).$$

Proof. Since S is a triangulation of the d -sphere, we have $h_k = h_{d+1-k}$, $0 \leq k \leq d+1$, and $h_0 = 1$. Moreover, we have $\tau_j = \tau_{d-1-j}$, $0 \leq j \leq d-1$ by Alexander duality (see [6, Lemma 2.2]). \square

For dimension three this takes the following simple form.

Corollary 5.6. *For every triangulated 3-sphere S we have $\tau_0 = \tau_2$ and*

$$(12) \quad 2\tau_0 - \tau_1 = \frac{h_1(h_1+1)}{10(h_1+5)} - \frac{h_2}{30}.$$

Equivalently, in terms of the f - and g -vector

$$(13) \quad 2\tau_0 - \tau_1 = \frac{f_0^2 - 4f_0 + 5}{5(f_0 + 1)} - \frac{f_1}{30}, \quad 2\tau_0 - \tau_1 = \frac{g_1(g_1+1)}{15(g_1+6)} - \frac{g_2}{30}.$$

Proof. For the first formula, Corollary 5.5 implies that

$$-\tau_1 + 2 \sum_{i=-1}^0 (-1)^i \tau_i(C) = \frac{1}{5} \left(2 \sum_{k=0}^1 \frac{(-1)^{k+1} h_k}{\binom{4}{k}} + \frac{(-1)^1 h_2}{\binom{4}{2}} \right).$$

Taking into account that $\tau_{-1} = 1/(f_0 + 1) = 1/(h_1 + 5)$, this gives

$$-\tau_1 - \frac{2}{h_1 + 5} + 2\tau_0 = \frac{1}{5} \left(-2 + \frac{h_1}{2} - \frac{h_2}{6} \right),$$

$$2\tau_0 - \tau_1 = \frac{h_1(h_1+1)}{10(h_1+5)} - \frac{h_2}{30}.$$

For the second formula we substitute $h_1 = f_0 - 4$ and $h_2 = f_1 - 3f_0 + 6$ to obtain

$$\begin{aligned} \frac{h_1(h_1+1)}{10(h_1+5)} - \frac{h_2}{30} &= \frac{(f_0-4)(f_0-3)}{10(f_0+1)} + \frac{f_0-2}{10} - \frac{f_1}{30} \\ &= \frac{2f_0^2 - 8f_0 + 10}{10(f_0+1)} - \frac{f_1}{30} = \frac{f_0}{5} - \frac{f_0-1}{f_0+1} - \frac{f_1}{30}. \end{aligned}$$

\square

5.2. Stacked spheres. A closed d -manifold is called k -stacked if it is the boundary of a $(d+1)$ -manifold whose interior faces all have dimension greater than $d-k$. The only 0-stacked manifold is the boundary of a $(d+1)$ -simplex, and 1-stacked manifolds are simply called stacked. In this section we explore bounds and exact values for the entries of the τ -vector for k -stacked d -spheres.

Stackedness is stronger (and more interesting) for small values of k . For *polytopal* spheres and for $k \leq (d-1)/2$, being k -stacked is equivalent to $g_{k+1}(S) = 0$. For *non-polytopal* spheres, the forward implication of this statement still holds, but the reverse direction is only conjectured.

For the τ -vector, we can prove the following statement.

Theorem 5.7. *Let S be a k -stacked d -sphere. Then $\tau_i(S) = 0$ for $k \leq i \leq d-k-1$.*

Proof. Let V be the vertex set of S . Let B_1 be a k -stacked $(d+1)$ -ball with $\partial B_1 = S$, and let $B_2 = S * \{v\}$. Let $T = B_1 \cup B_2$ be the $(d+1)$ -sphere obtained gluing B_1 and B_2 along S . (That is, T is obtained coning a vertex v over the boundary of B_1).

Since B_1 is k -stacked we have that S and B_1 have the same $(d-k)$ -skeleton, and the same for T and B_2 . Thus, for every $i \in [k, d-k-1]$ and every $W \subset V$

$$\tilde{\beta}_i(S[W]) = \tilde{\beta}_i(B_1[W]) = \tilde{\beta}_i(T[W])$$

and

$$\tilde{\beta}_{d-i}(T[V \setminus W \cup \{v\}]) \leq \tilde{\beta}_{d-i}(B_2[V \setminus W \cup \{v\}]) = \tilde{\beta}_{d-i}(S[V \setminus W] * \{v\}) = 0.$$

By Alexander duality on T these two Betti numbers coincide, so $\tau_i(S) = 0$. \square

Remark 5.8. Every l -neighborly d -sphere S is automatically $(d-l+1)$ -stacked. Hence Theorem 5.7 implies $\tau_i(S) = 0$ for $i = d-l+1, \dots, l-2$. But this is not new, since Proposition 4.9 already gave $\tau_i(S) = 0$ for $i \leq l-2$ (without the need for S to be a sphere). Observe also that Theorem 5.7 is void unless $k \leq (d-1)/2$.

Remark 5.9. A statement similar to Theorem 5.7 but weaker is proved by Bagchi and Datta in [6, Theorem 3.5a]. There S is assumed not only to be k -stacked but also to be the boundary of a shellable ball with no interior $(d-k)$ -faces. Equivalently, S is assumed to be obtained from the boundary of the simplex by only (i, j) -flips with $i \leq k$ (Bagchi and Datta call such spheres *k-stellated*). In this case, $\tau_k = \dots = \tau_{d-k-1} = 0$ also follows from Corollary 4.4.

For *polytopal* spheres, Adiprasito [1, Corollary 6.5] proves that for every induced subcomplex $S[W]$ of a polytopal d -sphere S and for every $k < d/2$,

$$\tilde{\beta}_k(S[W]) \leq g_{k+1}(S).$$

In the case of polytopal spheres, our Theorem 5.7 is a special case of this statement, since by the Generalized Lower Bound Theorem being k -stacked is equivalent to $g_{k+1}(S) = 0$ and implies $g_{i+1}(S) = 0$ for all $i \in [k+1, d-k]$.

Spheres which are 1-stacked are often simply referred to as *stacked*. Stacked d -spheres are precisely those triangulated d -spheres which can be obtained from the boundary of the $(d+1)$ -simplex by a number of bistellar $(1, d+1)$ -moves, or, equivalently, which can be written as the connected sum of boundaries of the $(d+1)$ -simplex. This implies that the stacking order is a perfect elimination order with in-degree sequence $(0, 1, 2, \dots, d+1, \dots, d+1)$. Thus if S is a stacked sphere, we have $\tau_{-1}(S) = \tau_d(S) = \frac{1}{f_0+1}$ and, due to Corollary 4.7,

$$\tau_0(S) = \tau_{d-1}(S) = \frac{n-d-2}{(d+2)(d+3)} - \frac{1}{d+3} + \frac{1}{n+1} = \frac{n-2d-4}{(d+2)(d+3)} + \frac{1}{n+1}.$$

Moreover, Theorem 5.7 implies that $\tau_i(S) = 0$ for all other entries. In particular, the τ -vector of a stacked sphere is completely determined by its dimension and its number of vertices. Note that for $d = 2$ this gives the equality case of Theorem 1.2.

5.3. Nearly stacked spheres. We now focus on complexes which are not stacked, but *nearly stacked*, meaning that $g_2 = 1$ instead of 0. That is, we call a d -sphere with n vertices *nearly stacked* if it has $f_1(S) = (d+1)n - \binom{d+2}{2} + 1$ edges, one more than the number of edges of a stacked sphere [26]. These spheres have the following complete characterisation by Nevo and Novinsky [41]: a d -sphere S satisfies $g_2(S) = 1$ if and only if it is obtained either from $\partial\Delta_i * \partial\Delta_j$, $i+j = d+1$, $i, j \geq 2$, or from $C_k * \partial\Delta_{d-1}$

by an arbitrary number of stacking operations. Here $\partial\Delta_i$ denotes the boundary of an i -simplex and C_k is the boundary of a k -gon.¹

In the following two lemmas we compute the τ -vectors of these two minimal cases.

Lemma 5.10. *Let $S = \partial\Delta_i * \partial\Delta_j$, $i, j \geq 2$, $d = i + j - 1$. Then all entries of the τ -vector of S are 0 except*

$$\begin{aligned}\tau_{-1} &= \tau_d = \frac{1}{d+4} \\ \tau_{i-1} &= \tau_{j-1} = \frac{1}{(d+4)\binom{d+3}{i+1}}, \text{ if } i \neq j \\ \tau_{i-1} &= \frac{2}{(d+4)\binom{d+3}{i+1}}, \text{ if } i = j.\end{aligned}$$

Proof. Let V_1 and V_2 be the subsets of vertices of $\partial\Delta_i$ and $\partial\Delta_j$ respectively. Since induced subcomplexes of the join are joins of the induced subcomplexes and since all induced subcomplexes of $\partial\Delta_i$ are contractible except for the empty one and the full one, the only W 's that contribute to the τ -vector are $W \in \{\emptyset, V_1, V_2, V_1 \cup V_2\}$, which add respectively to τ_{-1} , τ_{i-1} , τ_{j-1} , and τ_{i+j-1} , as stated. \square

Lemma 5.11. *Let $S_m = C_m * \partial\Delta_{d-1}$, with $d \geq 4$. Then all entries of the τ -vector of S_m are 0 except*

$$\begin{aligned}\tau_{-1} &= \tau_d = \frac{1}{m+d+1} \\ \tau_0 &= \tau_{d-1} = \frac{m-3}{(d+3)(d+2)} + \frac{1}{m+d+1} + \frac{1}{(d+1)\binom{d+m+1}{d+1}} - \left(\frac{1}{d+4} + \frac{1}{(d+1)\binom{d+4}{3}} \right) \\ \tau_1 &= \tau_{d-2} = \frac{1}{(m+d+1)\binom{m+d}{m}}.\end{aligned}$$

For $d = 3$ the same holds except the value of τ_1 is multiplied by a factor of 2.

Proof. Using again that the only W 's that can contribute are those that contain either all or none of the vertices of Δ_{d-1} , all values are straightforward except for τ_0 and τ_{d-1} , which coincide by Alexander duality. Hence, we concentrate on τ_0 for the rest of the proof.

First observe that, since any vertex of $\partial\Delta^{d-1}$ is joined to all other vertices of S_m by an edge, for a subcomplex induced by a subset of vertices of S_m to have non-trivial 0-homology it must only contain vertices of C_m . Let v be one vertex of S_m which belongs to the cycle C_m . Then if V is the vertex set of S_{m-1} , we let $V \cup \{v\}$ be the vertex set of S_m , and let x, y be the neighbors of v in C_m . By Theorem 2.7 we can consider S_{m-1} and S_m as having the same ground set $V \cup \{v\}$ and obtain

$$\begin{aligned}\tau_0(S_{m-1}) &= \frac{1}{m+d+1} \sum_{X \subset V} \left(\frac{\tilde{\beta}_0(S_{m-1}[X])}{\binom{m+d}{|X|}} + \frac{\tilde{\beta}_0(S_{m-1}[X \cup v])}{\binom{m+d}{|X \cup v|}} \right) \\ &= \frac{1}{m+d+1} \sum_{X \subset V} \left(\frac{\tilde{\beta}_0(S_{m-1}[X])}{\binom{m+d}{|X|}} + \frac{\tilde{\beta}_0(S_{m-1}[X])}{\binom{m+d}{|X \cup v|}} \right)\end{aligned}$$

Then, the difference between τ_0 of S_m and of S_{m-1} is given by

¹Nevo and Novinsky state this result in a more specialized version for simplicial $(d+1)$ -polytopes, but they prove it for homology d -spheres.

$$\begin{aligned}\tau_0(S_m) - \tau_0(S_{m-1}) &= \frac{1}{m+d+1} \sum_{X \subset V} \frac{\tilde{\beta}_0(S_m[X]) - \tilde{\beta}_0(S_{m-1}[X])}{\binom{m+d}{|X|}} + \\ &\quad + \frac{1}{m+d+1} \sum_{X \subset V} \frac{\tilde{\beta}_0(S_m[X \cup v]) - \tilde{\beta}_0(S_{m-1}[X \cup v])}{\binom{m+d}{|X \cup v|}}.\end{aligned}$$

The numerator of the first sum is equal to 1 if $\{x, y\} \subset X$ and $X \neq V(C_m)$, otherwise it is 0. The numerator in the second sum is equal to 1 if $X \cap \{x, y\} = \emptyset$ and $X \neq \emptyset$, otherwise it is 0. Thus the above equation turns into

$$\begin{aligned}\tau_0(S_m) - \tau_0(S_{m-1}) &= \frac{1}{m+d+1} \sum_{\substack{X \subset V \\ \{x, y\} \subset X}} \frac{1}{\binom{m+d}{|X|}} + \frac{1}{m+d+1} \sum_{\substack{X \subset V \setminus \{x, y\} \\ X \neq \emptyset}} \frac{1}{\binom{m+d}{|X|+1}} \\ &= \frac{1}{m+d+1} \left(\sum_{k=0}^{m-4} \frac{\binom{m-3}{k}}{\binom{m+d}{k+2}} + \sum_{k=1}^{m-3} \frac{\binom{m-3}{k}}{\binom{m+d}{k+1}} \right) = \frac{1}{m+d+1} \sum_{k=1}^{m-3} \frac{\binom{m-2}{k}}{\binom{m+d}{k+1}} \\ &= \frac{1}{m+d+1} \left(\sum_{k=0}^{m-2} \frac{\binom{m-2}{k}}{\binom{m+d}{k+1}} - \frac{1}{m+d} - \frac{1}{\binom{m+d}{d+1}} \right) \\ &\stackrel{*}{=} \frac{1}{m+d+1} \left(\frac{m+d+1}{(d+3)(d+2)} - \frac{1}{m+d} - \frac{1}{\binom{m+d}{d+1}} \right) \\ &= \frac{1}{(d+3)(d+2)} - \frac{1}{m+d+1} \left(\frac{1}{m+d} + \frac{1}{\binom{m+d}{d+1}} \right).\end{aligned}$$

The equality $\stackrel{*}{=}$ is a consequence of Lemma 3.3 with $n = m - 2$, $a = 1$ and $b = d + 2$. Let us now introduce

$$f(m) := \frac{m+d+1}{(d+3)(d+2)} + \frac{1}{m+d+1} + \frac{1}{(d+1)\binom{d+m+1}{d+1}}$$

and observe that

$$\begin{aligned}f(m) - f(m-1) &= \frac{1}{(d+3)(d+2)} - \frac{1}{(m+d+1)(m+d)} + \left(\frac{1}{(d+1)\binom{d+m+1}{d+1}} - \frac{1}{(d+1)\binom{d+m}{d+1}} \right) \\ &= \frac{1}{(d+3)(d+2)} - \frac{1}{(m+d+1)(m+d)} - \left(\frac{\binom{d+m+1}{d+1} - \binom{d+m}{d+1}}{(d+1)\binom{d+m+1}{d+1}\binom{d+m}{d+1}} \right) \\ &= \frac{1}{(d+3)(d+2)} - \frac{1}{(m+d+1)(m+d)} - \left(\frac{\frac{d+m+1}{m}\binom{d+m}{d+1} - \binom{d+m}{d+1}}{(d+1)\binom{d+m+1}{d+1}\binom{d+m}{d+1}} \right) \\ &= \frac{1}{(d+3)(d+2)} - \frac{1}{(m+d+1)(m+d)} - \frac{1}{m\binom{d+m+1}{d}} \\ &= \frac{1}{(d+3)(d+2)} - \frac{1}{(m+d+1)(m+d)} - \frac{1}{(m+d+1)\binom{m+d}{d+1}} \\ &= \tau_0(S_m) - \tau_0(S_{m-1}).\end{aligned}$$

Since $\tau_0(S_3) = 0$ by the previous lemma, for every $m \geq 3$ we have

$$\tau_0(S_m) = \sum_{t=4}^m (\tau_0(S_t) - \tau_0(S_{t-1})) = \sum_{t=4}^m (f(t) - f(t-1)) = f(m) - f(3),$$

as stated. \square

Stacking operations only change two (pairs of) entries in the τ -vector: $\tau_0 = \tau_{d-1}$ and $\tau_1 = \tau_{d-2}$. The following two statements give the exact minimum and maximum value for these entries among all nearly stacked spheres. The other entries are as described in Lemmas 5.10 and 5.11.

Theorem 5.12. (*τ_0 of nearly stacked spheres.*) *Let T be an n -vertex triangulation of the d -sphere with $g_2(T) = 1$ and let*

$$c_d(n) := \frac{n-d-3}{(d+2)(d+3)} + \frac{1}{n+1} - \frac{1}{d+4}.$$

Then

$$c_d(n) - \frac{1}{(d+1)} \left(\frac{1}{\binom{d+4}{3}} - \frac{1}{\binom{n+1}{d+1}} \right) \leq \tau_0(T) = \tau_{d-1}(T) \leq c_d(n).$$

Equality in the lower bound is attained only by $C_{n-d} * \Delta_{d-1}$. Equality in the upper bound is attained only by applying $(n-3-d)$ stacking operations to $\Delta_i * \Delta_j$ for any $i, j \geq 2$ with $i+j = d-1$.

Proof. The spheres of the form $\partial\Delta_i * \partial\Delta_j$ all have $\tau_0 = 0$ and $d+3$ vertices. By Corollary 4.7, all d -spheres with n vertices obtained by stacking $n-d-3$ times have

$$\tau_0 = \frac{n-d-3}{(d+2)(d+3)} + \frac{1}{n+1} - \frac{1}{d+4}.$$

Observe that d -spheres that are stacked over $C_3 * \partial\Delta_{d-1}$ are a particular case of this.

On the other extreme, by substituting $m = n-d$ in Lemma 5.11, the d -sphere $C_{n-d} * \partial\Delta_{d-1}$ satisfies

$$\tau_0(C_{n-d} * \partial\Delta_{d-1}) = \frac{n-d-3}{(d+3)(d+2)} + \frac{1}{n+1} + \frac{1}{(d+1)\binom{n+1}{d+1}} - \left(\frac{1}{d+4} + \frac{1}{(d+1)\binom{d+4}{3}} \right)$$

Between these two cases are the intermediate ones where we stack k times over $C_m * \partial\Delta_{d-1}$, for $m+k = n-d$. We only need to show that the value we obtain for τ_0 monotonically increases with k . For this it suffices to show that if S'_m denotes the sphere obtained by a single stacking of $C_{m-1} * \partial\Delta_{d-1}$, we have $\tau_0(S'_m) > \tau_0(C_m * \partial\Delta_{d-1})$. This follows from the fact that, by Corollary 4.7,

$$\tau_0(S'_m) - \tau_0(C_{m-1} * \partial\Delta_{d-1}) = \frac{1}{(d+2)(d+3)} - \frac{1}{(m+d)(m+d+1)},$$

while, as seen in the proof of Lemma 5.11,

$$\tau_0(C_m * \partial\Delta_{d-1}) - \tau_0(C_{m-1} * \partial\Delta_{d-1}) = \frac{1}{(d+3)(d+2)} - \frac{1}{m+d+1} \left(\frac{1}{m+d} + \frac{1}{\binom{m+d}{d+1}} \right).$$

□

Theorem 5.13. (τ_1 of nearly stacked spheres.) *Let T be an n -vertex triangulation of the d -sphere with $g_2(T) = 1$. Then*

$$\begin{aligned} \frac{1}{2\binom{n+1}{4}} &\leq \tau_1(T) \leq \frac{1}{70}, && \text{for } d = 3, \\ \frac{1}{5\binom{n+1}{5}} &\leq \tau_1(T) = \tau_2(T) \leq \frac{1}{280} && \text{for } d = 4, \\ 0 &\leq \tau_1(T) = \tau_{d-2}(T) \leq \frac{1}{4\binom{d+4}{4}}. && \text{for } d \geq 5. \end{aligned}$$

*Equality in the lower bound is attained only by $C_{n-d} * \partial\Delta_{d-1}$ ($d = 3, 4$) and $\partial\Delta_i * \partial\Delta_j$ ($i, j \geq 3, d \geq 5$). Equality in the upper bound is attained only by applying $(n - 3 - d)$ stacking operations to $C_3 * \partial\Delta_{d-1}$.*

Proof. By [41, Theorem 1.3], every sphere with $g_2 = 1$ can be obtained from either $\partial\Delta_i * \partial\Delta_j$, $i + j = d + 1$, $i, j \geq 2$, or from $C_k * \partial\Delta_{d-1}$ or stacking, and Lemmas 5.10 and 5.11 give us the τ -vector in this case.

Since τ_1 does not change under stacking, Lemmas 5.10 and 5.11 directly produce the possible values for τ_1 . Moreover, for $d = 3, 4$ we have that the only case in Lemma 5.10 is the special case $C_3 * \partial\Delta_{d-1}$ of Lemma 5.11. It follows that

$$\begin{aligned} \tau_1(T) &= \frac{2}{(m+4)\binom{m+3}{3}} = \frac{1}{2\binom{m+4}{4}}, && \text{if } d = 3, \\ \tau_1(T) &= \frac{1}{(m+5)\binom{m+4}{4}} = \frac{1}{5\binom{m+5}{5}}, && \text{if } d = 4. \end{aligned}$$

The extremal cases are $m = 3$ and $m = n - d$, which give

$$\begin{aligned} \frac{1}{2\binom{n+1}{4}} &\leq \tau_1(T) \leq \frac{1}{70}, && \text{if } d = 3, \\ \frac{1}{5\binom{n+1}{5}} &\leq \tau_1(T) \leq \frac{1}{280}, && \text{if } d = 4. \end{aligned}$$

For $d > 4$ the same arguments apply except Lemma 5.10 is no longer a special case of Lemma 5.11. In fact, what now happens is that both Lemmas give the same upper bound, obtained by $S = \partial\Delta_2 * \partial\Delta_{d-1} = C_3 * \partial\Delta_{d-1}$. The lower bound is zero, obtained by $\partial\Delta_i * \partial\Delta_j$ with $i, j > 2$. That is, we have

$$0 \leq \tau_1(S) \leq \frac{1}{(d+4)\binom{d+3}{3}} = \frac{4}{\binom{d+4}{4}}, \quad \text{for } d > 4.$$

□

Remark 5.14. A classification for d -spheres with $g_2 = 2$ is known due to Zheng [46]. In particular, all such d -spheres are polytopal. However, the classification itself is significantly more involved than the one for $g_2 = 1$.

6. THE τ -VECTOR OF 3-SPHERES AND THE μ -VECTOR OF 4-MANIFOLDS

In this section we take a closer look at the τ -vector of 3-spheres and the μ -vector of 4-manifolds. Thus, before we start, let us first recall the definition of the μ -vector and its relation with the τ -vector. Let M be a manifold, then Theorem 1.1 states that $\tilde{\beta}_i(M) \leq \mu_i(M) := \sum_v \tau_{i-1} \text{lk}_M(v)$, with equality if and only if M is \mathbb{F} -tight.

This statement provides two reasons for studying the τ -vector:

- The τ - and μ -vectors can be used to check whether a simplicial complex is tight. This motivated Bagchi and Datta's work in [6].

- Upper bounds on τ_{i-1} of the links can provide lower bounds on the size of triangulations of a manifold in terms of its Betti numbers. This technique was successfully used by Burton, Datta, Singh and Spreer in [14] to obtain an alternative proof for Theorem 6.5 below.

The case of 4-manifolds and 3-spheres is particularly interesting because here the τ -vector of the links (hence the μ -vector of the manifold) is determined uniquely by τ_0 .

In Section 6.1 we give experimental data on the τ -vector for 3-spheres with up to 10 vertices. This provides further evidence for Conjecture 3.11. In Section 6.2 we review the concept of \mathbb{F} -tightness. In Section 6.3 we discuss a conjectured and an actual lower bound for transitive triangulations of 4-manifolds based on Conjecture 3.11 and Theorem 3.10 respectively.

6.1. The τ -vector of 3-spheres. By Corollary 5.6, the τ -vector of a sphere S of dimension up to three and with a given f -vector is determined by τ_0 ; that is, by counting connected components of induced subcomplexes of S , see Equation (1). Since, moreover, τ_1 monotonically depends on $\tau_2 = \tau_0$, this value seems to be a good parameter to estimate the “combinatorial complexity” of a triangulated 3-sphere with prescribed f -vector. In higher dimensions we can still use τ_0 as a measure of complexity, but it no longer captures the entire information of the τ -vector.

Figures 2 and 3 present the values of τ_0 for all triangulated 3-spheres with $g_1 + 5 = f_0 \in \{7, \dots, 10\}$, plotted in terms of $g_2 = f_1 - (4f_0 - 10)$. Observe that once the number of vertices is fixed, g_2 completely characterizes the f -vector and it ranges between 0 (stacked 3-spheres) and $\binom{g_1+1}{2}$ (neighborly spheres). We do not include plots for $f_0 \leq 6$ because those have $g_2 \in \{0, 1\}$ and are completely classified, as described in detail in Section 5.3. In particular, there is only one sphere for each f -vector, the Billera-Lee sphere.

The enumeration of 3-spheres up to 10 vertices is due to [4, 34], and our data is taken from [32]. The data-points corresponding to Billera-Lee spheres are marked by red triangles and in all cases they coincide with the maximum value of τ_0 , in agreement with Conjecture 3.11.

In addition, using the classification of simplicial 4-polytopes up to 10 vertices due to Firsching [22], we mark every τ_0 -value of a polytopal 3-sphere with a square. The data for this experiment is taken from [21].

Apart from providing experimental evidence for Conjecture 3.11, these calculations produce additional insight into the nature of the τ -vector:

- The data gives an idea of the *range* of values to expect for the entries of the τ -vector for a set of triangulations of the 3-sphere with fixed f -vector.
- While the maximal values of the entries of the τ -vector are all realized by polytopal spheres (the Billera-Lee spheres), this is not true for the *minimal* entries. This can be seen for the first time in the case of 10-vertex 3-spheres, but we conjecture it to happen also for larger number of vertices.

These observations suggest that finding good *lower bounds* for the entries of the τ -vector is interesting in its own right.

Remark 6.1. In general there are many spheres with both the same τ - and f -vectors as a given Billera-Lee sphere. For example, there is more than one neighborly 3-sphere and more than one stacked 3-sphere for each number of vertices. By Proposition 4.9 and Theorem 5.7 these 3-spheres all have the same τ -vector as the corresponding Billera-Lee spheres. Even more, there are examples of 3-spheres with the τ -vector of a Billera-Lee sphere but with different in-degree sequences. (Recall that the τ -vector only depends on the in-degree sequence as a multiset, not on its order.) As an example, the 8-vertex 24-edge Billera-Lee sphere has a perfect elimination order with in-degree sequence $(0, 1, 2, 3, 4, 5, 5, 4)$. On the other hand, the connected sum of two copies of

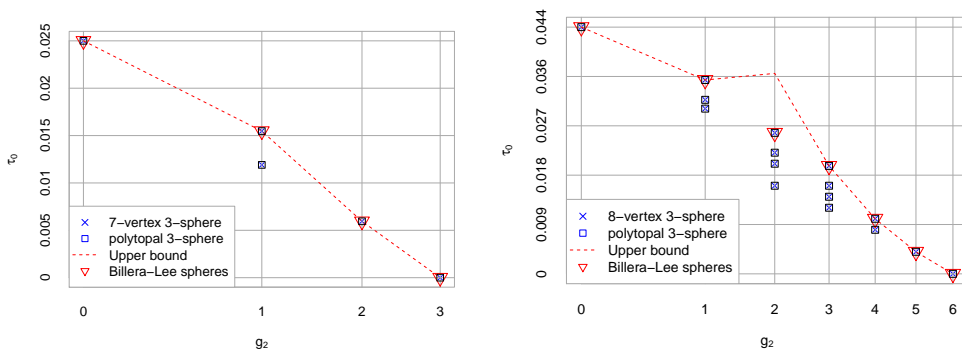


FIGURE 2. Values of τ_0 for all triangulated 3-spheres with 7 and 8 vertices with the upper bound from Theorem 3.10 (dashed line) and the conjectured upper bound from Conjecture 3.11 (red triangles, the τ_0 -values for the Billera-Lee spheres) green. All τ_0 -values can be realized by polytopal spheres.

the cyclic 4-polytope $C_4(6)$ has a perfect elimination order with in-degree sequence $(0, 1, 2, 3, 4, 5, 4, 5)$. (There are actually two isomorphism types of such connected sums, none of them having the 1-skeleton of a Billera-Lee sphere.)

Figure 4 lists the number of 3-spheres up to 10 vertices with the τ -vector of the corresponding Billera-Lee sphere.

6.2. Tightness. Tightness was defined in a geometric context by Alexandrov in 1938 in [3] and it is a generalization of convexity. Loosely speaking, a manifold is tight in Alexandrov's sense if it does not exhibit any dents or holes other than the ones required by its topology. For example, a sphere is tight if and only if it is the boundary of a convex ball.

The combinatorial version of tightness we are interested in was introduced by Kühnel in 1995 in [31] (Banchoff in 1970 introduced an intermediate version for embedded polyhedral manifolds in [9, 10]):

Definition 6.2. Let C be a simplicial complex of dimension d with vertex set V and let \mathbb{F} be a field. We say that C is \mathbb{F} -tight if for every $W \subset V$ the inclusion

$$i_W : C[W] \rightarrow C$$

induces an injective homomorphism in homology for every dimension. C is called *tight* if it is tight for at least one field. Equivalently, if it is tight for some prime field \mathbb{F}_p ; see, for instance, [7, Lemma 2.8(b)].

One source of interest for tight triangulations lies within the following conjecture.

Conjecture 6.3 (Lutz and Kühnel [31]). *Tight triangulations of a manifold M minimize every entry of the f -vector among all triangulations of M .*

Conjecture 6.3 is trivially true in dimension 2, and it has recently been proven in dimension 3 by Bagchi, Datta and Spreer [8].

Every tight triangulation of a d -manifold is 2-neighborly, since otherwise there is a subset $W = \{v_1, v_2\}$ which does not inject in 0-dimensional homology. For the converse, there are the following two cases known: (a) Every $(k+1)$ -neighborly triangulation of a $2k$ -manifold is tight (This follows from [28, Corollary 4.7]). We call such triangulations *tight of neighborly type* (this class includes all tight 2-manifolds); (b) Every 2-neighborly

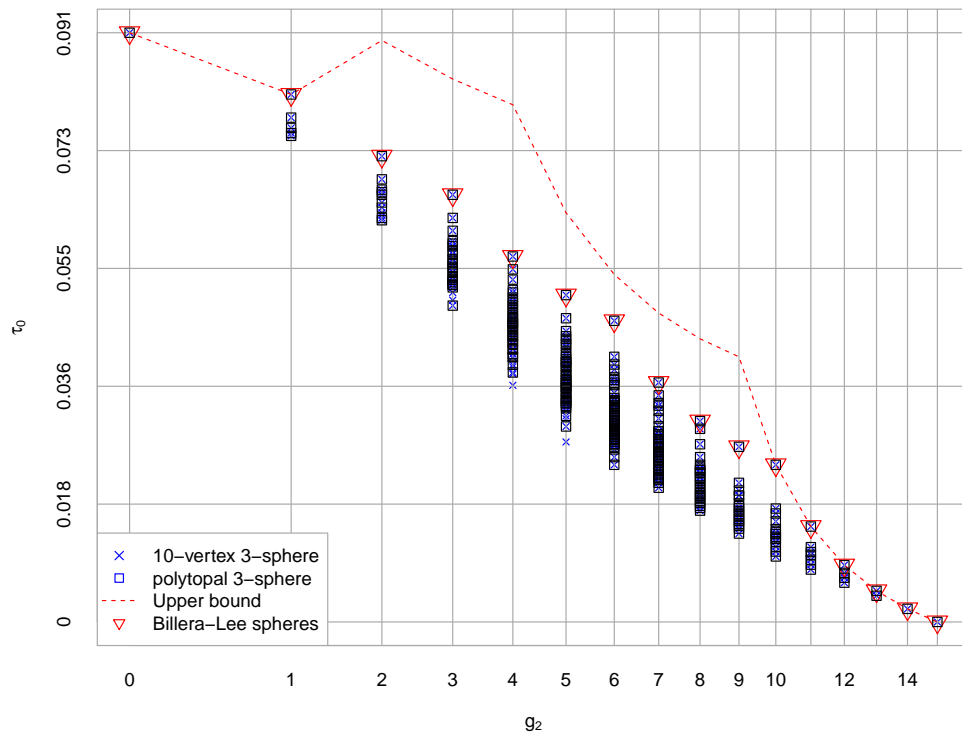
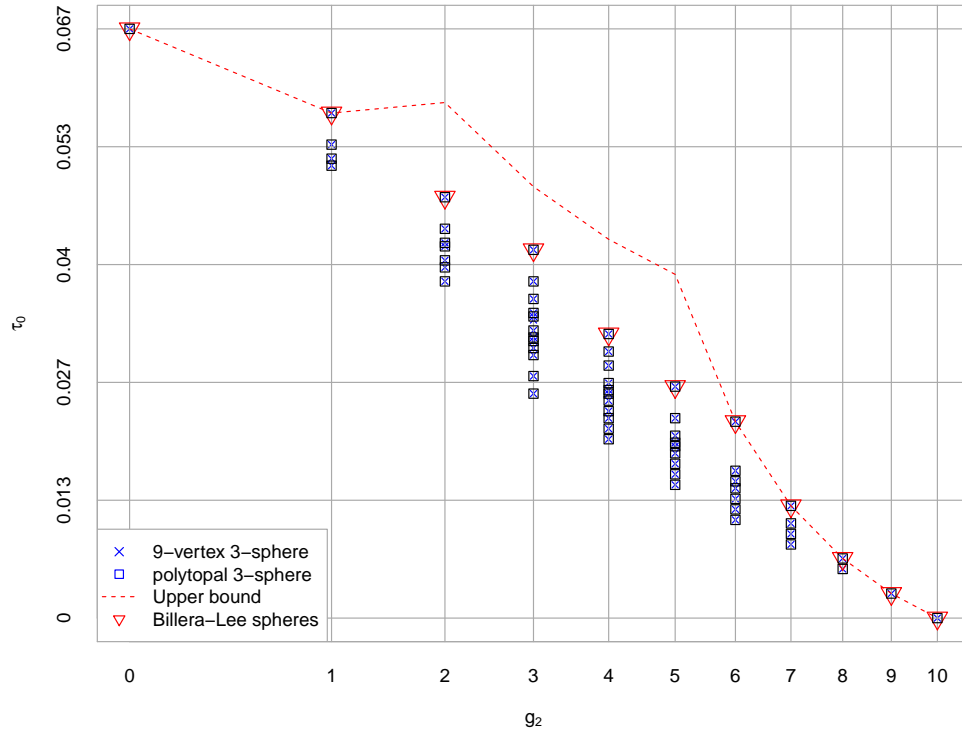


FIGURE 3. Values of τ_0 for all triangulated 3-spheres with 9 and 10 vertices. Note that minimal values for τ_0 are not always realizable by polytopal spheres.

$\begin{array}{c} g_2 \\ \backslash \\ f_0 \end{array}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Σ
5	1																1
6	1	1															2
7	1	1	1	1													4
8	3	3	5	2	3	5	4										25
9	7	14	29	14	44	48	26	42	111	121	51						507
10	30	82	235	108	541	600	287	1019	2505	2890	993	1938	5175	13241	14057	3677	47378

FIGURE 4. Number of isomorphism types of triangulated 3-spheres with the same τ -vector as the corresponding Billera-Lee sphere.

and stacked triangulation is tight [26, 42]. We call such triangulations *tight of stacked type* (this class includes all tight 3-manifolds by [8]).

The following two statements characterize these two classes of tight triangulations in terms of their f -vector (and the homology of the manifold), providing further justification for Conjecture 6.3:

Theorem 6.4 (Kühnel [28] for $k = 2$; Novik and Swartz [42, Theorem 4.4] for higher k). *Let C be a triangulation of a $2k$ -dimensional manifold. Then we have*

$$\binom{f_0(C) - k - 2}{k + 1} \geq (-1)^k \binom{2k + 1}{k + 1} (\chi(C) - 2),$$

with equality if and only if C is $(k + 1)$ -neighborly (and thus tight).

Theorem 6.4 was first conjectured as [28, Conjecture B].

Theorem 6.5 (Novik and Swartz [42], Bagchi [5]). *Let C be a triangulation of an \mathbb{F} -orientable d -manifold with n vertices, $d \geq 3$. Then we have*

$$g_2 \geq \binom{d + 2}{2} \beta_1(C, \mathbb{F}).$$

with equality if and only if C is stacked.

In Theorem 6.5, the case of equality in dimension ≥ 4 is due to Novik and Swartz [42], and in dimension $d = 3$ to Bagchi [5] (confirming [42, Problem 5.3]).

Naturally, for any triangulation we have that $\binom{h_1}{2} \geq g_2$, with equality being equivalent to 2-neighborliness. Hence, triangulations achieving the equality $\binom{h_1}{2} = \binom{d+2}{2} \beta_1(C, \mathbb{F})$ are the ones that are both stacked and 2-neighborly. These are called *tight-neighborly* and were conjectured to be \mathbb{F} -tight by Lutz, Sulanke and Swartz [35]. This conjecture was confirmed in dimension three by Burton, Datta, Singh and Spreer [14] and in $d \geq 4$ by Effenberger [20].

Corollary 6.6. *Let C be a triangulation of an \mathbb{F} -orientable d -manifold with n vertices, $d \geq 3$. Then we have*

$$\binom{n - d - 1}{2} = \binom{h_1}{2} \geq \binom{d + 2}{2} \beta_1(C, \mathbb{F}).$$

with equality if and only if C is tight-neighborly (which implies \mathbb{F} -tightness).

An infinite family of tight triangulations of stacked type is described by Kühnel in [28]. Many more such examples – including an infinite family as well as numerous sporadic examples – are constructed in [15] by a systematic search using a set of conditions first formulated in [17].

Tight triangulations of stacked and neighborly types are opposite in the following sense. Assume the manifold satisfies Poincaré duality, that is, \mathbb{F} is arbitrary if M is orientable and it is of characteristic two if not. In the neighborly case, the underlying d -manifold M must be $(d/2 - 1)$ -connected and thus the only non-zero Betti numbers are $\tilde{\beta}_0(M) = \beta_{2k}(M) = 1$ and $\beta_k = (-1)^k(\chi(C) - 2)$. In the stacked case, on the other hand, we must have all Betti numbers equal to zero except $\tilde{\beta}_0(M) = \beta_{2k}(M) = 1$ and $\beta_1 = \beta_{d-1}$. Nonetheless, both cases can be unified by Theorem 1.1, which has been the initial motivation to study the τ - and μ -vectors of a simplicial complex.

Since in Theorem 1.1 the Betti numbers of a tight triangulation must attain an upper bound in form of its μ -vector, and since tight triangulations are conjectured to be minimal (see Conjecture 6.3), one might conjecture that the vertex links of all tight combinatorial d -manifolds must have τ -vectors of Billera-Lee spheres with matching f -vectors. However, this is not the case, as the following example shows. In particular, tight triangulations do not always have vertex links with maximal τ -vector entries. This fact is somewhat surprising, especially in light of Conjecture 6.3.

Example 6.7. Consider the 15-vertex triangulation of $(S^1 \sim S^3) \# (\mathbb{C}P^2) \#^5$ constructed in [31]. Its automorphism group is transitive on its vertices and its vertex links have f -vector $(1, 14, 64, 100, 50)$ and τ -vector $(1/15, 1/15, 1/3, 1/15, 1/15)$. It follows that the μ -vector equals $(1, 1, 5, 1, 1)$ and hence the triangulation is \mathbb{F}_2 -tight due to Theorem 1.1.

However, the Billera-Lee sphere with f -vector $(1, 14, 64, 100, 50)$ has $\tau_0 = 71/792$ which is considerably higher than $1/15$.

6.3. Triangulated 4-manifolds with transitive automorphism group. For spheres in dimension (up to) three, Theorem 3.10 provides a global upper bound for all entries of the τ -vector in terms of the f -vector.

More precisely, let S and T be n -vertex 3-spheres, S stacked and T 2-neighbourly. Then we have that $\tau_1(S) = 0 = \tau_0(T)$, and thus $\tau_0(S) = \binom{n-4}{2}/(n+1)\binom{6}{2}$ and $\tau_1(T) = 2\binom{n-3}{3}/(n+1)\binom{6}{3}$. This has the following implications for triangulated 4-manifolds:

Let M be an $(n+1)$ -vertex, 2-neighbourly 4-manifold with m triangles. A simple calculation shows that then

$$\frac{2}{3}(2n^2 - 3n - 5) \leq m \leq \binom{n+1}{3}.$$

Whenever the lower bound is satisfied, all vertex links are $(n-1)$ -vertex stacked 3-spheres and we necessarily have $\mu_1(M) = \binom{n-4}{2}/\binom{6}{2}$ and $\mu_2 = 0$. Similarly, whenever the upper bound is attained, all vertex links are $(n-1)$ -vertex 2-neighbourly 3-spheres, and we have $\mu_1(M) = 0$ and $\mu_2 = 2\binom{n-3}{3}/\binom{6}{3}$. By Corollary 5.6 this transforms into

$$\tilde{\beta}_1(M) \leq \binom{n-4}{2}/\binom{6}{2}; \quad \tilde{\beta}_2(M) = 0$$

for a 2-neighbourly 4-manifold M with only stacked vertex links, and

$$\tilde{\beta}_1(M') = 0; \quad \tilde{\beta}_2(M') \leq 2\binom{n-3}{3}/\binom{6}{3}$$

for a 3-neighbourly 4-manifold M' . Both bounds coincide with existing bounds on 4-manifolds, see Theorem 6.4 and Corollary 6.6. Moreover, equality in these bounds here always implies by Theorem 1.1 that the triangulation is both minimal and tight. Note, however, that bounds obtained this way rely on the triangulated 4-manifold to have a prescribed f -vector of a certain kind.

Here we want to generalize these results using the upper bound from Theorem 3.10 and the conjectured upper bound from Conjecture 3.11. To keep calculations simple we only consider the case where all vertex links have the same f -vector (as is the case for 2-neighborly and stacked, as well as for 3-neighborly triangulations, as explained above).

This also includes the case when M has vertex-transitive automorphism group (see [33] for a classification of such triangulations for small numbers of vertices).

Suppose that M is an orientable connected 4-manifold with Betti numbers $\tilde{\beta}_0 + 1 = \tilde{\beta}_4 = 1$, $\tilde{\beta}_1 = \tilde{\beta}_3 = k$ and $\tilde{\beta}_2 = \ell$ and with vertex set V . A simple calculation shows that $f_i(M)$ satisfies the following identity:

$$f_i(M) = \frac{1}{i+1} \sum_{v \in V} f_{i-1}(\text{lk}_v(M)); \quad 0 \leq i \leq 4.$$

Assuming that all links have equal f -vector $f(\text{lk}(M)) = (f_{-1}, \dots, f_3)$ this simplifies to

$$f_i(M)/f_0(M) = \frac{f_{i-1}}{i+1}; \quad 0 \leq i \leq 4.$$

Since the links of M are 3-spheres we have $f_2 = 2(f_1 - f_0)$ and $f_3 = f_1 - f_0$ and we set $n := f_0$ and $e := f_1$.

Now combining the inequality $\tilde{\beta}_i(M) \leq \mu_i(M)$ with the definition of the μ -vector and the conjectured and proven upper bounds from Conjecture 3.11 and Theorem 3.10 we have the following statements.

$$\begin{aligned} \tilde{\beta}_1(M) &\leq \mu_1(M) \\ &= f_0(M) \cdot \tau_0(\text{lk}(M)) \\ &\leq f_0(M) \cdot \tau_0(G(n, e, d)) \\ (14) \quad &= f_0(M) \left(\frac{1}{n+1} - \frac{1}{k+1} + \frac{1}{(j+1)(j+2)} + \frac{n-k-1}{(d+2)(d+3)} \right) \end{aligned}$$

where $k, j \in \mathbb{Z}$ are given by the fact that k is the largest integer such that $e = j + \binom{k}{2} + \frac{n-k-1}{d+1}$, and $d = 2$ for Theorem 3.10, and $d = 3$ if Conjecture 3.11 is assumed to be true. By Corollary 5.6 this yields:

$$\begin{aligned} \tilde{\beta}_2(M) &\leq \mu_2(M) \\ &= f_0(M) \cdot \tau_1(\text{lk}(M)) \\ &\leq f_0(M) \cdot \tau_1(G(n, e, d)) \\ &= f_0(M) \left(2\tau_0(G(n, e, d)) - \frac{n^2 - 4n + 5}{5(n+1)} + \frac{e}{30} \right) \\ (15) \quad &= f_0(M) \left(2 \left(\frac{1}{n+1} - \frac{1}{k+1} + \frac{1}{(j+1)(j+2)} + \frac{n-k-1}{(d+2)(d+3)} \right) - \frac{n^2 - 4n + 5}{5(n+1)} + \frac{e}{30} \right). \end{aligned}$$

While these bounds are quite difficult to analyse by hand, they provide the basis for an algorithm to find a lower bound on the number of faces necessary to triangulate a PL 4-manifold M with fixed first and second Betti numbers (here subject to the additional condition that all vertex links in the triangulation have the same f -vector):

- (1) Given $\tilde{\beta}_1(M)$ and $\tilde{\beta}_2(M)$, go through all theoretically possible f -vectors of a triangulated 4-manifold of Euler characteristic $\chi(M) = 2 - 2\tilde{\beta}_1 + \tilde{\beta}_2$ (with all vertex links sharing the same f -vector) in lexicographically increasing order.
- (2) For each such f -vector, check whether the upper bound on μ_1 and μ_2 (Equations (14) and (15)) exceeds $\tilde{\beta}_1(M)$ and $\tilde{\beta}_2(M)$ componentwise.
- (3) The first f -vector satisfying this condition acts as a lower bound to triangulate M .

In the case of 2-neighborly triangulations of 4-manifolds (i.e., the only case where triangulations are not trivially non-tight) feasible f -vectors together with their Euler characteristic and an upper bound on their first and second Betti numbers are listed in Figure 5.

$f(\text{lk})$	$f(M)$	$\chi(M)$	Thm. 3.10 $(\tilde{\beta}_1, \tilde{\beta}_2) \leq$	Conj. 3.11 $(\tilde{\beta}_1, \tilde{\beta}_2) \leq$	Triangulations
(5, 10)	(6, 15, 20, 15, 6)	2	(0, 0)	(0, 0)	$\partial\Delta^4$
(8, 28)	(9, 36, 84, 90, 36)	3	(0, 1)	(0, 1)	$\mathbb{C}P^2$ [29]
(9, 36)	(10, 45, 120, 135, 54)	4	(0, 2)	(0, 2)	Does not exist [30]
(10, 30)	(11, 55, 110, 110, 44)	0	(1, 0)	(1, 0)	$S^1 \times S^3$ [28]
(11, 36)	(12, 66, 144, 150, 60)	0	(1, 1)	(1, 0)	
(11, 41)	(12, 66, 164, 180, 72)	2	(1, 2)	(0, 1)	
(11, 46)	(12, 66, 184, 210, 84)	4	(0, 3)	(0, 2)	
(11, 51)	(12, 66, 204, 240, 96)	6	(0, 4)	(0, 4)	
(13, 48)	(14, 91, 224, 245, 98)	0	(2, 2)	(1, 1)	
(13, 63)	(14, 91, 294, 350, 140)	7	(0, 6)	(0, 6)	
(13, 78)	(14, 91, 364, 455, 182)	14	(0, 12)	(0, 12)	
(14, 46)	(15, 105, 230, 240, 96)	-4	(3, 1)	(3, 0)	
(14, 50)	(15, 105, 250, 270, 108)	-2	(3, 2)	(2, 0)	
(14, 52)	(15, 105, 260, 285, 114)	-1	(3, 3)	(2, 1)	
(14, 54)	(15, 105, 270, 300, 120)	0	(2, 3)	(2, 2)	
(14, 56)	(15, 105, 280, 315, 126)	1	(2, 4)	(2, 3)	
(14, 60)	(15, 105, 300, 345, 138)	3	(2, 5)	(1, 4)	
(14, 62)	(15, 105, 310, 360, 144)	4	(2, 6)	(1, 4)	
(14, 64)	(15, 105, 320, 375, 150)	5	(2, 7)	(1, 5)	$(S^1 \sim S^3) \# (\mathbb{C}P^2)^{\#5}$ [31]
(14, 66)	(15, 105, 330, 390, 156)	6	(1, 7)	(1, 6)	
(14, 68)	(15, 105, 340, 405, 162)	7	(1, 7)	(1, 7)	
(14, 70)	(15, 105, 350, 420, 168)	8	(1, 8)	(0, 7)	
(14, 72)	(15, 105, 360, 435, 174)	9	(1, 9)	(0, 8)	
(14, 74)	(15, 105, 370, 450, 180)	10	(0, 9)	(0, 9)	
(14, 76)	(15, 105, 380, 465, 186)	11	(0, 10)	(0, 10)	
(14, 78)	(15, 105, 390, 480, 192)	12	(0, 11)	(0, 11)	
(14, 80)	(15, 105, 400, 495, 198)	13	(0, 12)	(0, 11)	
(14, 82)	(15, 105, 410, 510, 204)	14	(0, 12)	(0, 12)	
(14, 84)	(15, 105, 420, 525, 210)	15	(0, 13)	(0, 13)	
(14, 86)	(15, 105, 430, 540, 216)	16	(0, 14)	(0, 14)	
(14, 88)	(15, 105, 440, 555, 222)	17	(0, 15)	(0, 15)	
(14, 90)	(15, 105, 450, 570, 228)	18	(0, 16)	(0, 16)	
(15, 60)	(16, 120, 320, 360, 144)	0	(3, 5)	(2, 3)	
(15, 75)	(16, 120, 400, 480, 192)	8	(2, 10)	(1, 8)	
(15, 105)	(16, 120, 560, 720, 288)	24	(0, 22)	(0, 22)	$K3$ [16]

FIGURE 5. Upper bound on Betti numbers coming from Theorem 3.10 and Conjecture 3.11 for 2-neighborly triangulated 4-manifolds with vertex links of constant f -vectors.

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