Isomorphism between the R-matrix and Drinfeld presentations of Yangian in types B, C and D

Naihuan Jing, Ming Liu and Alexander Molev

Abstract

It is well-known that the Gauss decomposition of the generator matrix in the R-matrix presentation of the Yangian in type A yields generators of its Drinfeld presentation. Defining relations between these generators are known in an explicit form thus providing an isomorphism between the presentations. It has been an open problem since the pioneering work of Drinfeld to extend this result to the remaining types. We give a solution for the classical types B, C and D by constructing an explicit isomorphism between the R-matrix and Drinfeld presentations of the Yangian. It is based on an embedding theorem which allows us to consider the Yangian of rank $n-1$ as a subalgebra of the Yangian of rank n of the same type.

1 Introduction

According to the original definition of Drinfeld [5], the *Yangian* associated to a simple Lie algebra $\mathfrak{g}$ is a Hopf algebra with a finite set of generators. Another presentation of the Yangian was given by him in [6] and it is known as the *new realization* or *Drinfeld presentation*; see also book by Chari and Pressly [4, Chapter 12] for an exposition. The Hopf algebra which coincides with the Yangian in type A was considered previously in the work of Faddeev and the St. Petersburg (Leningrad) school; see the expository paper [11] by Kulish and Sklyanin. The defining relations of this algebra are written in the form of a single RTT relation involving the Yang R-matrix R. An explicit isomorphism between the R-matrix and Drinfeld presentations of the Yangian in type A is constructed with the use of the Gauss decomposition of the generator matrix $T(u)$. Complete proofs were given by Brundan and Kleshchev [3]; see also [13, Section 3.1] for an exposition.

At least for the classical types, the R-matrix presentation is convenient for describing the coproduct of the Yangian and allows one to develop tensor techniques to investigate its algebraic structure; cf. Arnaudon *et al.* [1], [2] (types B, C and D) and [13, Chapter 1] (type A). On the other hand, finite-dimensional irreducible representations of the Yangian associated with any simple Lie algebra $\mathfrak{g}$ are classified uniformly in terms of its Drinfeld presentation. An explicit isomorphism between the presentations is therefore important for bringing together the two approaches and enhancing algebraic tools for understanding the Yangian and its representations. Our main result is a construction of such an isomorphism in the remaining classical types B, C and D which thus solves the open problem going back to Drinfeld's work [6].

To explain our construction, suppose that the simple Lie algebra $\mathfrak{g}$ is associated with the Cartan matrix $A = [a_{ij}]_{i,j=1}^n$. Let $\alpha_1, \dots, \alpha_n$ be the corresponding simple roots (normalized as in (5.1) and (5.2) below for types B_n, C_n and D_n). In accordance to [6], the *Drinfeld Yangian* $\mathrm{Y}^D(\mathfrak{g})$ is generated by elements κ_{ir}, ξ_{ir}^+ and ξ_{ir}^- with $i = 1, \dots, n$ and $r = 0, 1, \dots$ subject to the defining relations

$$
\begin{aligned}
[\kappa_{i\,r}, \kappa_{j\,s}] &= 0, \\
[\xi_{i\,r}^+, \xi_{j\,s}^-] &= \delta_{ij}\, \kappa_{i\,r+s}, \\
[\kappa_{i\,0}, \xi_{j\,s}^\pm] &= \pm\, (\alpha_i, \alpha_j)\, \xi_{j\,s}^\pm, \\
[\kappa_{i\,r+1}, \xi_{j\,s}^\pm] - [\kappa_{i\,r}, \xi_{j\,s+1}^\pm] &= \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\kappa_{i\,r}\, \xi_{j\,s}^\pm + \xi_{j\,s}^\pm\, \kappa_{i\,r}\big), \\
[\xi_{i\,r+1}^\pm, \xi_{j\,s}^\pm] - [\xi_{i\,r}^\pm, \xi_{j\,s+1}^\pm] &= \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\xi_{i\,r}^\pm\, \xi_{j\,s}^\pm + \xi_{j\,s}^\pm\, \xi_{i\,r}^\pm\big), \\
\sum_{p \in \mathfrak{S}_m} [\xi_{i\,r_{p(1)}}^\pm, [\xi_{i\,r_{p(2)}}^\pm, \dots [\xi_{i\,r_{p(m)}}^\pm, \xi_{j\,s}^\pm] \dots]] &= 0,
\end{aligned}
\tag{1.1}
$$

where the last relation holds for all $i \neq j$, and we denoted $m = 1 - a_{ij}$.

If $\mathfrak{g} = \mathfrak{g}_N$ is the orthogonal Lie algebra $\mathfrak{o}_N$ (with $N = 2n$ or $N = 2n+1$) or symplectic Lie algebra $\mathfrak{sp}_N$ (with even $N = 2n$) then the algebra $\mathrm{Y}^R(\mathfrak{g}_N)$ (the *Yangian in the R-matrix or RTT presentation*) can be defined with the use of the rational R-matrix first discovered in [14]. The defining relations take the form of the *RTT relation*

$$
R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v) \tag{1.2}
$$

together with the *unitarity condition*

$$
T'(u+\kappa)\, T(u) = 1, \tag{1.3}
$$

with the notation explained below in Section 2. Here $T(u)$ is a square matrix of size N whose (i,j) entry is the formal series

$$
t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r}
$$

so that the algebra $\mathrm{Y}^R(\mathfrak{g}_N)$ is generated by all coefficients $t_{ij}^{(r)}$ subject to the defining relations (1.2) and (1.3). Apply the Gauss decomposition to the matrix $T(u)$,

$$
T(u) = F(u)\, H(u)\, E(u), \tag{1.4}
$$

where $F(u)$, $H(u)$ and $E(u)$ are uniquely determined matrices of the form

$$F(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{21}(u) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{N1}(u) & f_{N2}(u) & \dots & 1 \end{bmatrix}, \qquad E(u) = \begin{bmatrix} 1 & e_{12}(u) & \dots & e_{1N}(u) \\ 0 & 1 & \dots & e_{2N}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

and $H(u) = \text{diag}\,\big[h_1(u), \dots, h_N(u)\big]$. Define the series with coefficients in $\text{Y}^R(\mathfrak{g}_N)$ by

$$\kappa_i(u) = h_i\big(u - (i-1)/2\big)^{-1}\, h_{i+1}\big(u - (i-1)/2\big)$$

for $i = 1, \dots, n-1$, and

$$\kappa_n(u) = \begin{cases} h_n\big(u - (n-1)/2\big)^{-1}\, h_{n+1}\big(u - (n-1)/2\big) & \text{for} \quad \mathfrak{o}_{2n+1} \\ 2\,h_n\big(u - n/2\big)^{-1}\, h_{n+1}\big(u - n/2\big) & \text{for} \quad \mathfrak{sp}_{2n} \\ h_{n-1}\big(u - (n-2)/2\big)^{-1}\, h_{n+1}\big(u - (n-2)/2\big) & \text{for} \quad \mathfrak{o}_{2n}. \end{cases}$$

Furthermore, set

$$\xi_i^+(u) = f_{i+1\,i}\big(u - (i-1)/2\big), \qquad \xi_i^-(u) = e_{i\,i+1}\big(u - (i-1)/2\big)$$

for $i = 1, \dots, n-1$,

$$\xi_n^+(u) = \begin{cases} f_{n+1\,n}\big(u - (n-1)/2\big) & \text{for} \quad \mathfrak{o}_{2n+1} \\ f_{n+1\,n}\big(u - n/2\big) & \text{for} \quad \mathfrak{sp}_{2n} \\ f_{n+1\,n-1}\big(u - (n-2)/2\big) & \text{for} \quad \mathfrak{o}_{2n} \end{cases}$$

and

$$\xi_n^-(u) = \begin{cases} e_{n\,n+1}\big(u - (n-1)/2\big) & \text{for} \quad \mathfrak{o}_{2n+1} \\ e_{n\,n+1}\big(u - n/2\big) & \text{for} \quad \mathfrak{sp}_{2n} \\ e_{n-1\,n+1}\big(u - (n-2)/2\big) & \text{for} \quad \mathfrak{o}_{2n}. \end{cases}$$

Introduce elements of $\text{Y}^R(\mathfrak{g}_N)$ by the respective expansions into power series in u^{-1},

$$\kappa_i(u) = 1 + \sum_{r=0}^{\infty} \kappa_{i\,r}\, u^{-r-1} \qquad \text{and} \qquad \xi_i^{\pm}(u) = \sum_{r=0}^{\infty} \xi_{i\,r}^{\pm}\, u^{-r-1} \tag{1.5}$$

for $i = 1, \dots, n$. Our main result is the following.

Main Theorem. *The mapping which sends the generators $\kappa_{i\,r}$ and $\xi_{i\,r}^{\pm}$ of $\text{Y}^D(\mathfrak{g}_N)$ to the elements of $\text{Y}^R(\mathfrak{g}_N)$ with the same names defines an isomorphism $\text{Y}^D(\mathfrak{g}_N) \cong \text{Y}^R(\mathfrak{g}_N)$.*

As was pointed out in [1], the existence of such an isomorphism follows from the fact that the classical limits of the Hopf algebras $\mathrm{Y}^D(\mathfrak{g}_N)$ and $\mathrm{Y}^R(\mathfrak{g}_N)$ define the same bialgebra structure on the Lie algebra of polynomial currents $\mathfrak{g}_N[x]$. Therefore, the Hopf algebras must be isomorphic due to Drinfeld's uniqueness theorem on quantization. The main obstacle for constructing an isomorphism explicitly in types B, C and D is that, unlike type A, there is no *natural* embedding of the Yangian of rank $n-1$ into the Yangian of rank n in their R-matrix presentations. To overcome this difficulty, we first prove an embedding theorem allowing us to consider the Yangian $\mathrm{Y}^R(\mathfrak{g}_{N-2})$ as a subalgebra of $\mathrm{Y}^R(\mathfrak{g}_N)$. When restricted to the universal enveloping algebras, this coincides with the natural embedding $\mathrm{U}(\mathfrak{g}_{N-2}) \hookrightarrow \mathrm{U}(\mathfrak{g}_N)$. This theorem effectively reduces the isomorphism problem to the case of rank 2.

The second ingredient of the proof of the Main Theorem is another presentation (called *minimalistic*) of the Yangian $\mathrm{Y}^D(\mathfrak{g}_N)$ which goes back to Levendorskiĭ [12] and was recently given in a modified form by Guay *et al.* [8]. Its use eliminates the need to verify complicated Serre-type relations in $\mathrm{Y}^D(\mathfrak{g})$ for the proof that our map is a homomorphism.

We will mainly work with the *extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$ which is defined by the relation (1.2) omitting (1.3). We give a Drinfeld presentation for $\mathrm{X}(\mathfrak{g}_N)$ which we believe is of independent interest. This presentation given in Theorem 5.14 is analogous to the one in type A; see [3]. Furthermore, we describe the center $\mathrm{ZX}(\mathfrak{g}_N)$ of the extended Yangian in its Drinfeld presentation by providing explicit formulas for generators of $\mathrm{ZX}(\mathfrak{g}_N)$ (Theorem 5.8) which are then used in the proof of the Main Theorem.

We acknowledge the support of the School of Mathematical Sciences at the South China University of Technology and the Australian Research Council.

2 Notation and definitions

Define the orthogonal Lie algebras $\mathfrak{o}_N$ with $N = 2n+1$ and $N = 2n$ (corresponding to types B and D, respectively) and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ (of type C) as subalgebras of $\mathfrak{gl}_N$ spanned by all elements F_{ij},

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{and} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'}, \tag{2.1}$$

respectively, for $\mathfrak{o}_N$ and $\mathfrak{sp}_N$, where the E_{ij} denote the standard basis elements of $\mathfrak{gl}_N$. Here we use the notation $i' = N - i + 1$, and in the symplectic case we set $\varepsilon_i = 1$ for $i = 1, \ldots, n$ and $\varepsilon_i = -1$ for $i = n+1, \ldots, 2n$. To consider the three cases B, C and D simultaneously we will use the notation $\mathfrak{g}_N$ for any of the Lie algebras $\mathfrak{o}_N$ or $\mathfrak{sp}_N$.

To introduce the R-matrix presentation of the Yangian, we will need a standard tensor notation. By taking the canonical basis $e_1, \ldots, e_N$ of $\mathbb{C}^N$, we will identify the endomorphism algebra $\mathrm{End}\,\mathbb{C}^N$ with the algebra of $N \times N$ matrices. The matrix units e_{ij} with $i, j \in \{1, \ldots, N\}$ form a basis of $\mathrm{End}\,\mathbb{C}^N$. We will work with tensor product algebras of the form

$$\mathrm{End}\,(\mathbb{C}^N)^{\otimes m} \otimes \mathcal{A} = \underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathcal{A}, \tag{2.2}$$

where $\mathcal{A}$ is a unital associative algebra. For any element

$$X = \sum_{i,j=1}^{N} e_{ij} \otimes X_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathcal{A} \tag{2.3}$$

and any $a \in \{1, \dots, m\}$ we will denote by X_a the element (2.3) associated with the a-th copy of $\operatorname{End} \mathbb{C}^N$ so that

$$X_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes X_{ij} \in \operatorname{End} (\mathbb{C}^N)^{\otimes m} \otimes \mathcal{A}, \tag{2.4}$$

where 1 is the identity endomorphism. Moreover, given any element

$$C = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N,$$

for any two indices $a, b \in \{1, \dots, m\}$ such that $a < b$, we set

$$C_{ab} = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)} \in \operatorname{End} (\mathbb{C}^N)^{\otimes m}. \tag{2.5}$$

We will keep the same notation C_{ab} for the element $C_{ab} \otimes 1$ of the algebra (2.2).

Set

$$\kappa = \begin{cases} N/2 - 1 & \text{in the orthogonal case,} \\ N/2 + 1 & \text{in the symplectic case.} \end{cases}$$

As defined in [14], the *R-matrix* $R(u)$ is a rational function in a complex parameter u with values in the tensor product algebra $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}, \tag{2.6}$$

where

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}, \tag{2.7}$$

while Q is defined by the formulas

$$Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{i'j'} \qquad \text{and} \qquad Q = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j\, e_{ij} \otimes e_{i'j'}, \tag{2.8}$$

in the orthogonal and symplectic case, respectively. Note the relations $P^2 = 1,\ Q^2 = N\,Q$ and

$$PQ = QP = \begin{cases} Q & \text{in the orthogonal case,} \\ -Q & \text{in the symplectic case.} \end{cases}$$

The rational function (2.6) satisfies the Yang–Baxter equation

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{2.9}$$

The *extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$ is a unital associative algebra with generators $t_{ij}^{(r)}$, where $1\leqslant i,j\leqslant N$ and $r=1,2,\dots$, satisfying certain quadratic relations. Introduce the formal series

$$t_{ij}(u)=\delta_{ij}+\sum_{r=1}^{\infty}t_{ij}^{(r)}\,u^{-r}\in\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]] \tag{2.10}$$

and set

$$T(u)=\sum_{i,j=1}^{N}e_{ij}\otimes t_{ij}(u)\in\mathrm{End}\,\mathbb{C}^N\otimes\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]].$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{g}_N)$ are then written in the form

$$R_{12}(u-v)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,R_{12}(u-v). \tag{2.11}$$

The *Yangian*[1] $\mathrm{Y}(\mathfrak{g}_N)$ is defined as the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ which consists of the elements stable under the automorphisms

$$\mu_f: T(u)\mapsto f(u)\,T(u), \tag{2.12}$$

for all series $f(u)=1+f_1u^{-1}+f_2u^{-2}+\cdots$ with $f_i\in\mathbb{C}$.

The following tensor product decomposition holds

$$\mathrm{X}(\mathfrak{g}_N)=\mathrm{ZX}(\mathfrak{g}_N)\otimes\mathrm{Y}(\mathfrak{g}_N), \tag{2.13}$$

where $\mathrm{ZX}(\mathfrak{g}_N)$ is the center of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$. The center is generated by the coefficients of the series

$$z_N(u)=1+\sum_{r=1}^{\infty}z_N^{(r)}\,u^{-r} \tag{2.14}$$

found by

$$T'(u+\kappa)\,T(u)=T(u)\,T'(u+\kappa)=z_N(u)1, \tag{2.15}$$

where the prime denotes the matrix transposition defined by

$$(X')_{ij}=\begin{cases}X_{j'i'} & \text{in the orthogonal case,}\\ \varepsilon_i\varepsilon_j\,X_{j'i'} & \text{in the symplectic case.}\end{cases} \tag{2.16}$$

Equivalently, the Yangian $\mathrm{Y}(\mathfrak{g}_N)$ is the quotient of the algebra $\mathrm{X}(\mathfrak{g}_N)$ by the relation $z_N(u)=1$, that is,

$$T'(u+\kappa)\,T(u)=1; \tag{2.17}$$

[1] Since we will work with this R-matrix presentation of the Yangian most of the time, we will suppress the superscript R in the notation $\mathrm{Y}^R(\mathfrak{g}_N)$ used in the Introduction.

see [1] and [2] for more details on the structure of the Yangian.

In terms of the series (2.10) the defining relations (2.11) can be written as

$$\begin{aligned}[t_{ij}(u),t_{kl}(v)] &= \frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v)-t_{kj}(v)\,t_{il}(u)\Big) \\ &-\frac{1}{u-v-\kappa}\Big(\delta_{ki'}\sum_{p=1}^{N}\theta_{ip}\,t_{pj}(u)\,t_{p'l}(v)-\delta_{lj'}\sum_{p=1}^{N}\theta_{jp}\,t_{kp'}(v)\,t_{ip}(u)\Big),\end{aligned} \tag{2.18}$$

where we set $\theta_{ij}\equiv 1$ in the orthogonal case, and $\theta_{ij}=\varepsilon_i\varepsilon_j$ in the symplectic case. Similarly, relation (2.17) reads as

$$\sum_{i=1}^{N}\theta_{ki}\,t_{i'k'}(u+\kappa)\,t_{il}(u)=\delta_{kl}. \tag{2.19}$$

3 Embedding theorems

Let $A=[a_{ij}]$ be an $N\times N$ matrix over a ring with 1. Denote by A^{ij} the matrix obtained from A by deleting the i-th row and j-th column. Suppose that the matrix A^{ij} is invertible. The *ij-th quasideterminant of A* is defined by the formula

$$|A|_{ij}=a_{ij}-r_i^{j}\,(A^{ij})^{-1}\,c_j^{i},$$

where r_i^j is the row matrix obtained from the i-th row of A by deleting the element a_{ij}, and c_j^i is the column matrix obtained from the j-th column of A by deleting the element a_{ij}; see [7]. In particular, the four quasideterminants of a 2×2 matrix A are

$$|A|_{11}=a_{11}-a_{12}\,a_{22}^{-1}\,a_{21},\qquad |A|_{12}=a_{12}-a_{11}\,a_{21}^{-1}\,a_{22},$$
$$|A|_{21}=a_{21}-a_{22}\,a_{12}^{-1}\,a_{11},\qquad |A|_{22}=a_{22}-a_{21}\,a_{11}^{-1}\,a_{12}.$$

The quasideterminant $|A|_{ij}$ is also denoted by boxing the entry a_{ij},

$$|A|_{ij}=\begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1N}\\ & \dots & & \dots & \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{iN}\\ & \dots & & \dots & \\ a_{N1} & \dots & a_{Nj} & \dots & a_{NN}\end{vmatrix}.$$

Now suppose that $n\geqslant 1$ in the case B, and $n\geqslant 2$ in the cases C and D. With the given value of n, consider the algebra $\mathrm{X}(\mathfrak{g}_{N-2})$ and let the indices of the generators $t_{ij}^{(r)}$ range over the sets $2\leqslant i,j\leqslant 2'$ and $r=1,2,\dots$ (the prime refers to $\mathfrak{g}_N$ so that $i'=N-i+1$).

The following is our first main result.

Theorem 3.1. *The mapping*

$$t_{ij}(u)\mapsto\begin{vmatrix} t_{11}(u) & t_{1j}(u)\\ t_{i1}(u) & \boxed{t_{ij}(u)}\end{vmatrix},\qquad 2\leqslant i,j\leqslant 2', \tag{3.1}$$

defines an injective algebra homomorphism $\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{X}(\mathfrak{g}_N)$*. Moreover, its restriction to the subalgebra* $\mathrm{Y}(\mathfrak{g}_{N-2})$ *defines an injective algebra homomorphism* $\mathrm{Y}(\mathfrak{g}_{N-2}) \to \mathrm{Y}(\mathfrak{g}_N)$*.*

Proof. Denote by $s_{ij}(u)$ the quasideterminant appearing in (3.1) so that

$$s_{ij}(u) = t_{ij}(u) - t_{i1}(u)t_{11}(u)^{-1}t_{1j}(u). \tag{3.2}$$

We start by verifying that the series $s_{ij}(u)$ satisfy the defining relations for $\mathrm{X}(\mathfrak{g}_{N-2})$. We will do this by connecting the quasideterminants with quantum minors of the matrix $T(u)$. Introduce power series $\tau^{a_1a_2}_{b_1b_2}(u)$ in u^{-1} with coefficients in $\mathrm{X}(\mathfrak{g}_N)$ as the matrix elements of either side of (2.11) with $v = u - 1$:

$$R_{12}(1)\, T_1(u)\, T_2(u-1) = \sum_{a_i,b_i} e_{a_1b_1} \otimes e_{a_2b_2} \otimes \tau^{a_1a_2}_{b_1b_2}(u). \tag{3.3}$$

Lemma 3.2. *(i) If* $a_1 \neq a_2'$ *then* $\tau^{a_1a_2}_{b_1b_2}(u) = -\tau^{a_2a_1}_{b_1b_2}(u)$*.*

(ii) If $b_1 \neq b_2'$ *then* $\tau^{a_1a_2}_{b_1b_2}(u) = -\tau^{a_1a_2}_{b_2b_1}(u)$*.*

Proof. The operator $R_{12}(1)$ remains unchanged if we multiply it from the left or from the right by $(1-P_{12})/2 + Q_{12}/N$ in the orthogonal case, and by $(1-P_{12})/2$ in the symplectic case. By applying multiplication from the left to the left hand side of (3.3) we derive part (i). Applying (2.11) and using the respective multiplications of the left hand side of (3.3) from the right we get part (ii). □

Remark 3.3. As the proof of Lemma 3.2 shows, the assumptions are not necessary in the symplectic case for the skew-symmetry properties to hold. □

Lemma 3.4. *For any* $2 \leqslant i, j \leqslant 2'$ *we have*

$$s_{ij}(u) = t_{11}(u+1)^{-1}\,\tau^{1i}_{1j}(u+1). \tag{3.4}$$

Moreover,

$$\big[t_{11}(u), \tau^{1i}_{1j}(v)\big] = 0. \tag{3.5}$$

Proof. By (3.2) we have

$$t_{11}(u+1)\, s_{ij}(u) = t_{11}(u+1)t_{ij}(u) - t_{11}(u+1)t_{i1}(u)t_{11}(u)^{-1}t_{1j}(u).$$

However, $t_{11}(u+1)t_{i1}(u) = t_{i1}(u+1)t_{11}(u)$ by (2.18), so that

$$t_{11}(u+1)\, s_{ij}(u) = t_{11}(u+1)t_{ij}(u) - t_{i1}(u+1)t_{1j}(u).$$

The definition (3.3) implies that this coincides with $\tau^{1i}_{1j}(u+1)$ hence (3.4) follows. Relation (3.5) follows easily from (2.18). It can also be derived by noting that the commutation relations between the series involved in this calculation are the same as for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. Therefore, (3.5) holds due to the corresponding properties of the quantum minors for $\mathrm{Y}(\mathfrak{gl}_N)$; see, e.g., [13, Section 1.7]. □

Now we will need some simplified expressions for both sides of (2.9) when $v = u - 1$.

Lemma 3.5. *We have the relations*

$$R_{12}(1)\,R_{13}(u)\,R_{23}(u-1)$$
$$= R_{12}(1)\,\Big(1 - \frac{P_{13}+P_{23}}{u-1} + \frac{Q_{13}+Q_{23}}{u-\kappa} - \frac{P_{23}\,Q_{12}}{(u-1)(u-\kappa)} - P_{13}\,Q_{23}\,\varphi(u)\Big) \qquad (3.6)$$

and

$$R_{23}(u-1)\,R_{13}(u)\,R_{12}(1)$$
$$= \Big(1 - \frac{P_{13}+P_{23}}{u-1} + \frac{Q_{13}+Q_{23}}{u-\kappa} - \frac{Q_{12}\,P_{23}}{(u-1)(u-\kappa)} - Q_{23}\,P_{13}\,\varphi(u)\Big)\,R_{12}(1), \qquad (3.7)$$

where

$$\varphi(u) = \begin{cases} \dfrac{1-4/N}{(u-1)(u-\kappa)} & \text{in the orthogonal case,} \\ \dfrac{1}{u(u-\kappa-1)} & \text{in the symplectic case.} \end{cases}$$

Proof. The product of R-matrices $R_{13}(u)\,R_{23}(u-1)$ on the left hand side of (3.6) equals

$$1 - \frac{P_{13}}{u} - \frac{P_{23}}{u-1} + \frac{Q_{13}}{u-\kappa} + \frac{Q_{23}}{u-\kappa-1} + \frac{P_{13}\,P_{23}}{u(u-1)}$$
$$- \frac{P_{13}\,Q_{23}}{u(u-\kappa-1)} - \frac{Q_{13}\,P_{23}}{(u-1)(u-\kappa)} + \frac{Q_{13}\,Q_{23}}{(u-\kappa)(u-\kappa-1)}.$$

We have

$$(1-P_{12})\,P_{13}\,P_{23} = (1-P_{12})\,P_{12}\,P_{13} = -(1-P_{12})\,P_{13}$$

and

$$(1-P_{12})\,Q_{13}\,Q_{23} = (1-P_{12})\,Q_{13}\,P_{12} = -(1-P_{12})\,Q_{23}.$$

Now continuing with the symplectic case, we also have

$$Q_{12}\,P_{13}\,P_{23} = Q_{12}\,P_{12}\,P_{13} = -Q_{12}\,P_{13}$$

and

$$Q_{12}\,Q_{13}\,Q_{23} = Q_{12}\,Q_{13}\,P_{12} = -Q_{12}\,Q_{23}.$$

Since $Q_{13}\,P_{23} = P_{23}\,Q_{12}$, the expression simplifies to the right hand side of (3.6).

In the orthogonal case, the last two relations take the different form

$$Q_{12}\,P_{13}\,P_{23} = Q_{12}\,P_{13} \qquad \text{and} \qquad Q_{12}\,Q_{13}\,Q_{23} = Q_{12}\,Q_{23}.$$

Write $P_{13}Q_{23} = Q_{12}P_{13}$ and use the relations $(1-P_{12})Q_{12} = 0$ and $Q_{12}^2 = NQ_{12}$ together with $Q_{12}Q_{23} = Q_{12}P_{13}$ to express the left hand side of (3.6) as

$$(1-P_{12})\Big(1-\frac{P_{13}+P_{23}}{u-1}+\frac{Q_{13}+Q_{23}}{u-\kappa}-\frac{Q_{13}P_{23}}{(u-1)(u-\kappa)}\Big)$$
$$+\frac{Q_{12}}{1-\kappa}\Big(1-\frac{P_{13}+P_{23}}{u-1}+\frac{Q_{13}+Q_{23}}{u-\kappa}-\frac{Q_{13}P_{23}+2(\kappa-1)P_{13}}{(u-1)(u-\kappa)}\Big).$$

This coincides with the right hand side of (3.6).

The proof of (3.7) is obtained by using the same arguments and writing the products of the P and Q operators in the reverse order. □

Lemma 3.6. *The mapping*

$$t_{ij}(u) \mapsto \tau_{1j}^{1i}(u), \qquad 2 \leqslant i,j \leqslant 2',$$

defines a homomorphism $\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{X}(\mathfrak{g}_N)$.

Proof. Consider the tensor product algebra (2.2) with $m = 4$, which is associated with the extended Yangian. We have the relation

$$\begin{aligned} &R_{23}(a-1)\,R_{13}(a)\,R_{24}(a)\,R_{14}(a+1)\,R_{12}(1)\,T_1(u)\,T_2(u-1)R_{34}(1)\,T_3(v)\,T_4(v-1)\\ &= R_{34}(1)\,T_3(v)\,T_4(v-1)\,R_{12}(1)\,T_1(u)\,T_2(u-1)\,R_{14}(a+1)\,R_{24}(a)\,R_{13}(a)\,R_{23}(a-1), \end{aligned} \tag{3.8}$$

where $a = u - v$. It follows easily by a repeated application of the Yang–Baxter relation (2.9) and the RTT relation (2.11). We will transform the operators on both sides of (3.8) by using Lemma 3.5 and then equate some matrix elements. We begin with the right hand side and apply first (2.11) with $v = u-1$ to write the product $R_{12}(1)\,T_1(u)\,T_2(u-1)$ in the reverse order. Next, use (2.9) to write

$$R_{12}(1)\,R_{14}(a+1)\,R_{24}(a)\,R_{13}(a)\,R_{23}(a-1) = R_{24}(a)\,R_{14}(a+1)\,R_{12}(1)\,R_{13}(a)\,R_{23}(a-1)$$

and apply (3.6) with u replaced by a to the last three factors. Then use (2.9) again, and apply (3.6) with u replaced by $a+1$ to the product $R_{12}(1)\,R_{14}(a+1)\,R_{24}(a)$. As a result, the right hand side of (3.8) is transformed in such a way that the last four factors are replaced with the product

$$\begin{aligned} &\Big(1-\frac{P_{14}+P_{24}}{a}+\frac{Q_{14}+Q_{24}}{a-\kappa+1}-\frac{P_{24}Q_{12}}{a(a-\kappa+1)}-P_{14}Q_{24}\,\varphi(a+1)\Big)\\ &\qquad\times\Big(1-\frac{P_{13}+P_{23}}{a-1}+\frac{Q_{13}+Q_{23}}{a-\kappa}-\frac{P_{23}Q_{12}}{(a-1)(a-\kappa)}-P_{13}Q_{23}\,\varphi(a)\Big). \end{aligned} \tag{3.9}$$

Now apply the operator on the right hand side of (3.8) to a basis vector of the form $e_1 \otimes e_j \otimes e_1 \otimes e_l$ for certain $j, l \in \{2, \dots, 2'\}$. Each of the operators Q_{12}, Q_{13} and Q_{23} annihilates the vector, so that the application of the second factor in (3.9) gives

$$\frac{a-2}{a-1}\, e_1 \otimes e_j \otimes e_1 \otimes e_l - \frac{1}{a-1}\, e_1 \otimes e_1 \otimes e_j \otimes e_l. \tag{3.10}$$

Next apply the first factor in (3.9) to each of the vectors occurring in (3.10). The operators Q_{12} and Q_{14} annihilate the vector $e_1 \otimes e_j \otimes e_1 \otimes e_l$, while

$$P_{14}\,(e_1 \otimes e_j \otimes e_1 \otimes e_l) = e_l \otimes e_j \otimes e_1 \otimes e_1.$$

The vector $e_l \otimes e_j \otimes e_1 \otimes e_1$ will be annihilated by a subsequent application of the operator $R_{34}(1)T_3(v)T_4(v-1)$ due to Lemma 3.2(ii). The same property holds for the vector $P_{14}Q_{24}\,(e_1 \otimes e_j \otimes e_1 \otimes e_l)$. The application of the first factor in (3.9) to the second vector $e_1 \otimes e_1 \otimes e_j \otimes e_l$ in (3.10) gives

$$e_1 \otimes e_1 \otimes e_j \otimes e_l - \frac{1}{a}\,(e_l \otimes e_1 \otimes e_j \otimes e_1 + e_1 \otimes e_l \otimes e_j \otimes e_1).$$

By Lemma 3.2(ii), this expression will be annihilated by a subsequent application of the operator $R_{12}(1)T_1(u)T_2(u-1)$.

We may conclude that the restriction of the operator on the right hand side of (3.8) to the subspace spanned by the basis vectors of the form $e_1 \otimes e_j \otimes e_1 \otimes e_l$ with $j, l \in \{2, \dots, 2'\}$ coincides with the operator

$$\frac{a-2}{a-1}\, R_{34}(1)T_3(v)T_4(v-1)\, R_{12}(1)T_1(u)T_2(u-1)\Big(1 - \frac{P_{24}}{a} + \frac{Q_{24}}{a-\kappa+1}\Big). \tag{3.11}$$

Now consider the operator on the left hand side of (3.8). We will apply it to basis vectors of $(\mathbb{C}^N)^{\otimes 4}$ and look at the coefficients of the basis vectors of the form $e_1 \otimes e_i \otimes e_1 \otimes e_k$ with $i, k \in \{2, \dots, 2'\}$ in the image. The same argument as for the right hand side, with the use of (3.7) and Lemma 3.2(i) instead, and with reversed factors in the operators, implies that the coefficients of such basis vectors coincide with those of the operator

$$\frac{a-2}{a-1}\Big(1 - \frac{P_{24}}{a} + \frac{Q_{24}}{a-\kappa+1}\Big)\, R_{12}(1)T_1(u)T_2(u-1)\, R_{34}(1)T_3(v)T_4(v-1). \tag{3.12}$$

Furthermore, the application to the vectors $e_1 \otimes e_j \otimes e_1 \otimes e_l$ with $j, l \in \{2, \dots, 2'\}$ gives

$$R_{12}(1)T_1(u)T_2(u-1)\, R_{34}(1)T_3(v)T_4(v-1)\,(e_1 \otimes e_j \otimes e_1 \otimes e_l)$$
$$\equiv \sum_{c,d=1}^{1'} \tau_{1j}^{1c}(u)\,\tau_{1l}^{1d}(v)\,(e_1 \otimes e_c \otimes e_1 \otimes e_d), \tag{3.13}$$

where we only keep the basis vectors which can give a nonzero contribution to the coefficient of $e_1 \otimes e_i \otimes e_1 \otimes e_k$ after the subsequent application of the operators $1, P_{24}$ or Q_{24}. Moreover, by Lemma 3.2(i), the values $c = 1$ and $d = 1$ can also be excluded from the range of the summation indices. This implies that the values $c = 1'$ and $d = 1'$ can be excluded as well, and so we can write an operator equality

$$1 - \frac{P_{24}}{a} + \frac{Q_{24}}{a - \kappa + 1} = R_{24}(u - v),$$

which is the R-matrix associated with the algebra $\mathrm{X}(\mathfrak{g}_{N-2})$. The same argument with the use of Lemma 3.2(ii) shows that this equality can also be used for the operator (3.11). In other words, by equating the matrix elements of the operators (3.11) and (3.12) we get the R-matrix form of the defining relations for the algebra $\mathrm{X}(\mathfrak{g}_{N-2})$ satisfied by the series $\tau^{1i}_{1j}(u)$, as required. □

Returning to the proof of the theorem, we can now show that the map (3.1) defines a homomorphism. By taking the composition of the homomorphism of Lemma 3.6 with the shift automorphism $T(u) \mapsto T(u+1)$ we get another homomorphism $\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{X}(\mathfrak{g}_N)$ defined by $t_{ij}(u) \mapsto \tau^{1i}_{1j}(u+1)$. It remains to apply Lemma 3.4 and note the commutation relations $\big[t_{11}(u), t_{11}(v)\big] = 0$.

Next we show that the homomorphism (3.1) is injective. For each N introduce an ascending filtration on the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ by setting $\deg t^{(r)}_{ij} = r - 1$ for all $r \geqslant 1$. Denote by $\bar{t}^{(r)}_{ij}$ the image of $t^{(r)}_{ij}$ in the $(r-1)$-th component of the associated graded algebra $\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N)$. The map (3.1) defines a homomorphism of the associated graded algebras $\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N)$. It takes the generator $\bar{t}^{(r)}_{ij} \in \mathrm{gr}\,\mathrm{X}(\mathfrak{g}_{N-2})$ to the element of $\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N)$ denoted by the same symbol. As shown in the proof of the Poincaré–Birkhoff–Witt theorem for the extended Yangian [2, Corollary 3.10], the mapping

$$\bar{t}^{(r)}_{ij} \mapsto F_{ij}\, x^{r-1} + \frac{1}{2}\, \delta_{ij}\, \zeta_r \tag{3.14}$$

defines an isomorphism

$$\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N) \cong \mathrm{U}(\mathfrak{g}_N[x]) \otimes \mathbb{C}[\zeta_1, \zeta_2, \dots],$$

where $\mathbb{C}[\zeta_1, \zeta_2, \dots]$ is the algebra of polynomials in variables ζ_i. These variables correspond to the images of the central elements $z^{(r)}_N$ defined in (2.14),

$$\bar{z}^{(r)}_N \mapsto \zeta_r. \tag{3.15}$$

Therefore the homomorphism $\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N)$ is injective and so is the homomorphism (3.1).

Finally, observe that the homomorphism (3.1) commutes with the automorphism μ_f defined in (2.12) associated with an arbitrary series $f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{X}(\mathfrak{g}_{N-2}) & \longrightarrow & \mathrm{X}(\mathfrak{g}_N) \\ {\scriptstyle \mu_f}\downarrow & & \downarrow{\scriptstyle \mu_f} \\ \mathrm{X}(\mathfrak{g}_{N-2}) & \longrightarrow & \mathrm{X}(\mathfrak{g}_N), \end{array}$$

where the horizontal arrows denote the homomorphism (3.1). Therefore, the image of the restriction of this homomorphism to the subalgebra $\mathrm{Y}(\mathfrak{g}_{N-2})$ of $\mathrm{X}(\mathfrak{g}_{N-2})$ is contained in the subalgebra $\mathrm{Y}(\mathfrak{g}_N)$ of $\mathrm{X}(\mathfrak{g}_N)$. This restriction thus defines an injective homomorphism $\mathrm{Y}(\mathfrak{g}_{N-2}) \to \mathrm{Y}(\mathfrak{g}_N)$. □

The following generalization of Theorem 3.1 will be useful for our arguments below. Fix a positive integer m such that $m \leqslant n$ for type B and $m \leqslant n-1$ for types C and D. Suppose that the generators $t_{ij}^{(r)}$ of the algebra $\mathrm{X}(\mathfrak{g}_{N-2m})$ are labelled by the indices $m+1 \leqslant i, j \leqslant (m+1)'$ and $r = 1, 2, \dots$ with $i' = N - i + 1$ as before.

Theorem 3.7. *The mapping*

$$\psi_m : t_{ij}(u) \mapsto \begin{vmatrix} t_{11}(u) & \dots & t_{1m}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{m1}(u) & \dots & t_{mm}(u) & t_{mj}(u) \\ t_{i1}(u) & \dots & t_{im}(u) & \boxed{t_{ij}(u)} \end{vmatrix}, \qquad m+1 \leqslant i, j \leqslant (m+1)', \tag{3.16}$$

defines an injective homomorphism $\mathrm{X}(\mathfrak{g}_{N-2m}) \to \mathrm{X}(\mathfrak{g}_N)$. *Moreover, its restriction to the subalgebra* $\mathrm{Y}(\mathfrak{g}_{N-2m})$ *defines an injective homomorphism* $\mathrm{Y}(\mathfrak{g}_{N-2m}) \to \mathrm{Y}(\mathfrak{g}_N)$.

Proof. We argue by induction on m. The case $m = 1$ is Theorem 3.1. Suppose that $m \geqslant 2$. By the Sylvester theorem for quasideterminants [7] (see also [10] for a proof), we have the identity

$$\begin{vmatrix} t_{11}(u) & \dots & t_{1m}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{m1}(u) & \dots & t_{mm}(u) & t_{mj}(u) \\ t_{i1}(u) & \dots & t_{im}(u) & \boxed{t_{ij}(u)} \end{vmatrix} = \begin{vmatrix} s_{22}(u) & \dots & s_{2m}(u) & s_{2j}(u) \\ \dots & \dots & \dots & \dots \\ s_{m2}(u) & \dots & s_{mm}(u) & s_{mj}(u) \\ s_{i2}(u) & \dots & s_{im}(u) & \boxed{s_{ij}(u)} \end{vmatrix},$$

where

$$s_{ab}(u) = \begin{vmatrix} t_{11}(u) & t_{1b}(u) \\ t_{a1}(u) & \boxed{t_{ab}(u)} \end{vmatrix}.$$

By Theorem 3.1, the mapping $t_{ab}(u) \mapsto s_{ab}(u)$ with $2 \leqslant a, b \leqslant 2'$ defines a homomorphism $\mathrm{X}(\mathfrak{g}_{N-2}) \to \mathrm{X}(\mathfrak{g}_N)$. Furthermore, by the induction hypothesis, the map

$$t_{ij}(u) \mapsto \begin{vmatrix} s_{22}(u) & \dots & s_{2m}(u) & s_{2j}(u) \\ \dots & \dots & \dots & \dots \\ s_{m2}(u) & \dots & s_{mm}(u) & s_{mj}(u) \\ s_{i2}(u) & \dots & s_{im}(u) & \boxed{s_{ij}(u)} \end{vmatrix}, \qquad m+1 \leqslant i, j \leqslant (m+1)',$$

defines a homomorphism $\mathrm{X}(\mathfrak{g}_{N-2m}) \to \mathrm{X}(\mathfrak{g}_{N-2})$ thus proving that (3.16) is a homomorphism. Its injectivity and the remaining parts of the theorem are verified in the same way as for Theorem 3.1. □

We will point out a consistence property of the embeddings (3.16) whose particular case $l = 1$ was already used in the proof of Theorem 3.7; cf. [13, eq. (1.85)] for its counterpart in type A. We will write $\psi_m = \psi_m^{(N)}$ to indicate the dependence of N. For a parameter l we have the corresponding embedding

$$\psi_m^{(N-2l)} : \mathrm{X}(\mathfrak{g}_{N-2m-2l}) \hookrightarrow \mathrm{X}(\mathfrak{g}_{N-2l})$$

provided by (3.16).

Proposition 3.8. *We have the equality of maps*

$$\psi_l^{(N)} \circ \psi_m^{(N-2l)} = \psi_{l+m}^{(N)}.$$

Proof. For all $l+1 \leqslant a, b \leqslant (l+1)'$ introduce the series $s_{ab}(u)$ with coefficients in $\mathrm{X}(\mathfrak{g}_N)$ by

$$s_{ab}(u) = \begin{vmatrix} t_{11}(u) & \dots & t_{1l}(u) & t_{1b}(u) \\ \dots & \dots & \dots & \dots \\ t_{l1}(u) & \dots & t_{ll}(u) & t_{lb}(u) \\ t_{a1}(u) & \dots & t_{al}(u) & \boxed{t_{ab}(u)} \end{vmatrix}.$$

The desired equality amounts to the identity for series with coefficients in $\mathrm{X}(\mathfrak{g}_N)$,

$$\begin{vmatrix} s_{l+1\,l+1}(u) & \dots & s_{l+1\,l+m}(u) & s_{l+1\,j}(u) \\ \dots & \dots & \dots & \dots \\ s_{l+m\,l+1}(u) & \dots & s_{l+m\,l+m}(u) & s_{l+m\,j}(u) \\ s_{i\,l+1}(u) & \dots & s_{i\,l+m}(u) & \boxed{s_{ij}(u)} \end{vmatrix} = \begin{vmatrix} t_{11}(u) & \dots & t_{1\,l+m}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{l+m\,1}(u) & \dots & t_{l+m\,l+m}(u) & t_{l+m\,j}(u) \\ t_{i1}(u) & \dots & t_{i\,l+m}(u) & \boxed{t_{ij}(u)} \end{vmatrix}$$

which holds for all $l+m+1 \leqslant i, j \leqslant (l+m+1)'$ due to the Sylvester theorem for quasideterminants [7], [10]. □

For subsets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ of $\{1, \dots, N\}$ introduce A-type quantum minors by the formula

$$t^{\,a_1 \dots a_k}_{\,b_1 \dots b_k}(u) = \sum_{p \in \mathfrak{S}_k} \operatorname{sgn} p \cdot t_{a_{p(1)} b_1}(u) \dots t_{a_{p(k)} b_k}(u - k + 1).$$

These are formal series in u^{-1} with coefficients in $\mathrm{X}(\mathfrak{g}_N)$.

Proposition 3.9. *For all $m+1 \leqslant i, j \leqslant (m+1)'$ we have the identity*

$$\begin{vmatrix} t_{11}(u) & \dots & t_{1m}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{m1}(u) & \dots & t_{mm}(u) & t_{mj}(u) \\ t_{i1}(u) & \dots & t_{im}(u) & \boxed{t_{ij}(u)} \end{vmatrix} = t^{\,1 \dots m}_{\,1 \dots m}(u+m)^{-1} \cdot t^{\,1 \dots m\, i}_{\,1 \dots m\, j}(u+m).$$

Proof. This identity holds for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$; see, e.g., [13, Section 1.11]. The commutation relations between the generator series $t_{ab}(u)$ of $\mathrm{X}(\mathfrak{g}_N)$ occurring in this identity are the same as for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, with a possible exception of the commutators $[t_{ij}(u), t_{ij}(v)]$, and in addition in the B case, the commutators of the series in the last row or last column of the quasideterminant. However, since the quasideterminant depends linearly on such generators, the identity does not depend on such commutators. Hence it holds for $\mathrm{X}(\mathfrak{g}_N)$ as well. □

The following is a counterpart of the corresponding property of the Yangian for $\mathfrak{gl}_N$; see, e.g., [3].

Corollary 3.10. *We have the relations*

$$\big[t_{ab}(u), \psi_m(t_{ij}(v))\big] = 0$$

for all $1 \leqslant a, b \leqslant m$ *and* $m+1 \leqslant i, j \leqslant (m+1)'$.

Proof. By Proposition 3.9 we only need to verify that $t_{ab}(u)$ commutes with the quantum minors. This follows by the same argument as for the proof of the proposition. □

4 Gauss decomposition

As we pointed out in the Introduction, the Gauss decomposition (1.4) will play a key role in constructing the Drinfeld generators. We will assume that $T(u)$ is the generator matrix for the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ (that is, we ignore relation (1.3)) and recall the well-known formulas for the entries of the matrices $F(u)$, $H(u)$ and $E(u)$ which occur in (1.4); see, e.g., [13, Sec. 1.11]. We have

$$h_i(u) = \begin{vmatrix} t_{11}(u) & \dots & t_{1\,i-1}(u) & t_{1\,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,i}(u) \\ t_{i\,1}(u) & \dots & t_{i\,i-1}(u) & \boxed{t_{i\,i}(u)} \end{vmatrix}, \qquad i = 1, \dots, N, \tag{4.1}$$

whereas

$$e_{ij}(u) = h_i(u)^{-1} \begin{vmatrix} t_{11}(u) & \dots & t_{1\,i-1}(u) & t_{1\,j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,j}(u) \\ t_{i\,1}(u) & \dots & t_{i\,i-1}(u) & \boxed{t_{i\,j}(u)} \end{vmatrix} \tag{4.2}$$

and

$$f_{ji}(u) = \begin{vmatrix} t_{11}(u) & \dots & t_{1\,i-1}(u) & t_{1\,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,i}(u) \\ t_{j\,1}(u) & \dots & t_{j\,i-1}(u) & \boxed{t_{j\,i}(u)} \end{vmatrix} h_i(u)^{-1} \tag{4.3}$$

for $1 \leqslant i < j \leqslant N$. Obviously, the algebra $\mathrm{X}(\mathfrak{g}_N)$ is generated by the coefficients of the series $f_{ji}(u)$, $e_{ij}(u)$ and $h_i(u)$ which we will refer to as the *Gaussian generators*.

Suppose that $0 \leqslant m < n$ if $N = 2n$ and $0 \leqslant m \leqslant n$ if $N = 2n+1$. We will use the superscript $[m]$ to indicate square submatrices corresponding to rows and columns labelled by $m+1, m+2, \dots, (m+1)'$. In particular, we set

$$F^{[m]}(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{m+2\,m+1}(u) & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ f_{(m+1)'\,m+1}(u) & \dots & f_{(m+1)'\,(m+2)'}(u) & 1 \end{bmatrix},$$

$$E^{[m]}(u) = \begin{bmatrix} 1 & e_{m+1\,m+2}(u) & \dots & e_{m+1\,(m+1)'}(u) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & e_{(m+2)'\,(m+1)'}(u) \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

and $H^{[m]}(u) = \mathrm{diag}\,\big[h_{m+1}(u), \dots, h_{(m+1)'}(u)\big]$. Furthermore, introduce the product of these matrices by

$$T^{[m]}(u) = F^{[m]}(u)\, H^{[m]}(u)\, E^{[m]}(u).$$

Accordingly, the entries of $T^{[m]}(u)$ will be denoted by $t^{[m]}_{ij}(u)$ with $m+1 \leqslant i, j \leqslant (m+1)'$.

Proposition 4.1. *The series $t^{[m]}_{ij}(u)$ coincides with the image of the generator series $t_{ij}(u)$ of the extended Yangian $\mathrm{X}(\mathfrak{g}_{N-2m})$ under the embedding (3.16),*

$$t^{[m]}_{ij}(u) = \psi_m\big(t_{ij}(u)\big), \qquad m+1 \leqslant i, j \leqslant (m+1)'.$$

Proof. Set $s_{ij}(u) = \psi_m\big(t_{ij}(u)\big)$. Since the Gauss decomposition (1.4) uniquely determines the matrices $F(u)$, $H(u)$ and $E(u)$, it suffices to verify that such matrices obtained by the Gauss decomposition of the matrix

$$S(u) = \big[s_{ij}(u)\big], \qquad m+1 \leqslant i, j \leqslant (m+1)',$$

coincide with $F^{[m]}(u)$, $H^{[m]}(u)$ and $E^{[m]}(u)$, respectively. Let $S(u) = \widetilde{F}(u)\, \widetilde{H}(u)\, \widetilde{E}(u)$ be the Gauss decomposition of $S(u)$. By formulas (4.1), (4.2) and (4.3), the entries of the matrices $\widetilde{F}(u)$, $\widetilde{H}(u)$ and $\widetilde{E}(u)$ are found as the quasideterminants of certain submatrices of $S(u)$. However, as we pointed out in the proof of Proposition 3.8, such quasideterminants coincide with the corresponding quasideterminants of submatrices of the matrix $T(u)$. □

We record the following as an immediate consequence of Proposition 4.1.

Corollary 4.2. *The subalgebra $\mathrm{X}^{[m]}(\mathfrak{g}_N)$ generated by the coefficients of all series $t^{[m]}_{ij}(u)$ with $m+1 \leqslant i, j \leqslant (m+1)'$ is isomorphic to the extended Yangian $\mathrm{X}(\mathfrak{g}_{N-2m})$. In particular, we have the relation*

$$R^{[m]}_{12}(u-v)\, T^{[m]}_1(u)\, T^{[m]}_2(v) = T^{[m]}_2(v)\, T^{[m]}_1(u)\, R^{[m]}_{12}(u-v), \tag{4.4}$$

where $R^{[m]}(u)$ *is the R-matrix associated with* $\mathrm{X}(\mathfrak{g}_{N-2m})$. *Moreover,*

$$T^{[m]\prime}(u+\kappa^{[m]})T^{[m]}(u) = T^{[m]}(u)T^{[m]\prime}(u+\kappa^{[m]}) = z_N^{[m]}(u)\,1, \tag{4.5}$$

with $\kappa^{[m]} = \kappa - m$, *for a certain series* $z_N^{[m]}(u)$ *whose coefficients generate the center of the subalgebra* $\mathrm{X}^{[m]}(\mathfrak{g}_N)$. □

Introduce the coefficients of the series defined in (4.1), (4.2) and (4.3) by

$$e_{ij}(u) = \sum_{r=1}^{\infty} e_{ij}^{(r)}\, u^{-r}, \qquad f_{ji}(u) = \sum_{r=1}^{\infty} f_{ji}^{(r)}\, u^{-r}, \qquad h_i(u) = 1 + \sum_{r=1}^{\infty} h_i^{(r)}\, u^{-r}.$$

Furthermore, define the series by

$$k_i(u) = h_i(u)^{-1}h_{i+1}(u), \qquad e_i(u) = e_{i\,i+1}(u), \qquad f_i(u) = f_{i+1\,i}(u) \tag{4.6}$$

for $i = 1, \dots, n-1$, and set

$$e_n(u) = \begin{cases} e_{n\,n+1}(u) & \text{for} \quad \mathfrak{o}_{2n+1} \\ e_{n\,n+1}(u) & \text{for} \quad \mathfrak{sp}_{2n} \\ e_{n-1\,n+1}(u) & \text{for} \quad \mathfrak{o}_{2n} \end{cases}, \qquad f_n(u) = \begin{cases} f_{n+1\,n}(u) & \text{for} \quad \mathfrak{o}_{2n+1} \\ f_{n+1\,n}(u) & \text{for} \quad \mathfrak{sp}_{2n} \\ f_{n+1\,n-1}(u) & \text{for} \quad \mathfrak{o}_{2n} \end{cases} \tag{4.7}$$

and

$$k_n(u) = \begin{cases} h_n(u)^{-1}\, h_{n+1}(u) & \text{for} \quad \mathfrak{o}_{2n+1} \\ 2\,h_n(u)^{-1}\, h_{n+1}(u) & \text{for} \quad \mathfrak{sp}_{2n} \\ h_{n-1}(u)^{-1}\, h_{n+1}(u) & \text{for} \quad \mathfrak{o}_{2n}. \end{cases} \tag{4.8}$$

Lemma 4.3. *For the parameter* m *chosen as above, suppose that the indices* j, k, l *satisfy* $m+1 \leqslant j, k, l \leqslant (m+1)'$ *and* $j \neq l'$. *Then the following relations hold in the extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$,

$$\big[e_{m\,j}(u), t_{kl}^{[m]}(v)\big] = \frac{1}{u-v} t_{kj}^{[m]}(v)\big(e_{m\,l}(v) - e_{m\,l}(u)\big), \tag{4.9}$$

$$\big[f_{j\,m}(u), t_{kl}^{[m]}(v)\big] = \frac{1}{u-v}\big(f_{k\,m}(u) - f_{k\,m}(v)\big)\, t_{j\,l}^{[m]}(v). \tag{4.10}$$

Proof. It is sufficient to verify the relations for $m = 1$; the general case will follow by the application of the homomorphism ψ_m; see Proposition 4.1. Both relations follow by similar arguments so we only verify (4.9). Since

$$h_1(v) = t_{11}(v), \qquad f_{k1}(v) = t_{k1}(v)t_{11}(v)^{-1}, \qquad e_{1l}(v) = t_{11}(v)^{-1}t_{1l}(v),$$

we can write

$$t_{kl}^{[1]}(v) = t_{kl}(v) - t_{k1}(v)t_{11}(v)^{-1}t_{1l}(v) = t_{kl}(v) - f_{k1}(v)h_1(v)e_{1l}(v).$$

The defining relations (2.18) imply

$$[t_{1j}(u), t_{kl}(v)] = \frac{1}{u-v}\big(t_{kj}(u)t_{1l}(v) - t_{kj}(v)t_{1l}(u)\big)$$

and so

$$\begin{aligned}\big[t_{1j}(u), t^{[1]}_{kl}(v)\big] + \big[t_{1j}(u), f_{k1}(v)h_1(v)e_{1l}(v)\big] &= \frac{1}{u-v}\big(t^{[1]}_{kj}(u)t_{1l}(v) - t^{[1]}_{kj}(v)t_{1l}(u)\big)\\ &+ \frac{1}{u-v}\big(f_{k1}(u)h_1(u)e_{1j}(u)t_{1l}(v) - f_{k1}(v)h_1(v)e_{1j}(v)t_{1l}(u)\big).\end{aligned}$$

The second commutator on the left hand side can be transformed as

$$\begin{aligned}\big[t_{1j}(u), f_{k1}(v)h_1(v)e_{1l}(v)\big] &= \big[t_{1j}(u), t_{k1}(v)\big]\, e_{1l}(v) + t_{k1}(v)\big[t_{1j}(u), e_{1l}(v)\big]\\ &= \big[t_{1j}(u), t_{k1}(v)\big]e_{1l}(v) + t_{k1}(v)\big[t_{1j}(u), t_{11}(v)^{-1}t_{1l}(v)\big],\end{aligned}$$

which equals

$$\begin{aligned}&\big[t_{1j}(u), t_{k1}(v)\big]e_{1l}(v) + t_{k1}(v)\big[t_{1j}(u), t_{11}(v)^{-1}\big]t_{1l}(v) + f_{k1}(v)\big[t_{1j}(u), t_{1l}(v)\big]\\ &\qquad = \big[t_{1j}(u), t_{k1}(v)\big]e_{1l}(v) - f_{k1}(v)\big[t_{1j}(u), t_{11}(v)\big]e_{1l}(v) + f_{k1}(v)\big[t_{1j}(u), t_{1l}(v)\big].\end{aligned}$$

Hence, calculating the commutators by (2.18), we come to the relation

$$\begin{aligned}\big[t_{1j}(u), f_{k1}(v)h_1(v)e_{1l}(v)\big] &= \frac{1}{u-v}\big(t_{kj}(u)h_1(v)e_{1l}(v) - t_{kj}(v)h_1(u)e_{1l}(v)\big)\\ &- \frac{1}{u-v}\big(f_{k1}(v)h_1(u)e_{1j}(u)h_1(v)e_{1l}(v) - f_{k1}(v)h_1(v)e_{1j}(v)h_1(u)e_{1l}(v)\big)\\ &+ \frac{1}{u-v}\big(f_{k1}(v)t_{1j}(u)h_1(v)e_{1l}(v) - f_{k1}(v)t_{1j}(v)h_1(u)e_{1l}(u)\big).\end{aligned}$$

This gives

$$\big[t_{1j}(u), t^{[1]}_{kl}(v)\big] = \frac{1}{u-v}\big(t^{[1]}_{kj}(v)h_1(u)e_{1l}(v) - t^{[1]}_{kj}(v)h_1(u)e_{1l}(u)\big).$$

By Corollary 3.10, $t_{11}(u)$ commutes with $t^{[1]}_{kj}(v)$ and so (4.9) with $m = 1$ follows. □

5 Drinfeld presentation of extended Yangian

Here we will give a Drinfeld presentation for the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ analogous to that of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ [3]; see also [13, Sec. 3.1]. Isomorphisms between classical Lie algebras in low ranks lead to corresponding Yangian isomorphisms; see [2] and [9]. We begin by reviewing them in the context of Drinfeld presentations.

5.1 Low rank isomorphisms

We will follow the notation of [13, Sec. 3.1] for the Gauss decomposition of the generator matrix of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, but use the corresponding capital letters $H_i(u)$, $E_{ij}(u)$ and $F_{ji}(u)$ for the entries of the respective matrices occurring in the $\mathrm{Y}(\mathfrak{gl}_N)$ counterpart of (1.4). The next lemmas are implied by the results of [2, Sec. 4].

Lemma 5.1. *In terms of the Gaussian generators, the isomorphism* $\mathrm{X}(\mathfrak{sp}_2) \to \mathrm{Y}(\mathfrak{gl}_2)$ *has the form*

$$\begin{aligned} h_1(u) &\mapsto H_1(u/2), & e_{12}(u) &\mapsto E_{12}(u/2), \\ h_2(u) &\mapsto H_2(u/2), & f_{21}(u) &\mapsto F_{21}(u/2). \end{aligned}$$

Lemma 5.2. *In terms of the Gaussian generators, the isomorphism* $\mathrm{X}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{gl}_2)$ *has the form*

$$\begin{aligned} h_1(u) &\mapsto H_1(2u)H_1(2u+1), & e_{12}(u) &\mapsto \sqrt{2}\,E_{12}(2u+1), & f_{21}(u) &\mapsto \sqrt{2}\,F_{21}(2u+1), \\ h_2(u) &\mapsto H_1(2u)H_2(2u+1), & e_{23}(u) &\mapsto -\sqrt{2}\,E_{12}(2u), & f_{32}(u) &\mapsto -\sqrt{2}\,F_{21}(2u), \\ h_3(u) &\mapsto H_2(2u)H_2(2u+1), & e_{13}(u) &\mapsto -E_{12}(2u+1)^2, & f_{31}(u) &\mapsto -F_{21}(2u+1)^2. \end{aligned}$$

Lemma 5.3. *In terms of the Gaussian generators, the embedding* $\mathrm{X}(\mathfrak{o}_4) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$ *is given by*

$$\begin{aligned} h_1(u) &\mapsto H_1(u)H_1'(u), & h_{2'}(u) &\mapsto H_2(u)H_1'(u), \\ h_2(u) &\mapsto H_1(u)H_2'(u), & h_{1'}(u) &\mapsto H_2(u)H_2'(u), \end{aligned}$$

together with

$$\begin{aligned} e_{12}(u) &\mapsto E_{12}'(u), & e_{12'}(u) &\mapsto E_{12}(u), \\ e_{11'}(u) &\mapsto -E_{12}(u)E_{12}'(u), & e_{22'}(u) &\mapsto 0, \\ e_{21'}(u) &\mapsto -E_{12}(u), & e_{2'1'}(u) &\mapsto -E_{12}'(u), \end{aligned}$$

and

$$\begin{aligned} f_{21}(u) &\mapsto F_{21}'(u), & f_{2'1}(u) &\mapsto F_{21}(u), \\ f_{1'1}(u) &\mapsto -F_{21}(u)F_{21}'(u), & f_{2'2}(u) &\mapsto 0, \\ f_{1'2}(u) &\mapsto -F_{21}(u), & f_{1'2'}(u) &\mapsto -F_{21}'(u), \end{aligned}$$

where $H_1'(u)$, $H_2'(u)$, $E_{12}'(u)$ *and* $F_{21}'(u)$ *denote the Gaussian generators of the second copy of* $\mathrm{Y}(\mathfrak{gl}_2)$ *in the tensor product.*

It will be convenient to use a uniform root notation for all three cases, so we will assume that the simple roots of $\mathfrak{g}_N$ are $\alpha_1, \dots, \alpha_n$ with

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \dots, n-1, \tag{5.1}$$

and

$$\alpha_n = \begin{cases} \epsilon_n & \text{for} \quad \mathfrak{o}_{2n+1} \\ 2\epsilon_n & \text{for} \quad \mathfrak{sp}_{2n} \\ \epsilon_{n-1} + \epsilon_n & \text{for} \quad \mathfrak{o}_{2n}, \end{cases} \tag{5.2}$$

where $\epsilon_1, \dots, \epsilon_n$ is an orthonormal basis of an Euclidian space with the bilinear form $(.\,,.)$.

Proposition 5.4. *We have the relations in* $\mathrm{X}(\mathfrak{g}_N)$,

$$\begin{aligned}
\big[h_n(u), e_n(v)\big] &= -(\epsilon_n, \alpha_n)\frac{h_n(u)\big(e_n(u) - e_n(v)\big)}{u-v}, \\
\big[h_n(u), f_n(v)\big] &= (\epsilon_n, \alpha_n)\frac{\big(f_n(u) - f_n(v)\big)h_n(u)}{u-v}, \\
\big[e_n(u), e_n(v)\big] &= \frac{(\alpha_n, \alpha_n)}{2}\frac{\big(e_n(u) - e_n(v)\big)^2}{u-v}, \\
\big[f_n(u), f_n(v)\big] &= -\frac{(\alpha_n, \alpha_n)}{2}\frac{\big(f_n(u) - f_n(v)\big)^2}{u-v}, \\
\big[e_n(u), f_n(v)\big] &= \frac{k_n(u) - k_n(v)}{u-v}.
\end{aligned}$$

Moreover, for $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ *we have*

$$\begin{aligned}
[h_{n+1}(u), e_n(v)] = {}& \frac{1}{2(u-v)}h_{n+1}(u)\big(e_n(u) - e_n(v)\big) \\
& - \frac{1}{2(u-v-1)}\big(e_n(u-1) - e_n(v)\big)h_{n+1}(u)
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
[h_{n+1}(u), f_n(v)] = {}& -\frac{1}{2(u-v)}h_{n+1}(u)\big(f_n(u) - f_n(v)\big) \\
& + \frac{1}{2(u-v-1)}\big(f_n(u-1) - f_n(v)\big)h_{n+1}(u),
\end{aligned} \tag{5.4}$$

whereas for $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ *and* $\mathfrak{o}_{2n}$ *we have*

$$\big[h_{n+1}(u), e_n(v)\big] = (\epsilon_n, \alpha_n)\frac{h_{n+1}(u)\big(e_n(u) - e_n(v)\big)}{u-v}$$

and

$$\big[h_{n+1}(u), f_n(v)\big] = -(\epsilon_n, \alpha_n)\frac{\big(f_n(u) - f_n(v)\big)h_{n+1}(u)}{u-v}.$$

Proof. By Corollary 4.2, the subalgebra $\mathrm{X}^{[n-1]}(\mathfrak{g}_N)$ of $\mathrm{X}(\mathfrak{g}_N)$ is isomorphic to $\mathrm{X}(\mathfrak{o}_3)$ and $\mathrm{X}(\mathfrak{sp}_2)$ in types B and C, respectively, while the subalgebra of $\mathrm{X}^{[n-2]}(\mathfrak{o}_{2n})$ is isomorphic to $\mathrm{X}(\mathfrak{o}_4)$. Hence the relations are implied by Lemmas 5.1, 5.2, 5.3, and the Drinfeld presentation of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$; see [13, Sec. 3.1]. For instance, to verify the first relation in type B, use Lemma 5.2 to get

$$\begin{aligned}\big[h_1(u), e_1(v)\big] &= \big[H_1(2u)H_1(2u+1), \sqrt{2}\,E_{12}(2v+1)\big]\\ &= \big[H_1(2u), \sqrt{2}\,E_{12}(2v+1)\big]H_1(2u+1) + H_1(2u)\big[H_1(2u+1), \sqrt{2}\,E_{12}(2v+1)\big].\end{aligned}$$

Applying the commutator formula

$$(u-v)\big[H_1(u), E_{12}(v)\big] = -H_1(u)\big(E_{12}(u) - E_{12}(v)\big)$$

from [13, Lemma 3.1.1], we bring the right hand side to the required form. □

5.2 Type A relations

Since the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ contains a subalgebra isomorphic to the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$, some relations between the Gaussian generators of $\mathrm{X}(\mathfrak{g}_N)$ can be obtained from those of the Drinfeld presentation of $\mathrm{Y}(\mathfrak{gl}_n)$. We record them in the next proposition, where we use the generating functions

$$e_i^{\circ}(u) = \sum_{r=2}^{\infty} e_i^{(r)} u^{-r} \qquad \text{and} \qquad f_i^{\circ}(u) = \sum_{r=2}^{\infty} f_i^{(r)} u^{-r}. \tag{5.5}$$

Proposition 5.5. *The following relations hold in* $\mathrm{X}(\mathfrak{g}_N)$*, with the conditions on the indices* $1 \leqslant i, j \leqslant n-1$ *and* $1 \leqslant k, l \leqslant n$*,*

$$\big[h_k(u), h_l(v)\big] = 0, \tag{5.6}$$

$$\big[e_i(u), f_j(v)\big] = \delta_{ij}\,\frac{k_i(u) - k_i(v)}{u-v}, \tag{5.7}$$

$$\big[h_k(u), e_j(v)\big] = -(\epsilon_k, \alpha_j)\,\frac{h_k(u)\big(e_j(u) - e_j(v)\big)}{u-v}, \tag{5.8}$$

$$\big[h_k(u), f_j(v)\big] = (\epsilon_k, \alpha_j)\,\frac{\big(f_j(u) - f_j(v)\big)\,h_k(u)}{u-v}. \tag{5.9}$$

Moreover, for $1 \leqslant i \leqslant n-2$ *we have*

$$u\big[e_i^{\circ}(u), e_{i+1}(v)\big] - v\big[e_i(u), e_{i+1}^{\circ}(v)\big] = e_i(u)\,e_{i+1}(v),$$

$$u\big[f_i^{\circ}(u), f_{i+1}(v)\big] - v\big[f_i(u), f_{i+1}^{\circ}(v)\big] = -f_{i+1}(v)\,f_i(u),$$

and for $1 \leqslant i \leqslant n-1$ *we have*

$$\big[e_i(u), e_i(v)\big] = \frac{(e_i(u) - e_i(v))^2}{u-v} \qquad \textit{and}$$

$$\big[f_i(u), f_i(v)\big] = -\frac{(f_i(u) - f_i(v))^2}{u-v}.$$

For $1 \leqslant i, j \leqslant n-1$ we have

$$\big[e_i(u), e_j(v)\big] = 0 \qquad \textit{and} \qquad \big[f_i(u), f_j(v)\big] = 0, \qquad \textit{if} \quad (\alpha_i, \alpha_j) = 0,$$

whereas

$$\big[e_i(u), [e_i(v), e_j(w)]\big] + \big[e_i(v), [e_i(u), e_j(w)]\big] = 0 \qquad \textit{and}$$
$$\big[f_i(u), [f_i(v), f_j(w)]\big] + \big[f_i(v), [f_i(u), f_j(w)]\big] = 0 \qquad \textit{if} \quad |i-j| = 1.$$

Proof. The coefficients of the series $t_{ij}(u)$ with $i, j \in \{1, \dots, n\}$ generate a subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ isomorphic to the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. Hence, the upper left $n \times n$ submatrices of the matrices $F(u)$, $H(u)$ and $E(u)$ defined by the Gauss decomposition (1.4) are given by the same formulas as the corresponding elements of $\mathrm{Y}(\mathfrak{gl}_n)$. Therefore they satisfy the relations as described in [3, Section 5]; see also [13, Section 3.1]. □

We point out two useful consequences of (5.8) and (5.9):

$$h_i(u)e_i(u) = e_i(u-1)h_i(u) \qquad \text{and} \qquad h_i(u)f_i(u-1) = f_i(u)h_i(u), \tag{5.10}$$

which hold for all $i = 1, \dots, n-1$.

Another subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ isomorphic to the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is generated by the coefficients of the series $t_{ij}(u)$ with $n' \leqslant i, j \leqslant 1'$. This corresponds to the lower right $n \times n$ submatrix of $T(u)$. It is known that the map

$$\varsigma : T(u) \mapsto T(-u)^{-1} \tag{5.11}$$

defines an automorphism of $\mathrm{X}(\mathfrak{g}_N)$; see [2, Sec. 2]. Hence, the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ generated by the coefficients of the series $\varsigma\big(t_{ij}(u)\big)$ with $n' \leqslant i, j \leqslant 1'$ is isomorphic to the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. On the other hand, by inverting the matrices in the Gauss decomposition (1.4) we get

$$T(u)^{-1} = E(u)^{-1} H(u)^{-1} F(u)^{-1}. \tag{5.12}$$

Since the upper triangular matrix $E(u)^{-1}$ appears on the left and the lower triangular matrix $F(u)^{-1}$ appears on the right, the lower right $n \times n$ submatrix of $T(u)^{-1}$ will only involve the corresponding submatrices of $E(u)^{-1}$, $H(u)^{-1}$ and $F(u)^{-1}$. Regarding this lower right submatrix of $T(-u)^{-1}$ as the generator matrix for $\mathrm{Y}(\mathfrak{gl}_n)$, apply now the automorphism of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ defined by the same formula (5.11). We can thus conclude that the product of the matrices

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{(n-1)'\,n'}(u) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{1'\,n'}(u) & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} h_{n'}(u) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & h_{1'}(u) \end{bmatrix} \begin{bmatrix} 1 & e_{n'\,(n-1)'}(u) & \dots & e_{n'\,1'}(u) \\ 0 & 1 & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

yields the Gauss decomposition of the generator matrix for $\mathrm{Y}(\mathfrak{gl}_n)$. So we have derived the following.

Proposition 5.6. *All relations in $\mathrm{X}(\mathfrak{g}_N)$ given in Proposition 5.5 remain valid under the replacement of the indices of all series by $i \mapsto (n-i+1)'$ for $1 \leqslant i \leqslant n$.* □

5.3 Central elements in terms of Gaussian generators

We note some relations implied by Corollary 4.2 and low rank isomorphisms pointed out in Section 5.1. Write relation (2.15) in the form

$$T'(u+\kappa) = z_N(u)\, T(u)^{-1}. \tag{5.13}$$

By the Gauss decomposition (1.4) we have $t_{11}(u) = h_1(u)$ so that using (5.12) and taking the (N,N)-entry on both sides of (5.13) we get

$$h_1(u+\kappa) = z_N(u)\, h_{1'}(u)^{-1}. \tag{5.14}$$

Taking $N = 2n+1$ and applying this relation to the subalgebra $\mathrm{X}^{[n-1]}(\mathfrak{o}_N) \cong \mathrm{X}(\mathfrak{o}_3)$ (see Corollary 4.2), we get

$$z_N^{[n-1]}(u) = h_n(u+\kappa-n+1)\, h_{n'}(u).$$

Lemma 5.2 allows us to bring this to the form

$$z_N^{[n-1]}(u) = h_n\big(u+1/2\big)\, h_n\big(u-1/2\big)^{-1}\, h_{n+1}(u)\, h_{n+1}\big(u-1/2\big). \tag{5.15}$$

Similarly, the application of (5.14) to the subalgebra $\mathrm{X}^{[n-1]}(\mathfrak{sp}_N) \cong \mathrm{X}(\mathfrak{sp}_2)$ with $N = 2n$ gives

$$z_N^{[n-1]}(u) = h_n(u+\kappa-n+1)\, h_{n+1}(u) = h_n(u+2)\, h_{n+1}(u). \tag{5.16}$$

If $\mathfrak{g}_N = \mathfrak{o}_N$ with $N = 2n$ we apply (5.14) to the subalgebra $\mathrm{X}^{[n-2]}(\mathfrak{o}_N) \cong \mathrm{X}(\mathfrak{o}_4)$ to get

$$z_N^{[n-2]}(u) = h_{n-1}(u+\kappa-n+2)\, h_{(n-1)'}(u) = h_{n-1}(u+1)\, h_{(n-1)'}(u).$$

We have $h_{n-1}(u)\, h_{(n-1)'}(u) = h_n(u)\, h_{n'}(u)$ by Lemma 5.3 and so

$$z_N^{[n-2]}(u) = h_{n-1}(u+1)\, h_{n-1}(u)^{-1}\, h_n(u)\, h_{n'}(u). \tag{5.17}$$

We will need some symmetry properties for the entries of the matrices in the Gauss decomposition (1.4).

Proposition 5.7. *The following relations hold in* $\mathrm{X}(\mathfrak{g}_N)$,

$$e_{(i+1)'\, i'}(u) = -e_i(u+\kappa-i) \qquad \text{and} \qquad f_{i'\,(i+1)'}(u) = -f_i(u+\kappa-i) \tag{5.18}$$

for $i = 1, \dots, n-1$. *Furthermore, if* $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ *then we have*

$$e_{n+1\, n+2}(u) = -e_n\big(u+1/2\big) \qquad \text{and} \qquad f_{n+2\, n+1}(u) = -f_n\big(u+1/2\big),$$

and if $\mathfrak{g}_N = \mathfrak{o}_{2n}$ *then*

$$e_{n\,(n-1)'}(u) = -e_n(u) \qquad \text{and} \qquad f_{(n-1)'\, n}(u) = -f_n(u).$$

Proof. Suppose that $1 \leqslant i \leqslant n-1$. By Corollary 4.2, we have the following consequence of relation (4.5),

$$T^{[i-1]}(u)^{-1} \, z_N^{[i-1]}(u) = T^{[i-1]\,\prime}(u + \kappa^{[i-1]}). \tag{5.19}$$

As with the above derivation of (5.14), take the (i', i')-entry on both sides of (5.19) to get

$$h_{i'}(u)^{-1} \, z_N^{[i-1]}(u) = h_i(u + \kappa - i + 1). \tag{5.20}$$

Similarly, by taking the $\big((i+1)', i'\big)$-entry in (5.19) we obtain

$$-e_{(i+1)'\,i'}(u) \, h_{i'}(u)^{-1} \, z_N^{[i-1]}(u) = h_i(u + \kappa - i + 1) \, e_i(u + \kappa - i + 1).$$

Together with (5.20) this gives

$$-e_{(i+1)'\,i'}(u) = h_i(u + \kappa - i + 1) \, e_i(u + \kappa - i + 1) \, h_i(u + \kappa - i + 1)^{-1}.$$

The first relation in (5.18) now follows from (5.10) and a similar argument verifies the second. The additional relations in types B and D follow from Lemmas 5.2 and 5.3, respectively. □

We are now in a position to give explicit formulas for the series $z_N(u)$ in terms of the Gaussian generators $h_i(u)$. Recall that by Proposition 5.5 the coefficients of the series $h_i(u)$ pairwise commute for $i = 1, \dots, n$; see (5.6).

Theorem 5.8. *We have the identities in* $\mathrm{X}(\mathfrak{g}_N)$:

$$z_N(u) = \prod_{i=1}^{n} h_i(u + \kappa - i)^{-1} \prod_{i=1}^{n} h_i(u + \kappa - i + 1) \cdot h_{n+1}(u) \, h_{n+1}\big(u - 1/2\big)$$

for $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$,

$$z_N(u) = \prod_{i=1}^{n-1} h_i(u + \kappa - i)^{-1} \prod_{i=1}^{n} h_i(u + \kappa - i + 1) \cdot h_{n+1}(u)$$

for $\mathfrak{g}_N = \mathfrak{o}_{2n}$ *and* $\mathfrak{g}_N = \mathfrak{sp}_{2n}$.

Proof. Take the $(2', 2')$ entries on both sides of (5.13). Expressing the entries of the matrices $T'(u+\kappa)$ and $T(u)^{-1}$ in terms of the Gaussian generators from (1.4) and (5.12), we get

$$h_2(u+\kappa) + f_1(u+\kappa) \, h_1(u+\kappa) \, e_1(u+\kappa) = z_N(u) \big(h_{2'}(u)^{-1} + e_{2'\,1'}(u) \, h_{1'}(u)^{-1} f_{1'\,2'}(u)\big).$$

Since $z_N(u)$ is central in $\mathrm{X}(\mathfrak{g}_N)$, using (5.14) we can rewrite this as

$$h_{2'}(u)^{-1} z_N(u) = h_2(u+\kappa) + f_1(u+\kappa) \, h_1(u+\kappa) \, e_1(u+\kappa) - e_{2'\,1'}(u) \, h_1(u+\kappa) \, f_{1'\,2'}(u).$$

Now apply (5.18) to get

$$h_{2'}(u)^{-1}z_N(u) = h_2(u+\kappa) + f_{21}(u+\kappa)\,h_1(u+\kappa)\,e_{12}(u+\kappa) \\ - e_{12}(u+\kappa-1)\,h_1(u+\kappa)\,f_{21}(u+\kappa-1).$$

Due to (5.10), this simplifies to

$$h_{2'}(u)^{-1}z_N(u) = h_2(u+\kappa) + \big[f_{21}(u+\kappa-1), e_{12}(u+\kappa)\big]\,h_1(u+\kappa).$$

Calculating the commutator by (5.7), we bring this relation to the form

$$h_{2'}(u)^{-1}z_N(u) = h_1(u+\kappa-1)^{-1}\,h_1(u+\kappa)\,h_2(u+\kappa-1).$$

Finally, use (5.20) with $i=2$ to get the recurrence formula

$$z_N(u) = h_1(u+\kappa-1)^{-1}\,h_1(u+\kappa)\,z_N^{[1]}(u).$$

By Corollary 4.2, the desired identities now follow from the respective base cases (5.15), (5.16) and (5.17). □

Remark 5.9. The expansions provided by Theorem 5.8 are analogous to the multiplicative formula for the central series $z(u)$ for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ implied by the quantum Liouville formula and a decomposition of the quantum determinant [13, Theorem 1.9.5 and Corollary 1.11.8]. □

5.4 Relations for Gaussian generators

In addition to the type A relations in $\mathrm{X}(\mathfrak{g}_N)$ described in Section 5.2, we will now derive some root system specific relations for each of the types B, C and D.

Proposition 5.10. *We have the relations in* $\mathrm{X}(\mathfrak{g}_N)$:

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = 0 \qquad \text{and} \qquad \big[h_{n+1}(u), f_{n-1}(v)\big] = 0$$

for $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$,

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = \frac{h_{n+1}(u)\big(e_{n-1}(v) - e_{n-1}(u+2)\big)}{u-v+2} \qquad \text{and}$$

$$\big[h_{n+1}(u), f_{n-1}(v)\big] = -\frac{\big(f_{n-1}(v) - f_{n-1}(u+2)\big)\,h_{n+1}(u)}{u-v+2}$$

for $\mathfrak{g}_N = \mathfrak{sp}_{2n}$, *and*

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = \frac{h_{n+1}(u)\,\big(e_{n-1}(v) - e_{n-1}(u)\big)}{u-v} \qquad \text{and}$$

$$\big[h_{n+1}(u), f_{n-1}(v)\big] = -\frac{\big(f_{n-1}(v) - f_{n-1}(u)\big)\,h_{n+1}(u)}{u-v}$$

for $\mathfrak{g}_N = \mathfrak{o}_{2n}$.

Proof. First take $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$. Corollary 3.10 implies that $h_{n+1}(u)$ commutes with each element of the subalgebra generated by the $t_{ij}(u)$ with $1 \leqslant i, j \leqslant n$ and so the relations follow. Now let $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ or $\mathfrak{g}_N = \mathfrak{o}_{2n}$. By Corollary 4.2, the subalgebra $\mathrm{X}^{[n-2]}(\mathfrak{g}_N)$ of $\mathrm{X}(\mathfrak{g}_N)$ is isomorphic to $\mathrm{X}(\mathfrak{g}_4)$. Applying Proposition 5.6 to this subalgebra, we get

$$\big[h_{n+1}(u), e_{n+1\,n+2}(v)\big] = \frac{h_{n+1}(u)\big(e_{n+1\,n+2}(v) - e_{n+1\,n+2}(u)\big)}{u-v}$$

and

$$\big[h_{n+1}(u), f_{n+2\,n+1}(v)\big] = -\frac{\big(f_{n+2\,n+1}(v) - f_{n+2\,n+1}(u)\big)\, h_{n+1}(u)}{u-v}.$$

It remains to apply Proposition 5.7. □

In the next proposition we use the root notation (5.1) and (5.2).

Proposition 5.11. *For $i = 1, \dots, n$ in the algebra $\mathrm{X}(\mathfrak{g}_N)$ we have*

$$\big[h_i(u), h_{n+1}(v)\big] = 0, \tag{5.21}$$

$$\big[h_i(u), e_n(v)\big] = -(\epsilon_i, \alpha_n)\, \frac{h_i(u)\big(e_n(u) - e_n(v)\big)}{u-v}, \tag{5.22}$$

$$\big[h_i(u), f_n(v)\big] = (\epsilon_i, \alpha_n)\, \frac{\big(f_n(u) - f_n(v)\big)h_i(u)}{u-v}, \tag{5.23}$$

and

$$\big[e_i(u), f_n(v)\big] = \big[e_n(u), f_i(v)\big] = \delta_{i\,n} \frac{k_n(u) - k_n(v)}{u-v}. \tag{5.24}$$

Moreover, if $i < n-1$ then

$$\big[h_{n+1}(u), e_i(v)\big] = 0 \qquad \textit{and} \qquad \big[h_{n+1}(u), f_i(v)\big] = 0. \tag{5.25}$$

Proof. Note that relations (5.22) and (5.23) for $i = n$ were already verified in Proposition 5.4. Now suppose that $\mathfrak{g}_N = \mathfrak{o}_{2n}$. By the defining relations (2.18), the subalgebra generated by the coefficients of the series $t_{ij}(u)$ with i, j running over the set $J = \{1, \dots, n-1, n+1\}$ is isomorphic to $\mathrm{Y}(\mathfrak{gl}_n)$. Furthermore, Lemma 5.3 implies that

$$e_{n\,n+1}(u) = f_{n+1\,n}(u) = 0.$$

Hence, we have the Gauss decomposition

$$T_J(u) = F_J(u)\, H_J(u)\, E_J(u), \tag{5.26}$$

where the subscript J indicates the submatrices in (1.4) whose rows and columns are labelled by the elements of the set J. Therefore, the Gaussian generators which occur as the entries of the matrices $F_J(u)$, $H_J(u)$ and $E_J(u)$ satisfy the type A relations as described in Proposition 5.5. This completed the proof for type D.

Now let $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ or $\mathfrak{g}_N = \mathfrak{sp}_{2n}$. Almost all of the relations (5.22) and (5.23) with $i < n$, as well as (5.24) and (5.25) with $i < n-1$ follow from Corollary 3.10. For instance, (5.22) and (5.23) are immediate from the observation that all elements of the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ generated by $t_{ij}(u)$ with $i, j = 1, \dots, n-1$ commute with the subalgebra $\mathrm{X}^{[n-1]}(\mathfrak{g}_N)$. In addition, we use Lemma 5.1 to see that $[h_n(u), h_{n+1}(v)] = 0$ in type C.

Furthermore, the case $i = n$ of (5.24) was already pointed out in Proposition 5.4. To verify the remaining cases $i = n-1$ of (5.24), apply Lemma 4.3 with $m = n-1$. Since $t^{[n-1]}_{n+1\,n}(v) = f_n(v)h_n(v)$, relation (4.9) gives

$$\big[e_{n-1}(u), f_n(v)h_n(v)\big] = \frac{1}{u-v}\, f_n(v)h_n(v)\,\big(e_{n-1}(v) - e_{n-1}(u)\big).$$

On the other hand,

$$\big[e_{n-1}(u), f_n(v)h_n(v)\big] = \big[e_{n-1}(u), f_n(v)\big]\, h_n(v) + f_n(v)\,\big[e_{n-1}(u), h_n(v)\big],$$

whereas

$$\big[e_{n-1}(u), h_n(v)\big] = \frac{1}{u-v}\, h_n(v)\,\big(e_{n-1}(v) - e_{n-1}(u)\big) \tag{5.27}$$

by Proposition 5.5. This gives $\big[e_{n-1}(u), f_n(v)\big] = 0$. The other case of (5.24) is verified in the same way. □

Lemma 5.12. *In the algebra* $\mathrm{X}(\mathfrak{o}_{2n+1})$ *we have*

$$\big[e_{n-1}(u), e_n(v)\big] = \frac{e_{n-1\,n+1}(v) - e_{n-1\,n+1}(u) - e_{n-1}(v)e_n(v) + e_{n-1}(u)e_n(v)}{u-v}, \tag{5.28}$$

$$\big[f_{n-1}(u), f_n(v)\big] = \frac{f_{n+1\,n-1}(u) - f_{n+1\,n-1}(v) - f_{n-1}(u)f_n(v) + f_{n-1}(v)f_n(v)}{u-v}. \tag{5.29}$$

In the algebra $\mathrm{X}(\mathfrak{o}_{2n})$ *we have*

$$\big[e_{n-1}(u), e_n(v)\big] = 0, \qquad \big[f_{n-1}(u), f_n(v)\big] = 0, \tag{5.30}$$

and

$$\big[e_{n-2}(u), e_n(v)\big] = \frac{e_{n-2\,n+1}(v) - e_{n-2\,n+1}(u) - e_{n-2}(v)e_n(v) + e_{n-2}(u)e_n(v)}{u-v}, \tag{5.31}$$

$$\big[f_{n-2}(u), f_n(v)\big] = \frac{f_{n+1\,n-2}(u) - f_{n+1\,n-2}(v) - f_{n-2}(u)f_n(v) + f_{n-2}(v)f_n(v)}{u-v}. \tag{5.32}$$

In the algebra $\mathrm{X}(\mathfrak{sp}_{2n})$ *we have*

$$\big[e_{n-1}(u), e_n(v)\big] = \frac{2\,\big(e_{n-1\,n+1}(v) - e_{n-1\,n+1}(u) - e_{n-1}(v)e_n(v) + e_{n-1}(u)e_n(v)\big)}{u-v}, \tag{5.33}$$

$$\big[f_{n-1}(u), f_n(v)\big] = \frac{2\,\big(f_{n+1\,n-1}(u) - f_{n+1\,n-1}(v) - f_{n-1}(u)f_n(v) + f_{n-1}(v)f_n(v)\big)}{u-v}. \tag{5.34}$$

Proof. By Lemma 4.3, in $\mathrm{X}(\mathfrak{o}_{2n+1})$ we have

$$\big[e_{n-1\,n}(u), t^{[n-1]}_{n\,n+1}(v)\big] = \frac{1}{u-v} t^{[n-1]}_{nn}(v)\big(e_{n-1\,n+1}(v) - e_{n-1\,n+1}(u)\big).$$

Using the Gauss decomposition for $T^{[n-1]}(u)$, we can write the left hand side as

$$\big[e_{n-1\,n}(u), h_n(v)\, e_{n\,n+1}(v)\big] = \big[e_{n-1\,n}(u), h_n(v)\big]\, e_{n\,n+1}(v) + h_n(v)\, \big[e_{n-1\,n}(u), e_{n\,n+1}(v)\big].$$

Now observe that $t^{[n-1]}_{nn}(v) = h_n(v)$ so that (5.28) follows by the application of (5.27). A similar argument yields (5.29).

Now turn to the case $\mathfrak{g}_N = \mathfrak{o}_{2n}$. Relation (5.30) follows from Lemma 5.3. As we pointed out in the proof of Proposition 5.11, the Gaussian generators which occur as the entries of the matrices $F_J(u)$, $H_J(u)$ and $E_J(u)$ in (5.26) satisfy the type A relations as described in Proposition 5.5. Therefore, (5.31) and (5.32) follow from the Drinfeld presentation of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$; see [13, Lemma 3.1.2].

To prove (5.33) and (5.34), note that by Corollary 4.2, the subalgebra $\mathrm{X}^{[n-2]}(\mathfrak{sp}_{2n})$ is isomorphic to $\mathrm{X}(\mathfrak{sp}_4)$. Hence, we may take $n = 2$ and will work with this case throughout the rest of the argument.

The defining relations (2.18) for $\mathrm{X}(\mathfrak{sp}_4)$ give

$$\begin{aligned}\big[t_{12}(u), t_{23}(v)\big] &= \frac{1}{u-v}\Big(t_{22}(u)t_{13}(v) - t_{22}(v)t_{13}(u)\Big) \\ &\quad + \frac{1}{u-v-3}\Big(t_{24}(v)t_{11}(u) + t_{23}(v)t_{12}(u) - t_{22}(v)t_{13}(u) - t_{21}(v)t_{14}(u)\Big).\end{aligned} \tag{5.35}$$

Applying the Gauss decomposition (1.4), for the left hand side of (5.35) we can write

$$\begin{aligned}\big[t_{12}(u), t_{23}(v)\big] &= \big[h_1(u)e_{12}(u), h_2(v)e_{23}(v) + f_{21}(v)h_1(v)e_{13}(v)\big] = \\ &= h_1(u)\big[e_{12}(u), h_2(v)e_{23}(v)\big] + \big[h_1(u), h_2(v)e_{23}(v)\big]e_{12}(u) \\ &\quad + \big[h_1(u)e_{12}(u), f_{21}(v)h_1(v)\big]e_{13}(v) + f_{21}(v)h_1(v)\big[h_1(u)e_{12}(u), e_{13}(v)\big].\end{aligned}$$

Corollary (3.10) implies that $\big[h_1(u), h_2(v)e_{23}(v)\big] = 0$. Furthermore, by (2.18)

$$\big[h_1(u)e_{12}(u), f_{21}(v)h_1(v)\big] = \big[t_{12}(u), t_{21}(v)\big] = \frac{1}{u-v}\big(t_{22}(u)t_{11}(v) - t_{22}(v)t_{11}(u)\big)$$

$$= \frac{1}{u-v}\Big(h_2(u)h_1(v) - h_2(v)h_1(u) + f_{21}(u)h_1(u)e_{12}(u)h_1(v) - f_{21}(v)h_1(v)e_{12}(v)h_1(u)\Big).$$

Similarly, we have

$$\big[h_1(u)e_{12}(u), e_{13}(v)\big] = \big[t_{12}(u), t_{11}(v)^{-1}t_{13}(v)\big],$$

and (2.18) gives

$$\big[t_{12}(u), t_{11}(v)^{-1}\big] = \frac{1}{u-v} t_{11}(v)^{-1}\Big(t_{12}(v)t_{11}(u) - t_{12}(u)t_{11}(v)\Big)t_{11}(v)^{-1}$$

together with

$$\big[t_{12}(u), t_{13}(v)\big] = \frac{1}{u-v}\big(t_{12}(u)t_{13}(v) - t_{12}(v)t_{13}(u)\big) + \frac{1}{u-v-3}\big(t_{14}(v)t_{11}(u) + t_{13}(v)t_{12}(u) - t_{12}(v)t_{13}(u) - t_{11}(v)t_{14}(u)\big).$$

Writing the resulting expression back in terms of the Gaussian generators and applying (5.8) with $k = 2$ and $j = 1$ we find that the left hand side of (5.35) equals

$$h_1(u)h_2(v)[e_{12}(u), e_{23}(v)] + \frac{1}{u-v}\Big(h_1(u)h_2(v)e_{12}(v)e_{23}(v) - h_1(u)h_2(v)e_{12}(u)e_{23}(v)\Big)$$
$$+ \frac{1}{u-v}\Big(h_2(u)h_1(v)e_{13}(v) - h_2(v)h_1(u)e_{13}(v) + f_{21}(u)h_1(u)e_{12}(u)h_1(v)e_{13}(v) - f_{21}(v)h_1(v)e_{12}(v)h_1(u)e_{13}(u)\Big) +$$
$$\frac{1}{u-v-3}\, f_{21}(v)h_1(v)\Big(e_{14}(v)h_1(u) + e_{13}(v)h_1(u)e_{12}(u) - e_{12}(v)h_1(u)e_{13}(u) - h_1(u)e_{14}(u)\Big).$$

As a next step, write the right hand side of (5.35) in terms of the Gaussian generators. Cancelling common terms on both sides, we bring the relation to the form

$$h_1(u)h_2(v)\,\big[e_{12}(u), e_{23}(v)\big] = \frac{1}{u-v}\, h_1(u)h_2(v)\Big(e_{13}(v) - e_{13}(u) - e_{12}(v)e_{23}(v) + e_{12}(u)e_{23}(v)\Big) + \frac{1}{u-v-3}\, h_2(v)\Big(e_{24}(v)h_1(u) + h_1(u)e_{23}(v)e_{12}(u) - h_1(u)e_{13}(u)\Big),$$

where we also used the property $\big[h_1(u), e_{23}(v)\big] = 0$ implied by Corollary 3.10. Since the series $h_1(u)$ and $h_2(v)$ are invertible, we thus get

$$\big[e_{12}(u), e_{23}(v)\big] = \frac{1}{u-v}\Big(e_{13}(v) - e_{13}(u) - e_{12}(v)e_{23}(v) + e_{12}(u)e_{23}(v)\Big) + \frac{1}{u-v-3}\Big(h_1(u)^{-1}e_{24}(v)h_1(u) + e_{23}(v)e_{12}(u) - e_{13}(u)\Big). \tag{5.36}$$

Now we need an expression for the commutator $\big[h_1(u), e_{24}(v)\big]$ which is obtained by a calculation similar to the above derivation of (5.36). Namely, we begin with the following analogue of (5.35),

$$\big[t_{11}(u), t_{24}(v)\big] = \frac{1}{u-v}\Big(t_{21}(u)t_{14}(v) - t_{21}(v)t_{14}(u)\Big) + \frac{1}{u-v-3}\Big(t_{24}(v)t_{11}(u) + t_{23}(v)t_{12}(u) - t_{22}(v)t_{13}(u) - t_{21}(v)t_{14}(u)\Big). \tag{5.37}$$

Then write the left hand side in terms of the Gaussian generators so that it equals

$$h_2(v)\big[h_1(u),e_{24}(v)\big]+\big[h_1(u),f_{21}(v)\big]h_1(v)e_{14}(v)+f_{21}(v)\big[h_1(u),h_1(v)e_{14}(v)\big].$$

Furthermore, use (5.9) with $k=j=1$ and expand

$$\big[h_1(u),h_1(v)e_{14}(v)\big]=\big[t_{11}(u),t_{14}(v)\big]$$

by the defining relations (2.18). Now writing the resulting expressions on both sides of (5.37) in terms of the Gaussian generators and simplifying as with the derivation of (5.36), we come to the desired commutation relation

$$\big[h_1(u),e_{24}(v)\big]=\frac{1}{u-v-3}\Big(e_{24}(v)h_1(u)+h_1(u)e_{23}(v)e_{12}(u)-h_1(u)e_{13}(u)\Big).$$

Applying it to (5.36), we can transform the latter as

$$\begin{aligned}\big[e_{12}(u),e_{23}(v)\big]&=\frac{1}{u-v}\Big(e_{13}(v)-e_{13}(u)-e_{12}(v)e_{23}(v)+e_{12}(u)e_{23}(v)\Big)\\&\quad+\frac{1}{u-v-2}\Big(e_{24}(v)+e_{23}(v)e_{12}(u)-e_{13}(u)\Big).\end{aligned}$$

By rearranging the terms, write it in an equivalent form,

$$\begin{aligned}\big[e_{12}(u),e_{23}(v)\big]&=\frac{2}{u-v}\Big(e_{13}(v)-e_{13}(u)-e_{12}(v)e_{23}(v)+e_{12}(u)e_{23}(v)\Big)\\&\quad+\frac{1}{u-v-1}\Big(e_{24}(v)+e_{12}(v)e_{23}(v)-e_{13}(v)\Big).\end{aligned}$$

Finally, multiply both sides by $u-v-1$ and set $u=v+1$ to see that the second summand vanishes. This yields (5.33). Relation (5.34) is verified by a similar argument. □

In the next proposition we use notation (5.5) for generating functions of elements of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$.

Proposition 5.13. *Suppose that* $1\leqslant i\leqslant n-1$. *If* $(\alpha_i,\alpha_n)=0$ *then*

$$\big[e_i(u),e_n(v)\big]=0\qquad\textit{and}\qquad\big[f_i(u),f_n(v)\big]=0. \tag{5.38}$$

If $(\alpha_i,\alpha_n)\neq 0$ *then*

$$u\big[e_i^{\circ}(u),e_n(v)\big]-v\big[e_i(u),e_n^{\circ}(v)\big]=-(\alpha_i,\alpha_n)\,e_i(u)\,e_n(v)\qquad\textit{and} \tag{5.39}$$

$$u\big[f_i^{\circ}(u),f_n(v)\big]-v\big[f_i(u),f_n^{\circ}(v)\big]=(\alpha_i,\alpha_n)\,f_n(v)\,f_i(u). \tag{5.40}$$

Proof. If $i\leqslant n-2$ in types B and C or $i\leqslant n-3$ in type D, then (5.38) is implied by Corollary 3.10. If $i=n-1$ in type D, then (5.38) follows from Lemma 5.3 with $e_1(u)=e_{12}(u)$ and $e_2(u)=e_{12'}(u)$ by taking into account Corollary 4.2. To get (5.39) and (5.40), consider the expressions for $(u-v)\big[e_i(u),e_n(v)\big]$ and $(u-v)\big[f_i(u),f_n(v)\big]$ provided by Lemma 5.12 and take the coefficients of $u^{-r}v^{-s}$ for $r,s\geqslant 1$. □

Note that relations (5.39) and (5.40) hold in the case $(\alpha_i,\alpha_n)=0$ as well; they are implied by (5.38).

5.5 Theorem on the Drinfeld presentation

We will now prove the theorem on the Drinfeld presentation for $\mathrm{X}(\mathfrak{g}_N)$. We use notation (4.6), (4.7), (4.8) and (5.5) for the generating series and the root notation (5.1) and (5.2).

Theorem 5.14. *The extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$ *is generated by the coefficients of the series* $h_i(u)$ *with* $i = 1, \dots, n+1$, *and* $e_i(u)$ *and* $f_i(u)$ *with* $i = 1, \dots, n$, *subject only to the following sets of relations, where the indices take all admissible values unless specified otherwise. We have*

$$\big[h_i(u), h_j(v)\big] = 0, \tag{5.41}$$

$$\big[e_i(u), f_j(v)\big] = \delta_{ij}\, \frac{k_i(u) - k_i(v)}{u - v}. \tag{5.42}$$

For $i \leqslant n$ *we have*

$$\big[h_i(u), e_j(v)\big] = -(\epsilon_i, \alpha_j)\, \frac{h_i(u)\big(e_j(u) - e_j(v)\big)}{u - v}, \tag{5.43}$$

$$\big[h_i(u), f_j(v)\big] = (\epsilon_i, \alpha_j)\, \frac{\big(f_j(u) - f_j(v)\big)\, h_i(u)}{u - v}. \tag{5.44}$$

For $j \leqslant n-2$ *we have*

$$\big[h_{n+1}(u), e_j(v)\big] = 0, \qquad \big[h_{n+1}(u), f_j(v)\big] = 0. \tag{5.45}$$

For $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ *we have*

$$\begin{aligned}[h_{n+1}(u), e_n(v)] = {} & \frac{1}{2(u-v)} h_{n+1}(u)\big(e_n(u) - e_n(v)\big) \\ & - \frac{1}{2(u-v-1)}\big(e_n(u-1) - e_n(v)\big) h_{n+1}(u)\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}[h_{n+1}(u), f_n(v)] = {} & -\frac{1}{2(u-v)} h_{n+1}(u)\big(f_n(u) - f_n(v)\big) \\ & + \frac{1}{2(u-v-1)}\big(f_n(u-1) - f_n(v)\big) h_{n+1}(u),\end{aligned} \tag{5.47}$$

whereas for $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ *and* $\mathfrak{o}_{2n}$ *we have*

$$\big[h_{n+1}(u), e_n(v)\big] = (\epsilon_n, \alpha_n)\, \frac{h_{n+1}(u)\big(e_n(u) - e_n(v)\big)}{u - v} \tag{5.48}$$

and

$$\big[h_{n+1}(u), f_n(v)\big] = -(\epsilon_n, \alpha_n)\, \frac{\big(f_n(u) - f_n(v)\big) h_{n+1}(u)}{u - v}. \tag{5.49}$$

Moreover, for $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ we have

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = 0 \qquad \textit{and} \qquad \big[h_{n+1}(u), f_{n-1}(v)\big] = 0, \tag{5.50}$$

for $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ we have

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = \frac{h_{n+1}(u)\big(e_{n-1}(v) - e_{n-1}(u+2)\big)}{u - v + 2} \qquad \textit{and} \tag{5.51}$$
$$\big[h_{n+1}(u), f_{n-1}(v)\big] = -\frac{\big(f_{n-1}(v) - f_{n-1}(u+2)\big)\, h_{n+1}(u)}{u - v + 2}$$

and for $\mathfrak{g}_N = \mathfrak{o}_{2n}$ we have

$$\big[h_{n+1}(u), e_{n-1}(v)\big] = \frac{h_{n+1}(u)\,\big(e_{n-1}(v) - e_{n-1}(u)\big)}{u - v} \qquad \textit{and} \tag{5.52}$$
$$\big[h_{n+1}(u), f_{n-1}(v)\big] = -\frac{\big(f_{n-1}(v) - f_{n-1}(u)\big)\, h_{n+1}(u)}{u - v}.$$

In all three cases we have

$$\big[e_i(u), e_i(v)\big] = \frac{(\alpha_i, \alpha_i)}{2} \frac{(e_i(u) - e_i(v))^2}{u - v} \qquad \textit{and} \tag{5.53}$$

$$\big[f_i(u), f_i(v)\big] = -\frac{(\alpha_i, \alpha_i)}{2} \frac{(f_i(u) - f_i(v))^2}{u - v}. \tag{5.54}$$

Furthermore,

$$\big[e_i(u), e_j(v)\big] = 0 \qquad \textit{and} \qquad \big[f_i(u), f_j(v)\big] = 0 \qquad \textit{if} \quad (\alpha_i, \alpha_j) = 0, \tag{5.55}$$

whereas for $i \neq j$ we have

$$u\big[e_i^{\circ}(u), e_j(v)\big] - v\big[e_i(u), e_j^{\circ}(v)\big] = -(\alpha_i, \alpha_j)\, e_i(u)\, e_j(v) \qquad \textit{and} \tag{5.56}$$

$$u\big[f_i^{\circ}(u), f_j(v)\big] - v\big[f_i(u), f_j^{\circ}(v)\big] = (\alpha_i, \alpha_j)\, f_j(v)\, f_i(u). \tag{5.57}$$

Finally, for $i \neq j$ we have the Serre relations

$$\sum_{p \in \mathfrak{S}_m} \big[e_i(u_{p(1)}), \big[e_i(u_{p(2)}), \dots, \big[e_i(u_{p(m)}), e_j(v)\big] \dots\big]\big] = 0 \qquad \textit{and} \tag{5.58}$$

$$\sum_{p \in \mathfrak{S}_m} \big[f_i(u_{p(1)}), \big[f_i(u_{p(2)}), \dots, \big[f_i(u_{p(m)}), f_j(v)\big] \dots\big]\big] = 0, \tag{5.59}$$

where $m = 1 - a_{ij}$.

Proof. Apart from the Serre relations where i or j takes the value n, all the relations are satisfied in the algebra $\mathrm{X}(\mathfrak{g}_N)$ due to Propositions 5.4, 5.5, 5.10, 5.11 and 5.13. To derive (5.58) and (5.59) we use a theorem of Levendorskiĭ [12] which provides a simplified presentation of the Drinfeld Yangian $\mathrm{Y}^D(\mathfrak{g})$; see also [8]. In particular, the theorem implies that the Serre relations (1.1) in the definition of $\mathrm{Y}^D(\mathfrak{g})$ (see Section 1) can be replaced by their level zero case $r_1 = \cdots = r_m = s = 0$. As we demonstrate in Section 6 below, the defining relations of $\mathrm{Y}^D(\mathfrak{g}_N)$ are satisfied by the elements of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ which are defined by the respective coefficients of the series $\kappa_i(u)$, $\xi_i^{\pm}(u)$ in Section 1. This includes the level zero case of (1.1) which is implied by the embedding $\mathrm{U}(\mathfrak{g}_N) \hookrightarrow \mathrm{X}(\mathfrak{g}_N)$; see [2, Proposition 3.11]. Hence, by [12], relations (1.1) hold for the coefficients of the series $\xi_i^{\pm}(u)$. However, the series $\xi_i^-(u)$ and $\xi_i^+(u)$ coincide with $e_i(u)$ and $f_i(u)$, respectively, up to a shift of the variable u by a constant. This shift does not affect the Serre relations, and so we can conclude that (5.58) and (5.59) hold as well for all $i \neq j$.

Now consider the algebra $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ with generators and relations as in the statement of the theorem. The above argument shows that there is a homomorphism

$$\widehat{\mathrm{X}}(\mathfrak{g}_N) \to \mathrm{X}(\mathfrak{g}_N) \tag{5.60}$$

which takes the generators $h_i^{(r)}$, $e_i^{(r)}$ and $f_i^{(r)}$ of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ to the elements of $\mathrm{X}(\mathfrak{g}_N)$ denoted by the same symbols. We need to demonstrate that this homomorphism is surjective and injective. To prove the surjectivity we need a lemma.

Lemma 5.15. *For all $1 \leqslant i < j \leqslant n$ in the algebra $\mathrm{X}(\mathfrak{o}_{2n+1})$ we have*

$$\begin{aligned} e_{i\,j+1}^{(r)} &= \big[e_{i\,j}^{(r)}, e_j^{(1)}\big], & f_{j+1\,i}^{(r)} &= \big[f_j^{(1)}, f_{j\,i}^{(r)}\big],\\ e_{i\,j'}^{(r)} &= -\big[e_{i\,j'-1}^{(r)}, e_j^{(1)}\big], & f_{j'\,i}^{(r)} &= -\big[f_j^{(1)}, f_{j'-1\,i}^{(r)}\big]. \end{aligned}$$

For all $1 \leqslant i < j \leqslant n-1$ in the algebra $\mathrm{X}(\mathfrak{sp}_{2n})$ we have

$$\begin{aligned} e_{i\,j+1}^{(r)} &= \big[e_{i\,j}^{(r)}, e_j^{(1)}\big], & f_{j+1\,i}^{(r)} &= \big[f_j^{(1)}, f_{j\,i}^{(r)}\big],\\ e_{i\,j'}^{(r)} &= -\big[e_{i\,j'-1}^{(r)}, e_j^{(1)}\big], & f_{j'\,i}^{(r)} &= -\big[f_j^{(1)}, f_{j'-1\,i}^{(r)}\big]. \end{aligned}$$

Moreover, for $1 \leqslant i \leqslant n-1$ we have

$$\begin{aligned} e_{i\,n'}^{(r)} &= \frac{1}{2}\,\big[e_{i\,n'-1}^{(r)}, e_n^{(1)}\big], & f_{n'\,i}^{(r)} &= \frac{1}{2}\,\big[f_n^{(1)}, f_{n'-1\,i}^{(r)}\big],\\ e_{i\,i'}^{(r)} &= -\big[e_{i\,i'-1}^{(r)}, e_i^{(1)}\big], & f_{i'\,i}^{(r)} &= -\big[f_i^{(1)}, f_{i'-1\,i}^{(r)}\big]. \end{aligned}$$

For all $1 \leqslant i < j \leqslant n-1$ in the algebra $\mathrm{X}(\mathfrak{o}_{2n})$ we have

$$\begin{aligned} e_{i\,j+1}^{(r)} &= \big[e_{i\,j}^{(r)}, e_j^{(1)}\big], & f_{j+1\,i}^{(r)} &= \big[f_j^{(1)}, f_{j\,i}^{(r)}\big],\\ e_{i\,j'}^{(r)} &= -\big[e_{i\,j'-1}^{(r)}, e_j^{(1)}\big], & f_{j'\,i}^{(r)} &= -\big[f_j^{(1)}, f_{j'-1\,i}^{(r)}\big], \end{aligned}$$

and for $1 \leqslant i \leqslant n-2$ we have

$$e_{i\,n'}^{(r)} = -\big[e_{i\,n'-1}^{(r)}, e_n^{(1)}\big], \qquad f_{n'\,i}^{(r)} = -\big[f_n^{(1)}, f_{n'-1\,i}^{(r)}\big].$$

Proof. All relations follow easily from the Gauss decomposition (1.4) and defining relations (2.18). To illustrate, consider the case $\mathfrak{g}_N = \mathfrak{sp}_{2n}$. By taking the coefficients of v^{-1} on both sides of (2.18), for $1 < j \leqslant n-1$ we get $[t_{1j}(u), t^{(1)}_{j\,j+1}] = t_{1\,j+1}(u)$. Writing this in terms of the Gaussian generators we come to the relation $h_1(u)[e_{1j}(u), e^{(1)}_j] = h_1(u)e_{1\,j+1}(u)$, which gives $[e_{1j}(u), e^{(1)}_j] = e_{1\,j+1}(u)$ so that $e^{(r)}_{1\,j+1} = [e^{(r)}_{1\,j},\, e^{(1)}_j]$.

By a similar argument, for $1 \leqslant j \leqslant n-1$ we find that $e^{(r)}_{1\,j'} = [e^{(r)}_{1\,j},\, e^{(1)}_{(j+1)'\,j'}]$. Now apply Proposition 5.7 to write this as $e^{(r)}_{1\,j'} = -[e^{(r)}_{1\,j},\, e^{(1)}_j]$. The remaining cases with $i = 1$ are treated in the same way. The extension to arbitrary values of i follows by the application of Corollary 4.2. □

By Lemma 5.15, all elements $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ with $r \geqslant 1$ and the conditions $i < j$ and $i < j'$ in the orthogonal case, and $i < j$ and $i \leqslant j'$ in the symplectic case, belong to the subalgebra $\widetilde{\mathrm{X}}(\mathfrak{gl}_N)$ of $\mathrm{X}(\mathfrak{g}_N)$ generated by the coefficients of the series $h_i(u)$ with $i = 1, \dots, n+1$, and $e_i(u)$, $f_i(u)$ with $i = 1, \dots, n$. Hence, the Gauss decomposition (1.4) implies that all coefficients of the series $t_{ij}(u)$ with the same respective conditions on the indices i and j also belong to the subalgebra $\widetilde{\mathrm{X}}(\mathfrak{gl}_N)$. Furthermore, by Theorem 5.8, all coefficients of the series $z_N(u)$ are also in $\widetilde{\mathrm{X}}(\mathfrak{gl}_N)$. Finally, taking the coefficients of u^{-r} for $r = 1, 2, \dots$ in (2.15) and using induction on r, we conclude that the coefficients of all series $t_{ij}(u)$ belong to $\widetilde{\mathrm{X}}(\mathfrak{gl}_N)$ so that $\widetilde{\mathrm{X}}(\mathfrak{gl}_N) = \mathrm{X}(\mathfrak{gl}_N)$. This proves that the homomorphism (5.60) is surjective.

In the rest of the proof we will show that this homomorphism is injective. We will follow the argument of [3] dealing with type A, and adapt it to the orthogonal and symplectic Lie algebras. As a first step, observe that the set of monomials in the generators $h^{(r)}_i$ with $i = 1, \dots, n+1$ and $r \geqslant 1$, and $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ with $r \geqslant 1$ and the conditions $i < j$ and $i < j'$ in the orthogonal case, and $i < j$ and $i \leqslant j'$ in the symplectic case, taken in some fixed order, is linearly independent in the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$. Indeed, under the isomorphism (3.14), the images of the elements $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ in the $(r-1)$-th component of the graded algebra $\mathrm{gr}\,\mathrm{X}(\mathfrak{g}_N)$ respectively correspond to $F_{ij}\,x^{r-1}$ and $F_{ji}\,x^{r-1}$. Similarly, the image of $h^{(r)}_i$ correspond to $F_{ii}\,x^{r-1} + \zeta_r/2$ for $i = 1, \dots, n$, while for the image of $h^{(r)}_{n+1}$ we have

$$\bar{h}^{(r)}_{n+1} \mapsto \begin{cases} \zeta_r/2 & \text{for} \quad \mathfrak{o}_{2n+1} \\ -F_{n\,n}\,x^{r-1} + \zeta_r/2 & \text{for} \quad \mathfrak{sp}_{2n} \\ -F_{n-1\,n-1}\,x^{r-1} - F_{n\,n}\,x^{r-1} + \zeta_r/2 & \text{for} \quad \mathfrak{o}_{2n}, \end{cases}$$

which follows from (3.15) and Theorem 5.8. Hence the claim is implied by the Poincaré–Birkhoff–Witt theorem for $\mathrm{U}(\mathfrak{g}_N[x])$.

Define elements $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ inductively as follows. For $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ set $e^{(r)}_{i\,i+1} = e^{(r)}_i$ and $f^{(r)}_{i+1\,i} = f^{(r)}_i$, and

$$\begin{aligned} e^{(r)}_{i\,j+1} &= [e^{(r)}_{i\,j},\, e^{(1)}_j], & f^{(r)}_{j+1\,i} &= [f^{(1)}_j,\, f^{(r)}_{j\,i}], \\ e^{(r)}_{i\,j'} &= -[e^{(r)}_{i\,j'-1},\, e^{(1)}_j], & f^{(r)}_{j'\,i} &= -[f^{(1)}_j,\, f^{(r)}_{j'-1\,i}], \end{aligned} \tag{5.61}$$

for $1 \leqslant i < j \leqslant n$. For $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ set $e^{(r)}_{i\,i+1} = e^{(r)}_i$ and $f^{(r)}_{i+1\,i} = f^{(r)}_i$, and

$$\begin{aligned} e^{(r)}_{i\,j+1} &= [e^{(r)}_{i\,j}, e^{(1)}_j], & f^{(r)}_{j+1\,i} &= [f^{(1)}_j, f^{(r)}_{j\,i}], \\ e^{(r)}_{i\,j'} &= -[e^{(r)}_{i\,j'-1}, e^{(1)}_j], & f^{(r)}_{j'\,i} &= -[f^{(1)}_j, f^{(r)}_{j'-1\,i}], \end{aligned} \tag{5.62}$$

for $1 \leqslant i < j \leqslant n-1$. Furthermore, set $e^{(r)}_{n\,n'} = e^{(r)}_n$ and $f^{(r)}_{n'\,n} = f^{(r)}_n$, and

$$\begin{aligned} e^{(r)}_{i\,n'} &= \frac{1}{2}\,[e^{(r)}_{i\,n'-1}, e^{(1)}_n], & f^{(r)}_{n'\,i} &= \frac{1}{2}\,[f^{(1)}_n, f^{(r)}_{n'-1\,i}], \\ e^{(r)}_{i\,i'} &= -[e^{(r)}_{i\,i'-1}, e^{(1)}_i], & f^{(r)}_{i'\,i} &= -[f^{(1)}_i, f^{(r)}_{i'-1\,i}], \end{aligned} \tag{5.63}$$

for $1 \leqslant i \leqslant n-1$. For $\mathfrak{g}_N = \mathfrak{o}_{2n}$ set $e^{(r)}_{i\,i+1} = e^{(r)}_i$ and $f^{(r)}_{i+1\,i} = f^{(r)}_i$, and

$$\begin{aligned} e^{(r)}_{i\,j+1} &= [e^{(r)}_{i\,j}, e^{(1)}_j], & f^{(r)}_{j+1\,i} &= [f^{(1)}_j, f^{(r)}_{j\,i}], \\ e^{(r)}_{i\,j'} &= -[e^{(r)}_{i\,j'-1}, e^{(1)}_j], & f^{(r)}_{j'\,i} &= -[f^{(1)}_j, f^{(r)}_{j'-1\,i}], \end{aligned} \tag{5.64}$$

for $1 \leqslant i < j \leqslant n-1$. Furthermore, set $e^{(r)}_{n-1\,n'} = e^{(r)}_n$ and $f^{(r)}_{n'\,n-1} = f^{(r)}_n$, and

$$e^{(r)}_{i\,n'} = -[e^{(r)}_{i\,n'-1}, e^{(1)}_n], \qquad f^{(r)}_{n'\,i} = -[f^{(1)}_n, f^{(r)}_{n'-1\,i}] \tag{5.65}$$

for $1 \leqslant i \leqslant n-2$.

By Lemma 5.15, these definitions are consistent with those of the elements of the algebra $\mathrm{X}(\mathfrak{g}_N)$ in the sense that the images of the elements $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ of the algebra $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ under the homomorphism (5.60) coincide with the elements of $\mathrm{X}(\mathfrak{g}_N)$ with the same name.

The injectivity property of the homomorphism (5.60) will follow if we prove that the algebra $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ is spanned by monomials in $h^{(r)}_i$, $e^{(r)}_{ij}$ and $f^{(r)}_{ji}$ taken in some fixed order. Denote by $\widehat{\mathcal{E}}$, $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{H}}$ the subalgebras of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ respectively generated by all elements of the form $e^{(r)}_i$, $f^{(r)}_i$ and $h^{(r)}_i$. Define an ascending filtration on $\widehat{\mathcal{E}}$ by setting $\deg e^{(r)}_i = r-1$. Denote by $\operatorname{gr}\widehat{\mathcal{E}}$ the corresponding graded algebra. Let $\bar{e}^{(r)}_{ij}$ be the image of $e^{(r)}_{ij}$ in the $(r-1)$-th component of the graded algebra $\operatorname{gr}\widehat{\mathcal{E}}$. Extend the range of subscripts of $\bar{e}^{(r)}_{ij}$ to all values $1 \leqslant i < j \leqslant 1'$ by using the skew-symmetry conditions

$$\bar{e}^{(r)}_{ij} = -\theta_{ij}\bar{e}^{(r)}_{j'\,i'}. \tag{5.66}$$

The desired spanning property of the monomials in the $e^{(r)}_{ij}$ clearly follows from the relations

$$[\bar{e}^{(r)}_{ij}, \bar{e}^{(s)}_{kl}] = \delta_{kj}\,\bar{e}^{(r+s-1)}_{il} - \delta_{il}\,\bar{e}^{(r+s-1)}_{kj} - \theta_{ij}\,\delta_{ki'}\,\bar{e}^{(r+s-1)}_{j'l} + \theta_{ij}\,\delta_{j'l}\,\bar{e}^{(r+s-1)}_{ki'}. \tag{5.67}$$

We will be verifying these relations separately for each of the three cases.

Type B_n. If $i,j,k,l \in \{1,\dots,n\}$, then (5.67) are essentially type A relations and they were already verified in [3]. We will often use these particular cases of (5.67) in the arguments below.

By relation (5.56) (in the algebra $\widehat{\mathrm{X}}(\mathfrak{o}_{2n+1})$) we have $[\bar e^{(r)}_{i\,i+1}, \bar e^{(s)}_{n\,n+1}] = \delta_{i+1,n}\,\bar e^{(r+s-1)}_{i\,n+1}$. For $i<j<n$, using the definition (5.61) we obtain

$$[\bar e^{(r)}_{i\,j}, \bar e^{(s)}_{n\,n+1}] = \big[[\bar e^{(r)}_{i\,j-1}, \bar e^{(1)}_{j-1\,j}], \bar e^{(s)}_{n\,n+1}\big]$$
$$= \big[\bar e^{(r)}_{i\,j-1}, [\bar e^{(1)}_{j-1\,j}, \bar e^{(s)}_{n\,n+1}]\big] + \big[\bar e^{(1)}_{j-1\,j}, [\bar e^{(s)}_{n\,n+1}, \bar e^{(r)}_{i\,j-1}]\big] = \big[\bar e^{(1)}_{j-1\,j}, [\bar e^{(s)}_{n\,n+1}, \bar e^{(r)}_{i\,j-1}]\big].$$

Hence, an obvious induction gives $[\bar e^{(r)}_{i\,j}, \bar e^{(s)}_{n\,n+1}] = 0$ for $i<j<n$. Now we verify $[\bar e^{(r)}_{i\,n}, \bar e^{(s)}_{n\,n+1}] = \bar e^{(r+s-1)}_{i\,n+1}$. Using (5.61), we get

$$[\bar e^{(r)}_{i\,n}, \bar e^{(s)}_{n\,n+1}] = \big[[\bar e^{(r)}_{i\,n-1}, \bar e^{(1)}_{n-1\,n}], \bar e^{(s)}_{n\,n+1}\big]$$
$$= \big[\bar e^{(r)}_{i\,n-1}, [\bar e^{(1)}_{n-1\,n}, \bar e^{(s)}_{n\,n+1}]\big] + \big[\bar e^{(1)}_{n-1\,n}, [\bar e^{(s)}_{n\,n+1}, \bar e^{(r)}_{i\,n-1}]\big] = [\bar e^{(r)}_{i\,n-1}, \bar e^{(s)}_{n-1\,n+1}],$$

where the last equality holds by $[\bar e^{(r)}_{i\,n-1}, \bar e^{(s)}_{n\,n+1}] = 0$ and $[\bar e^{(s)}_{n-1\,n}, \bar e^{(1)}_{n\,n+1}] = \bar e^{(s)}_{n-1\,n+1}$. Furthermore, $\bar e^{(s)}_{n-1\,n+1} = [\bar e^{(s)}_{n-1\,n}, \bar e^{(1)}_{n\,n+1}]$ by (5.61), and so

$$[\bar e^{(r)}_{i\,n}, \bar e^{(s)}_{n\,n+1}] = \big[\bar e^{(r)}_{i\,n-1}, [\bar e^{(s)}_{n-1\,n}, \bar e^{(1)}_{n\,n+1}]\big]$$
$$= \big[\bar e^{(s)}_{n-1\,n}, [\bar e^{(r)}_{i\,n-1}, \bar e^{(1)}_{n\,n+1}]\big] + \big[\bar e^{(1)}_{n\,n+1}, [\bar e^{(s)}_{n-1\,n}, \bar e^{(r)}_{i\,n-1}]\big] = -[\bar e^{(1)}_{n\,n+1}, \bar e^{(r+s-1)}_{i\,n}] = \bar e^{(r+s-1)}_{i\,n+1}.$$

Thus, we have verified that $[\bar e^{(r)}_{i\,j}, \bar e^{(s)}_{n\,n+1}] = \delta_{jn}\,\bar e^{(r+s-1)}_{i\,n+1}$ for $1 \leqslant i < j \leqslant n$.

Next we will check

$$[\bar e^{(r)}_{i\,j}, \bar e^{(s)}_{k\,n+1}] = \delta_{jk}\,\bar e^{(r+s-1)}_{i\,n+1} \tag{5.68}$$

for $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < n$. Suppose first that $1 \leqslant i < j < n$. We have

$$[\bar e^{(r)}_{i\,j}, \bar e^{(s)}_{k\,n+1}] = \big[\bar e^{(r)}_{i\,j}, [\bar e^{(s)}_{k\,n}, \bar e^{(1)}_{n\,n+1}]\big]$$
$$= \big[\bar e^{(s)}_{k\,n}, [\bar e^{(r)}_{i\,j}, \bar e^{(1)}_{n\,n+1}]\big] + \big[\bar e^{(1)}_{n\,n+1}, [\bar e^{(s)}_{k\,n}, \bar e^{(r)}_{i\,j}]\big] = -\delta_{kj}\,[\bar e^{(1)}_{n\,n+1}, \bar e^{(r+s-1)}_{i\,n}] = \delta_{kj}\,\bar e^{(r+s-1)}_{i\,n+1}.$$

Now let $1 \leqslant i < j = n$. Note first that

$$[\bar e^{(r)}_{n-1\,n}, \bar e^{(s)}_{n-1\,n+1}] = \big[\bar e^{(r)}_{n-1\,n}, [\bar e^{(1)}_{n-1\,n}, \bar e^{(s)}_{n\,n+1}]\big] = \big[\bar e^{(1)}_{n-1\,n}, [\bar e^{(r)}_{n-1\,n}, \bar e^{(s)}_{n\,n+1}]\big]$$
$$+ \big[\bar e^{(s)}_{n\,n+1}, [\bar e^{(1)}_{n-1\,n}, \bar e^{(r)}_{n-1\,n}]\big] = \big[\bar e^{(1)}_{n-1\,n}, [\bar e^{(r)}_{n-1\,n}, \bar e^{(s)}_{n\,n+1}]\big] = \big[\bar e^{(1)}_{n-1\,n}, [\bar e^{(1)}_{n-1\,n}, \bar e^{(r+s)}_{n\,n+1}]\big] = 0,$$

where the last equality holds by the Serre relations (5.58) with $a_{n-1\,n} = -1$. As a next step, verify $[\bar e^{(r)}_{n-2\,n}, \bar e^{(s)}_{n-1\,n+1}] = 0$. Indeed, we have

$$[\bar e^{(r)}_{n-2\,n}, \bar e^{(s)}_{n-1\,n+1}] = \big[[\bar e^{(r)}_{n-2\,n-1}, \bar e^{(1)}_{n-1\,n}], [\bar e^{(1)}_{n-1\,n}, \bar e^{(s)}_{n\,n+1}]\big]$$
$$= \big[\bar e^{(1)}_{n-1\,n}, [[\bar e^{(r)}_{n-2\,n-1}, \bar e^{(1)}_{n-1\,n}], \bar e^{(s)}_{n\,n+1}]\big] + \big[\bar e^{(s)}_{n\,n+1}, [\bar e^{(1)}_{n-1\,n}, [\bar e^{(r)}_{n-2\,n-1}, \bar e^{(1)}_{n-1\,n}]]\big].$$

Since the second term is zero, this equals

$$\big[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(s)}_{n\,n+1}, \bar{e}^{(r)}_{n-2\,n-1}]]\big] + \big[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(r)}_{n-2\,n-1}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n+1}]]\big],$$

where the first term is zero, so the expression equals

$$\big[[\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n+1}], [\bar{e}^{(r)}_{n-2\,n-1}, \bar{e}^{(1)}_{n-1\,n}]\big] + \big[\bar{e}^{(r)}_{n-2\,n-1}, [\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n+1}]]\big] = [\bar{e}^{(s)}_{n-1\,n+1}, \bar{e}^{(r)}_{n-2\,n}] = 0.$$

Furthermore, for $k < n-1$ write the commutator $[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,n+1}]$ as

$$\begin{aligned}\big[\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(1)}_{k\,k+1}, \bar{e}^{(s)}_{k+1\,n+1}]\big] &= \big[\bar{e}^{(1)}_{k\,k+1}, [\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k+1\,n+1}]\big] + \big[\bar{e}^{(s)}_{k+1\,n+1}, [\bar{e}^{(1)}_{k\,k+1}, \bar{e}^{(r)}_{n-1\,n}]\big] \\ &= \big[\bar{e}^{(1)}_{k\,k+1}, [\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k+1\,n+1}]\big] + \delta_{k\,n-2}\,[\bar{e}^{(s)}_{n-1\,n+1}, \bar{e}^{(r)}_{n-2\,n}] = \big[\bar{e}^{(1)}_{k\,k+1}, [\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k+1\,n+1}]\big].\end{aligned}$$

Hence, by induction, the relation $[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,n+1}] = 0$ holds for $k \leqslant n-1$. Finally,

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n+1}] = \big[[\bar{e}^{(r)}_{i\,n-1}, \bar{e}^{(1)}_{n-1\,n}], \bar{e}^{(s)}_{k\,n+1}\big] = \big[\bar{e}^{(r)}_{i\,n-1}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{k\,n+1}]\big] + \big[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(s)}_{k\,n+1}, \bar{e}^{(r)}_{i\,n-1}]\big] = 0,$$

which completes the verification of (5.68).

For the next case of (5.67) to verify we take the relations

$$[\bar{e}^{(r)}_{i\,n+1}, \bar{e}^{(s)}_{k\,n+1}] = \bar{e}^{(r+s-1)}_{k\,i'}. \tag{5.69}$$

We may assume that $i \geqslant k$. If $i = k = n$ then the relations follow from (5.53). In its turn, this implies (5.69) for $i = n$ and arbitrary k by an argument similar to the one used in the proof of (5.68). Furthermore, a reverse induction on i, beginning with $i = n$ shows that (5.69) holds for $i > k$. For the remaining case of (5.69) with $i = k < n$ use induction on i starting with $i = n$. For $i < n$ we have

$$\begin{aligned}[\bar{e}^{(r)}_{in+1}, \bar{e}^{(s)}_{in+1}] &= \big[[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(r)}_{i+1\,n+1}], [\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n+1}]\big] \\ &= \big[[\bar{e}^{(1)}_{i\,i+1}, [\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n+1}]], \bar{e}^{(r)}_{i+1\,n+1}\big] + \big[\bar{e}^{(1)}_{i\,i+1}, [\bar{e}^{(r)}_{i+1\,n+1}, [\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n+1}]]\big].\end{aligned}$$

The first term vanishes, while by the induction hypothesis, the second term equals

$$\big[\bar{e}^{(1)}_{i\,i+1}, [[\bar{e}^{(r)}_{i+1\,n+1}, \bar{e}^{(1)}_{i\,i+1}], \bar{e}^{(s)}_{i+1\,n+1}]\big] = \big[[\bar{e}^{(r)}_{i+1\,n+1}, \bar{e}^{(1)}_{i\,i+1}], [\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n+1}]\big] = -[\bar{e}^{(r)}_{in+1}, \bar{e}^{(s)}_{in+1}]$$

so that $[\bar{e}^{(r)}_{in+1}, \bar{e}^{(s)}_{in+1}] = 0$, completing the proof of (5.69). As its consequence, we derive a more general relation

$$[\bar{e}^{(r)}_{ij}, \bar{e}^{(s)}_{k\,l'}] = \delta_{kj}\,\bar{e}^{(r+s-1)}_{i\,l'} + \delta_{jl}\,\bar{e}^{(r+s-1)}_{k\,i'} \tag{5.70}$$

which holds for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < l \leqslant n$. Indeed, using (5.69) we get

$$\begin{aligned}[\bar{e}^{(r)}_{ij}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(r)}_{ij}, [\bar{e}^{(1)}_{l\,n+1}, \bar{e}^{(s)}_{k\,n+1}]\big] &= \big[\bar{e}^{(1)}_{l\,n+1}, [\bar{e}^{(r)}_{ij}, \bar{e}^{(s)}_{k\,n+1}]\big] \\ &+ \big[\bar{e}^{(s)}_{k\,n+1}, [\bar{e}^{(1)}_{l\,n+1}, \bar{e}^{(r)}_{ij}]\big] = [\bar{e}^{(1)}_{l\,n+1}, \delta_{kj}\,\bar{e}^{(r+s-1)}_{i\,n+1}] + [\delta_{jl}\,\bar{e}^{(r)}_{i\,n+1}, \bar{e}^{(s)}_{k\,n+1}]\end{aligned}$$

which equals the right hand side of (5.70). Our next goal is to verify the relations

$$[\bar{e}^{(r)}_{i\,n+1}, \bar{e}^{(s)}_{k\,l'}] = 0 \tag{5.71}$$

for all admissible i, k, l with $k < l \leqslant n$. We begin with the particular case

$$[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{k\,n'}] = 0 \tag{5.72}$$

which we check by a reverse induction on k. Using the previously checked cases of (5.67), for $k = n-1$ we obtain

$$\begin{aligned}[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n'}] &= \big[\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n+1}]\big] \\ &= \big[\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(s)}_{n-1\,n}, \bar{e}^{(1)}_{n\,n+1}]]\big] = -\big[\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n}]]\big].\end{aligned}$$

Now apply the Serre relations (5.58) with $a_{n\,n-1} = -2$ to write this commutator as

$$\big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n}]]\big] + \big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n}]]\big].$$

Hence,

$$\begin{aligned}[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n'}] &= 2\big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n}]]\big] + \big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(s)}_{n-1\,n}, [\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(1)}_{n\,n+1}]]\big] \\ &= -2\big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n+1}]\big] = -2[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{n-1\,n'}] = 0\end{aligned}$$

establishing the induction base. The induction step is a straightforward application of (5.68), which completes the proof of (5.72). Now assume that $i = n$ in (5.71) to show that

$$[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{k\,l'}] = 0 \tag{5.73}$$

for all $k < l \leqslant n$. By writing

$$[\bar{e}^{(r)}_{n\,n+1}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(r)}_{n\,n+1}, [\bar{e}^{(1)}_{l\,n+1}, \bar{e}^{(s)}_{k\,n+1}]\big]$$

and relying on the already verified cases of (5.67), we reduce checking of (5.73) to the particular case $r = 1$ where we may also assume $l \leqslant n-1$. Proceed by

$$\big[\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{k\,l'}\big] = \big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(1)}_{l\,n}, \bar{e}^{(s)}_{k\,n'}]\big] = \big[\bar{e}^{(1)}_{l\,n}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{k\,n'}]\big] + \big[\bar{e}^{(s)}_{k\,n'}, [\bar{e}^{(1)}_{l\,n}, \bar{e}^{(1)}_{n\,n+1}]\big]$$

and use (5.72) to see that the expression is zero. Now (5.71) follows from (5.70) and (5.73):

$$[\bar{e}^{(r)}_{i\,n+1}, \bar{e}^{(s)}_{k\,l'}] = \big[[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(1)}_{n\,n+1}], \bar{e}^{(s)}_{k\,l'}\big] = \big[\bar{e}^{(1)}_{n\,n+1}, [\bar{e}^{(s)}_{k\,l'}, \bar{e}^{(r)}_{i\,n}]\big] + \big[\bar{e}^{(r)}_{i\,n}, [\bar{e}^{(1)}_{n\,n+1}, \bar{e}^{(s)}_{k\,l'}]\big] = 0.$$

The proof of (5.67) will be completed by checking that $[\bar{e}^{(r)}_{i\,j'}, \bar{e}^{(s)}_{k\,l'}] = 0$ for all admissible values with $i < j$ and $k < l$. This relation follows from (5.71) by

$$[\bar{e}^{(r)}_{i\,j'}, \bar{e}^{(s)}_{k\,l'}] = \big[[\bar{e}^{(r)}_{j\,n+1}, \bar{e}^{(1)}_{i\,n+1}], \bar{e}^{(s)}_{k\,l'}\big] = \big[\bar{e}^{(r)}_{j\,n+1}, [\bar{e}^{(1)}_{i\,n+1}, \bar{e}^{(s)}_{k\,l'}]\big] + \big[\bar{e}^{(1)}_{i\,n+1}, [\bar{e}^{(s)}_{k\,l'}, \bar{e}^{(r)}_{j\,n+1}]\big] = 0.$$

Type C_n. We will now verify (5.67) for $\mathfrak{g}_N = \mathfrak{sp}_{2n}$, where the arguments are quite similar to type B_n. We will outline the sequence of steps. By (5.55), we have $[\bar{e}^{(r)}_{i\,i+1}, \bar{e}^{(s)}_{n\,n'}] = 0$ for $i < n-1$. This implies

$$[\bar{e}^{(r)}_{i\,j}, \bar{e}^{(s)}_{n\,n'}] = 0 \tag{5.74}$$

for $i < j < n$ by an easy induction. Using (5.56) and the definition (5.63), we derive

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n'}] = \bar{e}^{(r+s-1)}_{n-1\,n'} + \bar{e}^{(r+s-1)}_{n\,(n-1)'} \tag{5.75}$$

which then implies a more general relation

$$\big[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{n\,n'}\big] = 2\,\bar{e}^{(r+s-1)}_{i\,n'} = \bar{e}^{(r+s-1)}_{i\,n'} + \bar{e}^{(r+s-1)}_{n\,i'} \tag{5.76}$$

for $i < n-1$. Using (5.63) we extend it further to

$$\big[\bar{e}^{(r)}_{i\,j}, \bar{e}^{(s)}_{k\,n'}\big] = \delta_{kj}\bar{e}^{(r+s-1)}_{i\,n'}, \tag{5.77}$$

for all $i < j < n$ and $k < n$, and then apply (5.75) to check

$$\big[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,n'}\big] = \bar{e}^{(r+s-1)}_{k\,(n-1)'}, \tag{5.78}$$

for $k < n$. Next, we verify

$$\big[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}\big] = \bar{e}^{(r+s-1)}_{k\,i'}, \tag{5.79}$$

for $k \leqslant n-1$ and $i < n-1$. We use the reverse induction on i, beginning with $i = n-1$ to show first that this relation holds for $i \geqslant k$. The induction base is (5.78), while for $i < n-1$ write $\big[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}\big] = \big[[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(r)}_{i+1\,n}], \bar{e}^{(s)}_{k\,n'}\big]$ then proceed by using the definition (5.62) and (5.74). The case (5.79) with $i < k$ now follows by an application of (5.63). Furthermore, assuming now that $i < j \leqslant n$ and $k \leqslant l < n$ we get from (5.79):

$$[\bar{e}^{(r)}_{ij}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(r)}_{ij}, [\bar{e}^{(1)}_{l\,n}, \bar{e}^{(s)}_{k\,n'}]\big] = [\delta_{jl}\,\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}] + [\bar{e}^{(1)}_{l\,n}, \delta_{kj}\,\bar{e}^{(r+s-1)}_{i\,n'}]$$

thus proving

$$[\bar{e}^{(r)}_{i\,j}, \bar{e}^{(s)}_{k\,l'}] = \delta_{kj}\,\bar{e}^{(r+s-1)}_{i\,l'} + \delta_{jl}\,\bar{e}^{(r+s-1)}_{k\,i'}. \tag{5.80}$$

Our next goal is to verify

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,l'}] = 0 \tag{5.81}$$

for all admissible i, k, l with $k \leqslant l \leqslant n-1$. We begin with the particular case

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,(n-1)'}] = 0. \tag{5.82}$$

By (5.79), we have

$$\begin{aligned}[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,(n-1)'}] &= \big[\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,n'}]\big] = \frac{1}{2}\,[\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n'}]]] \\ &= -\frac{1}{2}\,[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n'}]]] - \frac{1}{2}\,[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n'}]]],\end{aligned}$$

where we also used the Serre relation (5.58) with $a_{n-1\,n} = -2$. Hence

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,(n-1)'}] = -\big[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{n\,n'}]]\big] = -2\,[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,(n-1)'}]$$

and so (5.82) follows. By a reverse induction on k we then derive

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,(n-1)'}] = 0 \tag{5.83}$$

for $k < n-1$. This relations extends to all values $k \leqslant l \leqslant n-1$,

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,l'}] = 0. \tag{5.84}$$

The verification of (5.84) reduces to the case $r = 1$ with the use of (5.80), while in this particular case it follows from (5.79), (5.80) and (5.83). Furthermore, by (5.80) and (5.84),

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,l'}] = \big[[\bar{e}^{(r)}_{i\,n-1}, \bar{e}^{(1)}_{n-1\,n}], \bar{e}^{(s)}_{k\,l'}\big] = \big[\bar{e}^{(r)}_{i\,n-1}, [\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(s)}_{k\,l'}]\big] + \big[\bar{e}^{(1)}_{n-1\,n}, [\bar{e}^{(s)}_{k\,l'}, \bar{e}^{(r)}_{i\,n-1}]\big] = 0$$

thus completing the proof of (5.81). As a next case of (5.67), we prove

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,n'}] = 0. \tag{5.85}$$

If $i = k = n$ then this follows from (5.53). If $i = n-1$ and $k = n$ write

$$[\bar{e}^{(r)}_{n-1\,n'}, \bar{e}^{(s)}_{n\,n'}] = \frac{1}{2}\,\big[[\bar{e}^{(1)}_{n-1\,n}, \bar{e}^{(r)}_{n\,n'}], \bar{e}^{(s)}_{n\,n'}\big] = \frac{1}{2}\,\big[\bar{e}^{(s)}_{n\,n'}, [\bar{e}^{(r)}_{n\,n'}, \bar{e}^{(1)}_{n-1\,n}]\big].$$

Applying the Serre relation (5.58) with $a_{n\,n-1} = -1$ we get

$$[\bar{e}^{(r)}_{n-1\,n'}, \bar{e}^{(s)}_{n\,n'}] = -\frac{1}{2}\,\big[\bar{e}^{(r)}_{n\,n'}, [\bar{e}^{(s)}_{n\,n'}, \bar{e}^{(1)}_{n-1\,n}]\big] = -\frac{1}{2}\,\big[\bar{e}^{(s)}_{n\,n'}, [\bar{e}^{(r)}_{n\,n'}, \bar{e}^{(1)}_{n-1\,n}]\big] = \big[\bar{e}^{(s)}_{n\,n'}, \bar{e}^{(r)}_{n-1\,n'}\big]$$

and so (5.85) with $i = n-1$ and $k = n$ follows. Together with (5.74) this implies

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{n\,n'}] = 0 \tag{5.86}$$

for all i. For the remaining values of the indices, (5.85) follows by writing

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,n'}] = \frac{1}{2}\,\big[[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(1)}_{n\,n'}], \bar{e}^{(s)}_{k\,n'}\big]$$

and applying (5.76), (5.79) and (5.86). Furthermore, we will verify

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,l'}] = 0 \tag{5.87}$$

for $k \leqslant l \leqslant n-1$. By the definition (5.62),

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(r)}_{i\,n'}, [\bar{e}^{(1)}_{l\,l+1}, \bar{e}^{(s)}_{k\,(l+1)'}]\big] = \big[\bar{e}^{(1)}_{l\,l+1}, [\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,(l+1)'}]\big] + \big[\bar{e}^{(s)}_{k\,(l+1)'}, [\bar{e}^{(1)}_{l\,l+1}, \bar{e}^{(r)}_{i\,n'}]\big]$$

so that applying (5.77) we get

$$[\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(1)}_{l\,l+1}, [\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,(l+1)'}]\big].$$

Now (5.87) follows by induction on l with the use of (5.85). The last remaining case of (5.67) is $[\bar{e}^{(r)}_{i\,j'}, \bar{e}^{(s)}_{k\,l'}] = 0$ which holds for all $i \leqslant j \leqslant n-1$ and $k \leqslant l \leqslant n-1$. Indeed, we have

$$[\bar{e}^{(r)}_{i\,j'}, \bar{e}^{(s)}_{k\,l'}] = \big[[\bar{e}^{(1)}_{j\,n}, \bar{e}^{(r)}_{i\,n'}], \bar{e}^{(s)}_{k\,l'}\big] = \big[\bar{e}^{(1)}_{j\,n}, [\bar{e}^{(r)}_{i\,n'}, \bar{e}^{(s)}_{k\,l'}]\big] + \big[\bar{e}^{(r)}_{i\,n'}, [\bar{e}^{(s)}_{k\,l'}, \bar{e}^{(1)}_{j\,n}]\big]$$

which is zero by (5.81) and (5.87).

Type D_n. Now let $\mathfrak{g}_N = \mathfrak{o}_{2n}$. The arguments follow the same pattern as for types B_n and C_n above. We will assume throughout the rest of the proof that the indices i, j, k, l run over the set $\{1, \dots, n\}$. By (5.55) we have $[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{n-1\,n'}] = 0$. Therefore, for $i < n-1$, using the definition (5.64) we obtain

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{n-1\,n'}] = \big[[\bar{e}^{(r)}_{i\,n-1}, \bar{e}^{(1)}_{n-1\,n}], \bar{e}^{(s)}_{n-1\,n'}\big] = [\bar{e}^{(r+s-1)}_{i\,n'}, \bar{e}^{(1)}_{n-1\,n}] = -\bar{e}^{(r+s-1)}_{i\,(n-1)'}.$$

Hence, for $k < n-1$ we derive

$$[\bar{e}^{(r)}_{n-1\,n}, \bar{e}^{(s)}_{k\,n'}] = \big[\bar{e}^{(r)}_{n-1\,n}, [\bar{e}^{(s)}_{k\,n-1}, \bar{e}^{(1)}_{n-1\,n'}]\big] = -[\bar{e}^{(r+s-1)}_{k\,n}, \bar{e}^{(1)}_{n-1\,n'}] = \bar{e}^{(r+s-1)}_{k\,(n-1)'}. \tag{5.88}$$

Next we verify that

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}] = \bar{e}^{(r+s-1)}_{k\,i'} \tag{5.89}$$

by the reverse induction on i, beginning with $i = n-1$. Show first that this relation holds for $i > k$ by taking (5.88) as the induction base and using (5.64). To verify (5.89) for $i = k$ write

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{i\,n'}] = \big[[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(r)}_{i+1\,n}], [\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n'}]\big]$$

and proceed by induction. As a next step, we point out the relations

$$\bar{e}^{(r)}_{i\,j'} = [\bar{e}^{(1)}_{i\,i+1},\ \bar{e}^{(r)}_{i+1\,j'}] \tag{5.90}$$

which hold for $j > i+1$. Indeed, they follow by taking consecutive brackets of both sides of the first relation in (5.88) with the elements $\bar{e}^{(1)}_{n-1\,n}$, $\bar{e}^{(1)}_{n-2\,n-1}, \dots, \bar{e}^{(1)}_{j\,j+1}$ with the use of (5.88). Now we check (5.89) for $i \leqslant k$ by induction on $k-i$ taking the case $i = k$ considered above as the induction base. Suppose that $i < k$ and write

$$[\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}] = \big[[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(r)}_{i+1\,n}], \bar{e}^{(s)}_{k\,n'}\big]. \tag{5.91}$$

If $i+1 < k$, then by the induction hypothesis and (5.90) this equals

$$-[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(r+s-1)}_{i+1\,k'}] = -\bar{e}^{(r+s-1)}_{i\,k'}.$$

If $i+1 = k$, then (5.91) equals

$$\big[[\bar{e}^{(1)}_{i\,i+1}, \bar{e}^{(s)}_{i+1\,n'}], \bar{e}^{(r)}_{i+1\,n}\big] = [\bar{e}^{(s)}_{i\,n'}, \bar{e}^{(r)}_{i+1\,n}] = -[\bar{e}^{(r)}_{i+1\,n}, \bar{e}^{(s)}_{i\,n'}] = -\bar{e}^{(r+s-1)}_{i\,(i+1)'}$$

by (5.89) with $i > k$ verified above. Thus, (5.89) holds for all admissible values of i and k.

By employing (5.89) we can derive a more general relation: for $j < n$ we have

$$[\bar{e}^{(r)}_{i\,j}, \bar{e}^{(s)}_{k\,l'}] = \delta_{k\,j}\,\bar{e}^{(r+s-1)}_{i\,l'} + \delta_{j\,l}\,\bar{e}^{(r+s-1)}_{k\,i'}. \tag{5.92}$$

Indeed,

$$[\bar{e}^{(r)}_{i\,j}, \bar{e}^{(s)}_{k\,l'}] = \big[\bar{e}^{(r)}_{i\,j}, [\bar{e}^{(1)}_{l\,n}, \bar{e}^{(s)}_{k\,n'}]\big] = [\delta_{j\,l}\,\bar{e}^{(r)}_{i\,n}, \bar{e}^{(s)}_{k\,n'}] + [\bar{e}^{(1)}_{l\,n}, \delta_{k\,j}\,\bar{e}^{(r+s-1)}_{i\,n'}]$$

which gives (5.92). Our next goal is to verify the relations

$$[\bar{e}_{i\,n}^{(r)}, \bar{e}_{k\,l'}^{(s)}] = 0 \tag{5.93}$$

for all admissible i, k, l with $k < l \leqslant n-1$. We begin with the particular case

$$[\bar{e}_{n-1\,n}^{(r)}, \bar{e}_{k\,(n-1)'}^{(s)}] = 0 \tag{5.94}$$

which we check by reverse induction on k. For $k = n-2$ by (5.89) we have

$$[\bar{e}_{n-1\,n}^{(r)}, \bar{e}_{n-2\,(n-1)'}^{(s)}] = \big[\bar{e}_{n-1\,n}^{(r)}, [\bar{e}_{n-1\,n}^{(1)}, [\bar{e}_{n-2\,n-1}^{(1)}, \bar{e}_{n-1\,n'}^{(s)}]]\big] = -\big[\bar{e}_{n-1\,n}^{(1)}, [\bar{e}_{n-2\,n}^{(r)}, \bar{e}_{n-1\,n'}^{(s)}]\big] = 0.$$

For $k < n-2$ use (5.90) to write

$$[\bar{e}_{n-1\,n}^{(r)}, \bar{e}_{k\,(n-1)'}^{(s)}] = \big[\bar{e}_{n-1\,n}^{(r)}, [\bar{e}_{k\,k+1}^{(1)}, \bar{e}_{k+1\,(n-1)'}^{(s)}]\big]$$

which is zero by the induction hypothesis. This verifies (5.94). We will now extend (5.94) to all values $k < l \leqslant n-1$:

$$[\bar{e}_{n-1\,n}^{(r)}, \bar{e}_{k\,l'}^{(s)}] = 0. \tag{5.95}$$

First, verify it for $r = 1$. Assume that $l \leqslant n-2$. By using (5.92) with $j = l = n-1$ and (5.94) we get

$$\begin{aligned}[\bar{e}_{n-1\,n}^{(1)}, \bar{e}_{k\,l'}^{(s)}] &= \big[\bar{e}_{n-1\,n}^{(1)}, [\bar{e}_{l\,n-1}^{(1)}, \bar{e}_{k\,(n-1)'}^{(s)}]\big] = -[\bar{e}_{l\,n}^{(1)}, \bar{e}_{k\,(n-1)'}^{(s)}]\\ &= -\big[\bar{e}_{l\,n}^{(1)}, [\bar{e}_{n-1\,n}^{(1)}, \bar{e}_{k\,n'}^{(s)}]\big] = -[\bar{e}_{n-1\,n}^{(1)}, \bar{e}_{k\,l'}^{(s)}],\end{aligned}$$

where we also used (5.89). Hence (5.95) holds for $r = 1$. Furthermore, by a similar transformation we find

$$\begin{aligned}[\bar{e}_{n-1\,n}^{(r)}, \bar{e}_{k\,l'}^{(s)}] &= \big[\bar{e}_{n-1\,n}^{(r)}, [\bar{e}_{l\,n}^{(1)}, \bar{e}_{k\,n'}^{(s)}]\big] = [\bar{e}_{l\,n}^{(1)}, \bar{e}_{k\,(n-1)'}^{(r+s-1)}]\\ &= \big[\bar{e}_{l\,n}^{(1)}, [\bar{e}_{n-1\,n}^{(1)}, \bar{e}_{k\,n'}^{(r+s-1)}]\big] = [\bar{e}_{n-1\,n}^{(1)}, \bar{e}_{k\,l'}^{(r+s-1)}] = 0,\end{aligned}$$

thus proving (5.95). Consider (5.92) with $j = n-1$ and $k < l \leqslant n-1$ and take the bracket of both sides with $\bar{e}_{n-1\,n}^{(1)}$. This finally gives (5.93).

To complete checking (5.67), it remains to show that

$$[\bar{e}_{i\,j'}^{(r)}, \bar{e}_{k\,l'}^{(s)}] = 0 \tag{5.96}$$

for all admissible values with $i < j$ and $k < l$. To this end, observe that the mapping $\sigma : \widehat{\mathrm{X}}(\mathfrak{o}_{2n}) \to \widehat{\mathrm{X}}(\mathfrak{o}_{2n})$ which swaps generators of each pair $(e_{n-1}^{(r)}, e_n^{(r)})$, $(f_{n-1}^{(r)}, f_n^{(r)})$ and $(h_n^{(r)}, h_{n'}^{(r)})$, and leaves all other generators unchanged, defines an involutive automorphism of $\widehat{\mathrm{X}}(\mathfrak{o}_{2n})$. The subalgebra $\widehat{\mathcal{E}}$ is invariant under σ and we have the induced automorphism $\bar{\sigma}$ of the associated graded algebra $\mathrm{gr}\,\widehat{\mathcal{E}}$. The definitions (5.64) and (5.65) imply that $\bar{\sigma}$ acts by

$$\bar{e}_{i\,n}^{(r)} \mapsto \bar{e}_{i\,n'}^{(r)}, \qquad \bar{e}_{i\,n'}^{(r)} \mapsto \bar{e}_{i\,n}^{(r)}$$

for $i < n$, and leaves each element $\bar{e}_{ij}^{(r)}$ and $\bar{e}_{ij'}^{(r)}$ unchanged for $i < j \leqslant n-1$. Hence, applying $\bar{\sigma}$ to (5.93) we get

$$[\bar{e}_{in'}^{(r)}, \bar{e}_{kl'}^{(s)}] = 0 \tag{5.97}$$

for all admissible i, k, l with $k < l \leqslant n-1$. Finally, if $i < j \leqslant n-1$ and $k < l \leqslant n-1$, then using (5.93) and (5.97) we get

$$[\bar{e}_{ij'}^{(r)}, \bar{e}_{kl'}^{(s)}] = \big[[\bar{e}_{jn}^{(1)}, \bar{e}_{in'}^{(r)}], \bar{e}_{kl'}^{(s)}\big] = 0$$

thus verifying the remaining cases of (5.67) for type D_n.

To complete the proof of the theorem, note that in all three types relations (5.67) imply that the graded algebra $\mathrm{gr}\,\widehat{\mathcal{E}}$ is spanned by the set of monomials in the elements $\bar{e}_{ij}^{(r)}$ taken in some fixed order. Hence the algebra $\widehat{\mathcal{E}}$ is spanned by the corresponding monomials in the elements $e_{ij}^{(r)}$. It is immediate from the defining relation of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$ that the mapping

$$h_i^{(r)} \mapsto h_i^{(r)}, \qquad e_i^{(r)} \mapsto f_i^{(r)}, \qquad f_i^{(r)} \mapsto e_i^{(r)}$$

defines an anti-automorphism of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$. By applying this anti-automorphism, we deduce that the ordered monomials in the elements $f_{ji}^{(r)}$ span the subalgebra $\widehat{\mathcal{F}}$. Note also that the ordered monomials in $h_i^{(r)}$ span $\widehat{\mathcal{H}}$. Furthermore, by the defining relations of $\widehat{\mathrm{X}}(\mathfrak{g}_N)$, the multiplication map

$$\widehat{\mathcal{F}} \otimes \widehat{\mathcal{H}} \otimes \widehat{\mathcal{E}} \to \widehat{\mathrm{X}}(\mathfrak{g}_N)$$

is surjective. Thus, ordering the elements $h_i^{(r)}$, $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ in such a way that the elements of $\widehat{\mathcal{F}}$ precede the elements of $\widehat{\mathcal{H}}$, and the latter precede the elements of $\widehat{\mathcal{E}}$, we can conclude that the ordered monomials in these elements span $\widehat{\mathrm{X}}(\mathfrak{g}_N)$. This proves that (5.60) is an isomorphism. □

We point out another version of the Poincaré–Birkhoff–Witt theorem for $\mathrm{X}(\mathfrak{g}_N)$ which is implied by the proof of Theorem 5.14. Denote by $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{H}$ the subalgebras of $\mathrm{X}(\mathfrak{g}_N)$ respectively generated by all elements of the form $e_i^{(r)}$, $f_i^{(r)}$ and $h_i^{(r)}$. Consider the generators $h_i^{(r)}$ with $i = 1, \dots, n+1$ and $r \geqslant 1$, and $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ with $r \geqslant 1$ and the conditions $i < j$ and $i < j'$ in the orthogonal case, and $i < j$ and $i \leqslant j'$ in the symplectic case. Order the elements $h_i^{(r)}$, $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ in such a way that the elements of $\mathcal{F}$ precede the elements of $\mathcal{H}$, and the latter precede the elements of $\mathcal{E}$.

Corollary 5.16. *The set of all ordered monomials in the elements $h_i^{(r)}$, $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ with the respective conditions on the indices forms a basis of $\mathrm{X}(\mathfrak{g}_N)$.* □

6 Isomorphism theorem for the Drinfeld Yangian

We will now prove the Main Theorem as stated in the Introduction. By the R-matrix definition of the Yangian $\mathrm{Y}(\mathfrak{g}_N) = \mathrm{Y}^R(\mathfrak{g}_N)$ in Section 2, this is the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$, whose elements are stable under all automorphisms (2.12). It is clear from the definition of the series $\kappa_i(u)$ and $\xi_i^{\pm}(u)$ with $i = 1, \dots, n$, and the explicit formulas (4.1), (4.2) and (4.3), that all the coefficients κ_{ir} and $\xi_{ir}^{\pm}$ defined in (1.5) belong to the subalgebra $\mathrm{Y}(\mathfrak{g}_N)$.

Proposition 6.1. *The subalgebra $\mathrm{Y}(\mathfrak{g}_N)$ of $\mathrm{X}(\mathfrak{g}_N)$ is generated by the elements κ_{ir}, ξ_{ir}^{+} and ξ_{ir}^{-} with $i = 1, \dots, n$ and $r = 0, 1, \dots$.*

Proof. Due to the tensor decomposition (2.13) of $\mathrm{X}(\mathfrak{g}_N)$, it suffices to check that these elements together with the coefficients $z_N^{(r)}$ of the series $z_N(u)$ given in (2.14) generate the algebra $\mathrm{X}(\mathfrak{g}_N)$. Using the definition of the series $\kappa_i(u)$ and formulas (4.6) it is straightforward to express $h_1(u)h_{n+1}(u)^{-1}$ as a product of the series of the form $\kappa_i(u)^{-1}$ with some shifts of u by constants. On the other hand, Theorem 5.8 implies that $h_1(u+\kappa)h_{n+1}(u)$ equals $z_N(u)$ times the same kind of product of the shifted series $\kappa_i(u)^{-1}$. Therefore, all coefficients of $h_1(u)$ and hence all coefficients of the series $h_i(u)$ with $i = 1, \dots, n+1$ belong to the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ generated by the elements given in the proposition. Furthermore, for each i, the elements $e_i^{(r)}$ and $f_i^{(r)}$ are found as linear combinations of the ξ_{is}^{-} and ξ_{is}^{+}, respectively. By Theorem 5.14, the coefficients of the series $h_i(u)$ with $i = 1, \dots, n+1$, and $e_i(u)$ and $f_i(u)$ with $i = 1, \dots, n$ generate the algebra $\mathrm{X}(\mathfrak{g}_N)$ thus completing the proof. $\square$

Now we will verify that the generators κ_{ir}, ξ_{ir}^{+} and ξ_{ir}^{-} of the subalgebra $\mathrm{Y}(\mathfrak{g}_N)$ provided by Proposition 6.1 satisfy the defining relations of the Drinfeld Yangian $\mathrm{Y}^D(\mathfrak{g}_N)$ as given in the Introduction. We will do this in terms of the equivalent generating series relations for $\kappa_i(u)$ and $\xi_i^{\pm}(u)$. We use the notation $\{a, b\} = ab + ba$.

Proposition 6.2. *The following relations hold in $\mathrm{Y}(\mathfrak{g}_N)$:*

$$\big[\kappa_i(u), \kappa_j(v)\big] = 0, \tag{6.1}$$

$$\big[\xi_i^{+}(u), \xi_j^{-}(v)\big] = -\delta_{ij}\, \frac{\kappa_i(u) - \kappa_i(v)}{u - v}, \tag{6.2}$$

$$\big[\kappa_i(u), \xi_j^{\pm}(v)\big] = \mp \frac{1}{2}\,(\alpha_i, \alpha_j)\, \frac{\big\{\kappa_i(u), \xi_j^{\pm}(u) - \xi_j^{\pm}(v)\big\}}{u - v}, \tag{6.3}$$

$$\big[\xi_i^{\pm}(u), \xi_j^{\pm}(v)\big] + \big[\xi_j^{\pm}(u), \xi_i^{\pm}(v)\big] = \mp \frac{1}{2}\,(\alpha_i, \alpha_j) \frac{\big\{\xi_i^{\pm}(u) - \xi_i^{\pm}(v), \xi_j^{\pm}(u) - \xi_j^{\pm}(v)\big\}}{u - v}, \tag{6.4}$$

$$\sum_{p \in \mathfrak{S}_m} \big[\xi_i^{\pm}(u_{p(1)}), \big[\xi_i^{\pm}(u_{p(2)}), \dots, \big[\xi_i^{\pm}(u_{p(m)}), \xi_j^{\pm}(v)\big] \dots\big]\big] = 0, \tag{6.5}$$

where the last relation holds for all $i \neq j$, and we set $m = 1 - a_{ij}$.

Proof. The proof amounts to writing the relations of Theorem 5.14 in terms the series $\kappa_i(u)$ and $\xi_i^{\pm}(u)$. In particular, (6.1) and (6.2) are immediate from (5.6) and (5.7), respectively. Moreover, the Serre relations (6.5) are implied by (5.58) and (5.59). In the case where $i, j \leqslant n-1$, relations (6.3) and (6.4) hold due to the corresponding results in type A as shown in Section 5.2; see [3] and [13, Sec. 3.1] for the calculations. The remaining cases of (6.3) and (6.4) are dealt with in a way quite similar to type A, so we will only point out a few necessary modifications specific for types B, C and D.

Type B_n. To verify the $i=j=n$ case of (6.3) for $\xi_n^-(v)$ write

$$\big[h_n^{-1}(u)h_{n+1}(u), e_n(v)\big] = h_n^{-1}(u)\big[h_{n+1}(u), e_n(v)\big] + \big[h_n^{-1}(u), e_n(v)\big]h_{n+1}(u).$$

Calculating the commutators by (5.3) and (5.43), we get

$$(u-v)\big[h_n^{-1}(u)h_{n+1}(u), e_n(v)\big] = \frac{1}{2}h_n^{-1}(u)h_{n+1}(u)\big(e_n(u)-e_n(v)\big)$$
$$+\big(e_n(u)-e_n(v)\big)h_n^{-1}(u)h_{n+1}(u) - \frac{u-v}{2(u-v-1)}h_n(u)^{-1}\big(e_n(u-1)-e_n(v)\big)h_{n+1}(u).$$

Relation (5.43) gives

$$h_n(u)^{-1}e_n(u-1) = e_n(u)h_n(u)^{-1}$$

and

$$h_n(u)^{-1}e_n(v) = \frac{1}{u-v}\,e_n(u)h_n(u)^{-1} + \frac{u-v-1}{u-v}\,e_n(v)h_n(u)^{-1}.$$

Therefore, we derive

$$(u-v)[h_n^{-1}(u)h_{n+1}(u), e_n(v)] = \frac{1}{2}\,\big\{h_n^{-1}(u)h_{n+1}(u), e_n(u)-e_n(v)\big\}.$$

This yields (6.3) with $i=j=n$ by writing the relation in terms of the series $\kappa_n(u)$ and $\xi_n^-(v)$. The remaining cases of (6.3) follow by similar arguments.

Now choose the minus signs in (6.4) and verify it for $i=n-1$ and $j=n$. By (5.28) we have

$$(u-v+1/2)\big[e_{n-1}(u), e_n(v-1/2)\big] = e_{n-1\,n+1}(v-1/2) - e_{n-1\,n+1}(u) \tag{6.6}$$
$$- e_{n-1}(v-1/2)e_n(v-1/2) + e_{n-1}(u)e_n(v-1/2).$$

By swapping u and v we also get

$$(u-v-1/2)\big[e_n(u-1/2), e_{n-1}(v)\big] = e_{n-1\,n+1}(u-1/2)e_{n-1\,n+1}(v) \tag{6.7}$$
$$- e_{n-1}(u-1/2)e_n(u-1/2) + e_{n-1}(v)e_n(u-1/2)$$

which we can also write in the form

$$e_{n-1}(v)e_n(u-1/2) = \frac{u-v-1/2}{u-v+1/2}\,e_n(u-1/2)e_{n-1}(v) \tag{6.8}$$
$$-\frac{1}{u-v+1/2}\Big(e_{n-1\,n+1}(u-1/2) - e_{n-1\,n+1}(v) - e_{n-1}(u-1/2)e_n(u-1/2)\Big).$$

Now (6.7) and (6.8) give

$$(u-v+1/2)\big[e_n(u-1/2), e_{n-1}(v)\big] = e_{n-1\,n+1}(u-1/2) - e_{n-1\,n+1}(v) \tag{6.9}$$
$$- e_{n-1}(u-1/2)e_n(u-1/2) + e_n(u-1/2)e_{n-1}(v).$$

By (6.6) and (6.9), we have

$$
\begin{aligned}
&(u-v+1/2)\Big(\big[e_{n-1}(u),e_n(v-1/2)\big]+\big[e_n(u-1/2),e_{n-1}(v)\big]\Big)\\
&\ =e_{n-1\,n+1}(u-1/2)-e_{n-1\,n+1}(v)-e_{n-1}(u-1/2)e_n(u-1/2)+e_n(u-1/2)e_{n-1}(v)\\
&\ +e_{n-1\,n+1}(v-1/2)-e_{n-1\,n+1}(u)-e_{n-1}(v-1/2)e_n(v-1/2)+e_{n-1}(u)e_n(v-1/2).
\end{aligned}
$$

Setting $v=u$ in (6.8) we get

$$
e_{n-1}(u-1/2)e_n(u-1/2)=\frac{1}{2}\big\{e_{n-1}(u),e_n(u-1/2)\big\}+e_{n-1\,n+1}(u-1/2)-e_{n-1\,n+1}(u).
$$

Using also this relation with u replaced by v, we thus come to

$$
\begin{aligned}
&(u-v+1/2)\Big(\big[e_{n-1}(u),e_n(v-1/2)\big]+\big[e_n(u-1/2),e_{n-1}(v)\big]\Big)\\
&=e_n(u-1/2)e_{n-1}(v)+e_{n-1}(u)e_n(v-1/2)\\
&-\frac{1}{2}\big\{e_{n-1}(u),e_n(u-1/2)\big\}-\frac{1}{2}\big\{e_{n-1}(v),e_n(v-1/2)\big\}
\end{aligned}
$$

which is equivalent to (6.4) for $i=n-1$ and $j=n$. The case with the plus signs and all other remaining cases follow by similar calculations.

Type C_n. Since $[h_i(u),e_n(v)]=0$ and $[h_i(u),f_n(v)]=0$ for $i\leqslant n-1$ by (5.43) and (5.44), relation (6.3) holds for $i<n-1$ and $j=n$. Furthermore, (5.43) gives

$$
(u-v+1)\big[h_n(u),e_n(v-1)\big]=-2h_n(u)\big(e_n(u)-e_n(v-1)\big) \tag{6.10}
$$

which implies

$$
\begin{aligned}
(u-v+1)\big[h_{n-1}(u)^{-1}h_n(u),e_n(v-1)\big]&=(u-v+1)h_{n-1}(u)^{-1}\big[h_n(u),e_n(v-1)\big]\\
&=-2h_{n-1}(u)^{-1}h_n(u)\big(e_n(u)-e_n(v-1)\big).
\end{aligned}
$$

Taking $v=u$ in (6.10) we get

$$
-2h_{n-1}(u)^{-1}h_n(u)e_n(u)=-\big\{h_{n-1}(u)^{-1}h_n(u),e_n(u-1)\big\}.
$$

Hence,

$$
\begin{aligned}
(u-v+1)\big[h_{n-1}(u)^{-1}h_n(u),e_n(v-1)\big]&=(u-v+1)h_{n-1}(u)^{-1}\big[h_n(u),e_n(v-1)\big]\\
&=2h_{n-1}(u)^{-1}h_n(u)e_n(v-1)-\big\{h_{n-1}(u)^{-1}h_n(u),e_n(u-1)\big\}.
\end{aligned}
$$

This gives

$$
(u-v)\big[h_{n-1}(u)^{-1}h_n(u),e_n(v-1)\big]=-\big\{h_{n-1}(u)^{-1}h_n(u),e_n(u-1)-e_n(v-1)\big\}
$$

which implies (6.3) for $i = n-1$ and $j = n$ for the series $\xi_n^-(v)$. The remaining cases of (6.3) follow by similar arguments. In particular, the case $i = j = n$ for the same series is straightforward from (5.43) and (5.48), whereas the case $i = n$ and $j = n-1$ is derived from (5.43) and (5.51).

Relations (5.53) and (5.55) imply the respective cases of (6.4). The derivation of (6.4) for $i = n-1$ and $j = n$ relies on (5.33) and (5.34) and follows the same pattern as for type B_n above.

Type D_n. Applying (5.43) and (5.48) we get

$$\begin{aligned}\big[h_{n-1}(u)^{-1}h_{n+1}(u), e_n(v)\big] &= \frac{1}{u-v}h_{n-1}(u)^{-1}h_{n+1}(u)\big(e_n(u)-e_n(v)\big)\\ &+ \frac{1}{u-v}\big(e_n(u)-e_n(v)\big)h_{n-1}(u)^{-1}h_{n+1}(u)\end{aligned}$$

so that (6.3) holds for $i = j = n$ for the series $\xi_n^-(v)$. The remaining cases of (6.3) follow with the use of (5.43) by similar calculations. Relations (5.53), (5.54) and (5.55) imply the corresponding cases of (6.4). The remaining case of (6.4) requiring a longer calculation is $i = n-2$ and $j = n$. It is performed with the use of (5.31) and (5.32) in the same way as for the case $i = n-1$ and $j = n$ in type B_n above. □

Propositions 6.1 and 6.2 imply that the mapping $\mathrm{Y}^D(\mathfrak{g}_N) \to \mathrm{Y}(\mathfrak{g}_N)$ considered in the Main Theorem is a surjective homomorphism. Its injectivity follows from the decomposition

$$\mathrm{Y}(\mathfrak{g}_N) = \mathcal{E} \otimes (\mathrm{Y}(\mathfrak{g}_N) \cap \mathcal{H}) \otimes \mathcal{F} \tag{6.11}$$

and the corresponding arguments of the proof of Theorem 5.14. This completes the proof of the Main Theorem.

References

[1] D. Arnaudon, J. Avan, N. Crampé, L. Frappat and E. Ragoucy, *R-matrix presentation for super-Yangians $Y(\mathrm{osp}(m|2n))$*, J. Math. Phys. **44** (2003), 302–308.

[2] D. Arnaudon, A. Molev and E. Ragoucy, *On the R-matrix realization of Yangians and their representations*, Annales Henri Poincaré **7** (2006), 1269–1325.

[3] J. Brundan and A. Kleshchev, *Parabolic presentations of the Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$, Comm. Math. Phys. **254** (2005), 191–220.

[4] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[5] V. G. Drinfeld, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[6] V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[7] I. M. Gelfand and V. S. Retakh, *Determinants of matrices over noncommutative rings*, Funct. Anal. Appl. **25** (1991), 91–102.

[8] N. Guay, H. Nakajima and C. Wendlandt, *Coproduct for Yangians of affine Kac-Moody algebras*, arXiv:1701.05288.

[9] N. Jing and M. Liu, *Isomorphism between two realizations of the Yangian* $Y(\mathfrak{so}_3)$, J. Phys. A **46** (2013), 075201, 12 pp.

[10] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), 1–23.

[11] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method: recent developments*, in: "Integrable Quantum Field Theories", Lecture Notes in Phys. **151**, Springer, Berlin, 1982, pp. 61–119.

[12] S. Z. Levendorskiĭ, *On generators and defining relations of Yangians*, J. Geom. Phys., **12** (1993), 1–11.

[13] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI, 2007.

[14] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, Ann. Phys. **120** (1979), 253–291.

N.J.:
Department of Mathematics
North Carolina State University, Raleigh, NC 27695, USA
jing@math.ncsu.edu

N.J. & M.L.:
School of Mathematical Sciences
South China University of Technology
Guangzhou, Guangdong 510640, China
mamliu@scut.edu.cn

A.M.:
School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au