THE TOEPLITZ NONCOMMUTATIVE SOLENOID AND ITS KMS STATES

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Abstract. We use Katsura's topological graphs to define Toeplitz extensions of Latrémolière and Packer's noncommutative-solenoid C^*-algebras. We identify a natural dynamics on each Toeplitz noncommutative solenoid and study the associated KMS states. Our main result shows that the space of extreme points of the KMS simplex of the Toeplitz noncommutative torus at a strictly positive inverse temperature is homeomorphic to a solenoid; indeed, there is an action of the solenoid group on the Toeplitz noncommutative solenoid that induces a free and transitive action on the extreme boundary of the KMS simplex. With the exception of the degenerate case of trivial rotations, at inverse temperature zero there is a unique KMS state, and only this one factors through Latrémolière and Packer's noncommutative solenoid.

1. Introduction

In this paper, we describe the KMS states of Toeplitz extensions of the noncommutative solenoids constructed by Latrémolière and Packer [22]. We prove that the extreme boundary of the KMS simplex is homeomorphic to a topological solenoid. In recent years, following Bost and Connes' work [2] relating KMS theory to the Riemann zeta function, there has been a great deal of interest in the KMS structure of C^*-algebras associated to algebraic and combinatorial objects. In particular Laca and Raeburn's results [20] about the Toeplitz algebra of the $ax + b$-semigroup over $\mathbb{N}$ precipitated a surge of activity around computations of KMS states for Toeplitz-like extensions. Various authors have studied KMS states on Toeplitz algebras associated to algebraic objects [4, 21, 6], directed graphs [14, 12, 5], higher-rank graphs [28, 13, 9], C^*-correspondences [19, 15], and topological graphs [1]. The results suggest that the KMS structure of such algebras for their natural gauge actions frequently encodes key features of the generating object.

The noncommutative solenoids $\mathcal{A}_\theta^{\mathscr{S}}$ are C^*-algebras introduced by Latrémolière and Packer in [22]. They are among the first examples of twisted C^*-algebras of non-compactly-generated abelian groups to be studied in detail, and have interesting representation-theoretic properties [23, 24]. In addition to the definition of noncommutative solenoids as twisted group C^*-algebras, Latrémolière and Packer provide a number of equivalent descriptions. The one we are interested in realises them as direct limits of noncommutative tori. Specifically, given a positive integer N and a sequence θ_n of real numbers such that $N^2\theta_{n+1} - \theta_n$ is an integer for every n, there are homomorphisms $\mathcal{A}_{\theta_n} \to \mathcal{A}_{\theta_{n+1}}$ that send the canonical unitary generators of $\mathcal{A}_{\theta_n}$ to the Nth powers of the corresponding generators of $\mathcal{A}_{\theta_{n+1}}$. The noncommutative solenoid for the sequence $\theta = (\theta_n)$ is the direct limit of the $\mathcal{A}_{\theta_n}$ under these homomorphisms. Latrémolière and Packer's work focusses on features like simplicity, K-theory and classification of noncommutative solenoids.

Here we use Katsura's theory of topological graph C^*-algebras [16] to introduce a class of Toeplitz extensions $\mathcal{T}_\theta^{\mathscr{S}}$ of noncommutative solenoids, realised as direct limits of Toeplitz extensions $\mathcal{T}(E_{\theta_n})$ of noncommutative tori, and then study their KMS states. Our main result says that at inverse temperatures above zero, the extreme boundary of the KMS simplex of $\mathcal{T}_\theta^{\mathscr{S}}$ is homeomorphic to the classical solenoid $\mathscr{S}$, and there is an action of $\mathscr{S}$ on $\mathcal{T}_\theta^{\mathscr{S}}$ that induces a free and transitive action on the extreme KMS states. This is further evidence that KMS structure for Toeplitz-like algebras recovers key features of the underlying generating

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objects. Interestingly, this homeomorphism is subtler than one might expect: though the results of [1] show that the KMS simplex of each approximating subalgebra $\mathcal{T}(E_{\theta_n}) \subseteq \mathcal{T}_\theta^{\mathscr{S}}$ has extreme boundary homeomorphic to the circle, these homeomorphisms are not compatible with the connecting maps in the inductive system. In fact, none of the extreme points in the KMS simplex of any $\mathcal{T}(E_{\theta_n})$ extend to KMS states of $\mathcal{T}_\theta^{\mathscr{S}}$. Identifying the simplex of KMS states of a given $\mathcal{T}(E_{\theta_n})$ that do extend to KMS states of $\mathcal{T}_\theta^{\mathscr{S}}$ requires a careful analysis of the interaction between the subinvariance relation, described in [1], that characterises KMS states on the $\mathcal{T}(E_{\theta_n})$ and the compatibility relation imposed by the connecting maps $\mathcal{T}(E_{\theta_n}) \hookrightarrow \mathcal{T}(E_{\theta_{n+1}})$. We think the ideas involved in this analysis may be applicable to other investigations of KMS states on direct-limit C^*-algebras. Our main result also shows that at inverse temperature β there is a unique KMS state (unless all the θ_n are zero, a degenerate case that we discuss separately), and that there are no KMS states at inverse temperatures below zero. Perhaps surprisingly, for nonzero θ the structure of the KMS simplex of $\mathcal{T}_\theta^{\mathscr{S}}$ does not depend on whether the θ_n are rational.

We proceed as follows. After a brief preliminaries section, we begin in Section 3 by considering KMS states for actions on direct limits that preserve the approximating subalgebras. We record a general—and presumably well known—description of the KMS simplex as a projective limit of the KMS simplices of the approximating subalgebras. The connecting maps in this projective system need not be surjective, which is the cause of the subtleties that arise in computing the KMS states of Toeplitz noncommutative solenoids later in the paper. In Section 4, we consider the topological graph E_γ that encodes rotation on the circle $\mathbb{R}/\mathbb{Z}$ by angle $\gamma \in \mathbb{R}$. We describe the Toeplitz algebra $\mathcal{T}(E_\gamma)$ of this topological graph as universal for an isometry S and a representation π of $C(\mathbb{R}/\mathbb{Z})$, and its topological-graph C^*-algebra $\mathcal{O}(E_\gamma)$ as the quotient by the ideal generated by $1 - SS^*$. In particular, $\mathcal{O}(E_\gamma)$ is canonically isomorphic to the noncommutative torus $\mathcal{A}_\gamma$. In Section 5 we consider a sequence $\theta = (\theta_n)$ in $\mathbb{R}/\mathbb{Z}$ such that $N^2\theta_{n+1} = \theta_n$ for all n. We use our description of $\mathcal{T}(E_\gamma)$ from the preceding section to describe homomorphisms $\psi_n : \mathcal{T}(E_{\theta_n}) \to \mathcal{T}(E_{\theta_{n+1}})$ that descend through the quotient maps to the homomorphisms $\tau_n : \mathcal{O}(E_{\theta_n}) \to \mathcal{O}(E_{\theta_{n+1}})$ for which the noncommutative solenoid $\mathcal{A}_\theta^{\mathscr{S}}$ is isomorphic to $\varinjlim(\mathcal{O}(E_{\theta_n}), \tau_n)$.

In Section 6, we define the Toeplitz noncommutative solenoid as $\mathcal{T}_\theta^{\mathscr{S}} := \varinjlim(\mathcal{T}(E_{\theta_n}), \psi_n)$, by analogy with the description of $\mathcal{A}_\theta^{\mathscr{S}}$ outlined in Section 5. We describe a dynamics α on $\mathcal{T}_\theta^{\mathscr{S}}$ built from the gauge actions on the approximating subalgebras $\mathcal{T}(E_{\theta_n})$. Though the gauge actions on the $\mathcal{T}(E_{\theta_n})$ are all periodic $\mathbb{R}$-actions, the dynamics α is not. We are interested in the KMS states for this dynamics. The case $\theta = \mathbf{0} := (0, 0, 0, \dots)$ is a degenerate case, and we outline in Remark 6.5 how to describe the KMS states in this instance by decomposing both the algebra $\mathcal{T}_{\mathbf{0}}^{\mathscr{S}}$ and the dynamics α as tensor products. Since $\theta_n \neq 0$ implies $\theta_{n+1} \neq 0$, we can thereafter assume, without loss of generality, that every θ_n is nonzero. In the remainder of Section 6, we use our results about direct limits from Section 3 to realise the KMS simplex of $\mathcal{T}_\theta^{\mathscr{S}}$ for α at an inverse temperature $\beta > 0$ as a projective limit of spaces $\Omega_{\mathrm{sub}}^{r_n}$ of probability measures on $\mathbb{R}/\mathbb{Z}$ that satisfy a suitable subinvariance condition. This involves an interesting interplay between the subinvariance condition for KMS states on the $\mathcal{T}(E_{\theta_n})$ obtained from [1], and the compatibility condition coming from the connecting maps ψ_n. We believe that this analysis and our analysis of the space $\Omega_{\mathrm{sub}}^{r_n}$ in Section 7 may be of independent interest from the point of view of ergodic theory. The theorems in [1] are silent on the case $\beta = 0$, so we must argue this case separately, and our results for this case in Section 6 appear less sharp than for $\beta > 0$: they show only that the KMS_0-simplex embeds in the projective limit of the spaces Ω_{sub}^0. But we shall see later that the subinvariance condition at $\beta = 0$ has a unique solution, so that the projective limit in this case is a one-point set. So our embedding result for $\beta = 0$ is sufficient to show that there is a unique KMS_0 state.

In Section 7 we analyse the space Ω_{sub}^r for $r > 0$. We first construct a measure m_r satisfying the desired subinvariance relation, and then show that the measures obtained by composing this m_r with rotations are all of the extreme points of Ω_{sub}^r. This yields an isomorphism of Ω_{sub}^r

with the space of regular Borel probability measures on $\mathbb{R}/\mathbb{Z}$. A key step in our analysis is the characterisation in [12] of the subinvariant measures on the vertex set of a simple-cycle graph. We then turn in Section 8 to the proof of our main theorem. The key step is to establish that the connecting maps $\psi_n : \mathcal{T}(E_{\theta_n}) \to \mathcal{T}(E_{\theta_{n+1}})$ induce surjections $\Omega_{\rm sub}^{r_{n+1}} \twoheadrightarrow \Omega_{\rm sub}^{r_n}$ by showing that the induced maps carry extreme points to extreme points.

2. Preliminaries

In this section we recall the background that we need on topological graphs and their C^*-algebras, as introduced by Katsura in [16]. We then recall the notion of a KMS state for a C^*-algebra A and dynamics α.

Topological graphs and their C^*-algebras. For details of the following, see [16]. A *topological graph* $E = (E^0, E^1, r, s)$ consists of locally compact Hausdorff spaces E^0 and E^1, a continuous map $r : E^1 \to E^0$, and a local homeomorphism $s : E^1 \to E^0$. In [16] Katsura constructs from each topological graph E a Hilbert $C_0(E^0)$-bimodule $X(E)$ and two C^*-algebras: the Toeplitz algebra $\mathcal{T}(E)$ and the graph C^*-algebra $\mathcal{O}(E)$. In this article we only encounter topological graphs of the form $E = (Z, Z, \mathrm{id}, h)$, where $h : Z \to Z$ is a homeomorpism of a compact Hausdorff space Z, so we only discuss the details of $X(E)$, $\mathcal{T}(E)$ and $\mathcal{O}(E)$ in this setting.

When $E = (Z, Z, \mathrm{id}, h)$, where Z is compact, the module $X(E)$ is a copy of $C(Z)$ as a Banach space. The left and right actions are given by

$$g_1 \cdot f \cdot g_2(z) = g_1(z) f(z) g_2(h(z)), \quad \text{for } g_1, g_2 \in C(Z),\ f \in X(E),$$

and the inner product by $\langle f_1, f_2\rangle(z) = \overline{f_1(h^{-1}(z))} f_2(h^{-1}(z))$, for $f_1, f_2 \in X(E)$. We denote by φ the homomorphism $A \to \mathcal{L}(X(E))$ implementing the left action. In this case φ is injective.

A *representation* of $X(E)$ in a C^*-algebra B is a pair (ψ, π), consisting of a linear map $\psi : X(E) \to B$ and a homomorphism $\pi : C(Z) \to B$ satisfying

$$\psi(f \cdot h) = \psi(f)\pi(h), \quad \psi^*(f)\psi(g) = \pi(\langle f, g\rangle) \quad \text{and} \quad \psi(h \cdot f) = \pi(h)\psi(f)$$

for all $f, g \in X(E)$ and $h \in C(Z)$. The Toeplitz algebra $\mathcal{T}(E)$ is the Toeplitz algebra of $X(E)$, in the sense of [10], which is the universal C^*-algebra generated by a representation of $X(E)$. We denote by $(i^1_{X(E)}, i^0_{X(E)})$ the representation generating $\mathcal{T}(E)$.

For $f_1, f_2 \in X_E$ there is an adjointable operator $\Theta_{f_1,f_2} \in \mathcal{L}(X(E))$ given by $\Theta_{f_1,f_2}(g) = f_1\langle f_2, g\rangle_{C(Z)} = f_1 f_2^* g$. The algebra of generalised compact operators on $X(E)$ is

$$\mathcal{K}(X(E)) := \overline{\operatorname{span}}\{\Theta_{f_1,f_2} : f_1, f_2 \in X(E)\}.$$

Since $\Theta_{1,1} = 1_{\mathcal{L}(X(E))}$, we have $\mathcal{K}(X(E)) = \mathcal{L}(X(E))$. For a representation (ψ, π) of $X(E)$ in B there is a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X(E)) \to B$ satisfying $(\psi, \pi)^{(1)}(\Theta_{f_1,f_2}) = \psi(f_1)\psi(f_2)^*$ (see [26, page 202]).

The graph algebra $\mathcal{O}(E)$ is the Cuntz–Pimsner algebra of $X(E)$. So $\mathcal{O}(E)$ is the quotient of $\mathcal{T}(E)$ by the ideal generated by

$$\{(i^1_{X(E)}, i^0_{X(E)})^{(1)}(\varphi(h)) - i^0_{X(E)}(h) : h \in C(Z)\},$$

and is the universal C^*-algebra generated by a covariant representation of $X(E)$—that is, a representation (ψ, π) satisfying

$$(\psi, \pi)^{(1)}(\varphi(h)) = \pi(h) \quad \text{for all } h \in C(Z).$$

We denote the quotient map $\mathcal{T}(E) \to \mathcal{O}(E)$ by q, and we define $(j^1_{X(E)}, j^0_{X(E)}) := (q \circ i^1_{X(E)}, q \circ i^0_{X(E)})$, the covariant representation generating $\mathcal{O}(E)$.

KMS states. For details of the following, see [3]. Given a C^*-algebra A and an action $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, we say that $a \in A$ is *analytic* for α if the function $t \mapsto \alpha_t(a)$ is the restriction of an analytic function $z \mapsto \alpha_z(a)$ from $\mathbb{C}$ into A. The set of analytic elements is always norm dense in A. A state ϕ of A is a KMS_0-*state* if it is an α-invariant trace on A. For $\beta \in \mathbb{R} \setminus \{0\}$, a state ϕ of A is a KMS_β-*state*, or a KMS-state at inverse temperature β, for the system (A, α) if it satisfies the *KMS condition*

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \quad \text{for all analytic } a, b \in A.$$

It suffices to check this condition for all a, b in any α-invariant set of analytic elements that spans a dense subspace of A. The collection of KMS_β-states for a dynamics α on a unital C^*-algebra A forms a Choquet simplex, and we will denote it by $\mathrm{KMS}_\beta(A, \alpha)$.

3. KMS structure of direct limit C^*-algebras

The C^*-algebras of interest to us in this paper are examples of direct-limit C^*-algebras. In this short section we show that the simplex of KMS states of a direct-limit C^*-algebra, for an action that preserves the approximating subalgebras, is the projective limit of the simplicies of KMS states of the approximating subalgebras.

Proposition 3.1. *Suppose $\beta \in [0, \infty)$, and that $\{(A_j, \varphi_j, \alpha_j) : j \in \mathbb{N}\}$ is a sequence of unital C^*-algebras A_j, injective unital homomorphisms $\varphi_j : A_j \to A_{j+1}$, and strongly continuous actions $\alpha_j : \mathbb{R} \to \mathrm{Aut}\, A_j$ satisfying $\alpha_{j+1,t} \circ \varphi_j = \varphi_j \circ \alpha_{j,t}$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$. Denote by A_∞ the direct limit $\varinjlim(A_j, \varphi_j)$, and by $\varphi_{j,\infty}$ the canonical maps $A_j \to A_\infty$ satisfying $\varphi_{j+1,\infty} \circ \varphi_j = \varphi_{j,\infty}$ for each $j \in \mathbb{N}$. There is a strongly continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}\, A_\infty$ satisfying $\varphi_{j,\infty} \circ \alpha_{j,t} = \alpha_t \circ \varphi_{j,\infty}$ for each $j \in \mathbb{N}$ and $t \in \mathbb{R}$. Moreover, there is an affine isomorphism from $\mathrm{KMS}_\beta(A_\infty, \alpha)$ onto $\varprojlim(KMS_\beta(A_j, \alpha_j), \phi \mapsto \phi \circ \varphi_{j-1})$ that sends ϕ to $(\phi \circ \varphi_{j,\infty})_{j=0}^\infty$.*

Proof. For each $j \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$(\varphi_{j+1,\infty} \circ \alpha_{j+1,t}) \circ \varphi_j = \varphi_{j+1,\infty} \circ \varphi_j \circ \alpha_{j,t} = \varphi_{j,\infty} \circ \alpha_{j,t}.$$

So the universal property of A_∞ gives a homomorphism $\alpha_t : A_\infty \to A_\infty$ such that $\alpha_t \circ \varphi_{j,\infty} = \varphi_{j,\infty} \circ \alpha_{j,t}$ for all j.

It is straightforward to check that each α_t is an automorphism of A_∞ with inverse α_{-t}, and that $\alpha : \mathbb{R} \to \mathrm{Aut}\, A_\infty$ is an action satisfying $\varphi_{j,\infty} \circ \alpha_{j,t} = \alpha_t \circ \varphi_{j,\infty}$. An $\varepsilon/3$-argument using that each α_j is strongly continuous and that $\bigcup_j \varphi_{j,\infty}(A_j)$ is dense in A_∞ shows that α is strongly continuous.

For $j \in \mathbb{N}$ and $\phi \in \mathrm{KMS}_\beta(A_\infty, \alpha)$ define $h_j(\phi) := \phi \circ \varphi_{j,\infty}$. Since KMS_β states restrict to KMS_β states on invariant unital subalgebras, h_j maps $\mathrm{KMS}_\beta(A_\infty, \alpha)$ to $\mathrm{KMS}_\beta(A_j, \alpha_j)$ for each j. We have

$$h_{j+1} \circ \varphi_j = (\phi \circ \varphi_{j+1,\infty}) \circ \varphi_j = \phi \circ (\varphi_{j+1,\infty} \circ \varphi_j) = \phi \circ \varphi_{j,\infty} = h_j,$$

and so the universal property of $\varprojlim \mathrm{KMS}_\beta(A_j, \alpha_j)$ gives a map h from $\mathrm{KMS}_\beta(A_\infty, \alpha)$ into $\varprojlim \mathrm{KMS}_\beta(A_j, \alpha_j)$ satisfying $p_j \circ h = h_j$, where p_j denotes the canonical projection onto $\mathrm{KMS}_\beta(A_j, \beta_j)$. We claim that h is the desired affine isomorphism.

The map h is obviously affine. To see that h is surjective, fix $(\phi_j)_{j=0}^\infty \in \varprojlim(\mathrm{KMS}_\beta(A_j, \alpha_j))$, and take $j \leq k$, $a \in A_j$ and $b \in A_k$ with $\varphi_{j,\infty}(a) = \varphi_{k,\infty}(b)$. Then

$$0 = \varphi_{k,\infty}(b) - \varphi_{j,\infty}(a) = \varphi_{k,\infty}(b) - \varphi_{k,\infty}(\varphi_{k-1} \circ \cdots \circ \varphi_j(a)) = \varphi_{k,\infty}(b - \varphi_{k-1} \circ \cdots \circ \varphi_j(a)).$$

Since each φ_j is injective, each $\varphi_{j,\infty}$ is injective, and so $b = \varphi_{k-1} \circ \cdots \circ \varphi_j(a)$. Now

$$\phi_j(a) = \phi_k(\varphi_{k-1} \circ \cdots \circ \varphi_j(a)) = \phi_k(b),$$

and so there is a well-defined linear map $\phi_\infty : \bigcup_{j=0}^\infty \varphi_{j,\infty}(A_j) \to \mathbb{C}$ satisfying $\phi_\infty(\varphi_{j,\infty}(a)) = \phi_j(a)$ for all $j \in \mathbb{N}$ and $a \in A_j$. Since each $\varphi_{j,\infty}$ is isometric and each ϕ_j is norm-decreasing, each $\phi_\infty \circ \varphi_{j,\infty}$ is norm-decreasing, so ϕ_∞ is norm-decreasing. It therefore extends to a norm-decreasing $\phi_\infty : A_\infty \to \mathbb{C}$. Since $\|\phi_\infty\| \geq \|\phi_\infty \circ \varphi_j\| = \|\phi_j\| = 1$, we see that $\|\phi_\infty\| = 1$. Since

$\bigcup_j \varphi_{j,\infty}\big((A_j)_+\big)$ is dense in $(A_\infty)_+$ and since each $\phi_\infty \circ \varphi_{j,\infty} = \phi_j$ is positive, ϕ_∞ is positive, and therefore a state of A_∞.

To see that ϕ_∞ is KMS, observe that if $a \in A_j$ is α_j-analytic, then $\varphi_{j,\infty}(a)$ is α-analytic. Indeed, since $z \mapsto \varphi_{j,\infty}(\alpha_{j,z}(a))$ is an analytic extension of $t \mapsto \alpha_t(\varphi_{j,\infty}(a))$, the analytic extension of $t \mapsto \alpha_t(\varphi_{j,\infty}(a))$ is given by

$$\alpha_z(\varphi_{j,\infty}(a)) = \varphi_{j,\infty}(\alpha_{j,z}(a)).$$

So $\bigcup_j \{\varphi_{j,\infty}(a) : a \in A_j \text{ is analytic}\}$ is an α-invariant dense subspace of analytic elements in A_∞. So it suffices to show that $\phi_\infty\big(\varphi_{j,\infty}(a)\varphi_{k,\infty}(b)\big) = \phi_\infty\big(\varphi_{k,\infty}(b)\alpha_{i\beta}(\varphi_{j,\infty}(a))\big)$ whenever $a \in A_j$ and $b \in A_k$ are analytic. For this, let $l := \max\{j,k\}$ and observe that $a' := \varphi_{j,l}$ and $b' := \varphi_{k,l}(b)$ are α_l-analytic, and so

$$\begin{aligned}\phi_\infty(\varphi_{j,\infty}(a)\varphi_{k,\infty}(b)) &= \phi_\infty(\varphi_{l,\infty}(a'b') = \phi_l(a'b') = \phi_l(b'\alpha_{l,i\beta}(a')) \\ &= \phi_\infty\big(\varphi_{l,\infty}(b')\alpha_{i\beta}(\varphi_{l,\infty}(a'))\big) = \phi_\infty\big(\varphi_{j,\infty}(b)\alpha_{i\beta}(\varphi_{j,\infty}(a))\big).\end{aligned}$$

Since $h(\phi_\infty) = (\phi_\infty \circ \varphi_{j,\infty})_{j=0}^\infty = (\phi_j)_{j=0}^\infty$, we see that h is surjective.

Checking that h is injective is straightforward: if $h(\phi) = h(\psi)$, then $\phi \circ \varphi_{j,\infty} = \psi \circ \varphi_{j,\infty}$ for all $j \in \mathbb{N}$, which implies that ϕ and ψ agree on the dense subset $\bigcup_{j=0}^\infty \varphi_{j,\infty}(A_j)$, giving $\phi = \psi$.

To see that h is continuous, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathrm{KMS}_\beta(A_\infty, \alpha)$ converging weak* to $\phi \in \mathrm{KMS}_\beta(A_\infty, \alpha)$. Then $p_j(h(\phi_\lambda)) = \phi_\lambda \circ \varphi_{j,\infty}$ converges weak* to $p_j(h(\phi)) = \phi \circ \varphi_{j,\infty}$ for each $j \in \mathbb{N}$. Since the topology on the inverse limit is the initial topology induced by the projections p_j, this says that $h(\phi_\lambda)$ converges weak* to $h(\phi)$. Hence h is continuous. □

4. C^*-algebras from rotations on the circle

We are interested in topological graphs built from rotations on the circle. We write

$$\mathbb{S} := \mathbb{R}/\mathbb{Z}$$

for the circle, which we frequently identify with $[0,1)$ under addition modulo 1.

For $\gamma \in \mathbb{R}$, let R_γ denote clockwise rotation of the circle $\mathbb{S}$ by angle γ. So $R_\gamma(t) = t - \gamma \pmod 1$. Each R_γ is a homeomorphism of $\mathbb{S}$, and we denote by $E_\gamma := (\mathbb{S}, \mathbb{S}, \mathrm{id}_\mathbb{S}, R_\gamma)$ the corresponding topological graph. We denote the Hilbert bimodule $X(E_\gamma)$ by $C(\mathbb{S})_\gamma$, its inner product by $\langle\cdot,\cdot\rangle_\gamma$, and the homomorphism implementing the left action by $\phi_\gamma : C(\mathbb{S}) \to \mathcal{L}(C(\mathbb{R}/\mathbb{Z})_\gamma)$.

We can give alternative characterisations of the C^*-algebras $\mathcal{T}(E_\gamma)$ and $\mathcal{O}(E_\gamma)$. This is certainly not new: the description of $\mathcal{O}(E_\gamma)$ goes back to Pimsner [26, page 193, Example 3]. But we could not locate the exact formulation that we want for the description of $\mathcal{T}(E_\gamma)$ in the literature.

Definition 4.1. A *Toeplitz pair for E_γ in a C^*-algebra B* is a pair (π, S) consisting of a homomorphism π of $C(\mathbb{S})$ into B, and an isometry $S \in B$ satisfying

$$S\pi(f) = \pi(f \circ R_\gamma)S \quad \text{for all } f \in C(\mathbb{S}).$$

A *covariant pair for E_γ* is a Toeplitz pair (π, W) in which W is a unitary.

Proposition 4.2. *Let $\gamma \in \mathbb{R}$ and $E_\gamma = (\mathbb{S}, \mathbb{S}, \mathrm{id}_\mathbb{S}, R_\gamma)$.*

1. *The pair $(i_\gamma, s_\gamma) := (i^0_{X(E_\gamma)}, i^1_{X(E_\gamma)}(1))$ is a Toeplitz pair for E_γ that generates $\mathcal{T}(E_\gamma)$. Moreover, $\mathcal{T}(E_\gamma)$ is the universal C^*-algebra generated by a Toeplitz pair for E_γ: if (π, S) is a Toeplitz pair in a C^*-algebra B, then there is a homomorphism $\pi \times S : \mathcal{T}(E_\gamma) \to B$ satisfying $(\pi \times S) \circ i_\gamma = \pi$ and $(\pi \times S)(s_\gamma) = S$.*
2. *The pair $(j_\gamma, w_\gamma) := (j^0_{X(E_\gamma)}, j^1_{X(E_\gamma)}(1))$ is a covariant pair for E_γ that generates $\mathcal{O}(E_\gamma)$. Moreover, $\mathcal{O}(E_\gamma)$ is the universal C^*-algebra generated by a covariant pair for E_γ: if (π, W) is a covariant pair in a C^*-algebra B, then there is a homomorphism $\pi \times W : \mathcal{O}(E_\gamma) \to B$ satisfying $(\pi \times W) \circ j_\gamma = \pi$ and $(\pi \times W)(w_\gamma) = W$.*

Proof. We have $s_\gamma^* s_\gamma = i^0_{X(E_\gamma)}(\langle 1,1\rangle_\gamma) = i^0_{X(E_\gamma)}(1) = 1$, and so s_γ is an isometry. For each $f \in C(\mathbb{S})$ we have

$$\begin{aligned} i_\gamma(f\circ R_\gamma)s_\gamma &= i^0_{X(E_\gamma)}(f\circ R_\gamma) i^1_{X(E_\gamma)}(1) = i^1_{X(E_\gamma)}((f\circ R_\gamma)\cdot 1) \\ &= i^1_{X(E_\gamma)}(1\cdot f) = i^1_{X(E_\gamma)}(1) i^0_{X(E_\gamma)}(f) = s_\gamma i_\gamma(f), \end{aligned}$$

and so (i_γ, s_γ) is a Toeplitz pair. For $f \in C(\mathbb{S})_\gamma$ we have $i^1_{X(E_\gamma)}(f) = i_\gamma(f)s_\gamma$, so the pair $(i_\gamma, i^1_\eta(1))$ generates the ranges of both $i^0_{X(E_\gamma)}$ and $i^1_{X(E_\gamma)}$, and hence all of $\mathcal{T}(E_\gamma)$.

Now suppose B is a unital C^*-algebra and $\pi : C(\mathbb{S}) \to B$ and $S \in B$ form a Toeplitz pair (π, S) for E_γ in B. Define $\psi : C(\mathbb{S})_\gamma \to B$ by $\psi(f) = \pi(f)S$. We claim that (ψ,π) is a representation of $C(\mathbb{S})_\gamma$ in B. For each $f \in C(\mathbb{S})_\gamma$ and $g \in C(\mathbb{S})$ we have

$$\pi(g)\psi(f) = \pi(g)\pi(f)S = \pi(gf)S = \psi(gf) = \psi(g\cdot f)$$

and

$$\psi(f)\pi(g) = \pi(f)S\pi(g) = \pi(f(g\circ R_\gamma))S = \psi(f(g\circ R_\gamma)) = \psi(f\cdot g).$$

To check that the inner product is preserved, we let $f, h \in C(\mathbb{S})_\gamma$ and calculate

$$\begin{aligned} \psi(f)^*\psi(h) &= S^*\pi(f^*)\pi(h)S = S^*\pi(f^*\circ R_\gamma^{-1}\circ R_\gamma)\pi(h\circ R_\gamma^{-1}\circ R_\gamma)S \\ &= \pi(f^*\circ R_\gamma^{-1})S^*S\pi(h\circ R_\gamma^{-1}) = \pi((f^*h)\circ R_\gamma^{-1}). \end{aligned}$$

We have $\langle f,h\rangle_\gamma(z) = \overline{f(R_\gamma^{-1}(z))}g(R_\gamma^{-1}(z)) = (f^*g)\circ R_\gamma^{-1}(z)$. So $\langle f,h\rangle_\gamma = (f^*h)\circ R_\gamma^{-1}$, and hence $\psi(f)^*\psi(h) = \pi(\langle f,h\rangle_\gamma)$. This proves the claim.

The universal property of $\mathcal{T}(E_\gamma)$ yields a homomorphism $\psi\times\pi : \mathcal{T}(E_\gamma) \to B$ satisfying $(\psi\times\pi)\circ i^1_{X(E_\gamma)} = \psi$ and $(\psi\times\pi)\circ i^0_{X(E_\gamma)} = \pi$. Let $\pi\times S := \psi\times\pi$. Then

$$(\pi\times S)\circ i^0_{X(E_\gamma)} = (\psi\times\pi)\circ i^0_{X(E_\gamma)} = \pi,$$

and

$$(\pi\times S)(s_\gamma) = (\psi\times\pi)(i^1_{X(E_\gamma)}(1)) = \psi(1) = \pi(1)S = S.$$

Hence $\mathcal{T}(E_\gamma)$ is the universal C^*-algebra generated by a Toeplitz pair for E_γ.

To prove (2) it suffices to show that the ideal I generated by

$$\{(i^1_{X(E_\gamma)}, i^0_{X(E_\gamma)})^{(1)}(\varphi_\gamma(f)) - i^0_{X(E_\gamma)}(f) : f \in C(\mathbb{S})\}$$

is the ideal generated by the element $s_\gamma s_\gamma^* - 1$. We have

$$s_\gamma s_\gamma^* - 1 = (i^1_{X(E_\gamma)}, i^0_{X(E_\gamma)})^{(1)}(\Theta_{1,1}) - i^0_{X(E_\gamma)} = (i^1_{X(E_\gamma)}, i^0_{X(E_\gamma)})^{(1)}(\phi_\eta(1)) - i^0_{X(E_\gamma)}(1) \in I,$$

and hence the ideal generated by $s_\gamma s_\gamma^* - 1$ is contained in I. For the reverse containment we first note that $\varphi_\gamma(f) = \Theta_{f,1}$ for all $f \in C(\mathbb{S})$. Then

$$\begin{aligned} (i^1_{X(E_\gamma)}, i^0_{X(E_\gamma)})^{(1)}(\varphi_\eta(f)) - i^0_\eta(f) &= (i^1_{X(E_\gamma)}, i^0_{X(E_\gamma)})^{(1)}(\Theta_{f,1}) - i^0_\eta(f) \\ &= i^1_{X(E_\gamma)}(f) i^1_{X(E_\gamma)}(1)^* - i^0_\eta(f) \\ &= i^0_{X(E_\gamma)}(f) i^1_{X(E_\gamma)}(1) i^1_{X(E_\gamma)}(1)^* - i^0_{X(E_\gamma)}(f) \\ &= i^0_{X(E_\gamma)}(f)\big(s_\gamma s_\gamma^* - 1\big), \end{aligned}$$

and the result follows. □

Remarks 4.3. (1) We saw in the proof of Proposition 4.2 that a Toeplitz pair (π, S) for E_γ gives a representation (ψ,π) of $X(E_\gamma)$ such that $\psi(f) = \pi(f)S$. We denote the homomorphism $(\psi,\pi)^{(1)}$ of $\mathcal{K}(X(E_\gamma))$ by $(\pi,S)^{(1)}$; so $(\pi,S)^{(1)}(\Theta_{f,g}) = \pi(f)SS^*\pi(g)^*$.

(2) In [17, Theorem 6.2] Katsura proved a gauge-invariant uniqueness theorem for the Toeplitz algebra of a Hilbert bimodule. Suppose A is a C^*-algebra, X is a Hilbert A-bimodule, and (ψ,π) is a representation of X in a C^*-algebra B. The gauge-invariant

uniqueness theorem says that $\psi \times \pi : \mathcal{T}(X) \to B$ is injective if B carries a gauge action, $\psi \times \pi$ intertwines the gauge actions on $\mathcal{T}(X)$ and B, and the ideal

$$\{a \in A : \pi(a) \in (\psi, \pi)^{(1)}(\mathcal{K}(X))\}$$

of A is trivial. If (π, S) is a Toeplitz pair for E_γ, then this ideal is $\{f \in C(\mathbb{S}) : \pi(f) \in (\pi, S)^{(1)}(\mathcal{K}(X(E_\gamma)))\}$, which we can write as

$$\{f \in C(\mathbb{S}) : \pi(f) \in \overline{\operatorname{span}}\{\pi(g)SS^*\pi(h) : g, h \in C(\mathbb{S})\}\}.$$

We denote this ideal by $I_{(\pi,S)}$.

(3) Proposition 4.10 of [17] says that $I_{(i_\gamma, s_\gamma)} = 0$.

We can give spanning families for $\mathcal{T}(E_\gamma)$ and $\mathcal{O}(E_\gamma)$ using Toeplitz and covariant pairs.

Proposition 4.4. *Let $\gamma \in \mathbb{R}$ and $E_\gamma = (\mathbb{S}, \mathbb{S}, \mathrm{id}_{\mathbb{S}}, R_\gamma)$. Then*

$$\mathcal{T}(E_\gamma) = \overline{\operatorname{span}}\{s_\gamma^m i_\gamma(f) s_\gamma^{*n} : m, n \in \mathbb{N},\ f \in C(\mathbb{S})\},$$

and

$$\mathcal{O}(E_\gamma) = \overline{\operatorname{span}}\{w_\gamma^m j_\gamma(f) w_\gamma^{*n} : m, n \in \mathbb{N},\ f \in C(\mathbb{S})\}.$$

Proof. The set $\operatorname{span}\{s_\gamma^m i_\gamma(f) s_\gamma^{*n} : m, n \in \mathbb{N},\ f \in C(\mathbb{S})\}$ contains the generators of $\mathcal{T}(E_\gamma)$, so it suffices to show that it is a $*$-subalgebra. It is obviously closed under involution; that it is closed under multiplication follows from the calculation

$$\begin{aligned} s_\gamma^m i_\gamma(f) s_\gamma^{*n} s_\gamma^p i_\gamma(g) s_\gamma^{*q} &= \begin{cases} s_\gamma^m i_\gamma(f) s_\gamma^{*n-p} i_\gamma(g) s_\gamma^{*q} & \text{if } n \geq p \\ s_\gamma^m i_\gamma(f) s_\gamma^{p-n} i_\gamma(g) s_\gamma^{*q} & \text{if } n < p \end{cases} \\ &= \begin{cases} s_\gamma^m i_\gamma(f(g \circ R_\gamma^{-(n-p)})) s_\gamma^{*n-p+q} & \text{if } n \geq p \\ s_\gamma^{m+p-n} i_\gamma((f \circ R_\gamma^{-(p-n)})g) s_\gamma^{*q} & \text{if } n < p. \end{cases} \end{aligned}$$

Since each $s_\gamma^m i_\gamma(f) s_\gamma^{*n}$ is mapped to $w_\gamma^m j_\gamma(f) w_\gamma^{*n}$ under the quotient map $\mathcal{T}(E_\gamma) \to \mathcal{O}(E_\gamma)$, we have $\mathcal{O}(E_\gamma) = \overline{\operatorname{span}}\{w_\gamma^m j_\gamma(f) w_\gamma^{*n} : m, n \in \mathbb{N},\ f \in C(\mathbb{S})\}$. □

5. An alternative description of the noncommutative solenoid

Throughout the rest of this paper we fix a natural number $N \geq 2$. In [22], given a sequence $\theta = (\theta_n)_{n=1}^\infty$ in $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ such that $N^2\theta_{n+1} = \theta_n$ for all n, Latrémolière and Packer define the noncommutative solenoid $\mathcal{A}_\theta^{\mathscr{S}}$ as a twisted group C^*-algebra involving the N-adic rationals. In [22, Theorem 3.7] they give an equivalent characterisation of $\mathcal{A}_\theta^{\mathscr{S}}$. We will take this characterisation as our definition. We recall it now. Let

$$\Xi_N := \{(\theta_n)_{n=0}^\infty : \theta_n \in \mathbb{S} \text{ and } N^2\theta_{n+1} = \theta_n \text{ for each } n\}.$$

Recall that for $\gamma \in \mathbb{S}$ the rotation algebra $\mathcal{A}_\gamma$ is the universal C^*-algebra generated by unitaries U_γ and V_γ satisfying $U_\gamma V_\gamma = e^{2\pi i\gamma} V_\gamma U_\gamma$.

Definition 5.1. Let $\theta = (\theta_n)_{n=0}^\infty \in \Xi_N$, and for each $n \in \mathbb{N}$ let $\varphi_n : \mathcal{A}_{\theta_n} \to \mathcal{A}_{\theta_{n+1}}$ be the homomorphism satisfying

$$\varphi_n(U_{\theta_n}) = U_{\theta_{n+1}}^N \quad \text{and} \quad \varphi_n(V_{\theta_n}) = V_{\theta_{n+1}}^N.$$

The *noncommutative solenoid* $\mathcal{A}_\theta^{\mathscr{S}}$ is the direct limit $\varinjlim(\mathcal{A}_{\theta_n}, \varphi_n)$.

Remark 5.2. We have taken a slightly different point of view to [22] in describing $\mathcal{A}_\theta^{\mathscr{S}}$. In [22], Latrémolière and Packer consider collections of (θ_n) such that $N\theta_{n+1} - \theta_n \in \mathbb{Z}$, and take the direct limit $\varinjlim \mathcal{A}_{\theta_{2n}}$, with intertwining maps going from $\mathcal{A}_{\theta_{2n}}$ to $\mathcal{A}_{\theta_{2n+2}}$.

We now give an alternative characterisation of the noncommutative solenoid using topological graphs built from rotations of the circle as discussed in Section 4.

Notation 5.3. We denote by $\iota : \mathbb{S} \to \mathbb{T}$ the homeomorphism $t \mapsto e^{2\pi it}$, and by $p_N : \mathbb{S} \to \mathbb{S}$ the function $t \mapsto Nt$.

Proposition 5.4. *Let $N \geq 2$, and $\theta = (\theta_n)_{n=0}^{\infty} \in \Xi_N$. For each $n \in \mathbb{N}$ there is an injective homomorphism $\tau_n : \mathcal{O}(E_{\theta_n}) \to \mathcal{O}(E_{\theta_{n+1}})$ satisfying*

$$\tau_n(j_{\theta_n}(f)) = j_{\theta_{n+1}}(f \circ p_N) \quad and \quad \tau_n(w_{\theta_n}) = w_{\theta_{n+1}}^N,$$

for all $f \in C(\mathbb{S})$. Moreover $\varinjlim(\mathcal{O}(E_{\theta_n}), \tau_n) \cong \mathcal{A}_\theta^{\mathscr{S}}$.

We will prove the existence of the injective homomorphisms τ_n using the following result.

Lemma 5.5. *Let $N \in \mathbb{N}$ with $N \geq 2$, and take $\gamma, \eta \in \mathbb{S}$ with $N^2\eta - \gamma \in \mathbb{Z}$. Then there is an injective homomorphism $\psi : \mathcal{T}(E_\gamma) \to \mathcal{T}(E_\eta)$ satisfying*

$$\psi(i_\gamma(f)) = i_\eta(f \circ p_N) \quad and \quad \psi(s_\gamma) = s_\eta^N,$$

for all $f \in C(\mathbb{S})$. The map ψ descends to an injective homomorphism $\tau : \mathcal{O}(E_\gamma) \to \mathcal{O}(E_\eta)$ satisfying $\tau(j_\gamma(f)) = j_\eta(f \circ p_N)$ and $\tau(w_\gamma) = w_\eta^N$ for all $f \in C(\mathbb{S})$.

Proof. Consider $\pi : C(\mathbb{S}) \to \mathcal{T}(E_\eta)$ given by $\pi(f) = i_\eta(f \circ p_N)$ and let $S := s_\eta^N$. Since

$$R_\gamma \circ p_N = R_{N^2\eta} \circ p_N = R_\eta^{N^2} \circ p_N = p_N \circ R_\eta^N,$$

we have

$$\pi(f \circ R_\gamma)S = i_\eta(f \circ R_\gamma \circ p_N)s_\eta^N = i_\eta(f \circ p_N \circ R_\eta^N)s_\eta^N = s_\eta^N i_\eta(f \circ p_N) = S\pi(f).$$

So (π, S) is a Toeplitz pair for E_γ. The universal property of $\mathcal{T}(E_\gamma)$ now gives a homomorphism $\psi : \mathcal{T}(E_\gamma) \to \mathcal{T}(E_\eta)$ satisfying $\psi_n(i_\gamma(f)) = i_\eta(f \circ p_N)$ for all $f \in C(\mathbb{S})$, and $\psi(s_\gamma) = s_\eta^N$.

To see that ψ is injective, we aim to apply the gauge-invariant uniqueness theorem discussed in Remarks 4.3. We claim that $I_{(\pi,S)} \neq 0 \implies I_{(i_\eta,s_\eta)} \neq 0$. To see this, suppose that $0 \neq f \in I_{(\pi,S)}$. Fix $\epsilon > 0$, and choose $g_i, h_i \in C(\mathbb{S})$ with

$$\Big\|\pi(f) - \sum_{i=1}^k \pi(g_i)SS^*\pi(h_i)\Big\| < \epsilon.$$

So

$$\Big\|i_\eta(f \circ p_N) - \sum_{i=1}^k i_\eta(g_i \circ p_N)s_\eta^N s_\eta^{*N} i_\eta(h_i \circ p_N)\Big\| < \epsilon.$$

For every function $g \in C(\mathbb{S})$ we have

$$i_\eta(g \circ R_{-\eta}^{N-1}) = s_\eta^{*N-1} s_\eta^{N-1} i_\eta(g \circ R_{-\eta}^{N-1}) = s_\eta^{*N-1} i_\eta(g) s_\eta^{N-1}.$$

Hence

$$\begin{aligned}\Big\|i_\eta(f \circ p_N \circ R_{-\eta}^{N-1}) &- \sum_{i=1}^k i_\eta(g_i \circ p_N \circ R_{-\eta}^{N-1})s_\eta s_\eta^* i_\eta(h_i \circ p_N \circ R_{-\eta}^{N-1})\Big\| \\ &= \Big\|s_\eta^{*N-1} i_\eta(f \circ p_N){s_\eta}^{N-1} - \sum_{i=1}^k s_\eta^{*N-1} i_\eta(g_i \circ p_N)s_\eta^N s_\eta^{*N} i_\eta(h_i \circ p_N){s_\eta}^{N-1}\Big\| \\ &\leq \Big\|i_\eta(f \circ p_N) - \sum_{i=1}^k i_\eta(g_i \circ p_N)s_\eta^N s_\eta^{*N} i_\eta(h_i \circ p_N)\Big\| < \epsilon.\end{aligned}$$

It follows that $i_\eta(f \circ p_N \circ R_{-\eta}^{N-1}) \in (i_\eta, s_\eta)^{(1)}(\mathcal{K}(X(E_\eta)))$, and hence that $f \circ p_N \circ R_{-\eta}^{N-1} \in I_{(i_\eta,s_\eta)}$. This proves the claim.

By Remarks 4.3(3), $I_{(i_\eta,s_\eta)} = 0$, so the claim gives $I_{(\pi,S)} = 0$. We have $\psi(\mathcal{T}(E_\gamma)) \subseteq \overline{\operatorname{span}}\{s_\eta^{aN} i_\eta(f) s_\eta^{*bN} : f \in C(\mathbb{S}), a, b \in \mathbb{N}\}$. Hence the gauge action ρ^η of $\mathbb{T}$ on $\mathcal{T}(E_\gamma)$ satisfies $\rho_z^\eta \circ \psi = \rho_{z+e^{2\pi i/N}}^\eta \circ \psi$ for all $z \in \mathbb{T}$. So there is an action $\tilde{\rho}^\eta$ of $\mathbb{T}$ on $\psi(\mathcal{T}(E_\gamma))$ such that

$\tilde{\rho}^\eta_{e^{2\pi it}} \circ \psi = \rho e^{e\pi it/N}$ for all $t \in \mathbb{R}$. In particular,

$$\begin{aligned}\tilde{\rho}^\eta_{e^{2\pi it}} \circ \psi(s^a_\gamma i_\eta(f) s^{*b}_\gamma) &= \rho^\eta_{e^{2\pi it/N}}(s^{aN}_\eta i_\eta(f) s^{*bN}_\eta)\\ &= e^{2\pi it(a-b)} s^{aN}_\eta i_\eta(f) s^{*bN}_\eta = \psi \circ \rho^\gamma_{e^{2\pi it}}(s^{aN}_\eta i_\eta(f) s^{*bN}_\eta).\end{aligned}$$

So continuity and linearity gives $\tilde{\rho}^\eta_{e^{2\pi it}} = \psi \circ \rho^\gamma_{e^{2\pi it}}$. Hence the gauge-invariant uniqueness theorem [17, Theorem 6.2] shows that ψ is injective.

To see that ψ descends to the desired injective homomorphism $\tau : \mathcal{O}(E_\gamma) \to \mathcal{O}(E_\eta)$, it suffices to show that the image under ψ of the kernel of the quotient map $\mathcal{T}(E_\gamma) \to \mathcal{O}(E_\gamma)$ is contained in the kernel of $\mathcal{T}(E_\eta) \to \mathcal{O}(E_\eta)$. For this, it suffices to show that $\psi(1 - s_\gamma s^*_\gamma)$ is in the ideal generated by $1 - s_\eta s^*_\eta$, which it is becase

$$\psi(1 - s_\gamma s^*_\gamma) = 1 - s^N_\eta s^{*N}_\eta = \sum_{i=1}^N s^{N-i}_\eta (1 - s_\eta s_\eta) s^{*N-i}_\eta. \qquad \square$$

Proof of Proposition 5.4. For each $n \in \mathbb{N}$, Lemma 5.5 applied to $\gamma = \theta_n$ and $\eta = \theta_{n+1}$ gives the desired injective homomorphism τ_n.

Proposition 4.2 says that each $\mathcal{O}(E_\eta)$ is the crossed product $C(\mathbb{S}) \rtimes \mathbb{Z}$ for the automorphism $f \mapsto f \circ R_\eta$ of $C(\mathbb{S})$, which is the rotation algebra $\mathcal{A}_\eta$ (see [7, Example VIII.1.1] for details). So for each $n \in \mathbb{N}$ there is an isomorphism from $\mathcal{O}(E_{\theta_n})$ to $\mathcal{A}_{\theta_n}$ carrying $j_{\theta_n}(\iota)$ to U_{θ_n} and w_{θ_n} to V_{θ_n}. Since each τ_n satisfies

$$\tau_n(j_{\theta_n}(\iota)) = j_{\theta_{n+1}}(\iota \circ p_N) = j_{\theta_{n+1}}(\iota)^N \quad \text{and} \quad \tau_n(w_{\theta_n}) = w^N_{\theta_{n+1}},$$

the diagrams

$$\begin{array}{ccc} \mathcal{O}(E_{\theta_n}) & \xrightarrow{\tau_n} & \mathcal{O}(E_{\theta_{n+1}}) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{A}_{\theta_n} & \xrightarrow{\varphi_n} & \mathcal{A}_{\theta_{n+1}} \end{array}$$

commute. Hence $\varinjlim(\mathcal{O}(E_{\theta_n}), \tau_n) \cong \mathcal{A}^{\mathscr{S}}_\theta$. $\square$

Remark 5.6. In [18, Section 2], Katsura studies factor maps between topological-graph C^*-algebras, and the C^*-homomorphisms that they induce. He shows that the projective limit of a sequence (E_n) of topological graphs under factor maps is itself a topological graph. He then proves that the C^*-algebra $\mathcal{O}(\varprojlim E_n)$ of this topological graph is isomorphic to the direct limit $\varinjlim \mathcal{O}(E_n)$ of the C^*-algebras of the E_n under the homomorphisms induced by the factor maps. So it is natural to ask whether the maps $\tau_n : \mathcal{O}(E_{\theta_n}) \to \mathcal{O}(E_{\theta_{n+1}})$ correspond to factor maps. This is not the case: as observed on page 88 of [11], there is no factor map from $E_{\theta_{n+1}} \to E_{\theta_n}$ that induces the homomorphism of C^*-algebras described in Lemma 5.5.

6. The Toeplitz noncommutative solenoid and its KMS structure

In this section we introduce our Toeplitz noncommutative solenoids $\mathcal{T}^{\mathscr{S}}_\theta$. We introduce a natural dynamics on $\mathcal{T}^{\mathscr{S}}_\theta$ and apply Proposition 3.1 to begin to study its KMS structure.

Given $\theta = (\theta_n)_{n=0}^\infty \in \Xi_N$, Lemma 5.5 gives a sequence of injective homomorphisms $\psi_n : \mathcal{T}(E_{\theta_n}) \to \mathcal{T}(E_{\theta_{n+1}})$ satisfying

$$\psi_n(i_{\theta_n}(f)) = i_{\theta_{n+1}}(f \circ p_N) \quad \text{and} \quad \psi_n(s_{\theta_n}) = s^N_{\theta_{n+1}},$$

for all $f \in C(\mathbb{S})$.

Definition 6.1. We define $\mathcal{T}^{\mathscr{S}}_\theta := \varinjlim(\mathcal{T}(E_{\theta_n}), \psi_n)$ and call it the *Toeplitz noncommutative solenoid.* We write $\psi_{n,\infty} : \mathcal{T}(E_{\theta_n}) \to \mathcal{T}^{\mathscr{S}}_\theta$ for the canonical inclusions, so that $\psi_{n,\infty} = \psi_{n+1,\infty} \circ \psi_n$ for all n.

The following lemma indicates why it is sensible to regard $\mathcal{T}^{\mathscr{S}}_\theta$ as a natural Toeplitz extension of the noncommutative solenoid.

Lemma 6.2. *In the notation established in Proposition 5.4, there is a surjective homomorphism $q : \mathcal{T}_\theta^{\mathscr{S}} \to \mathcal{A}_\theta^{\mathscr{S}}$ such that $q(\psi_{n,\infty}(i_{\theta_n}(f))) = \tau_{n,\infty}(j_{\theta_n}(f))$ and $q(\psi_{n,\infty}(s_{\theta_n})) = \tau_{n,\infty}(w_{\theta_n})$ for all $n \in \mathbb{N}$ and all $f \in C(\mathbb{S})$. Moreover, $\ker(q)$ is generated as an ideal by $\psi_{1,\infty}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)$.*

Proof. For the first statement observe that the canonical homomorphisms $q_n : \mathcal{T}(E_{\theta_n}) \to \mathcal{O}(E_{\theta_n})$ intertwine the ψ_n with the τ_n. For the second statement, let I be the ideal of $\mathcal{T}_\theta^{\mathscr{S}}$ generated by $\psi_{1,\infty}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)$. Since $\ker(q)$ clearly contains $\psi_{1,\infty}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)$, we have $I \subseteq \ker(q)$. For the reverse inclusion, note that for $n \geq 1$,

$$\begin{aligned}\psi_{1,n}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*) &= i_{\theta_n}(1 \circ \iota_{N^n}) - s_{\theta_n}^{nN} s_{\theta_n}^{*nN} \\ &= i_{\theta_n}(1) - s_{\theta_n}(s_{\theta_n}^{nN-1} s_{\theta_n}^{*(nN-1)})s_{\theta_n}^* \geq i_{\theta_n}(1) - s_{\theta_n}s_{\theta_n}^*,\end{aligned}$$

so each $\psi_{n,\infty}(i_{\theta_n}(1) - s_{\theta_n}s_{\theta_n}^*) \leq \psi_{n,\infty}(\psi_{1,n}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)) = \psi_{1,\infty}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)$, which belongs to I. Thus $\psi_{n,\infty}(i_{\theta_n}(1) - s_{\theta_n}s_{\theta_n}^*) \in I$. Since $\ker(q) = \overline{\bigcup_n \ker(q) \cap \psi_{n,\infty}(\mathcal{T}(E_{\theta_n}))} = \overline{\bigcup_n \psi_{n,\infty}(\ker(q_n))}$, it therefore suffices to show that each $\ker(q_n)$ is generated by $i_{\theta_n}(1) - s_{\theta_n}s_{\theta_n}^*$, which follows from Proposition 4.2. □

Proposition 6.3. *There is a strongly continuous action $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}_\theta^{\mathscr{S}}$ satisfying*

$$\alpha_t(\psi_{j,\infty}(s_{\theta_j}^m i_{\theta_j}(f) s_{\theta_j}^{*n})) = e^{it(m-n)/N^j} \psi_{j,\infty}(s_{\theta_j}^m i_{\theta_j}(f) s_{\theta_j}^{*n}), \tag{6.1}$$

for each $j, m, n \in \mathbb{N}$ and $f \in C(\mathbb{S})$. This α descends to a strongly continuous action, also written α, on the noncommutative solenoid $\mathcal{A}_\theta^{\mathscr{S}}$.

Proof. For each $j \in \mathbb{N}$ we denote by ρ the gauge action on $\mathcal{T}(E_{\theta_j})$, and by ρ_j the strongly continuous action $t \mapsto \rho_{e^{it/N^j}}$ of $\mathbb{R}$ on $\mathcal{T}(E_{\theta_j})$; so $\rho_{j,t} \circ i_{\theta_j} = i_{\theta_j}$ and $\rho_{j,t}(s_{\theta_j}) = e^{it/N^j} s_{\theta_j}$ for each $t \in \mathbb{R}$. For each $j \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}\rho_{j+1,t} \circ \psi_j(s_{\theta_j}^m i_{\theta_j}(f) s_{\theta_j}^{*n}) &= e^{it(Nm-Nn)/N^{j+1}} s_{\theta_{j+1}}^{Nm} i_{\theta_{j+1}}(f) s_{\theta_{j+1}}^{*Nn} \\ &= e^{it(m-n)/N^j} s_{\theta_{j+1}}^{Nm} i_{\theta_{j+1}}(f) s_{\theta_{j+1}}^{*Nn} = \psi_j \circ \rho_{j,t}(s_{\theta_j}^m i_{\theta_j}(f) s_{\theta_j}^{*n}).\end{aligned}$$

Hence $\rho_{j+1,t} \circ \psi_j = \psi_j \circ \rho_{j,t}$, and Proposition 3.1 applied to each $(A_j, \alpha_j) = (\mathcal{T}(E_{\theta_j}), \rho_j)$ gives the desired action $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}_\theta^{\mathscr{S}}$.

For the final statement, observe that the α_t all fix $\psi_{1,\infty}(i_{\theta_1}(1) - s_{\theta_1}s_{\theta_1}^*)$, and so leave the ideal that it generates invariant; so they descend to $\mathcal{A}_\theta^{\mathscr{S}}$ by Lemma 6.2. □

Remark 6.4. The actions on graph C^*-algebras and their analogues studied in, for example, [5, 8, 1, 12] are lifts of circle actions, and so are periodic in the sense that $\alpha_t = \alpha_{t+2\pi}$ for all t. By contrast, while the action α of the preceding proposition restricts to a periodic action on each approximating subalgebra $\psi_{j,\infty}(\mathcal{T}(E_{\theta_j}))$, it is itself not periodic: $\alpha_t = \alpha_s \implies t = s$.

We now wish to study the KMS structure of the Toeplitz noncommutative solenoid $\mathcal{T}_\theta^{\mathscr{S}}$ under the dynamics α of Proposition 6.3.

Remark 6.5. The case $\theta = \mathbf{0} = (0, 0, \dots)$ is relatively easy to analyse. Let $\mathscr{S} = \varprojlim(\mathbb{S}, p_N)$ denote the classical solenoid, and $\mathcal{T}$ the Toeplitz algebra. Write s for the isometry generating $\mathcal{T}$, and $\kappa : \mathcal{T} \to \mathcal{T}$ for the homomorphism given by $\kappa(s) = s^N$. Then $\mathcal{T}_\mathbf{0}^{\mathscr{S}} \cong C(\mathscr{S}) \otimes \varinjlim(\mathcal{T}, \kappa)$. This isomorphism intertwines the quotient map $q : \mathcal{T}_\mathbf{0}^{\mathscr{S}} \to \mathcal{A}_\mathbf{0}^{\mathscr{S}}$ with the canonical quotient map $\mathrm{id} \otimes \tilde{q} : C(\mathscr{S}) \otimes \varinjlim(\mathcal{T}, \kappa) \to C(\mathscr{S}) \otimes C(\mathscr{S})$. It also intertwines α with $1 \otimes \tilde{\alpha}$ where $\tilde{\alpha}_t(\kappa_{j,\infty}(s)) = e^{it/N^j} \kappa_{j,\infty}(s)$. That is, $\tilde{\alpha}$ is equivariant over $\kappa_{j,\infty}$ with an action $\tilde{\alpha}_j$ on $\mathcal{T}$ that is a rescaling of the gauge dynamics studied in [12]. Theorems 3.1 and 4.3 of [12] imply that $(\mathcal{T}, \tilde{\alpha}_j)$ has a unique KMS_β state for every $\beta \geq 0$ and has no KMS_β states for $\beta < 0$, and that the KMS_0 state is the only one that factors through $C(\mathbb{S})$. So Proposition 3.1 implies that $(\varinjlim(\mathcal{T}, \kappa), \tilde{\alpha})$ has a unique KMS_β state ϕ_β for each $\beta \geq 0$ and has no KMS_β states for $\beta < 0$, and that the KMS_0 state is the only one that factors through $C(\mathscr{S})$. Hence the map $\psi \mapsto \psi \otimes \phi_\beta$ determines an affine isomorphism of the state space of $C(\mathscr{S})$ onto $\mathrm{KMS}_\beta(\mathcal{T}_\mathbf{0}^{\mathscr{S}}, \alpha)$

for each $\beta \geq 0$, there are no KMS_β states for $\beta < 0$, and the KMS_0 states are the only ones that factor through $\mathcal{A}_{\mathbf{0}}^{\mathscr{S}}$.

In light of Remark 6.5, we will from now on consider only those $\theta \in \Xi_N$ such that $\theta_j \neq 0$ for some j. Since $\theta_j \neq 0$ implies $\theta_{j+1} \neq 0$, and since $\varinjlim((\mathcal{A}_{\theta_n}, \varphi_n)_{n=1}^\infty) = \varinjlim((\mathcal{A}_{\theta_n}, \varphi_n)_{n=j}^\infty)$ for any j, we may therefore assume henceforth that $\theta_j \neq 0$ for all j.

Our main result is the following.

Theorem 6.6. *Take $N \in \{2, 3, \dots\}$, take $\theta = (\theta_j)_{j=0}^\infty \in \Xi_N$, and take $\beta \in (0, \infty)$. Suppose that $\theta_j \neq 0$ for all j. Then $\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ is isomorphic to the Choquet simplex of regular Borel probability measures on the solenoid $\mathscr{S} := \varprojlim(\mathbb{S}, p_N)$, and there is an action λ of $\mathscr{S}$ on $\mathcal{T}_\theta^{\mathscr{S}}$ that induces a free and transitive action of $\mathscr{S}$ on the extreme boundary of $\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$. There is a unique KMS_0-state on $\mathcal{T}_\theta^{\mathscr{S}}$ for α, and this is the only KMS state for α that factors through $\mathcal{A}_\theta^{\mathscr{S}}$. There are no KMS_β states for $\beta < 0$.*

The first step in proving Theorem 6.6 is to combine the results of [1] on KMS states of local homeomorphism C^*-algebras with Proposition 3.1 to characterise the KMS states of $\mathcal{T}_\theta^{\mathscr{S}}$ in terms of subinvariant probability measures on the circle. We start with some notation.

It is helpful to recall what the results of [1] say in the context of the topological graphs E_γ. Recall that ρ denotes tha gauge action on $\mathcal{T}(E_\gamma)$; we also use ρ for the lift of the gauge action to an action of $\mathbb{R}$ on $\mathcal{T}(E_\gamma)$. Combining Proposition 4.2 and Theorem 5.1 of [1], we see that for each regular Borel probability measure μ on $\mathbb{S}$ that is subinvariant in the sense that $\mu(R_\gamma(U)) \leq e^\beta \mu(U)$ for every Borel $U \subseteq \mathbb{S}$, there is a KMS_β-state $\phi_\mu \in \mathrm{KMS}_\beta(\mathcal{T}(E_\gamma), \rho)$ satisfying

$$\phi_\mu(s_\gamma^a i_\gamma(f) s_\gamma^{*b}) = \delta_{a,b} e^{-a\beta} \int_{\mathbb{S}} f \, d\mu; \tag{6.2}$$

and moreover, the map $\mu \mapsto \phi_\mu$ is an affine isomorphism of the simplex of subinvariant regular Borel probability measures on $\mathbb{S}$ to $\mathrm{KMS}_\beta(\mathcal{T}(E_\gamma), \rho)$.

Definition 6.7. Fix $r, s \in [0, \infty)$, and $\gamma \in \mathbb{S}$. Let $M(\mathbb{S})$ denote the set of regular Borel probability measures on $\mathbb{S}$. We define

$$M_{\mathrm{sub}}(s, \gamma) := \{m \in M(\mathbb{S}) : m(R_\gamma(U)) \leq e^s m(U) \text{ for all Borel } U \subseteq \mathbb{S}\}$$

and

$$\Omega_{\mathrm{sub}}^r := \{m \in M(\mathbb{S}) : m(R_t(U)) \leq e^{rt} m(U) \text{ for all } t \in [0, \infty) \text{ and Borel } U \subseteq \mathbb{S}\}. \tag{6.3}$$

Notation 6.8. For the rest of the section we fix $\theta = (\theta_j)_{j=0}^\infty \in \Xi_N$ such that $\theta_j \neq 0$ for all j, and $\beta \in [0, \infty)$. We define

$$r_j := \beta / N^j \theta_j \quad \text{for all } j \in \mathbb{N}.$$

Theorem 6.9. *Take $N \in \mathbb{N}$ with $N \geq 2$, $\theta = (\theta_j)_{j=0}^\infty \in \Xi_N$, and $\beta \in [0, \infty)$. Suppose that $\theta_j \neq 0$ for all j. Then there is an affine injection*

$$\omega : \mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha) \to \varprojlim(\Omega_{\mathrm{sub}}^{r_j}, m \mapsto m \circ p_N^{-1})$$

such that

$$\phi \circ \psi_{j,\infty}(s_{\theta_j}^a i_{\theta_j}(f) s_{\theta_j}^{*b}) = \delta_{a,b} e^{-a\beta/N^j} \int_{\mathbb{S}} f \, d\omega(\phi)_j \tag{6.4}$$

for each $\phi \in \mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ and $j \in \mathbb{N}$. If $\beta > 0$, then ω is an isomorphism.

Write ρ for the gauge action on $\mathcal{T}(E_{\theta_j})$, and ρ_j for the action $t \mapsto \rho_{e^{it/N^j}}$ of $\mathbb{R}$ on $\mathcal{T}(E_{\theta_j})$. Since the dynamics α on $\mathcal{T}_\theta^{\mathscr{S}}$ is induced by the ρ_j, Proposition 3.1 yields an affine isomorphism

$$\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha) \cong \varprojlim(\mathrm{KMS}_\beta(\mathcal{T}(E_{\theta_j}), \rho_j), \phi \mapsto \phi \circ \psi_{j-1}).$$

For each $j \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $\rho_{j,t} = \rho_{t/N^j}$, so $\mathrm{KMS}_\beta(\mathcal{T}(E_{\theta_j}), \rho_j) = \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho)$. For $\beta > 0$, the KMS_β simplex of each $(\mathcal{T}(E_{\theta_j}), \rho)$ is well understood by the results of [1] (see the discussion preceding (6.2)), and we use these results to prove the following.

Proposition 6.10. *With the hypotheses of Theorem 6.9, there is an affine injection*

$$\tau : \varprojlim(\mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho), \phi \mapsto \phi \circ \psi_{j-1}) \to \varprojlim(M_{\mathrm{sub}}(\beta/N^j, \theta_j), m \mapsto m \circ p_N^{-1})$$

such that $\phi_j = \phi_{\tau((\phi_k)_{k=0}^\infty)_j}$*, as defined at* (6.2)*, for all* $(\phi_k)_{k=0}^\infty$ *and* $j \in \mathbb{N}$*. If* $\beta > 0$ *then* τ *is an isomorphism.*

Throughout the rest of this section we suppress intertwining maps in projective limits.

Proof of Proposition 6.10. We first claim that for each $j \in \mathbb{N}$ there is an affine injection τ_j of $\mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho)$ onto $M_{\mathrm{sub}}(\beta/N^j, \theta_j)$ satisfying

$$\phi(i_{\theta_j}(f)) = \int_{\mathbb{S}} f \, d(\tau_j(\phi)) \quad \text{for all } \phi \in \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho) \text{ and } f \in C(\mathbb{S}),$$

and that for $\beta > 0$, this τ_j is an isomorphism. The statement for $\beta > 0$ follows directly from [1, Theorem 5.1] (see (6.2).

To prove the claim for $\beta = 0$, recall that the KMS_0 states on $\mathcal{T}(E_{\theta_j})$ for ρ are the ρ-invariant traces. Let $(i_{\theta_j}, s_{\theta_j})$ be the universal Toeplitz pair for E_{θ_j}. If ϕ is a KMS_0-state, then (with the convention that $s^n := s^{*|n|}$ for $n < 0$),

$$\begin{aligned}\phi(s_{\theta_j}^n i_{\theta_j}(f) s_{\theta_j}^{*m}) &= \phi(i_{\theta_j}(f) s_{\theta_j}^{*m} s_{\theta_j}^n) = \phi(i_{\theta_j}(f) s_{\theta_j}^{n-m}) \\ &= \int_{\mathbb{S}} \phi(\rho_t(i_{\theta_j}(f) s_{\theta_j}^{n-m})) \, d\mu(t) = \delta_{m,n} \phi(i_{\theta_j}(f)).\end{aligned} \tag{6.5}$$

So the Riesz–Markov–Kakutani representation theorem gives a regular Borel probability measure m_ϕ on $\mathbb{S}$ such that $\phi(s_{\theta_j}^n i_{\theta_j}(f) s_{\theta_j}^{*m}) = \int_{\mathbb{S}} f(t) \, dm_\phi(t)$. For $f \in C(\mathbb{S})_+$, we have

$$\begin{aligned}\phi(i_{\theta_j}(f)) &\geq \phi\big(i_{\theta_j}(\sqrt{f}) s_{\theta_j} s_{\theta_j}^* i_{\theta_j}(\sqrt{f})\big) \\ &= \phi\big(s_{\theta_j}^* i_{\theta_j}(\sqrt{f}) i_{\theta_j}(\sqrt{f}) s_{\theta_j}\big) = \phi(s_{\theta_j}^* i_{\theta_j}(f) s_{\theta_j}) = \phi(i_{\theta_j}(f \circ R_{-\theta_j}).\end{aligned}$$

Hence $m_\phi(R_{\theta_j}(U)) \leq m_\phi(U)$ for all Borel U. So $\phi \mapsto m_\phi$ is an affine map from $\mathrm{KMS}_0(\mathcal{T}(E_{\theta_j}), \rho)$ and (6.5) shows that it is injective. This completes the proof of the claim.

For each $j \in \mathbb{N}$ let p_j be the projection from $\varprojlim \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho)$ to $\mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho)$, and π_j the projection from $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$ to $M_{\mathrm{sub}}(\beta/N^j, \theta_j)$. Fix an element $(\phi_j)_{j=0}^\infty$ of $\varprojlim \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho)$. For each $k \geq 1$ and $f \in C(\mathbb{S})$ we have

$$\begin{aligned}\int_{\mathbb{S}} f \, d(\tau_{k-1}(\phi_{k-1})) &= \phi_{k-1}(i_{\theta_{k-1}}(f)) = \phi_k(\psi_{k-1}(i_{\theta_{k-1}}(f))) \\ &= \phi_k(i_{\theta_k}(f \circ p_N)) = \int_{\mathbb{S}} (f \circ p_N) \, d(\tau_k(\phi_k)) = \int_{\mathbb{S}} f \, d(\tau_k(\phi_k) \circ p_N^{-1}),\end{aligned}$$

and hence $\tau_{k-1}(\phi_{k-1}) = \tau_k(\phi_k) \circ p_N^{-1}$. It follows that

$$\tau_{k-1} \circ p_{k-1}((\phi_j)_{j=0}^\infty) = \tau(\phi_{k-1}) = \tau_k(\phi_k) \circ p_N^{-1} = \tau_k \circ p_k((\phi_j)_{j=0}^\infty) \circ p_N^{-1},$$

for each $k \geq 1$. The universal property of $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$ yields a map

$$\tau : \varprojlim \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}), \rho) \to \varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j),$$

whose image is $\varprojlim \mathrm{range}(\tau_k)$, satisfying $\pi_k \circ \tau = \tau_k \circ p_k$ for each $k \in \mathbb{N}$. For $\beta > 0$, we have $\varprojlim \mathrm{range}(\tau_k) = \varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$, and otherwise it is a compact affine subset, so it now suffices to prove that τ is an affine isomorphism onto its range. Since τ is an injective map from a compact space to a Hausdorff space, it therefore suffices to show that it is affine and continuous.

Suppose $\sum_{i=1}^q \lambda_i(\phi^i_j)_{j=0}^\infty$ is a convex combination in $\varprojlim \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}),\rho)$. For each $k \in \mathbb{N}$ and $f \in C(\mathbb{S})$ we have

$$\begin{aligned}\int_{\mathbb{S}} f\, d\Big(\tau_k\Big(\sum_{i=1}^q \lambda_i\phi^i_k\Big)\Big) &= \Big(\sum_{i=1}^q \lambda_i\phi^i_k\Big)(i_{\theta_k}(f)) = \sum_{i=1}^q \lambda_i\phi^i_k(i_{\theta_k}(f)) \\ &= \sum_{i=1}^k \lambda_i \int_{\mathbb{S}} f\, d(\tau_k(\phi^i_k)) = \int_{\mathbb{S}} f\, d\Big(\sum_{i=1}^q \lambda_i\tau_k(\phi^i_k)\Big).\end{aligned}$$

So the Riesz–Markov–Kakutani representation theorem gives $\tau_k\Big(\sum_{i=1}^q \lambda_i\phi^i_k\Big) = \sum_{i=1}^q \lambda_i\tau_k(\phi^i_k)$, and it follows that τ is affine.

Straightforward arguments using that $\pi_k \circ \tau = \tau_k \circ p_k$ for each $k \in \mathbb{N}$, and that each τ_k is injective, show that τ is injective. We just need to show that τ is continuous. Let $((\phi^\lambda_j)_{j=0}^\infty)_{\lambda\in\Lambda}$ be a net in $\varprojlim \mathrm{KMS}_{\beta/N^j}(\mathcal{T}(E_{\theta_j}),\rho)$ converging in the initial topology to $(\phi_j)_{j=0}^\infty$. Then $p_k(((\phi^\lambda_j)_{j=0}^\infty)_{\lambda\in\Lambda}) = (\phi^\lambda_k)_{\lambda\in\Lambda}$ converges weak* to $p_k((\phi_j)_{j=0}^\infty) = \phi_k$ for each $k \in \mathbb{N}$. Since τ_k is continuous and $\pi_k\circ\tau = \tau_k\circ p_k$ for each $k \in \mathbb{N}$, we have that $\pi_k(\tau(((\phi^\lambda_j)_{j=0}^\infty)_{\lambda\in\Lambda})) = \tau_k((\phi^\lambda_k)_{\lambda\in\Lambda})$ converges weak* to $\tau_k(\phi_k) = \pi_k(\tau((\phi_j)_{j=0}^\infty))$. Hence $\tau(((\phi^\lambda_j)_{j=0}^\infty)_{\lambda\in\Lambda})$ converges in the initial topology to $\tau((\phi_j)_{j=0}^\infty)$. So τ is continuous. □

Remark 6.11. Fix $\beta > 0$. Let h be the affine isomorphism of Proposition 3.1 and let τ be the affine isomorphism of Proposition 6.10. Setting $\omega := \tau \circ h$ gives an affine isomorphism

$$\omega : \mathrm{KMS}_\beta(\mathcal{T}^{\mathscr{S}}_\theta, \alpha) \to \varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$$

satisfying $\phi \circ \psi_{j,\infty} = \phi_{\omega(\phi)_j}$ for each $\phi \in \mathrm{KMS}_\beta(\mathcal{T}^{\mathscr{S}}_\theta, \alpha)$ and $j \in \mathbb{N}$. So to prove Theorem 6.9 it now suffices to show that $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j) \cong \varprojlim \Omega^{r_j}_{\mathrm{sub}}$.

Fix $(m_j)_{j=0}^\infty \in \varprojlim \Omega^{r_j}_{\mathrm{sub}}$. Taking $t = \theta_j$ in the definition of $\Omega^{r_j}_{\mathrm{sub}}$ (see Definition 6.7) shows that $\Omega^{r_j}_{\mathrm{sub}} \subseteq M_{\mathrm{sub}}(\beta/N^j, \theta_j)$. Hence $\varprojlim \Omega^{r_j}_{\mathrm{sub}}$ is contained in $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$. So we need the reverse containment. We start with a lemma.

Lemma 6.12. *Let m be a regular Borel probability measure on $\mathbb{S}$, and fix $\gamma \in (0,1)$, $s \in [0,\infty)$ and $N \in \mathbb{N}$ with $N \geq 2$. Suppose that $m(R_{\gamma/N^k}(U)) \leq e^{s/N^k} m(U)$ for every $k \in \mathbb{N}$ and every Borel set $U \subseteq \mathbb{S}$. Then $m \in \Omega^{s/\gamma}_{\mathrm{sub}}$.*

Proof. We need to show that $m(R_t(U)) \leq e^{(s/\gamma)t} m(U)$ for all $t \geq 0$ and Borel $U \subseteq \mathbb{S}$; or equivalently, that $m(R_{t\gamma}(U)) \leq e^{st} m(U)$ for all $t \geq 0$ and Borel $U \subseteq \mathbb{S}$. By the Riesz–Markov–Kakutani representation theorem, it suffices to show that

$$\int_{\mathbb{S}} f \circ R_{-t\gamma}\, dm \leq e^{st} \int_{\mathbb{S}} f\, dm \tag{6.6}$$

for every $t \geq 0$ and every $f \in C(\mathbb{S})_+$. Furthermore, if (6.6) holds whenever $0 \leq t \leq 1$, then for arbitrary $T \in [0,\infty)$, we can iterate (6.6) $\lceil T \rceil$ times for $t = \frac{T}{\lceil T\rceil}$ to obtain (6.6) for T; so it suffices to establish (6.6) for $t \in [0,1]$.

Fix $t \in [0,1]$ and $f \in C(\mathbb{S})$. Write

$$t = \sum_{i=1}^\infty \frac{a_i}{N^i}$$

where each $a_i \in \{0,\dots,N-1\}$. For each $n \in \mathbb{N}$, let $t_n := \sum_{i=1}^n \frac{a_i}{N^i}$. So t_n is a monotone increasing sequence in $[0,1]$ converging to t. Since the action $s \mapsto R_s$ of $\mathbb{R}$ on $\mathbb{S}$ by rotations is uniformly continuous, we have $f \circ R_{-t_n\gamma} \to f \circ R_{-t\gamma}$ in $\big(C(\mathbb{S}), \|\cdot\|_\infty\big)$. Since m is a Borel probability measure, the functional $f \mapsto \int_{\mathbb{S}} f\, dm$ is a state, and so

$$\int_{\mathbb{S}} f \circ R_{-t_n\gamma}\, dm \to \int_{\mathbb{S}} f \circ R_{-t\gamma}\, dm.$$

So it suffices to show that each $\int_{\mathbb{S}} f \circ R_{-t_n\gamma} \leq e^{st} \int_{\mathbb{T}} f\,dm$. So fix $n \in \mathbb{N}$. Let $K := \sum_{i=1}^n a_i N^{n-i}$, so that $t > t_n = \frac{K}{N^n}$. By hypothesis, for every Borel U, we have

$$m(R_{\frac{K\gamma}{N^n}}(U)) \leq e^{\frac{s}{N^n}} m(R_{\frac{(K-1)\gamma}{N^n}}(U)) \leq \cdots \leq e^{\frac{sK}{N^n}} m(U) \leq e^{st} m(U),$$

and it follows that $\int_{\mathbb{S}} f \circ R_{-t_n\gamma} \leq e^{st} \int_{\mathbb{S}} f\,dm$ as required. □

Proof of Theorem 6.9. As described in Remark 6.11, it suffices to show that $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$ is contained in $\varprojlim \Omega^{r_j}_{\mathrm{sub}}$. For each $\gamma \in \mathbb{S}$ we have $p_N \circ R_\gamma = R_{N\gamma} \circ p_N$, which implies that $p_N^{-1}(R_{N\gamma}(U)) = R_\gamma(p_N^{-1}(U))$ for all Borel $U \subseteq \mathbb{S}$. An iterative argument shows that

$$p_N^{-k}(R_{N^k\gamma}(U)) = R_\gamma(p_N^{-k}(U)) \quad \text{for all Borel } U \subseteq \mathbb{S} \text{ and } k \in \mathbb{N}. \tag{6.7}$$

Fix $(m_j)_{j=0}^\infty \in \varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$. Since the connecting maps in $\varprojlim M_{\mathrm{sub}}(\beta/N^j, \theta_j)$ and $\varprojlim \Omega^{r_j}_{\mathrm{sub}}$ are the same, it suffices to show that $m_j \in \Omega^{r_j}_{\mathrm{sub}}$ for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$. For each $k \in \mathbb{N}$ we have $N^{2k}\theta_{j+k} = \theta_k$ and $m_{j+k} \circ p_N^{-k} = m_j$. These identities and (6.7) give

$$\begin{aligned} m_j(R_{\theta_j/N^k}(U)) &= m_j(R_{N^k\theta_{j+k}}(U)) = m_{j+k}(p_N^{-k}(R_{N^k\theta_{j+k}}(U))) \\ &= m_{j+k}(R_{\theta_{j+k}}(p_N^{-k}(U))) \leq e^{\beta/N^{j+k}} m_{j+k}(p_N^{-k}(U)) = e^{\beta/N^{j+k}} m_j(U), \end{aligned}$$

for every Borel $U \subseteq \mathbb{S}$. So Lemma 6.12 with $\gamma = \theta_j$ and $s = \beta/N^j$ gives $m_j \in \Omega^{r_j}_{\mathrm{sub}}$. □

7. Subinvariant measures on $\mathbb{S}$

Throughout the section we fix $r \in [0, \infty)$ and denote Lebesgue measure on $\mathbb{S}$ by μ. The main result of this section gives a concrete description of the simplex Ω^r_{sub} of (6.3). Define $W_r : \mathbb{S} \to [0, \infty)$ by

$$W_r(t) = \Big(\frac{r}{1 - e^{-r}}\Big) e^{-rt}.$$

For each Borel $U \subseteq \mathbb{S}$, define

$$m_r(U) := \int_U W_r(t)\,dt. \tag{7.1}$$

This defines a regular Borel probability measure m_r on $\mathbb{S}$.

Theorem 7.1. *The simplex Ω^r_{sub} is the weak*-closed convex hull $\overline{\mathrm{conv}}\{m_r \circ R_s : 0 \leq s < 1\}$. If $r = 0$, then $m_r = \mu$ and $\Omega^r_{\mathrm{sub}} = \{\mu\}$.*

We need a number of results to prove this theorem.

Lemma 7.2. *Let $m \in \Omega^r_{\mathrm{sub}}$ and $n \in \mathbb{N}$. For $0 \leq j < 2^n$, let $U^n_j = [j/2^n, (j+1)/2^n) \subseteq \mathbb{S}$, and let v^n_j be the vector*

$$v^n_j := \frac{1 - e^{-r/2^n}}{1 - e^{-r}} \big(e^{-(2^n-j)r/2^n}, \ldots, e^{-(2^n-1)r/2^n}, 1, e^{-r/2^n}, e^{-2r/2^n}, \ldots, e^{-(2^n-(j+1))r/2^n}\big) \in \mathbb{R}^{2^n}. \tag{7.2}$$

Then $\big(m(U^n_0), m(U^n_1), \ldots, m(U^n_{2^n-1})\big) \in \mathrm{conv}\{v^n_j : 0 \leq j < 2^n\}$.

Proof. Let $x = (x_0, x_1, \ldots, x_{2^n-1})$ be the vector $\big(m(U^n_0), m(U^n_1), \ldots, m(U^n_{2^n-1})\big)$. For each $0 \leq j < 2^n$ we have

$$x_j = m(U^n_j) = m(R_{2^{-n}}(U^n_{j+1})) \leq e^{r/2^n} m(U^n_{j+1}) = e^{r/2^n} x_{j+1},$$

where addition in indices is modulo 2^n. Let C^{2^n} denote the graph with vertices $\mathbb{Z}/2^n\mathbb{Z}$ and edges $\{e_j : j \in \mathbb{Z}/2^n\mathbb{Z}\}$ with $s(e_j) = j$ and $r(e_j) = j + 1 \pmod{2^n}$, and let $A_{C^{2^n}}$ denote the adjacency matrix of C^{2^n}. Then x satisfies $A_{C^{2^n}} x \leq e^{r/2^n} x$. So x is subinvariant for $A_{C^{2^n}}$ in the sense of [12, Theorem 3.1], and is a probability measure because m is. By [12, Theorem 3.1(a)], there is a vector $y \in [1, \infty)^{\mathbb{Z}/2^n\mathbb{Z}}$ such that

$$y_j = \sum_{\mu \in (C^{2^n})^*, s(\mu)=j} e^{-r/2^n|\mu|} = \sum_{k=0}^{\infty} e^{-kr/2^n} = \big(1 - e^{-r/2^n}\big)^{-1} \quad \text{for } j \in \mathbb{Z}/2^n\mathbb{Z}.$$

For $0 \le j < 2^n$, define $\epsilon_j \in [0, \infty)^{\mathbb{Z}/2^n\mathbb{Z}}$ by

$$\epsilon_j(k) = \begin{cases} 1 - e^{-r/2^n} & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
&(I - e^{-r/2^n} A_{C^{2^n}}) v_j^n \\
&= \frac{1 - e^{-r/2^n}}{1 - e^{-r}} (I - e^{-r/2^n} A_{C^{2^n}}) \big(e^{-(2^n - j)r/2^n}, \dots, e^{-(2^n-1)r/2^n}, 1, e^{-r/2^n}, \dots, e^{-(2^n-(j+1))r/2^n}\big) \\
&= \frac{1 - e^{-r/2^n}}{1 - e^{-r}} \Big(\big(e^{-(2^n - j)r/2^n}, \dots, e^{-(2^n-1)r/2^n}, 1, e^{-r/2^n}, \dots, e^{-(2^n-(j+1))r/2^n}\big) \\
&\qquad\qquad - e^{-r/2^n}\big(e^{-(2^n - (j+1))r/2^n}, \dots, e^{-(2^n-1)r/2^n}, 1, e^{-r/2^n}, \dots, e^{-(2^n-(j+2))r/2^n}\big)\Big) \\
&= \frac{1 - e^{-r/2^n}}{1 - e^{-r}} \big(0, \dots, 0, 1 - e^{-r}, 0, \dots, 0\big) \\
&= (1 - e^{-r/2^n}) \big(0, \dots, 0, 1, 0, \dots, 0\big) \\
&= \epsilon_j.
\end{aligned}$$

So $v_j^n = (I - e^{-r/2^n} A_{C^{2^n}})^{-1} \epsilon_j$. Since the ϵ_j are the extreme points of the simplex $\{\epsilon : \epsilon \cdot y = 1\}$, it follows from [12, Theorem 3.1(c)] that the v_j^n are the extreme points of the simplex of subinvariant probability measures on $\mathbb{Z}/2^n\mathbb{Z}$. Since x is a subinvariant probability measure, it follows that it is a convex combination of the v_j^n. □

We now approximate m_r by convex combinations of restrictions of Lebesgue measure.

Lemma 7.3. *For $n \in \mathbb{N}$ and $j \in \mathbb{Z}/2^n\mathbb{Z}$, let $U_j^n = [j/2^n, (j+1)/2^n) \subseteq \mathbb{S}$, and let $W_{n,r}$ be the simple function*

$$W_{n,r} = \sum_{j=0}^{2^n-1} 2^n (v_0^n)_j 1_{U_j^n}.$$

Let $m_{n,r}$ be the measure $m_{n,r}(U) = \int_U W_{n,r}(t)\, d\mu(t)$ for Borel $U \subseteq \mathbb{S}$. Then $\lim_{n\to\infty} \big\|m_r - m_{n,r}\big\|_1 = 0$.

Proof. Fix $n \in \mathbb{N}$ and $0 \le j < 2^{-n}$. Then the average value of W_r over the interval U_j^n is

$$\begin{aligned}
2^n \int_{U_j^n} W_r(t)\, d\mu(t) &= 2^n \int_{j/2^n}^{(j+1)/2^n} \Big(\frac{r}{1 - e^{-r}}\Big) e^{-rt}\, d\mu(t) = 2^n \Big[\Big(\frac{-1}{1 - e^{-r}}\Big) e^{-rt}\Big]_{j/2^n}^{(j+1)/2^n} \\
&= \Big(\frac{-2^n}{1 - e^{-r}}\Big)\Big(e^{-(j+1)r/2^n} - e^{-jr/2^n}\Big) = 2^n \Big(\frac{1 - e^{-r/2^n}}{1 - e^{-r}}\Big) e^{-jr/2^n} = 2^n (v_0^n)_j,
\end{aligned}$$

the constant value of $W_{n,r}$ on U_j^n. The Mean Value Theorem—applied to $\int W_r(t)\, d\mu(t)$—implies that there exists $c_j^n \in (j/2^n, (j+1)/2^n)$ such that $W_r(c_j^n) = W_{n,r}(c_j^n)$.

Fix $\epsilon > 0$. The function W_r is uniformly continuous on $[0, 1)$, and so there exists $N \in \mathbb{N}$ such that $|W_r(s) - W_r(t)| < \epsilon$ whenever $s, t \in [0, 1)$ satisfy $|s - t| < 2^{-N}$. In particular, for $n \ge N$

and $0 \le j < 2^n$, the point c_j^n of the preceding paragraph satisfies

$$\begin{aligned}\sup\{W_r(t) - W_{n,r}(t) : j/2^n \le t < (j+1)/2^n\}\\ &= \sup\{W_r(t) - W_{n,r}(c_j^n) : j/2^n \le t < (j+1)/2^n\}\\ &= \sup\{W_r(t) - W_r(c_j^n) : j/2^n \le t < (j+1)/2^n\} \le \epsilon.\end{aligned}$$

So for $n \ge N$,

$$\begin{aligned}\big\|m_r - m_{n,r}\big\|_1 &= \int_0^1 |W_r(t) - W_{n,r}(t)|\, d\mu(t)\\ &= \sum_{j=0}^{2^n-1} \int_{j/2^n}^{(j+1)/2^n} |W_r(t) - W_{n,r}(t)|\, d\mu(t) \le \sum_{j=0}^{2^n-1} \int_{j/2^n}^{(j+1)/2^n} \epsilon\, d\mu(t) = \epsilon,\end{aligned}$$

and hence $\lim_{n\to\infty} \big\|m_r - m_{n,r}\big\|_1 = 0$. □

Corollary 7.4. *Given a sequence $(\lambda^n)_{n=1}^\infty$ of vectors $\lambda^n \in [0,1]^{2^n}$ satisfying $\sum_{j=0}^{2^n-1} \lambda_j^n = 1$ for all n, we have*

$$\lim_{n\to\infty} \Big\| \sum_{j=0}^{2^n-1} \lambda_j^n (m_r \circ R_{j/2^n}) - \sum_{j=0}^{2^n-1} \lambda_j^n (m_{n,r} \circ R_{j/2^n}) \Big\|_1 = 0.$$

Proof. The triangle inequality gives

$$\begin{aligned}\Big\| \sum_{j=0}^{2^n-1} \lambda_j^n (m_r \circ R_{j/2^n}) - \sum_{j=0}^{2^n-1} \lambda_j^n (m_{n,r} \circ R_{j/2^n}) \Big\|_1 &\le \sum_{j=0}^{2^n-1} \lambda_j^n \big\|m_r \circ R_{j/2^n} - m_{n,r} \circ R_{j/2^n}\big\|_1\\ &= \big\|m_r - m_{n,r}\big\|_1,\end{aligned}$$

and so the result follows from Lemma 7.3. □

Proof of Theorem 7.1. We first have to show that each $m_r \circ R_s \in \Omega^r_{\text{sub}}$. To see that $m_r \in \Omega^r_{\text{sub}}$, it suffices to prove that $W_r(R_t(t_0)) \le e^{rt} W_r(t_0)$ for all $t_0 \in \mathbb{S}$ and $t \in [0,\infty)$. Fix such a t_0 and t, and write $t_0 - t = t_1 + k$ for $t_1 \in [0,1)$ and $0 \ge k \in \mathbb{Z}$. Then

$$\begin{aligned}W_r(R_t(t_0)) = W_r(t_1) &= \Big(\frac{r}{1-e^{-r}}\Big) e^{-rt_1} = \Big(\frac{r}{1-e^{-r}}\Big) e^{rk} e^{-r(t_1+k)}\\ &= \Big(\frac{r}{1-e^{-r}}\Big) e^{rk} e^{-r(t_0-t)} = e^{rk} e^{rt} W_r(t_0) \le e^{rt} W_r(t_0),\end{aligned}$$

where the inequality follows because $rk \le 0$. So $m_r \in \Omega^r_{\text{sub}}$. For $0 \le s < 1$ and Borel $U \subseteq \mathbb{S}$, we have $m_r \circ R_s(R_t(U)) = m_r(R_t(R_s(U))) \le e^{rt} m_r \circ R_s(U)$ for all $t \in [0,\infty)$ and Borel $U \subseteq \mathbb{S}$, and hence $m_r \circ R_s \in \Omega^r_{\text{sub}}$.

Since Ω^r_{sub} is convex and weak* closed, we have $\overline{\text{conv}}\{m_r \circ R_s : 0 \le s < 1\} \subseteq \Omega^r_{\text{sub}}$. For the reverse containment, fix $m \in \Omega^r_{\text{sub}}$. For each $n \in \mathbb{N}$ and $0 \le j < 2^n$ we let $U_j^n := [j/2^n, (j+1)/2^n)$, and

$$x_n := \big(m(U_j^n)\big)_{j=0}^{2^n-1} \in [0,1]^{2^n}.$$

By Lemma 7.2 we can express x_n as a convex combination $x_n = \sum_{j=1}^{2^n-1} \lambda_j^n v_j^n$ of the vectors $\{v_0^n, \dots, v_{2^n-1}^n\}$ described at (7.2). We claim that the measures

$$M_n := \sum_{j=0}^{2^n-1} \lambda_j^n (m_r \circ R_{j/2^n})$$

converge weak* to m. To see this, fix $f \in C(\mathbb{S})_+$. It suffices to prove that $\int f\, dM_n \to \int f\, dm$. For each n, let

$$M_n' := \sum_{j=0}^{2^n-1} \lambda_j^n (m_{n,r} \circ R_{j/2^n}).$$

Corollary 7.4 shows that $\|M_n - M_n'\|_1 \to 0$ and in particular, $\int f\,dM_n - \int f\,dM_n' \to 0$. So it suffices to prove that

$$\int f\,dM_n' \to \int f\,dm.$$

For each n, define $f_n : \mathbb{S} \to \mathbb{R}$ by

$$f_n = \sum_{j=0}^{2^n-1} f(j/2^n) 1_{U_j^n}.$$

Since f is uniformly continuous on $\mathbb{S}$ we have $f_n \to f$ pointwise on $\mathbb{S}$. Since $|f|$ and each $|f_n|$ are bounded above by $\|f\|_\infty$, the Dominated Convergence Theorem implies that $\int f_n\,dm \to \int f\,dm$. So it now suffices to prove that

$$\Big| \int f_n\,dm - \int f\,dM_n' \Big| \to 0.$$

Fix $j,k \in \mathbb{Z}/2\mathbb{Z}$. Then $(v_0^n)_{j-k} = (v_j^n)_k$, and hence

$$\int_{U_k^n} f\,d\big(m_{n,r} \circ R_{j/2^n}\big) = 2^n (v_0^n)_{j-k} \int_{U_k^n} f\,d\mu = 2^n (v_j^n)_k \int_{U_k^n} f\,d\mu.$$

Hence

$$\begin{aligned}
\Big| \int f_n\,dm - \int f\,dM_n' \Big| &= \Big| \sum_{i=0}^{2^n-1} f(i/2^n) m(U_i^n) - \sum_{j=0}^{2^n-1} \lambda_j^n \Big(\sum_{k=0}^{2^n-1} \int_{U_k^n} f\,d\big(m_{n,r} \circ R_{j/2^n}\big)\Big)\Big| \\
&= \Big| \sum_{i=0}^{2^n-1} f(i/2^n) \Big(\sum_{l=0}^{2^n-1} \lambda_l^n v_l^n \Big)_i - \sum_{j=0}^{2^n-1} \lambda_j^n \sum_{k=0}^{2^n-1} \Big(2^n (v_j^n)_k \int_{U_k^n} f\,d\mu \Big) \Big| \\
&= \Big| \sum_{l=0}^{2^n-1} \lambda_l^n \sum_{i=0}^{2^n-1} \big(f(i/2^n)(v_l^n)_i \big) - \sum_{j=0}^{2^n-1} \lambda_j^n \sum_{k=0}^{2^n-1} \Big(2^n (v_j^n)_k \int_{U_k^n} f\,d\mu \Big) \Big| \\
&= \Big| \sum_{j=0}^{2^n-1} \lambda_j^n \Big(\sum_{i=0}^{2^n-1} \big(f(i/2^n)(v_j^n)_i \big) - \sum_{k=0}^{2^n-1} \Big(2^n (v_j^n)_k \int_{U_k^n} f\,d\mu \Big)\Big) \Big| \\
&= \Big| \sum_{j=0}^{2^n-1} \lambda_j^n \Big(\sum_{i=0}^{2^n-1} \Big(f(i/2^n)(v_j^n)_i - 2^n (v_j^n)_k \int_{U_k^n} f\,d\mu \Big)\Big) \Big|.
\end{aligned}$$

Since each $\|v_j^n\|_1 = 1$ and each $\sum_j \lambda_j^n = 1$, the triangle inequality gives

$$\begin{aligned}
\Big| \int f_n\,dm - \int f\,dM_n' \Big| &\le \sum_{j=0}^{2^n-1} \lambda_j^n \Big| \sum_{i=0}^{2^n-1} \Big(f(i/2^n) - 2^n \int_{U_i^n} f\,d\mu \Big)(v_j^n)_i \Big) \Big| \\
&\le \max_{0 \le j < 2^n} \sum_{i=0}^{2^n-1} (v_j^n)_i \Big| f(i/2^n) - 2^n \int_{U_i^n} f\,d\mu \Big| \\
&\le \max_{0 \le j < 2^n} \Big(\max_{0 \le i < 2^n} \Big| f(i/2^n) - 2^n \int_{U_i^n} f\,d\mu \Big| \Big) \\
&= \max_{0 \le i < 2^n} \Big| f(i/2^n) - 2^n \int_{U_i^n} f\,d\mu \Big|.
\end{aligned}$$

Fix $0 \le i \le 2^n$. The quantity $2^n \int_{U_i^n} f\,d\mu$ is the average value of f over U_i^n. Since f is continuous, the Mean Value Theorem implies that there exists $c \in U_i^n$ such that $f(c) = 2^n \int_{U_i^n} f\,d\mu$. Hence

$$\Big| \int f_n\,dm - \int f\,dM_n' \Big| \le \max_{0 \le i < 2^n} \sup_{c \in U_i^n} |f(i/2^n) - f(c)|.$$

Fix $\epsilon > 0$. By uniform continuity of f there exists N such that $|x-y| < 2^{-N} \implies |f(x) - f(y)| < \epsilon$. For $n \geq N$ we have $\sup_{c \in U_i^n} |f(i/2^n) - f(c)| \leq \epsilon$ for all i, giving $\left|\int f_n\,dm - \int f\,dM_n'\right| \leq \epsilon$. Hence $\left|\int f_n\,dm - \int f\,d\rho_n\right| \to 0$. So $m \in \overline{\operatorname{conv}}\{m_r \circ R_s : 0 \leq s < 1\}$, giving $\Omega^r_{\text{sub}} \subseteq \overline{\operatorname{conv}}\{m_r \circ R_s : 0 \leq s < 1\}$ as required.

For the final statement, observe that

$$\Omega^0_{\text{sub}} = \{m \in M(\mathbb{S}) : m(R_t(U)) \leq m(U) \text{ for all } t \in [0,\infty) \text{ and Borel } U \subseteq \mathbb{S}\}.$$

So if $m \in \Omega^0_{\text{sub}}$, then $m(U) = m(R_{1-t}(R_t(U))) \leq m(R_t(U)) \leq m(U)$ for all U, t, forcing $m(U) = m(R_t(U))$ for all U, t. Uniqueness of the Haar measure μ on the compact group $\mathbb{S}$ therefore gives $m = \mu$. So $\Omega^0_{\text{sub}} \subseteq \{\mu\}$. The reverse containment is trivial. □

We can use Theorem 7.1 to describe the extreme points of Ω^r_{sub}.

Proposition 7.5. *The set $\{m_r \circ R_s : 0 \leq s < 1\}$ is the set of extreme points of Ω^r_{sub}.*

The first step in proving Proposition 7.5 will be to show that m_r itself is an extreme point of Ω^r_{sub}. The following lemma will help.

Lemma 7.6. *Let $m \in \Omega^r_{\text{sub}}$ and $n \in \mathbb{N}$ with $n \geq 1$. If $m([\frac{n-1}{n}, 1)) \leq m_r([\frac{n-1}{n}, 1))$, then $m([\frac{i}{n}, \frac{i+1}{n})) = m_r([\frac{i}{n}, \frac{i+1}{n}))$ for all $0 \leq i < n$.*

Proof. First observe that by definition of m_r, we have $m_r(R_t(U)) = e^{rt} m_r(U)$ whenever $U \cup U - t \subseteq [0, 1)$. Using this at the fourth equality, we note that if $m([\frac{n-1}{n}, 1)) \leq m_r([\frac{n-1}{n}, 1))$, then subinvariance forces

$$\begin{aligned}
1 = m(\mathbb{S}) = \sum_{i=0}^{n-1} m\Big(\Big[\frac{i}{n}, \frac{i+1}{n}\Big)\Big) &= \sum_{i=0}^{n-1} m\Big(R_{(n-1-j)/n}\Big(\Big[\frac{n-1}{n}, 1\Big)\Big)\Big) \\
&\leq \sum_{i=0}^{n-1} e^{(n-1-j)r/n} m\Big(\Big[\frac{n-1}{n}, 1\Big)\Big) \\
&\leq \sum_{i=0}^{n-1} e^{(n-1-j)r/n} m_r\Big(\Big[\frac{n-1}{n}, 1\Big)\Big) \\
&= \sum_{i=0}^{n-1} m_r\Big(\Big[\frac{i}{n}, \frac{i+1}{n}\Big)\Big) \\
&= 1.
\end{aligned}$$

So we have equality throughout. From this we deduce first that

$$\sum_{i=0}^{n-1} m\Big(\Big[\frac{i}{n}, \frac{i+1}{n}\Big)\Big) = \sum_{i=0}^{n-1} e^{(n-1-j)r/n} m\Big(\Big[\frac{n-1}{n}, 1\Big)\Big).$$

Since the subinvariance relation forces $m\big(\big[\frac{i}{n}, \frac{i+1}{n}\big)\big) \leq e^{(n-1-j)r/n} m\big(\big[\frac{n-1}{n}, 1\big)\big)$ for each i, we deduce that $m\big(\big[\frac{i}{n}, \frac{i+1}{n}\big)\big) = e^{(n-1-j)r/n} m\big(\big[\frac{n-1}{n}, 1\big)\big)$ for each i. Since

$$\sum_{i=0}^{n-1} e^{(n-1-j)r/n} m\Big(\Big[\frac{n-1}{n}, 1\Big)\Big) = \sum_{i=0}^{n-1} e^{(n-1-j)r/n} m_r\Big(\Big[\frac{n-1}{n}, 1\Big)\Big),$$

we also have $m\big(\big[\frac{n-1}{n}, 1\big)\big) = m_r\big(\big[\frac{n-1}{n}, 1\big)\big)$. Hence for each i we have

$$m\Big(\Big[\tfrac{i}{n}, \tfrac{i+1}{n}\Big)\Big) = e^{(n-1-j)r/n} m\Big(\Big[\tfrac{n-1}{n}, 1\Big)\Big) = e^{(n-1-j)r/n} m_r\Big(\Big[\tfrac{n-1}{n}, 1\Big)\Big) = m\Big(\Big[\tfrac{i}{n}, \tfrac{i+1}{n}\Big)\Big). \qquad \square$$

Proof of Proposition 7.5. We first show that m_r is an extreme point of Ω^r_{sub}. First suppose $m \in \Omega^r_{\text{sub}}$ satisfies $m([\frac{n-1}{n}, 1)) \leq m_r([\frac{n-1}{n}, 1))$ for all n. We claim that $m = m_r$. Fix $f \in C(\mathbb{S})_+$. For each n define $f_n : \mathbb{S} \to \mathbb{R}$ by

$$f_n = \sum_{i=0}^{n-1} f(i/n) 1_{[\frac{i}{n}, \frac{i+1}{n})}.$$

The Dominated Convergence Theorem gives $\int f_n\, dm \to \int f\, dm$. By Lemma 7.6, $m([\frac{i}{n}, \frac{i+1}{n})) = m_r([\frac{i}{n}, \frac{i+1}{n}))$ for all $n \geq 1$ and $0 \leq i < n$. Hence the Dominated Convergence Theorem gives $\int f_n\, dm = \int f_n\, dm_r \to \int f\, dm_r$. It follows that $m = m_r$.

Now suppose that $m_1, m_2 \in \Omega^r_{\text{sub}}$, $t \in (0,1)$ and that one of m_1 and m_2 is not equal to m_r; say $m_1 \neq m_r$. The above claim yields n such that $m_1([\frac{n-1}{n}, 1)) > m_r([\frac{n-1}{n}, 1))$. So

$$(tm_1 + (1-t)m_2)\Big(\Big[\frac{n-1}{n}, 1\Big)\Big) > (tm_r + (1-t)m_2)\Big(\Big[\frac{n-1}{n}, 1\Big)\Big) \geq m_r\Big(\Big[\frac{n-1}{n}, 1\Big)\Big),$$

and hence $tm_1 + (1-t)m_2 \neq m_r$. So m_r cannot be expressed as a nontrivial convex combination of subinvariant probability measures, and hence is an extreme point of Ω^r_{sub}.

For $s \in \mathbb{S}$, the map $m \mapsto m \circ R_s$ is an affine homeomorphism of Ω^r_{sub}, so each $m \circ R_s$ is an extreme point of Ω^r_{sub}. This gives $\{m_r \circ R_s : s \in \mathbb{S}\} \subseteq \partial\Omega^r_{\text{sub}}$.

For the reverse containment, observe that the space Ω^r_{sub} of all subinvariant probability measures on $\mathbb{S}$ is a compact convex subset of the Banach space of all signed Borel measures on $\mathbb{S}$. The map $s \mapsto m_r \circ R_s$ is a homeomorphism of $\mathbb{S}$ onto $Z := \{m_r \circ R_s : s \in \mathbb{S}\}$. So Z is compact and in particular closed. Since Ω^r_{sub} is the closed convex hull of Z it follows from [25, Proposition 1.5] that the set of extreme points of Ω^r_{sub} is contained in the closure of Z and therefore in Z itself. □

8. Proof of the main theorem

We are now almost ready to prove Theorem 6.6. We saw in Theorem 6.9 that the KMS_β simplex of $\mathcal{T}^{\mathscr{S}}_\theta$ is affine isomorphic to the projective limit of the $\Omega^{r_j}_{\text{sub}}$ under the maps induced by the covering maps $p_N : \mathbb{S} \to \mathbb{S}$. So we now show that these induced maps carry extreme points to extreme points.

Lemma 8.1. *Let $N \in \mathbb{N}$ with $N \geq 2$, $\theta = (\theta_j)_{j=0}^\infty \in \Xi_N$, and $\beta \in (0, \infty)$. Suppose that $\theta_j \neq 0$ for all j. For each $j \in \mathbb{N}$, let $r_j := \frac{\beta}{N^j \theta_j}$, and let m_{r_j} be the subinvariant measure on $\mathbb{S}$ defined by (7.1). For each $s \in [0,1)$, we have $m_{r_{j+1}} \circ R_s \circ p_N^{-1} = m_{r_j} \circ R_{Ns}$.*

Proof. We first establish the result with $s = 0$. Fix $0 \leq a < b \leq 1$. It suffices to prove that $m_{r_{j+1}} \circ \iota_N^{-1}\big((a,b)\big) = m_{r_j}\big((a,b)\big)$. We have

$$m_{r_j}\big((a,b)\big) = \int_a^b W_{r_j}(t)\, dt = \frac{r_j}{1 - e^{-r_j}} \int_a^b e^{-r_j t}\, dt = \frac{-1}{1 - e^{-r_j}}\big(e^{-r_j b} - e^{-r_j a}\big). \tag{8.1}$$

We also have

$$m_{r_{j+1}} \circ p_N^{-1}\big((a,b)\big) = \sum_{i=0}^{N} m_{r_{j+1}}\Big(\Big(\frac{a+i}{N}, \frac{b+i}{N}\Big)\Big) = \sum_{i=0}^{N} \int_{\frac{a+i}{N}}^{\frac{b+i}{N}} W_{r_{j+1}}(t)\, dt. \tag{8.2}$$

Since

$$\int W_{r_{j+1}}(t)\, dt = \int \Big(\frac{r_{j+1}}{1 - e^{-r_{j+1}}}\Big) e^{-r_{j+1} t}\, dt = \frac{-1}{1 - e^{-r_{j+1}}} e^{-r_{j+1} t},$$

Equation (8.2) gives

$$\begin{aligned} m_{r_j}\big((a,b)\big) &= \frac{-1}{1-e^{-r_{j+1}}}\sum_{i=0}^{N}\Big[e^{-r_{j+1}t}\Big]_{\frac{a+i}{N}}^{\frac{b+i}{N}} \\ &= \frac{-1}{1-e^{-r_{j+1}}}\sum_{i=0}^{N} e^{-\frac{i}{N}r_{j+1}}\big(e^{-\frac{b}{N}r_{j+1}} - e^{-\frac{a}{N}r_{j+1}}\big) \\ &= \frac{-1}{1-e^{-r_{j+1}}}\frac{1-e^{-r_{j+1}}}{1-e^{-\frac{r_{j+1}}{N}}}\big(e^{-\frac{b}{N}r_{j+1}} - e^{-\frac{a}{N}r_{j+1}}\big) \\ &= \frac{-1}{1-e^{-\frac{r_{j+1}}{N}}}\big(e^{-\frac{b}{N}r_{j+1}} - e^{-\frac{a}{N}r_{j+1}}\big). \end{aligned} \tag{8.3}$$

Since $N^2\theta_{j+1} = \theta_j$, we have

$$\frac{r_{j+1}}{N} = \frac{\beta/(N^{j+1}\theta_{j+1})}{N} = \beta/(N^j \cdot N^2\theta_{j+1}) = \beta/N^j\theta_j = r_j,$$

and so (8.3) is precisely (8.1).

Now for $s \neq 0$, observe that $p_N \circ R_s = R_{Ns} \circ p_N$ so that $R_s(p_N^{-1}(U)) = p_N^{-1}(R_{Ns}(U))$ for all $U \subseteq \mathbb{S}$. Hence

$$m_{r_{j+1}} \circ R_s \circ p_N^{-1} = m_{r_{j+1}} \circ p_N^{-1} \circ R_{Ns} = m_{r_j} \circ R_{Ns}. \qquad \square$$

We now describe the extreme points of the space $\varprojlim(\Omega_{\text{sub}}^{r_j}, m \mapsto m \circ p_N^{-1})$. Given a Borel map $\psi : X \to Y$, we write $\psi_* : M_1(X) \to M_1(Y)$ for the induced map $\psi_*(m)(U) = m(\psi^{-1}(U))$.

Lemma 8.2. *Take $N \in \{2, 3, \dots\}$, fix $\theta = (\theta_j)_{j=0}^\infty \in \Xi_N$, and fix $\beta \in (0,\infty)$. Suppose that $\theta_j \neq 0$ for all j. For each $j \in \mathbb{N}$, let $r_j := \frac{\beta}{N^j\theta_j}$, and let m_{r_j} be the subinvariant measure on $\mathbb{S}$ defined by (7.1). The map $\pi : (s_j)_{j=1}^\infty \mapsto (m_{r_j} \circ R_{s_j})_{j=1}^\infty$ is a homeomorphism of $\varprojlim(\mathbb{S}, p_N)$ onto the set of extreme points of $\varprojlim(\Omega_{\text{sub}}^{r_j}, (p_N)_*)$.*

Proof. Since the $\Omega_{\text{sub}}^{r_j}$ are compact convex sets and $(p_N)_*$ is affine and continuous, the projective limit $\varprojlim \Omega_{\text{sub}}^{r_j}$ is a compact convex set. The map π is continuous, so its range is compact and hence closed. So to see that the image of π contains all of the extreme points of $\varprojlim \Omega_{\text{sub}}^{r_j}$, it suffices by [25, Proposition 1.5] to show that $\varprojlim \Omega_{\text{sub}}^{r_j}$ is contained in the closed convex hull of the $\pi\big((s_j)_{j=1}^\infty\big)$.

For this, fix a point $(m_j)_{j=1}^\infty \in \varprojlim \Omega_{\text{sub}}^{r_j}$. Take an open neighbourhood U of (m_j). By definition of the projective-limit topology, there exist $k \in \mathbb{N}$ and $U_k \subseteq \Omega_{\text{sub}}^{r_k}$ open such that the cylinder set $Z(U_k)$ satisfies $(m_j)_{j=1}^\infty \in Z(U_k) \subseteq U$. By Theorem 7.1, there exist $t_1, \dots, t_L \in [0,1]$ with $\sum t_l = 1$ such that

$$\textstyle\sum_{l=1}^L t_l(m_{r_k} \circ R_{s_l}) \in U_k.$$

Now for each $j \in \mathbb{N}$, define $m'_j := \sum_{l=1}^L t_l(m_{r_l} \circ R_{N^{j-l}s_l})$. Lemma 8.1 shows that for $j \leq j' \in \mathbb{N}$ we have $m'_j = (p_N)_*^{j'-j}(m'_{j'})$, and so $(m'_j)_{j=1}^\infty \in \varprojlim \Omega_{\text{sub}}^{r_j}$. For $l \leq L$, we have $(m_{r_l} \circ R_{N^{j-l}s_l})_{j=1}^\infty = \pi\big((N^{j-l}s_l)_{j=1}^\infty\big)$, and so

$$(m'_j)_{j=1}^\infty \in \operatorname{conv}\pi\big(\varprojlim(\mathbb{S}, p_N)\big) \cap U.$$

That is, $\varprojlim \Omega_{\text{sub}}^{r_j} \subseteq \overline{\operatorname{conv}}\big(\pi\big(\varprojlim \mathbb{S}\big)\big)$. So the range of π contains all the extreme points of $\varprojlim(\Omega_{\text{sub}}^{r_j}, (p_N)_*)$.

For the reverse containment, it suffices to show that each $\pi\big((s_j)_{j=1}^\infty\big)$ is an extreme point of $\varprojlim \Omega_{\text{sub}}^{r_j}$. For this, suppose that $t \in (0,1)$ and $m', m'' \in \varprojlim \Omega_{\text{sub}}^{r_j}$ satisfy

$$\pi\big((s_j)_{j=1}^\infty\big) = tm' + (1-t)m''.$$

For each j,

$$m_{r_j} \circ R_{s_j} = \pi\big((s_j)_{j=1}^\infty\big)_j = (tm' + (1-t)m'')_j = tm'_j + (1-t)m''_j.$$

Proposition 7.5 shows that each $m_{r_j} \circ R_{s_j}$ is an extreme point of $\Omega^{r_j}_{\mathrm{sub}}$, forcing $m'_j = m''_j = m_{r_j} \circ R_{s_j}$. So $m' = m'' = \pi\big((s_j)_{j=1}^\infty\big)$.

Finally, π is a homeomorphism onto its range because it is a continuous injection from a compact space to a Hausdorff space. □

The final ingredient needed for the proof of Theorem 6.6 is a suitable action λ of $\mathscr{S}$ on $\mathcal{T}_\theta^{\mathscr{S}}$.

Lemma 8.3. *There is an action λ of $\mathscr{S} = \varprojlim(\mathbb{S}, p_N)$ on $\mathcal{T}_\theta^{\mathscr{S}}$ such that*

$$\lambda_{(s_j)_{j=1}^\infty}\big(\psi_{j,\infty}(s^a_{\theta_j} i_{\theta_j}(f) s^{*b}_{\theta_j})\big) = s^a_{\theta_j} i_{\theta_j}(f \circ R_{s_j}) s^{*b}_{\theta_j}$$

for all $j, a, b \geq 0$ and $f \in C(\mathbb{S})$.

Proof. For each $j \in \mathbb{N}$, and each $t \in \mathbb{S}$, there is an automorphism of the topological graph E_{θ_j} given by $s \mapsto s + t$ for $s \in E^0_{\theta_j} = \mathbb{S}$, and $s \mapsto s + t$ for $s \in E^1_{\theta_j} = \mathbb{S}$. This automorphism induces an automorphism $\lambda_{j,t}$ of $\mathcal{T}(E_{\theta_j})$ such that $\lambda_{j,t}(s^a_{\theta_j} i_{\theta_j}(f) s^{*b}_{\theta_j}) = s^a_{\theta_j} i_{\theta_j}(f \circ R_t) s^{*b}_{\theta_j}$ for all $j, a, b \geq 0$ and $f \in C(\mathbb{S})$.

Since $\lambda_{j,t}(s_{\theta_j}) = s_{\theta_j}$ and $\lambda_{j,t}(i_{\theta_j}(f)) = i_{\theta_j}(f \circ R_t)$ for all $f \in C(\mathbb{S})$, a routine calculation shows that for $(s_j)_{j=1}^\infty \in \mathscr{S}$, we have $\psi_j \circ \lambda_{j,s_j} = \lambda_{j+1,s_{j+1}} \circ \psi_j$, and so the universal property of the direct limit yields the desired action λ of $\mathscr{S}$ on $\varinjlim(\mathcal{T}(E_{\theta_j}), \psi_j) = \mathcal{T}_\theta^{\mathscr{S}}$. □

Proof of Theorem 6.6. Theorem 6.9 yields an affine isomorphism

$$\omega : \mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha) \to \varprojlim(\Omega^{r_j}_{\mathrm{sub}}, (p_N)_*).$$

Lemma 8.2 shows that the space of extreme points of $\varprojlim \Omega^{r_j}_{\mathrm{sub}}$ is homeomorphic to the solenoid $\varprojlim \mathbb{S}$, so the extreme boundary of $\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ is homeomorphic to $\varprojlim \mathbb{S}$. As discussed on pages 141 and 138 of [27], the set of KMS states for a given dynamics on a unital C^*-algebra at given inverse temperature β is a Choquet simplex. So $\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ is a Choquet simplex, and therefore affine isomorphic to the simplex of regular Borel probability measures on its extreme boundary.

We claim that the action λ of Lemma 8.3 induces a free and transitive action of $\mathscr{S}$ on the extreme boundary of the KMS_β-simplex. The formula (6.4) shows that for $l \in \mathbb{N}$, we have

$$\omega(\phi \circ \lambda_{(s_j)_{j=1}^\infty})_l = \omega(\phi)_l \circ R_{s_l}.$$

That is, for $(m_j)_{j=1}^\infty \in \varprojlim(\Omega^{r_j}_{\mathrm{sub}})$, we have $\omega^{-1}((m_j)_{j=1}^\infty) \circ \lambda_{(s_j)_{j=1}^\infty} = \omega^{-1}((m_j \circ R_{s_j})_{j=1}^\infty)$. In particular, if $\pi : \varprojlim \mathbb{S} \to \varprojlim \Omega^{r_j}_{\mathrm{sub}}$ is the map of Lemma 8.2, then

$$\omega^{-1}(\pi((t_j)_{j=1}^\infty)) \circ \lambda_{(s_j)_{j=1}^\infty} = \omega^{-1}(\pi((t_j - s_j)_{j=1}^\infty)).$$

That is, the homeomorphism $\omega^{-1} \circ \pi$ of $\mathscr{S}$ onto the extreme boundary of $\mathrm{KMS}_\beta(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ intertwines λ with the action of $\mathscr{S}$ on itself by translation, which is free and transitive.

Now suppose that $\beta = 0$. Then each $\Omega^{r_j}_{\mathrm{sub}} = \Omega^0_{\mathrm{sub}} = \{\mu\}$, and so Theorem 6.9 gives an affine injection of $\mathrm{KMS}_0(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$ into the 1-point space $\varprojlim(\{\mu\}, \mathrm{id})$. So there is at most one KMS_0-state. That there is one follows from a standard argument: Choose $\beta_n \in (0, \infty)$ converging to 0. For each n, fix $\phi_n \in \mathrm{KMS}_{\beta_n}(\mathcal{T}_\theta^{\mathscr{S}}, \alpha)$. Weak*-compactness of the state space ensures that the ϕ_n have a convergent subsequence. Its limit is a KMS_0-state by [3, Proposition 5.3.23].

It remains to show that the KMS_0 state is the only one that factors through $\mathcal{A}_\theta^{\mathscr{S}}$, and that there are no KMS_β states for $\beta < 0$. For any β, if ϕ is a KMS_β state of $\mathcal{T}_\theta^{\mathscr{S}}$, then in particular,

$$\phi(\psi_{1,\infty}(s_{\theta_1} s^*_{\theta_1})) = \phi(\psi_{1,\infty}(s^*_{\theta_1})\alpha_{i\beta}(\psi_{1,\infty}(s_{\theta_1}))) = e^{-\beta}\phi(\psi_{1,\infty}(s^*_{\theta_1} s_{\theta_1})) = e^{-\beta}\phi(1_{\mathcal{T}_\theta^{\mathscr{S}}}), \tag{8.4}$$

and since ϕ is a state, we deduce that $\phi(1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1} s^*_{\theta_1})) = 1 - e^{-\beta}$. Since s_{θ_1} is an isometry, we have $1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1} s^*_{\theta_1}) \geq 0$ forcing $1 - e^{-\beta} \geq 0$ and hence $\beta \geq 0$. So there are no KMS_β states for $\beta < 0$.

If $\beta > 0$, then (8.4) shows that $\phi(1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1} s^*_{\theta_1})) > 0$, whereas the image of $1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1} s^*_{\theta_1})$ in $\mathcal{A}_\theta^{\mathscr{S}}$ is equal to zero. Hence ϕ does not factor through $\mathcal{A}_\theta^{\mathscr{S}}$.

It remains to prove that if ϕ is a KMS_0 state, then ϕ factors through $\mathcal{A}_\theta^{\mathscr{S}}$. Equation 8.4 implies that $\phi(1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1}s_{\theta_1}^*)) = 0$. The projection $1_{\mathcal{T}_\theta^{\mathscr{S}}} - \psi_{1,\infty}(s_{\theta_1}s_{\theta_1}^*)$ is fixed by α, and Lemma 6.2 implies that it generates the kernel of the quotient map $q : \mathcal{T}_\theta^{\mathscr{S}} \to \mathcal{A}_\theta^{\mathscr{S}}$. So [12, Lemma 2.2] implies that ϕ factors through $\mathcal{A}_\theta^{\mathscr{S}}$. □

References

[1] Z. Afsar, A. an Huef and I. Raeburn, *KMS states on C^*-algebras associated to local homeomorphisms*, Internat. J. Math. **25** (2014), 1450066, 28pp.

[2] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (N.S.) **1** (1995), 411–457.

[3] O. Bratteli and D.W. Robinson, Operator algebras and quantum statistical mechanics. 2, Equilibrium states. Models in quantum statistical mechanics, Springer-Verlag, Berlin, 1997, xiv+519.

[4] N. Brownlowe, A. an Huef, M. Laca and I. Raeburn, *Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers*, Ergodic Theory Dynam. Systems **32** (2012), 35–62.

[5] J. Christensen and K. Thomsen, *Finite digraphs and KMS states*, J. Math. Anal. Appl. **433** (2016), 1626–1646.

[6] L.O. Clarke, A. an Huef and I. Raeburn, *Phase transitions on the Toeplitz algebras of Baumslag–Solitar semigroups*, Indiana University Mathematical Journal, to appear (arXiv:1503.04873 [math.OA]).

[7] K.R. Davidson, C^*-algebras by example, American Mathematical Society, Providence, RI, 1996, xiv+309.

[8] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on O_A*, Math. Japon. **29** (1984), 607–619.

[9] C. Farsi, E. Gillaspy, S. Kang and J.A. Packer, *Separable representations, KMS states, and wavelets for higher-rank graphs*, J. Math. Anal. Appl. **434** (2016), 241–270.

[10] N. J. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. **48** (1999), 155–181.

[11] M. Hawkins, *Applications of compact topological graph C^*-algebras to noncommutative solenoids*, PhD Thesis, University of Wollongong 2015.

[12] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C^*-algebras of finite graphs*, J. Math. Anal. Appl. **405** (2013), 388–399.

[13] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C^*-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. **268** (2015), 1840–1875.

[14] T. Kajiwara and Y. Watatani, *KMS states on finite-graph C^*-algebras*, Kyushu J. Math. **67** (2013), 83–104.

[15] E.T.A. Kakariadis, *KMS states on Pimsner algebras associated with C^*-dynamical systems*, J. Funct. Anal. **269** (2015), 325–354.

[16] T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras I, Fundamental results*, Trans. Amer. Math. Soc. **356** (2004), 4287–4322.

[17] T. Katsura, *On C^*-algebras associated with C^*-correspondences*, J. Funct. Anal. **217** (2004), 366–401.

[18] T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras II. Examples*, Internat. J. Math. **17** (2006), 791–833.

[19] M. Laca and S. Neshveyev, *KMS states of quasi-free dynamics on Pimsner algebras*, J. Funct. Anal. **211** (2004), 457–482.

[20] M. Laca and I. Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. **225** (2010), 643–688.

[21] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, *Equilibrium states on the Cuntz–Pimsner algebras of self-similar actions*, J. Funct. Anal. **266** (2014), 6619–6661.

[22] F. Latrémolière and J.A. Packer *Noncommutative solenoids*, New York J. Math., to appear (arXiv:1110.6227 [math.OA]).

[23] F. Latrémolière and J.A. Packer, *Noncommutative solenoids and their projective modules*, Contemp. Math., 603, Commutative and noncommutative harmonic analysis and applications, 35–53, Amer. Math. Soc., Providence, RI, 2013.

[24] F. Latrémolière and J.A. Packer, *Explicit construction of equivalence bimodules between noncommutative solenoids*, Contemp. Math., 650, Trends in harmonic analysis and its applications, 111–140, Amer. Math. Soc., Providence, RI, 2015.

[25] R.R. Phelps, Lectures on Choquet's theorem, Springer–Verlag, Berlin, 2001, viii+124.

[26] M.V. Pimsner, *A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by* **Z**, Fields Inst. Commun., 12, Free probability theory (Waterloo, ON, 1995), 189–212, Amer. Math. Soc., Providence, RI, 1997.

[27] M. Takesaki and M. Winnink, *Local normality in quantum statistical mechanics*, Comm. Math. Phys. **30** (1973), 129–152.

[28] D. Yang, *Endomorphisms and modular theory of 2-graph C^*-algebras*, Indiana Univ. Math. J. **59** (2010), 495–520.

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