# SOLVABLE NORMAL SUBGROUPS OF 2-KNOT GROUPS

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Abstract. If $X$ is an orientable, strongly minimal $PD_4$-complex and $\pi_1(X)$ has one end then it has no nontrivial locally-finite normal subgroup. Hence if $\pi$ is a 2-knot group then (a) if $\pi$ is virtually solvable then either $\pi$ has two ends or $\pi \cong \Phi$, with presentation $\langle a, t | ta = a^2 t \rangle$, or $\pi$ is torsion-free and polycyclic of Hirsch length 4; (b) either $\pi$ has two ends, or $\pi$ has one end and the centre $\zeta\pi$ is torsion-free, or $\pi$ has infinitely many ends and $\zeta\pi$ is finite; and (c) the Hirsch-Plotkin radical $\sqrt{\pi}$ is nilpotent.

The main result of this note (in §1) is that if $X$ is an orientable $PD_4$-complex such that $\pi_1(X)$ has one end and the equivariant intersection pairing on $\pi_2(X)$ is 0 then $\pi_1(X)$ has no nontrivial locally-finite normal subgroup. This has several applications to 2-knot groups and their subgroups.

In §2 we show that if a 2-knot group $\pi$ is virtually solvable then either $\pi'$ is finite or $\pi \cong \Phi = Z*_2$, with presentation $\langle a, t | ta = a^2 t \rangle$ (the group of Fox's Example 10), or $\pi$ is torsion-free polycyclic and of Hirsch length 4. Such groups are all known. (The final family was found in [6].) More generally, if $S$ is an infinite solvable normal subgroup and $\pi$ is not itself solvable then $S \cong \mathbb{Z}^2$ or is virtually torsion-free abelian of rank 1. We show also that the Hirsch-Plotkin radical $\sqrt{\pi}$ of every 2-knot group is nilpotent. Finally we consider the centre $\zeta\pi$. If $\pi$ has one end then $\zeta\pi \cong \mathbb{Z}^2$ or is torsion-free, of rank $\leq 1$. In particular, this is so if the commutator subgroup $\pi'$ is infinite and $\zeta\pi$ has rank $> 0$. If $\pi$ has two ends $\zeta\pi$ has rank 1, and may be either $\mathbb{Z}$ or $\mathbb{Z} \oplus Z/2Z$, while if $\pi$ has infinitely many ends $\zeta\pi$ is finite. We extend a construction of [12] to give examples with $\sqrt{\pi}$ cyclic of order $q$ or $2q$, with $q$ odd.

## 1. $PD_4$-complexes with $\chi = 0$

A $PD_4$-complex $X$ with fundamental group $\pi$ is *strongly minimal* if the equivariant intersection pairing on $\pi_2(X)$ is 0, equivalently, if

1991 *Mathematics Subject Classification.* 57Q45.

*Key words and phrases.* centre, coherent, Hirsch-Plotkin radical, 2-knot, torsion.

the homomorphism from $H^2(\pi;\mathbb{Z}[\pi])$ to $H^2(X;\mathbb{Z}[\pi])$ induced by the classifying map $c_X : X \to K(\pi,1)$ is an isomorphism [5].

**Lemma 1.** *Let $G$ be a group. If $T$ is a locally finite normal subgroup of $G$ then $T$ acts trivially on $H^j(G;\mathbb{Z}[G])$, for all $j \geq 0$.*

*Proof.* If $T$ is finite then $H^j(G;\mathbb{Z}[G]) \cong H^j(G/T;\mathbb{Z}[G/T]))$, for all $j$, and the result is clear. Thus we may assume that $T$ and $G$ are infinite. Hence $H^0(G;\mathbb{Z}[G]) = 0$, and $T$ acts trivially. We may write $T = \cup_{n\geq 1} T_n$ as a strictly increasing union of finite subgroups. Then there are short exact sequences [8]

$$0 \to \varprojlim{}^1 H^{s-1}(T_n;\mathbb{Z}[\pi]) \to H^s(T;\mathbb{Z}[\pi]) \to \varprojlim H^s(T_n;\mathbb{Z}[\pi]) \to 0.$$

Hence $H^s(T;\mathbb{Z}[\pi]) = 0$ if $s \neq 1$ and $H^1(T;\mathbb{Z}[\pi]) = \varprojlim{}^1 H^0(T_n;\mathbb{Z}[\pi])$, and so the LHS spectral sequence collapses to give $H^j(G;\mathbb{Z}[G]) \cong H^{j-1}(G/T;H^1(T;\mathbb{Z}[G]))$, for all $j \geq 1$. Let $g \in T$. We may assume that $g \in T_n$ for all $n$, and so $g$ acts trivially on $H^0(T_n;\mathbb{Z}G)$, for all $j$ and $n$. But then $g$ acts trivially on $\varprojlim{}^1 H^0(T_n;\mathbb{Z}[\pi])$, by the functoriality of the construction. Hence every element of $T$ acts trivially on $H^{j-1}(G/T;H^1(T;\mathbb{Z}[G]))$, for all $j \geq 1$. □

**Theorem 2.** *Let $X$ be an orientable $PD_4$-complex with fundamental group $\pi$. If $X$ is strongly minimal and $\pi$ has one end then $\pi$ has no non-trivial locally-finite normal subgroup.*

*Proof.* Since $\pi$ has one end, $H_s(X;\mathbb{Z}[\pi]) = 0$ for $s \neq 0$ or 2. Poincaré duality and $c_X$ give an isomorphism $\Pi = H_2(X;\mathbb{Z}[\pi]) \cong H^2(X;\mathbb{Z}[\pi])$. Since $X$ is strongly minimal, this in turn is isomorphic to $H^2(\pi;\mathbb{Z}[\pi])$.

Suppose that $\pi$ has a nontrivial locally-finite normal subgroup $T$. Let $g \in T$ have prime order $p$, and let $C = \langle g \rangle \cong Z/pZ$. We apply Lemma 2.10 of [4], to conclude that $H_{i+3}(C;\mathbb{Z}) \cong H_i(C;\Pi)$ for all $i \geq 2$. Since $C$ has cohomological period 2 and acts trivially on $\Pi$, by Lemma 1, there is an exact sequence

$$0 \to Z/pZ \to \Pi \to \Pi \to 0.$$

But $\Pi \cong H^2(\pi;\mathbb{Z}[\pi])$ is torsion-free, by Proposition 13.7.1 of [2], since $\pi$ is finitely presentable. Hence $T$ has no such element $g$ and so $\pi$ has no such finite normal subgroup. □

**Corollary 2.1.** *Every locally-finite ascending subgroup of $\pi$ is trivial.*

*Proof.* If $T$ is a locally-finite ascending subgroup of $\pi$ then a transfinite induction shows that the normal closure of $T$ in $\pi$ is locally finite. □

Let $\beta_i^{(2)}(X)$ be the $i$th $L^2$ Betti number of $X$. (See [9] for a comprehensive exposition of $L^2$-theory.)

**Lemma 3.** *Let $X$ be a finite $PD_4$-complex with fundamental group $\pi$. If $\chi(X) = 0$ and $\beta_1^{(2)}(\pi) = 0$ then $X$ is strongly minimal.*

*Proof.* Since $X$ is a finite complex the $L^2$-Euler characteristic formula holds, and so $\chi(X) = \beta_2^{(2)}(X) - 2\beta_1^{(2)}(X)$. Hence $\beta_2^{(2)}(X) = 0$ also. Since $\beta_1^{(2)}(X) = \beta_1^{(2)}(\pi) = 0$, $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi) \geq 0$ and $\chi(X) = 0$, it follows that $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$. Hence $H^2(c_X; \mathbb{Z}[\pi])$ is an isomorphism, by part (3) of Theorem 3.4 of [4]. □

We shall apply these results to 2-knot groups in the next section.

## 2. Centres, Hirsch-Plotkin radicals and virtually solvable 2-knot groups

Let $K$ be a 2-knot with exterior $X(K) = S^4 \backslash K$ and group $\pi = \pi K = \pi_1(X(K))$, and let $M(K) = X(K) \cup S^1 \times D^3$ be the closed 4-manifold obtained by elementary surgery on $K$ in $S^4$. Then $\pi_1(M(K)) \cong \pi$ and $\chi(M(K)) = 0$.

Let $\zeta G, G'$ and $\sqrt{G}$ denote the centre, commutator subgroup and Hirsch-Plotkin radical of a group $G$, respectively. If $G$ is elementary amenable then it has a well-defined Hirsch length $h(G) \in \mathbb{N} \cup \{\infty\}$. (See [7] or Chapter 1 of [4].)

**Theorem 4.** *Let $K$ be a 2-knot with group $\pi = \pi K$. If $\pi$ has normal subgroups $A \leq E$ with $A$ a nontrivial abelian group and $E$ an infinite elementary amenable group then either $\pi'$ is finite or $E$ is virtually torsion-free solvable. If $h(E) = 1$ then $E$ is abelian or virtually $\mathbb{Z}$; if $h(E) = 2$ then $E \cong \mathbb{Z}^2$, and if $h(E) > 2$ then $E$ is torsion-free polycyclic, and $h(E) = 3$ or $4$.*

*Proof.* We may assume that $\pi'$ is infinite. Then $\pi$ has one end, and $\beta_1^{(2)}(\pi) = 0$, since $\pi$ has an infinite elementary amenable normal subgroup. Since $M(K)$ is a closed 4-manifold, it is homotopy equivalent to a finite $PD_4$-complex, and so is strongly minimal, by Lemma 3. The torsion subgroup of $A$ is characteristic, and so is normal in $\pi$. Hence $A$ is torsion-free, by Theorem 2. Therefore either $\pi \cong \Phi$ or $M(K)$ is aspherical or $A \cong \mathbb{Z}$ and $\pi/A$ has infinitely many ends, by Theorem 15.8 of [4]. In all cases $E$ is in fact virtually torsion-free solvable. (This is Corollary 1.9.2 of [4] when $M(K)$ is aspherical.)

If $E$ is infinite then $\pi$ has one or two ends. Hence if $h(E) = 1$ and $\pi'$ is infinite then $E$ has no finite normal subgroup, and so must have a torsion-free abelian subgroup of index $\leq 2$. If $E$ is not finitely generated then $\pi \cong \Phi$ or $M(K)$ is aspherical, by Theorem 15.8 of [4], and so $\pi$ is torsion-free. Hence $S$ must be abelian.

If $h(E) = 2$ then $M(K)$ is aspherical and $E$ is torsion-free and virtually $\mathbb{Z}^2$, by Theorems 9.1 and 16.2 of [4]. Since $M(K)$ is orientable, $E$ cannot be the Klein bottle group, and so $E \cong \mathbb{Z}^2$. If $h(E) > 2$ then $\pi$ is torsion-free polycyclic and $h(\pi) = 4$, by Theorem 8.1 of [4], so $h(E) = 3$ or 4. □

We do not know whether $E$ must be abelian when $\pi'$ is infinite, $E$ is finitely generated and $h(E) = 1$ or 2.

**Corollary 4.1.** *If $\pi$ is virtually solvable then either $\pi'$ is finite or $\pi \cong \Phi$ or $\pi$ is torsion-free polycyclic and $h(\pi) = 4$.*

*Proof.* If $\pi$ is virtually solvable it has a solvable normal subgroup $S$ of finite index. The lowest nontrivial term of the derived series for $S$ is characteristic in $S$, and so normal in $\pi$. Hence the theorem applies. □

It is enough to assume that $\pi$ is elementary amenable and has a nontrivial abelian normal subgroup. Can we relax "virtually solvable" further to just "elementary amenable"?

If $\pi \cong \Phi$ then $K$ is TOP isotopic to Fox's Example 10 (or its reflection) [5], while if $\pi$ is torsion-free polycyclic then it determines $M(K)$ is determined up to homeomorphism, by Theorem 17.4 of [4].

The product of locally-nilpotent normal subgroups of a group $G$ is again a locally-nilpotent normal subgroup, by the Hirsch-Plotkin Theorem, and the Hirsch-Plotkin radical $\sqrt{G}$ is the (unique) maximal such subgroup. (See Proposition 12.1.2 of [10].) This subgroup contains all the nilpotent normal subgroups of $G$, and is clearly nilpotent if it is finitely generated. However in general $\sqrt{G}$ need not be nilpotent.

**Corollary 4.2.** *The Hirsch-Plotkin radical $\sqrt{\pi}$ is nilpotent.*

*Proof.* Since $\sqrt{\pi}$ is locally nilpotent it has a maximal locally-finite normal subgroup $T$ with torsion-free quotient. If $\sqrt{\pi}$ is finitely generated there is nothing to prove. Otherwise, $\pi$ has one end and so $T$ is trivial, by Theorem 3. If $T = 1$ and $h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi}$ is abelian; if $h(\sqrt{\pi}) > 2$ then $\pi$ is virtually polycyclic and $h(\pi) = 4$, by Theorem 15.8 of [4]. In each case $\sqrt{\pi}$ is nilpotent. □

Every finitely generated abelian group is the centre of some high-dimensional knot group [3]. On the other hand, the only classical knots whose groups have nontrivial abelian normal subgroups are the torus knots, for which $\sqrt{\pi} = \zeta\pi \cong \mathbb{Z}$ and $\zeta\pi \cap \pi' = 1$. The intermediate case of 2-knots is less clear. If $\zeta\pi$ has rank $> 1$ then it is $\mathbb{Z}^2$; most twist spins of torus knots have such groups. There are examples with centre 1, $Z/2Z$, $\mathbb{Z}$ or $\mathbb{Z} \oplus Z/2Z$. (See Chapters 15–17 of [4].)

**Corollary 4.3.** *Let $K$ be a 2-knot with group $\pi = \pi K$. Then*

*(1) if $\pi$ has two ends then $\zeta\pi \cong \mathbb{Z} \oplus Z/2Z$ if $\pi'$ has even order; otherwise $\zeta\pi \cong \mathbb{Z}$;*
*(2) if $\pi$ has one end then $\zeta\pi \cong \mathbb{Z}^2$, or is torsion-free of rank $\leq 1$;*
*(3) if $\pi$ has infinitely many ends then $\zeta\pi$ is finite.*

*Proof.* If $\pi$ has two ends then $\pi'$ is finite, and so $\zeta\pi$ is finitely generated and of rank 1. It follows from the classification of such 2-knot groups (see §4 of Chapter 15 of [4]) that $\zeta\pi \cong \mathbb{Z} \oplus Z/2Z$ if $\pi'$ has even order, and otherwise $\zeta\pi \cong \mathbb{Z}$.

Part (2) follows from Theorem 4, while part (3) is clear. □

Note that $\pi$ has finitely many ends if $\pi'$ is finitely generated, or if $\zeta\pi$ is infinite. When $\pi$ has more than one end Lemma 2.10 of [4] either does not lead to a contradiction or does not apply.

If $\zeta\pi$ is a nontrivial torsion group then it is finite. Yoshikawa constructed an example of a 2-knot whose group $\pi$ has centre of order 2 [13]. It is easy to see that $\sqrt{\pi} = \zeta\pi$ in this case. The construction may be extended as follows. Let $q > 0$ be odd and let $k_q$ be a 2-bridge knot such that the 2-fold branched cyclic cover of $S^3$, branched over $k_q$ is a lens space $L(3q, r)$, for some $r$ relatively prime to $q$. Let $K_1 = \tau_2 k_q$ be the 2-twist spin of $k_q$, and let $K_2 = \tau_3 k$ be the 3-twist spin of a nontrivial knot $k$. Let $\gamma$ be a simple closed curve in $X(K_1)$ with image $[\gamma] \in \pi K_1$ of order $3q$, and let $w$ be a meridian for $K_2$. Then $w^3$ is central in $\pi K_2$. The group of the satellite of $K_1$ about $K_2$ relative to $\gamma$ is the generalized free product

$$\pi = \pi K_2/\langle\langle w^{3q}\rangle\rangle *_{w=[\gamma]} \pi K_1.$$

(see §14.3 of [4].) Hence $\sqrt{\pi} = \langle w^3\rangle \cong Z/qZ$, while $\zeta\pi = 1$.

If we use a 2-knot $K_1$ with group $(Q(8) \times Z/3qZ) \rtimes_\theta \mathbb{Z}$ instead and choose $\gamma$ so that $[\gamma]$ has order $6q$ then we obtain examples with $\sqrt{\pi} \cong Z/2qZ$ and $\zeta\pi = Z/2Z$. (Knots $K_1$ with such groups may be constructed by surgery on sections of mapping tori of homeomorphisms of 3-manifolds with fundamental group $Q(8) \times Z/3qZ$ [12].)

If $\zeta\pi$ has rank 1 and nontrivial torsion then $\pi'$ is finite, and $\zeta\pi$ is finitely generated.

If $\zeta\pi$ has rank 1 but is not finitely generated then $M(K)$ is aspherical. It is not known whether there are such 2-knots (nor, more generally, whether abelian normal subgroups of rank 1 in $PD_n$-groups with $n > 3$ must be finitely generated). What little we know about this case is as follows. Since $\zeta\pi < \pi'$ and $\pi/\pi' \cong \mathbb{Z}$, we must have $\zeta\pi \leq \pi''$. Since $\zeta\pi$ is torsion-free of rank 1 but is not finitely generated, $c.d.\zeta\pi = 2$. Hence if $G$ is a nonabelian subgroup which contains $\zeta\pi$ then $c.d.G \geq 3$, by

Theorem 8.6 of [1]. If $H$ is a subgroup of $\pi$ such that $H \cap \zeta\pi = 1$ then $H.\zeta\pi \cong H \times \zeta\pi$ is not finitely generated, and so has infinite index in $\pi$. Hence $c.d.H \times \mathbb{Z} \leq c.d.H \times \zeta\pi \leq 3$ [11]. Theorem 5.5 of [1] gives, firstly, that $c.d.H \leq 2$, and then, that if $H$ is $FP_2$ then $c.d.H \leq 1$, and so $H$ is free. Thus if $\pi$ is almost coherent every subgroup either meets $\zeta\pi$ nontrivially or is locally free.

If $\zeta\pi$ has rank $> 1$ then $M(K)$ is aspherical and $\zeta\pi \cong \mathbb{Z}^2$, by Theorem 16.3 of [4].

The following questions remain open:

(i) if $\zeta\pi$ has rank 1, must it be finitely generated?
(ii) if $\zeta\pi$ is finite, must it be $Z/2Z$ or 1?
(iii) is there a 2-knot group $\pi$ with $\sqrt{\pi}$ a non-cyclic finite group?
(iv) if $\pi$ is elementary amenable is it virtually solvable?

In each case the answer is "yes" if $\pi'$ is finitely presentable, for then the infinite cyclic cover $M(K)'$ is homotopy equivalent to a $PD_3$-complex, by Theorem 4.5 of [4].

At the time of writing, the largest known class of groups for which 4-dimensional TOP surgery works is the class $SA$ obtained from subexponential groups by taking increasing unions and extensions. Are there 2-knot groups in this class which are not virtually solvable?

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