W-graph determining elements in type *A*

Van Minh Nguyen

Abstract Let (W,S) be a Coxeter system of type A, so that W can be identified with the symmetric group $\mathrm{Sym}(n)$ for some positive integer n and S with the set of simple transpositions $\{(i,i+1) \mid 1 \leqslant i \leqslant n-1\}$. Let $\leqslant_{\mathsf{L}}$ denote the left weak order on W, and for each $J \subseteq S$ let w_J be the longest element of the subgroup W_J generated by J. We show that the basic skew diagrams with n boxes are in bijective correspondence with the pairs (w,J) such that the set $\{x \in W \mid w_J \leqslant_{\mathsf{L}} x \leqslant_{\mathsf{L}} ww_J\}$ is a nonempty union of Kazhdan–Lusztig left cells. These are also the pairs (w,J) such that $\mathscr{I}(w) = \{v \in W \mid v \leqslant_{\mathsf{L}} w\}$ is a W-graph ideal with respect to J. Moreover, for each such pair the elements of $\mathscr{I}(w)$ are in bijective correspondence with the standard tableaux associated with the corresponding skew diagram.

Keywords Coxeter group · W-graph · Kazhdan–Lusztig cell

1 Introduction

Let (W,S) be a Coxeter system and denote by $\leqslant_{\mathsf{L}}$ the left weak order on W (defined by $y \leqslant_{\mathsf{L}} x$ if and only if $l(xy^{-1}) = l(x) - l(y)$, where l denotes length relative to S). In [5] an algorithm was given that takes as input a pair $(\mathscr{I},J)$, where $\mathscr{I}$ is an ideal of $(W,\leqslant_{\mathsf{L}})$ and J is a subset of S, and produces a graph with edges labelled by integers and vertices coloured with subsets of S. If $(\mathscr{I},J)$ is a W-graph ideal then the output is a W-graph. It was shown in [5] that W-graphs for the Specht modules can be produced in this way. In [9] it was shown, more generally, that W-graphs for the Kazhdan–Lusztig left cells that contain longest elements of standard parabolic subgroups can be obtained from W-graph ideals. Indeed, if $J \subseteq S$ and W_J is the subgroup generated by J, then the left cell containing w_J (the longest element of W_J) is equal to $\mathscr{I}w_J$, where $(\mathscr{I},J)$ is a W-graph ideal.

Our first main result says that if (W,S) is of type A and $\mathscr{I}$ is an ideal of $(W,\leqslant_{\mathsf{L}})$ then $\mathscr{I}w_J$ is a union of Kazhdan–Lusztig left cells whenever $(\mathscr{I},J)$ is a W-graph ideal. Note that, by [7, Theorem 9.5], this is not true for types other than A, even when (W,S) has rank 2. However, [7, Theorem 5.2] shows that, for all types, if $\mathcal{C}$ is a set of left cells that is upward closed, in the sense that $c \in \mathcal{C}$ and $c' \geqslant c$ implies $c' \in \mathcal{C}$, then $\bigcup_{c \in \mathcal{C}} c$ is a W-graph ideal.

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Our second main result is the classification, when (W,S) is of type A_{n-1}, of the pairs (w,J) such that $(\mathscr{I}(w),J)$ is a W-graph ideal, where $\mathscr{I}(w) = \{v \in W \mid v \leqslant_{\mathrm{L}} w\}$. These are exactly the pairs (w,J) such that $l(ws) > l(w)$ for all $s \in J$ and $\mathscr{I}(w)w_J$ is a union of Kahzdan–Lusztig left cells. Furthermore, they are parametrized by the skew partitions of n, and in each case the elements of $\mathscr{I}(w)$ are parametrized by the standard tableaux associated with the corresponding basic skew diagram.

Since the current work is a sequel to [7], we shall freely use the notation and terminology of that paper.

2 Relationship between W-graph ideals and Kazhdan–Lusztig left cells in type A

A complete classification of W-graph ideals of finite Coxeter groups of rank 2 is given in Theorem 9.5 of [7]. We shall make use of the following special case.

Lemma 2.1 *Let (W,S) be a Coxeter system of type $A_2 = I_2(2)$, and let $S = \{s,t\}$. Then $(\mathscr{I},J)$ is a W-graph ideal if and only if one of the following alternatives is satisfied:*

(i) $(\mathscr{I},J) = (\{1\},S)$,
(ii) $(\mathscr{I},J) = (\{1\},\emptyset)$,
(iii) $(\mathscr{I},J) = (\{1,t,st\},\{s\})$,
(iv) $(\mathscr{I},J) = (\{\{1,t\},\{s\})$,
(v) $(\mathscr{I},J) = (\{1,s,ts\},\{t\})$,
(vi) $(\mathscr{I},J) = (\{\{1,s\},\{t\})$,
(vii) $(\mathscr{I},J) = (\{1,s,t,ts,st,tst\},\emptyset)$,
(viii) $(\mathscr{I},J) = (\{1,s,t,ts,st\},\emptyset)$.

Remark 2.2 Let $(\mathscr{I},J)$ be one of the W-graph ideals in the above list. It is readily checked that the set $\mathscr{I}w_J$ contains the element t if and only if it also contains the element st. Similarly, $s \in \mathscr{I}w_J$ if and only if $ts \in \mathscr{I}w_J$. This amounts to saying that $\mathscr{I}w_J$ is a union of left cells of W, since $\{t,st\}$ and $\{s,ts\}$ are left cells of W, and the other left cells are singleton sets.

The following result was proved in [7, Theorem 8.4].

Theorem 2.3 *Let $(\mathscr{I},J)$ be a W-graph ideal. Suppose that $K \subseteq S$ and $d \in D_K^{-1} \cap \mathscr{I}$, and put $\mathscr{I}_d = \{\, y \in W_K \mid yd \in \mathscr{I} \,\}$. Then $(\mathscr{I}_d, K \cap dJd^{-1})$ is a W_K-graph ideal.*

Using the notation of [7], if $(\mathscr{I},J)$ is a W-graph ideal let $\Gamma(\mathscr{I},J)$ denote the corresponding W-graph. We shall identify $\mathscr{I}$ with the vertex set of $\Gamma(\mathscr{I},J)$. Continuing with the hypotheses of Theorem 2.3, let $\Gamma = \Gamma(\mathscr{I},J)$ and let Γ_K be the W_K-graph obtained from Γ by ignoring the elements of $S \setminus K$. By [7, Remark 8.6], the mapping $y \mapsto yd$ from $\mathscr{I}_d$ to $W_K d \cap \mathscr{I}$ induces an isomorphism from the W_K-graph $\Gamma(\mathscr{I}_d, K \cap dJd^{-1})$ to the full subgraph of Γ_K spanned by $W_K d \cap \mathscr{I}$. Moreover, this set is a union of cells of Γ_K. Since it is trivial that each cell X of Γ is a union of cells of Γ_K, it follows that $Xd^{-1} \cap W_K$ is a union of cells of $\Gamma(\mathscr{I}_d, K \cap dJd^{-1})$. Applying this in the case $\mathscr{I} = D_J$ gives the following result.

Proposition 2.4 *Let $J, K \subseteq S$ and $d \in D_{K,J}$, and let $M = K \cap dJd^{-1}$. If X is any cell of (D_J,J), then $Xd^{-1} \cap W_K$ is a union of cells of (D_M^K, M), where $D_M^K = W_K \cap D_M$.*

Remark 2.5 Proposition 2.4 also follows from [8, Proposition 5.7], taking Γ to be the single vertex W_J-graph corresponding to the trivial representation of W_J.

Assume now that (W,S) is of type A. Following the terminology of [6, Section 5], for $x \in W$ define $\mathscr{L}(x) = \{ s \in S \mid sx < x \}$, and let $\approx$ be the equivalence relation on W generated by the relations $x \approx sx$ for all $x \in W$ and $s \in S$ such that $x < sx$ and $\mathscr{L}(x) \not\subseteq \mathscr{L}(sx)$. Kazhdan and Lusztig show that $\approx$ coincides with Kazhdan–Lusztig left equivalence, so that the equivalence classes are precisely the left cells. Hence to show that a subset X of W is a union of left cells it is sufficient to show that whenever the relation $x \approx sx$ holds, $x \in X$ if and only if $sx \in X$.

Lemma 2.6 *Let (W,S) be a Coxeter system of type A and let $X \subseteq W$. Then X is a union of left cells if and only if for all $s,t \in S$ such that st has order 3 and all $d \in W$ such that $sd > d$ and $td > d$, the set $\{ y \in W_{\{s,t\}} \mid yd \in X \}$ is a union of left cells in $W_{\{s,t\}}$.*

Proof Suppose first that X is a union of left cells. Let $s,t \in S$ with st of order 3, and let $d \in W$ satisfy $sd > d$ and $td > d$. We must show that the set $X_d = \{ y \in W_{\{s,t\}} \mid yd \in X \}$ is a union of left cells of $W_{\{s,t\}}$. That is, we must show that $t \in X_d$ if and only if $st \in X_d$, and $s \in X_d$ if and only if $ts \in X_d$. But since $l(xd) = l(x) + l(d)$ for all $x \in W_{\{s,t\}}$ it is clear that $td \approx std$, since $l(td) < l(std)$ and $t \in \mathscr{L}(td) \setminus \mathscr{L}(std)$. So td and std are in the same left cell of W, and hence $td \in X$ if and only if $std \in X$. Similarly, $sd \in X$ if and only if $tsd \in X$. Thus $t \in X_d$ if and only if $st \in X_d$, and $s \in X_d$ if and only if $ts \in X_d$, giving the desired conclusion.

Conversely, assume that for all $s,t \in S$ with st of order 3 and all $d \in W$ with $sd > d$ and $td > d$, the set $\{ y \in W_{\{s,t\}} \mid yd \in X \}$ is a union of left cells in $W_{\{s,t\}}$. We must show that X is a union of $\approx$ equivalence classes. It suffices to show that if $x \in W$ and $s \in S$ satisfy $x < sx$ and $\mathscr{L}(x) \not\subseteq \mathscr{L}(sx)$, then $x \in X$ if and only if $sx \in X$. Given such elements x and s, choose $t \in S$ such that $t \in \mathscr{L}(x) \setminus \mathscr{L}(sx)$. Then $tx < x < sx < tsx$, so that $l((tst)(tx)) = l(tsx) = l(tx) + 3$. So st has order 3, and $d = tx$ is the shortest element of the coset $W_{\{s,t\}}d$ (and $tsx = tstd$ the longest). Since $X_d = \{ y \in W_{\{s,t\}} \mid yd \in X \}$ is a union of left cells of $W_{\{s,t\}}$, it follows that $t \in X_d$ if and only if $st \in X_d$. That is, $td \in X$ if and only $std \in X$, as required. □

Theorem 2.7 *Let (W,S) be a Coxeter system of type A, and let $(\mathscr{I},J)$ be a W-graph ideal. Then $\mathscr{I}w_J$ is a union of Kazhdan–Lusztig left cells.*

Proof We use Lemma 2.6. Accordingly, let $s,t \in S$ be such that st has order 3 and let $d \in W$ be such that $sd > d$ and $td > d$. Put $K = \{s,t\}$ and $Y = \{ y \in W_K \mid yd \in \mathscr{I}w_J \}$. We show that Y is a union of left cells in W_K, noting first that this certainly holds if $Y = \emptyset$.

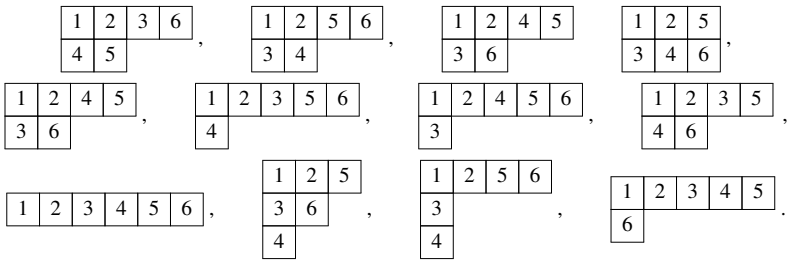
Assume that $Y \neq \emptyset$. Let e be the minimal length element in the right coset $W_K dw_J$. Since $\mathscr{I} \cap W_K dw_J = Ydw_J$ is nonempty and $\mathscr{I}$ is an ideal of $(W, \leqslant_{\mathrm{L}})$ it follows that $e \in \mathscr{I}$, and hence $e \in D_K^{-1} \cap \mathscr{I} \subseteq D_K^{-1} \cap D_J$. By Theorem 2.3 it follows that $(\mathscr{I}_e, L)$ is a W_K-graph ideal, where $\mathscr{I}_e = \{ y \in W_K \mid ye \in \mathscr{I} \}$ and $L = K \cap eJe^{-1}$. Hence $\mathscr{I}_e w_L$ is a union of left cells in W_K by Remark 2.2.

It remains to observe that $\mathscr{I}_e w_L = Y$, and since $\mathscr{I}_e w_L = \{ y \in W_K \mid yw_L e \in \mathscr{I} \}$ it suffices to show that $d = w_L e w_J$. Since e is the minimal length element in $W_K e W_J$ and $L = K \cap eJe^{-1}$ it follows that $w_L e w_J$ is the minimal length element in $W_K e w_J = W_K d$, and hence $d = w_L e w_J$, as required. □

Remark 2.8 It was shown in [7, Proposition 5.13] that if J is any subset of S then $\Gamma(D_J, J)$ is isomorphic to the full subgraph of $\Gamma(W,\emptyset)$ spanned by the vertices corresponding to elements of $D_J w_J$, via the obvious bijection $D_J \to D_J w_J$. Moreover, $D_J w_J$ is a union of left cells of W, and $X \subseteq D_J w_J$ is a left cell of W if and only if $\{ w \in D_J \mid ww_J \in X \}$ is a cell of (D_J, J). Thus Theorem 2.7 tells us that if (W,S) is of type A and $(\mathscr{I}, J)$ is a W-graph ideal then $\mathscr{I}$ is a union of cells of (D_J, J). Now by [7, Theorem 5.2], if $\mathcal{C}$ is a set of cells of (D_J, J) that is upward closed with respect to the Kazhdan–Lusztig partial order on cells, then the union of

the cells in $\mathcal{C}$ is an ideal of $(W,\leqslant_{\mathsf{L}})$ and is a W-graph ideal with respect to J. (These are the strong W-graph subideals of (D_J,J).) We conjecture that in type A all W-graph ideals have this form.

It has been shown by calculation that when W is of type A_5 there exist ideals of $(W,\leqslant_{\mathsf{L}})$ that are unions of left cells but are not W-graph ideals. It is well known that in this case the Robinson-Schensted map $w\mapsto(P(w),Q(w))$ is a bijection from W to the set of ordered pairs of standard tableaux on $\{1,2,3,4,5,6\}$, and that for each such standard tableau t the set $\{\,w\in W\mid Q(w)=t\,\}$ is a left cell of W. (See [1, Theorem A].) Using the computational algebra system Magma (see [3]), R. B. Howlett has shown that if $\mathscr{I}$ is the union of the left cells corresponding to the tableaux listed below then $\mathscr{I}$ is an ideal of $(W,\leqslant_{\mathsf{L}})$, but $(\mathscr{I},\emptyset)$ is not a W-graph ideal.



3 *W*-graphs derived from skew partitions

For each positive integer n we define W_n to be the symmetric group on the set $\{1,2,\ldots,n\}$ and $S_n=\{\,s_i\mid 1\leqslant i<n\,\}$, where s_i is the transposition $(i,i+1)$. Thus (W_n,S_n) is a Coxeter system of type A_{n-1}. If l and m are positive integers then we define $W_{[l,m]}$ to be the set of all permutations of $[l,m]=\{\,i\in\mathbb{Z}\mid l\leqslant i\leqslant m\,\}$. If $[l,m]\subseteq[1,n]$ then $W_{[l,m]}$ can be regarded as a standard parabolic subgroup of W_n (generated by $\{\,s_i\in S_n\mid l\leqslant i<m\,\}$). We use a left operator convention for permutations, writing wi for the image of i under the permutation w.

A *partition* of n is a sequence of positive integers $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k$ with $\sum_{i=1}^{k}\lambda_i=n$. The λ_i are called the *parts* of the partition. We adopt the convention that if λ is a partition with k parts then λ_i denotes the i-th part of λ if $i\in\{1,2,\ldots k\}$, and $\lambda_i=0$ if $i>k$. The *Young diagram* of λ is the set

$$[\lambda]=\{\,(i,j)\mid 1\leqslant i\leqslant k \text{ and } 1\leqslant j\leqslant\lambda_i\,\},$$

represented pictorially as a left-justified array of boxes with λ_i boxes in the i-th row from the top. We define $P(n)$ to be the set of all partitions of n.

If $\lambda\in P(n)$ then λ^* denotes the *conjugate* of λ, defined to be the partition whose diagram is the transpose of $[\lambda]$. That is, $[\lambda^*]=\{\,(j,i)\mid(i,j)\in[\lambda]\,\}$.

A *skew partition of n* is an ordered pair λ/μ such that $\lambda\in P(m+n)$ and $\mu\in P(m)$ for some nonnegative integer m, and $\lambda_i\geqslant\mu_i$ for all i. We write $\lambda/\mu\vdash n$ to indicate that λ/μ is a skew partition of n. In the case $m=0$ we identify λ/μ with λ.

The *skew diagram* $[\lambda/\mu]$ corresponding to a skew partition λ/μ is defined to be the complement of $[\mu]$ in $[\lambda]$. That is,

$$[\lambda/\mu]=\{\,(i,j)\mid(i,j)\in[\lambda]\text{ and }(i,j)\notin[\mu]\,\}.$$

We say that $[\lambda/\mu]$ has λ_1 columns and λ_1^* rows, and that $[\lambda/\mu]$ is *basic* if all rows and columns are nonempty. Thus $[\lambda/\mu]$ is basic if $\lambda_i>\mu_i$ and $\lambda_j^*>\mu_j^*$ for all $i\leqslant\lambda_1^*$ and $j\leqslant\lambda_1$.

If $\lambda/\mu \vdash n$ then a *skew tableau of shape* λ/μ, or (λ/μ)-tableau, is a bijective map $t\colon [\lambda/\mu] \to \mathcal{A}$, where $\mathcal{A}$ is a totally ordered set with n elements. We call $\mathcal{A}$ the *target* of t. In this paper the target will always be an interval $[m+1, m+n]$, with $m = 0$ unless otherwise specified. For each $a \in \mathcal{A}$ we define $\mathrm{row}(t,a)$ and $\mathrm{col}(t,a)$ to be the row index and column index of a in t, so that $t^{-1}(a) = (\mathrm{row}(t,a), \mathrm{col}(t,a))$. We say that t is *row standard* if its entries increase across the rows, *column standard* if its entries increase down the columns, and *standard* if it is both row standard and column standard.

We define $\mathrm{Tab}_m(\lambda/\mu)$ to be the set of all (λ/μ)-tableaux with target $[m+1, m+n]$, and $\mathrm{Std}_m(\lambda/\mu) = \{\, t \in \mathrm{Tab}_m(\lambda/\mu) \mid t \text{ is standard} \,\}$. The subscript m is usually omitted if $m = 0$. If $h \in \mathbb{Z}$ and $t \in \mathrm{Tab}_m(\lambda/\mu)$ then we define $h + t \in \mathrm{Tab}_{h+m}(\lambda/\mu)$ to be the tableau obtained by adding h to all entries of t. We define $\tau^{\lambda/\mu} \in \mathrm{Std}(\lambda/\mu)$ to be the (λ/μ)-tableau given by

$$\tau^{\lambda/\mu}(i,j) = j - \mu_i + \sum_{h=1}^{i-1} (\lambda_h - \mu_h) \tag{3.1}$$

for all $(i,j) \in [\lambda/\mu]$. That is, the numbers $1, 2, \ldots, (\lambda_1 - \mu_1)$ fill the first row of $\tau^{\lambda/\mu}$ in order from left to right, then the numbers $(\lambda_1 - \mu_1) + 1, (\lambda_1 - \mu_1) + 2, \ldots, (\lambda_1 - \mu_1) + (\lambda_2 - \mu_2)$ similarly fill the second row, and so on. We also define $\tau_{\lambda/\mu}$ to be the standard (λ/μ)-tableau that is the transpose of the (λ^*/μ^*)-tableau τ^{λ^*/μ^*} (so that the numbers 1 to n fill the columns of $\tau_{\lambda/\mu}$ in order from left to right).

It is clear that if $\lambda/\mu \vdash n$ then $W_{[m+1,m+n]}$ acts on $\mathrm{Tab}_m(\lambda/\mu)$, via $(wt)(i,j) = w(t(i,j))$ for all $(i,j) \in [\lambda/\mu]$, all $t \in \mathrm{Tab}_m(\lambda/\mu)$ and all $w \in W_{[m+1,m+n]}$. Thus $w \mapsto w(m + \tau_{\lambda/\mu})$ gives a bijective map from $W_{[m+1,m+n]}$ to $\mathrm{Tab}_m(\lambda/\mu)$. We define perm: $\mathrm{Tab}_m(\lambda/\mu) \to W_{[m+1,m+n]}$ to be the map inverse to $w \mapsto w(m + \tau_{\lambda/\mu})$, and use perm to transfer the Bruhat order and the left weak order from $W_{[m+1,m+n]}$ to $\mathrm{Tab}_m(\lambda/\mu)$. That is, for all $t_1, t_2 \in \mathrm{Tab}_m(\lambda/\mu)$ we write $t_1 \leqslant t_2$ if and only if $\mathrm{perm}(t_1) \leqslant \mathrm{perm}(t_2)$, and $t_1 \leqslant_{\mathrm{L}} t_2$ if and only if $\mathrm{perm}(t_1) \leqslant_{\mathrm{L}} \mathrm{perm}(t_2)$.

Remark 3.1 If $\lambda/\mu \vdash n$ and $t \in \mathrm{Std}(\lambda/\mu)$ then the *reading word* of t is defined to be the sequence $(a_1, a_2, \ldots, a_n)$ obtained by concatenating the rows of t in order from last row to first row. So there is a bijection $\mathrm{Std}(\lambda/\mu) \to W_n$ that maps each t to the permutation $\mathrm{word}(t)$ defined by $i \mapsto a_i$ for all $i \in \{1, 2, \ldots, n\}$. This bijection and our bijection perm are related by the equation $\mathrm{perm}(t) = \mathrm{word}(t) w^{-1}$, where $w = \mathrm{word}(\tau_{\lambda/\mu})$.

Whenever $\lambda/\mu \vdash n$ we define $J_{\lambda/\mu}$ to be the subset of S_n consisting of those s_i such that i and $i+1$ lie in the same column of $\tau_{\lambda/\mu}$, and we define $W_{\lambda/\mu}$ to be the standard parabolic subgroup of W_n generated by $J_{\lambda/\mu}$. Thus $W_{\lambda/\mu}$ is the column group of $\tau_{\lambda/\mu}$. Moreover, the set $D = \{\, d \in W_n \mid di < d(i+1) \text{ whenever } s_i \in J_{\lambda/\mu} \,\}$ is the set of minimal length representatives of the left cosets of $W_{\lambda/\mu}$ in W_n, since the condition $di < d(i+1)$ is equivalent to $l(ds_i) > l(d)$. Thus $\{\, d\tau_{\lambda/\mu} \mid d \in D \,\}$ is precisely the set of column standard (λ/μ)-tableaux.

We shall make use of the following result, which was was proved in [9].

Theorem 3.2 [9, Theorem 6.6] *Let $\lambda \in \mathcal{P}(n)$ and $J = J_\lambda$, so that $W_J = W_\lambda$ is the column group of τ_λ. Then $y \in W_n$ given by $y\tau_\lambda = \tau^\lambda$ is a W_n-graph determining element (with respect to J), and its W_n-graph is isomorphic to the W_n-graph of the left cell that contains w_J.*

Remark 3.3 In our present notation, Theorem 3.2 says that $(\mathscr{I}, J) = (\mathscr{I}(\mathrm{perm}(\tau^\lambda)), J_\lambda)$ is a W_n-graph ideal, and that $\mathscr{I}$ is isomorphic to the left cell of W_n containing w_J. As explained in Remark 2.8 above, the left cell of W_n containing w_J is isomorphic to the cell of (D_J, J) containing 1. By [9, Proposition 6.5], this cell coincides with $\mathscr{I}$. Now if X is an arbitrary cell of (D_J, J) then $X \leqslant \mathscr{I}$, since $1 \leqslant_{\mathrm{L}} d$ for all $d \in D_J$. Thus $\{\mathscr{I}\}$ is an upward-closed set of cells. So in fact the results of [9] show that $(\mathscr{I}, J)$ is a strong W-graph subideal of (D_J, J).

The following result is a straightforward generalization of [5, Lemma 6.2].

Lemma 3.4 *Let $\lambda/\mu \vdash n$ and let $\mathscr{I} = \mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu}))$, the ideal of $(W_n, \leqslant_{\mathsf{L}})$ generated by* $\mathrm{perm}(\tau^{\lambda/\mu})$. *Then* $\mathrm{Std}(\lambda/\mu) = \{ w\tau_{\lambda/\mu} \mid w \in \mathscr{I} \} = \{ t \in \mathrm{Tab}(\lambda/\mu) \mid t \leqslant_{\mathsf{L}} \tau^{\lambda/\mu} \}$.

Remark 3.5 The fact that the set of standard (λ/μ)-tableaux is in one-to-one correspondence with the ideal of $(W_n, \leqslant_{\mathsf{L}})$ generated by the element $\mathrm{perm}(\tau^{\lambda/\mu})$ is included, with other results, in Theorems 7.2 and 7.5 of [2].

Definition 3.6 We call $m + \tau^{\lambda/\mu}$ the *maximal tableau* in $\mathrm{Std}_m(\lambda/\mu)$.

Let $\lambda/\mu \vdash n$ and $t \in \mathrm{Std}_m(\lambda/\mu)$. For each $k \in [m+1, m+n]$ we define $t{\downarrow}k$ to be the skew tableau obtained by removing from t all boxes filled with entries greater than or equal to k. Thus $t{\downarrow}k \in \mathrm{Std}_m(\kappa/\mu)$, where $[\kappa] = [\lambda] \setminus \{ (i,j) \in [\lambda/\mu] \mid t(i,j) \geqslant k \}$, and $t{\downarrow}k$ is the restriction of t to $[\kappa/\mu] = \{ (i,j) \in [\lambda/\mu] \mid t(i,j) < k \}$. Clearly $\kappa/\mu \vdash (k-m-1)$. Similarly, we define $t{\uparrow}k$ to be the skew tableau obtained by removing all boxes with entries less than or equal to k, so that $t{\uparrow}k \in \mathrm{Std}_k(\lambda/\nu)$ where $[\nu] = [\mu] \cup \{ (i,j) \in [\lambda/\mu] \mid t(i,j) \leqslant k \}$, and $t{\uparrow}k$ is the restriction of t to $[\lambda/\nu] = \{ (i,j) \in [\lambda/\mu] \mid t(i,j) > k \}$. Clearly $\lambda/\nu \vdash (m+n-k)$.

We can now show that Theorem 3.2 generalizes to skew partitions in the natural way.

Theorem 3.7 *Let $\lambda/\mu \vdash n$ and $M = J_{\lambda/\mu}$. Then $(\mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu})), M)$ is a strong W_n-graph subideal of (D_M, M).*

Proof Recall that $\lambda \vdash (m+n)$ and $\mu \vdash m$ for some nonnegative integer m. Clearly there is an isomorphism $\varphi\colon W_n \to W_{[m+1,m+n]}$ given by $s_i \mapsto s_{m+i}$ for all $i \in [1, n-1]$, and it suffices to prove that $(\varphi(\mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu}))), \varphi(M))$ is a strong $W_{[m+1,m+n]}$-graph subideal of $(\varphi(D_M), \varphi(M))$. Note that $\mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu})) = \{ w \in W_n \mid w\tau_{\lambda/\mu} \in \mathrm{Std}(\lambda/\mu) \}$, by Lemma 3.4, and so $\varphi(\mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu}))) = \{ w \in W_{[m+1,m+n]} \mid w(m+\tau_{\lambda/\mu}) \in \mathrm{Std}_m(\lambda/\mu) \}$.

Write $W = W_{m+n}$, and note that $W_{[m+1,m+n]}$ can be identified with the standard parabolic subgroup W_K of W, where $K = \{ s_i \mid m+1 \leqslant i < m+n \}$. Let $t_0 \in \mathrm{Std}(\lambda)$ be defined by $t_0{\downarrow}(m+1) = \tau_\mu$ and $t_0{\uparrow}m = m + \tau_{\lambda/\mu}$, and let $d = \mathrm{perm}(t_0)$. Thus $t_0 = d\tau_\lambda$ and $d \in D_J$, where $J = J_\lambda$. Since $(d^{-1}(m+1), d^{-1}(m+2), \ldots, d^{-1}(m+n))$ is an increasing sequence, consecutive terms differing either by 1 or by $1 + \mu_j^*$ for some j, it follows that $sd > d$ for all $s \in K$. Thus $d \in D_K^{-1}$, and so $d \in D_{K,J} = D_K^{-1} \cap D_J$. By [7, Lemma 2.5], this implies that $\{ w \in W_K \mid wd \in D_J \} = D^K_{K \cap dJd^{-1}}$, the set of minimal length representatives of the left cosets of $W_{K \cap dJd^{-1}}$ in W_K.

Write $\mathscr{I} = \mathscr{I}(\mathrm{perm}(\tau^\lambda))$, so that $(\mathscr{I}, J)$ is a W-graph ideal by Theorem 3.2, and a strong W-graph subideal of (D_J, J) by Remark 3.3. Furthermore, $x \mapsto x\tau_\lambda$ gives a bijection from $\mathscr{I}$ to $\mathrm{Std}(\lambda)$. Since

$$\begin{aligned}
\{ w \in W_K \mid wd \in \mathscr{I} \} &= \{ w \in W_K \mid wd\tau_\lambda \in \mathrm{Std}(\lambda) \} \\
&= \{ w \in W_K \mid wt_0 \in \mathrm{Std}(\lambda) \} \\
&= \{ w \in W_K \mid (wt_0){\uparrow}m \in \mathrm{Std}_m(\lambda/\mu) \} \\
&= \{ w \in W_K \mid w(m+\tau_{\lambda/\mu}) \in \mathrm{Std}_m(\lambda/\mu) \} \\
&= \varphi(\mathscr{I}(\mathrm{perm}(\tau^{\lambda/\mu}))),
\end{aligned}$$

it follows from Theorem 2.3 that $(\mathscr{I}(\varphi(\mathrm{perm}(\tau^{\lambda/\mu}))), K \cap dJd^{-1})$ is a W_K-graph ideal. Moreover, by [7, Theorem 8.7], it is a strong W_K-graph subideal of $(D^K_{K \cap dJd^{-1}}, K \cap dJd^{-1})$. To complete the proof, it remains to show that $K \cap dJd^{-1} = \varphi(M)$. But this is clear from the fact that $d\tau_\lambda{\uparrow}m = m + \tau_{\lambda/\mu}$, which shows that if $m+1 \leqslant i < m+n$ then i and $i+1$ are in the same column of $d\tau_\lambda$ if and only if $i-m$ and $i-m+1$ are in the same column of $\tau_{\lambda/\mu}$. □

4 Converse of Theorem 3.7

Continuing with the notation used in the previous section, for each $J \subseteq \{s_1, s_2, \ldots, s_{n-1}\}$ define $\mathrm{Skew}(J)$ to be the set of all $\lambda/\mu \vdash n$ such that $J_{\lambda/\mu} = J$ and the skew diagram $[\lambda/\mu]$ is basic. For each $w \in D_J$ define $\mathrm{Skew}(J,w) = \{\, \lambda/\mu \in \mathrm{Skew}(J) \mid w\tau_{\lambda/\mu} \in \mathrm{Std}(\lambda/\mu) \,\}$, and $\mathrm{Std}(J,w) = \{\, w\tau_{\lambda/\mu} \mid \lambda/\mu \in \mathrm{Skew}(J,w) \,\}$. Observe that if $t \in \mathrm{Std}(J,w)$ then all other elements of $\mathrm{Std}(J,w)$ can be obtained from t by sliding columns vertically up or down. Clearly $\mathrm{Std}(J,w) \neq \emptyset$, since $\mathrm{Skew}(J,w)$ always contains the (unique) $\lambda/\mu \in \mathrm{Skew}(J)$ such that $[\lambda/\mu]$ has n rows, all of length 1. We define $\tau(J,w)$ to be the element of $\mathrm{Std}(J,w)$ with the least possible number of rows. It is clear that this element is unique.

In this section we shall show that if $w \in W_n$ has the property that $(\mathscr{I}(w), J)$ is a W_n-graph ideal then $\tau(J,w) = \tau^{\lambda/\mu}$ for some $\lambda/\mu \vdash n$.

For the purposes of this section it is convenient to make the following definition.

Definition 4.1 Let (W,S) be a Coxeter system, let $K \subseteq S$, and let W_K be the subgroup of W generated by K. If $w \in W$ then the *left W_K-component of w* is the element $x \in W_K$ such that $x^{-1}w$ is the minimal length element of $W_K w$, and $d = x^{-1}w$ is the *D_K^{-1}-component of w*.

Let $W = W_n$ and let $K = S_n \setminus \{s_{n-1}\}$ and $L = S_n \setminus \{s_1\}$, so that the parabolic subgroups W_K and W_L of W can be identified with W_{n-1} and $W_{[2,n]}$ respectively. Let $\lambda/\mu \vdash n$ and $t \in \mathrm{Std}(\lambda/\mu)$. Since t is standard, the number n must be at the bottom of its column and the right hand end of its row in t, and the number 1 must be at the top of its column and the left hand end of its row. Define κ and ν by $[\kappa] = [\lambda] \setminus \{t^{-1}(n)\}$ and $[\nu] = \mu \cup \{t^{-1}(1)\}$. The restriction of t to $[\kappa/\mu]$ is $t{\downarrow}n$, the (κ/μ)-tableau obtained by deleting from t the box containing n, and the restriction of t to $[\lambda/\nu]$ is $t{\uparrow}1$, the (λ/ν)-tableau obtained by deleting from t the box containing 1. Clearly $t{\downarrow}n \in \mathrm{Std}(\kappa/\mu)$ and $t{\uparrow}1 \in \mathrm{Std}_1(\lambda/\nu)$.

Lemma 4.2 *Let $W = W_n$ and let $K = S_n \setminus \{s_{n-1}\}$ and $L = S_n \setminus \{s_1\}$. Let λ/μ be a skew partition of n and $t \in \mathrm{Std}(\lambda/\mu)$, and let $w = \mathrm{perm}(t)$.*

(i) *The left W_K-component of w is $\mathrm{perm}(t{\downarrow}n)$, and the left W_L-component of w is $\mathrm{perm}(t{\uparrow}1)$.*

(ii) *Let $[\lambda] \setminus \{t^{-1}(n)\} = [\kappa]$ and $\mu \cup \{t^{-1}(1)\} = [\nu]$, and let $t', t'' \in \mathrm{Std}(\lambda/\mu)$ be defined by $t'{\downarrow}n = \tau_{\kappa/\mu}$ and $t'(n) = t(n)$, and $t''{\uparrow}1 = 1 + \tau_{\lambda/\nu}$ and $t''(1) = t(1)$. Then the D_K^{-1}-component of w is $d = \mathrm{perm}(t')$ and the D_L^{-1}-component of w is $e = \mathrm{perm}(t'')$. Furthermore, $K \cap dJ_{\lambda/\mu}d^{-1} = J_{\kappa/\mu}$ and $L \cap eJ_{\lambda/\mu}e^{-1} = \{\, s_i \in L \mid s_{i-1} \in J_{\lambda/\nu} \,\}$.*

Proof Let $d = \mathrm{perm}(t')$ and let $t^{-1}(n) = (i,j) \in [\lambda/\mu]$. Put $q = \tau_{\lambda/\mu}(i,j)$, and note that $w^{-1}(n) = d^{-1}(n) = q$ and $(d^{-1}(1), d^{-1}(2), \ldots, d^{-1}(n-1))$ is the sequence obtained by deleting q from the sequence $(1,2,\ldots,n)$. Thus $wd^{-1} \in W_K$ and $l(sd) > l(d)$ for all $s \in K$, which shows that d is the D_K^{-1}-component of w.

Since the D_K^{-1}-component of w is d, the left W_K-component of w is wd^{-1}. Since $wd^{-1}(t') = w\tau_{\lambda/\mu} = t$ it follows that $wd^{-1}(\tau_{\kappa/\mu}) = wd^{-1}(t'{\downarrow}n)$ is the tableau obtained from t by deleting the box containing $wd^{-1}(n) = n$. Hence $wd^{-1} = \mathrm{perm}(t{\downarrow}n)$.

The proof that $e = \mathrm{perm}(t'')$ is the D_L^{-1}-component of w and the proof that $\mathrm{perm}(t{\uparrow}1)$ is the left W_L-component of w are similar to the proofs that d is the D_K^{-1}-component and $\mathrm{perm}(t{\downarrow}n)$ the left W_K-component. So it remains to prove the last sentence of (ii).

Let $J = J_{\lambda/\mu}$, so that $W_J = W_{\lambda/\mu}$ is the column group of the tableau $\tau_{\lambda/\mu}$. Then eW_Je^{-1} is the column group of $e\tau_{\lambda/\mu} = t''$, and $W_L \cap eW_Je^{-1}$ is the column group of $t''{\uparrow}1 = 1 + \tau_{\lambda/\nu}$. That is, $W_L \cap eW_Je^{-1} = W_M$, where $M = \{\, s_i \in L \mid s_{i-1} \in J_{\lambda/\nu} \,\}$. But $W_L \cap eW_Je^{-1} = W_{L \cap eJe^{-1}}$, by [4, Theorem 2.7.4], since $e \in D_L^{-1} \cap D_J$, and hence $L \cap eJe^{-1} = M$, as required.

The proof that $K \cap dJd^{-1} = J_{\kappa/\mu}$ is similar. □

Definition 4.3 Given a tableau $t \in \mathrm{Std}(\lambda/\mu)$, for each $j \in [1,\lambda_1]$ let $M(t,j)$ be set of all integers k such that $k \leqslant \min(\lambda^*_j - \lambda^*_{j+1}, \mu^*_j - \mu^*_{j+1})$ and $t(i+k,j) < t(i,j+1)$ for all i such that $\mu^*_j - k < i \leqslant \lambda^*_{j+1}$. Define $m(t,j) = \max(M(t,j))$.

Remark 4.4 Intuitively, $m(t,j)$ is the maximal amount by which the j-th column of t can be slid up while keeping the tableau standard. Observe that when $k \in M(t,j)$, or, more generally, when $k \leqslant \min(\lambda^*_j - \lambda^*_{j+1}, \mu^*_j - \mu^*_{j+1})$, the condition $\mu^*_j - k < i \leqslant \lambda^*_{j+1}$ is satisfied if and only if $(i,j+1)$ and $(i+k,j)$ are both in $[\lambda/\mu]$. It is clear that $0 \in M(t,j)$ in all cases. Indeed, it is clear that $M(t,j) = [0, m(t,j)]$. Note also that if $\mu^*_j \geqslant \lambda^*_{j+1}$ then $\mu^*_j - \lambda^*_{j+1} \in M(t,j)$.

Let $\lambda/\mu \vdash n$ and $t \in \mathrm{Std}(\lambda/\mu)$, and for each $j \in [1,\lambda_1]$ define $\delta_j(t) = \sum_{l=j}^{\lambda_1} m(t,l)$. Then

$$(\lambda^*_j - \delta_j(t)) - (\lambda^*_{j+1} - \delta_{j+1}(t)) = (\lambda^*_j - \lambda^*_{j+1}) - m(t,j) \geqslant 0$$

and

$$(\mu^*_j - \delta_j(t)) - (\mu^*_{j+1} - \delta_{j+1}(t)) = (\mu^*_j - \mu^*_{j+1}) - m(t,j) \geqslant 0,$$

and it follows that $\{\, (i - \delta_j(t), j) \mid (i,j) \in [\lambda/\mu] \,\}$ is the diagram of a skew partition $\zeta/\eta \vdash n$ such that $\zeta^*_j = \lambda^*_j - \delta_j(t)$ and $\eta^*_j = \mu^*_j - \delta_j(t)$ for all $j \in [1,\lambda_1]$. Furthermore, if $t' \in \mathrm{Tab}_{\zeta/\eta}$ is defined by

$$t'(i,j) = t(i + \delta_j(t), j) \qquad \text{for all } (i,j) \in [\zeta/\eta]$$

then $t' \in \mathrm{Std}(\zeta/\eta)$, since if i and j are such that (i,j) and $(i,j+1)$ are both in $[\zeta/\eta]$ then

$$t'(i,j) = t(h + m(t,j), j) < t(h, j+1) = t'(i,j+1)$$

where $h = i + \delta_{j+1}(t)$. It is also clear that $\delta_j(t') = 0$ for all j.

Definition 4.5 Let $\lambda/\mu \vdash n$ and $t \in \mathrm{Std}(\lambda/\mu)$. We write $\mathrm{sqsh}(t)$ for the tableau t' defined in the above preamble, and we say that t is *squashed* if $\mathrm{sqsh}(t) = t$.

Remark 4.6 Let $\lambda/\mu \vdash n$ and $\zeta/\eta \vdash n$, and suppose that $\lambda^*_j - \mu^*_j = \zeta^*_j - \eta^*_j$ for all j. Note that this condition implies that $J_{\lambda/\mu} = J_{\zeta/\eta}$. It is easily shown that if $t \in \mathrm{Std}(\lambda/\mu)$ and $u \in \mathrm{Std}(\zeta/\eta)$ then $\mathrm{sqsh}(t) = \mathrm{sqsh}(u)$ if and only if $\mathrm{perm}(t) = \mathrm{perm}(u)$, or, equivalently, the tableau u can be obtained from the tableau t by sliding columns vertically. Hence, in the terminology introduced at the start of this section, if $J \subseteq \{s_1, s_2, \ldots, s_n\}$ and $w \in D_J$ then $\tau(J,w) = \mathrm{sqsh}(w\tau_{\lambda/\mu})$ for all $\lambda/\mu \in \mathrm{Skew}(J,w)$.

Lemma 4.7 *Let $\lambda/\mu \vdash n$. Then $\mathrm{sqsh}(\tau^{\lambda/\mu}) = \tau^{\zeta/\eta}$, where $[\zeta/\eta]$ is obtained from $[\lambda/\mu]$ by removing all empty rows.*

Proof Let $j \in [1,\lambda_1]$. Since all entries in any row of $\tau^{\lambda/\mu}$ exceed all entries in all earlier rows, it can be seen that $m(\tau^{\lambda/\mu}, j) = \max(0, \mu^*_j - \lambda^*_{j+1})$, from which the result follows. □

Corollary 4.8 *Let $\lambda/\mu \vdash n$, and let $r = \mathrm{sqsh}(\tau^{\lambda/\mu}\downarrow n)$ and $u = \mathrm{sqsh}(\tau^{\lambda/\mu}\uparrow 1)$. Let $q = \lambda^*_1$ and suppose that all rows of $[\lambda/\mu]$ are nonempty.*

(i) *If $\mu_1 < \lambda_1 - 1$ then $u = 1 + \tau^{\lambda/\nu}$, where $\nu_1 = \mu_1 + 1$ and $\nu_i = \mu_i$ for all $i > 1$.*
(ii) *If $\mu_1 = \lambda_1 - 1$ then $u = 1 + \tau^{\zeta/\nu}$, where $\nu_i = \mu_{i+1}$ and $\zeta_i = \lambda_{i+1}$ for all $i \geqslant 1$.*
(iii) *If $\mu_q < \lambda_q - 1$ then $r = \tau^{\kappa/\mu}$, where $\kappa_q = \lambda_q - 1$, and $\kappa_i = \lambda_i$ for all $i \neq q$.*
(iv) *If $\mu_q = \lambda_q - 1$ then $r = \tau^{\kappa/\xi}$, where $\kappa_i = \lambda_i$ and $\xi_i = \mu_i$ for all $j < q$, and $\kappa_i = \xi_i = 0$ for all $i \geqslant q$.* □

Definition 4.9 Let $W = W_n$, where n is a positive integer. A *W-graph determining tableau* is a tableau $t \in \mathrm{Std}(\lambda/\mu)$, where $\lambda/\mu \vdash n$, such that $(\mathscr{I}(\mathrm{perm}(t)), J_{\lambda/\mu})$ is a W-graph ideal.

Remark 4.10 If $t \in \mathrm{Std}(\lambda/\mu)$ is a W-graph determining tableau and $\mathrm{sqsh}(t) = t' \in \mathrm{Std}(\zeta/\eta)$, then t' is also a W-graph determining tableau, since $\mathrm{perm}(t') = \mathrm{perm}(t)$ and $J_{\zeta/\eta} = J_{\lambda/\mu}$. Moreover, if $w \in W$ and $(\mathscr{I}(w), J)$ is a W-graph ideal, then $\tau(J,w)$ is a W-graph determining tableau. So classifying W-graph determining tableaux is essentially the same as classifying W-graph determining elements in type A.

It follows from Remark 2.8 that if $t \in \mathrm{Std}(\lambda/\mu)$ is a W-graph determining tableau and $J = J_{\lambda/\mu}$ then $\mathscr{I}(\mathrm{perm}(t))$ is a union of cells of (D_J, J). This observation motivates the following definition.

Definition 4.11 Let $W = W_n$, where n is a positive integer, and let $t \in \mathrm{Std}(\lambda/\mu)$, where $\lambda/\mu \vdash n$. We say that t is a *cell ideal generating tableau* for W if $\mathscr{I}(\mathrm{perm}(t))$ is a union of cells of (D_J, J), where $J = J_{\lambda/\mu}$.

Remark 4.12 Observe that if $t \in \mathrm{Std}(\lambda/\mu)$ and $u \in \mathrm{Std}(\zeta/\eta)$ are such that $\mathrm{sqsh}(t) = \mathrm{sqsh}(u)$, then t is a cell ideal generating tableau if and only if u is a cell ideal generating tableau, since $\mathrm{perm}(t) = \mathrm{perm}(u)$ and $J_{\lambda/\mu} = J_{\zeta/\eta}$.

Remark 4.13 By Theorem 3.7 and Lemma 4.7, every squashed maximal tableau is a squashed W-graph determining tableau, and hence also a squashed cell ideal generating tableau. Our main theorem below will show that every squashed cell ideal generating tableau is maximal, so that these three classes of tableaux concide.

Recall that if $J \subseteq S_n$ and $w \in D_J$ then $t = \tau(J,w)$ is the unique squashed standard tableau such that $w = \mathrm{perm}(t)$ and $J = J_{\lambda/\mu}$, where λ/μ is the shape of t. Thus classifying the squashed cell ideal generating tableaux is the same as classifying the pairs (w,J) such that $\mathscr{I}(w)$ is a union of cells of (D_J, J), which in turn is the same as classifying the pairs (w,J) such that $\{\, x \in W \mid w_J \leqslant_{\mathrm{L}} x \leqslant_{\mathrm{L}} ww_J \,\}$ is a nonempty union of Kazhdan–Lusztig left cells.

Lemma 4.14 *Let $n > 1$ be a positive integer and suppose that t is a cell ideal generating tableau for W_n. Then $t{\downarrow}n$ and $-1 + (t{\uparrow}1)$ are both cell ideal generating tableaux for W_{n-1}.*

Proof Let λ/μ be the shape of t, and let κ be defined by $[\kappa] = [\lambda] \setminus \{t^{-1}(n)\}$, so that κ/μ is the shape of $t{\downarrow}n$. Let $K = S_n \setminus \{s_{n-1}\}$, so that W_{n-1} can be identified with the standard parabolic subgroup W_K of $W = W_n$ and let $J = J_{\lambda/\mu} \subseteq S_n$. Let $w = \mathrm{perm}(t)$ and $v = \mathrm{perm}(t{\downarrow}n)$, so that v is the left W_K-component of w, by Lemma 4.2.

Let d be the D_K^{-1}-component of w, so that $w = vd$ and $d \in D_{K,J}$. Since $l(vd) = l(v) + l(d)$ it follows that if $y \in W_K$ then $y \leqslant_{\mathrm{L}} v$ if and only if $yd \leqslant_{\mathrm{L}} vd$. Hence $\mathscr{I}(w)d^{-1} \cap W_K = \mathscr{I}(v)$. But since t is a cell ideal generating tableau it follows that $\mathscr{I}(w)$ is a union of cells of (D_J, J), and hence, by Proposition 2.4, $\mathscr{I}(w)d^{-1} \cap W_K$ is a union of cells of (D_M^K, M), where $M = K \cap dJd^{-1}$. Since $M = J_{\kappa/\mu}$, by Lemma 4.2, this shows that $\mathscr{I}(v)$ is a union of (D_M, M) cells in W_{n-1}, and hence $t{\downarrow}n$ is a cell ideal generating tableau for W_{n-1}, as claimed.

The proof of the other part is similar, using $L = S_n \setminus \{s_1\}$ instead of K, identifying W_L with $W_{[2,n]}$, and applying the obvious isomorphism $W_{[2,n]} \cong W_{n-1}$. □

Lemma 4.15 *Suppose that $n \geqslant 3$. The following two tableaux t and u, namely*

$$t = \begin{array}{|c|c|c|c|c|c|}\hline & 2 & 3 & \cdots & n-2 & n-1 \\ \hline 1 & n \\ \cline{1-2}\end{array}, \qquad u = \begin{array}{|c|c|c|c|c|c|}\hline & & & & 1 & n \\ \hline 2 & 3 & \cdots & n-2 & n-1 \\ \cline{1-5}\end{array},$$

are not cell ideal generating tableaux.

Proof Let $x = \mathrm{perm}(t)$ and $J = J_{\lambda/\mu}$, where λ/μ is the shape of t. Then $J = \{s_2\}$ and x is the $(n-2)$-cycle $(n, n-1, \ldots, 4, 3)$. That is, $x = s_{n-1}s_{n-2}\cdots s_3$. Clearly $s_1 \not\leqslant_{\mathrm{L}} x$, and hence $s_1 s_2 \notin \mathscr{I}(x)w_J = \mathscr{I}(x)s_2$. Since obviously $s_2 \in \mathscr{I}(x)w_J$, we see that $\mathscr{I}(x)w_J \cap W_{\{s_1,s_2\}}$ is not a union of cells of $W_{\{s_1,s_2\}}$. So it follows from Lemma 2.6 that $\mathscr{I}(x)w_J$ is not a union of cells of W, and so $\mathscr{I}(x)$ is not a union of cells of (D_J, J), by Remark 2.8.

Similarly, let $y = \mathrm{perm}(u)$ and $K = J_{\kappa/\nu}$, where κ/ν is the shape of u. Then $K = \{s_{n-2}\}$ and $y = s_1 s_2 \cdots s_{n-3}$, and $\mathscr{I}(y)w_K$ contains s_{n-2} but not $s_{n-1}s_{n-2}$. So $\mathscr{I}(y)w_K \cap W_{\{s_{n-2},s_{n-1}\}}$ is not a union of cells of $W_{\{s_{n-2},s_{n-1}\}}$, and $\mathscr{I}(y)$ is not a union of cells of (D_K, K). □

We now come to the main theorem of this paper.

Theorem 4.16 *If n is a positive integer and W is the symmetric group on $\{1, 2, \ldots, n\}$, then a squashed skew tableau is a W-graph ideal determining tableau if and only if it is a maximal skew tableau, and if and only if it is a cell ideal generating tableau for W.*

Proof As explained in Remark 4.13 above, it only remains to prove that every cell ideal generating tableau is maximal. Suppose that this is false, and let n be the minimal counterexample. Let $W = W_n$, and let t be a squashed cell ideal generating tableau that is not of the form $\tau^{\lambda/\mu}$. Let λ/μ be the shape of t, and let $J = J_{\lambda/\mu}$.

Note that modifying t by removing empty columns does not change J or $\mathrm{perm}(t)$, and so does not alter the fact that t is a cell ideal generating tableau. Nor does it alter the fact that t is squashed and $t \neq \tau^{\lambda/\mu}$. So we shall assume that t has no empty columns. Since t is squashed it has no empty rows.

If $n = 1$ there is only one skew diagram $[\lambda/\mu]$ with no empty rows or columns, and only one standard (λ/μ)-tableau. Since this contradicts $t \neq \tau^{\lambda/\mu}$, it follows that $n > 1$.

If $n = 2$ then there are exactly four standard skew tableaux with no empty rows or columns, namely

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\,, \qquad \begin{array}{c|c} & 1 \\ \hline 2 & \end{array} \qquad \text{and} \qquad \begin{array}{c|c} & 2 \\ \hline 1 & \end{array}\,,$$

the last of which is not squashed, and hence not equal to t. So t is one of the others, and in each case we see that t is maximal, contrary to the choice of t. So $n > 2$.

Write $t_n = \mathrm{sqsh}(t{\downarrow}n)$ and $t_1 = \mathrm{sqsh}(t{\uparrow}1)$. It follows from Lemma 4.14 and Remark 4.12 that t_n and $-1 + t_1$ are cell ideal generating tableaux, and so, by the minimality of our counterexample, it follows that $t_n = \tau^{\zeta/\eta}$ and $-1 + t_1 = \tau^{\theta/\xi}$ for some $\zeta/\eta \vdash n-1$ and $\theta/\xi \vdash n-1$. Let $g = \mathrm{col}(t, 1)$ and $h = \mathrm{col}(t, n)$.

Since $\mathrm{sqsh}(t_n{\uparrow}1) = \mathrm{sqsh}((t{\downarrow}n){\uparrow}1) = \mathrm{sqsh}((t{\uparrow}1){\downarrow}n) = \mathrm{sqsh}(t_1{\downarrow}n)$, by Remark 4.6, we deduce that

$$\mathrm{sqsh}(\tau^{\zeta/\eta}{\uparrow}1) = 1 + \mathrm{sqsh}(\tau^{\theta/\xi}{\downarrow}n). \tag{4.1}$$

Note that since t_n and t_1 are both squashed, neither $[\zeta/\eta]$ nor $[\theta/\xi]$ has empty rows. Let q be the number of rows of $[\theta/\xi]$. Our strategy is to compare the shapes of the tableaux on the left and right sides of Eq. (4.1), using Corollary 4.8. There are four cases.

Case 1.

Suppose that 1 is the unique entry in the first row of t_n and n is not the unique entry in the last row of t_1. Thus $\eta_1 = \zeta_1 - 1$ and $\xi_q < \theta_q - 1$. For the left hand side of Eq. (4.1), case (ii) of Corollary 4.8 applies; for the right hand side, case (iii) of Corollary 4.8 applies. We find that $\zeta_i = \theta_{i-1}$ for $2 \leqslant i \leqslant q$, while $\zeta_{q+1} = \theta_q - 1$. Note also that $\theta_q = \mathrm{col}(t_1, n) = \mathrm{col}(t, n) = h$.

Suppose first that $q \geqslant 2$. Then $\zeta_q = \theta_{q-1} \geqslant \theta_q$, and so we can define a partition ζ' by setting $\zeta'_i = \zeta_i$ for $i \leqslant q$ and $\zeta'_{q+1} = \theta_q = \zeta_{q+1} + 1$. Note that $\zeta'_i \geqslant \eta_i$ for all i, and that

$[\zeta'/\eta] = [\zeta/\eta] \cup \{(q+1,h)\}$. Writing $t' = \tau^{\zeta'/\eta}$, the maximal tableau of shape ζ'/η, we see that $t_n = \tau^{\zeta/\eta}$ is the restriction of t' to $[\zeta/\eta]$, and hence $\mathrm{col}(t',a) = \mathrm{col}(t_n,a) = \mathrm{col}(t,a)$ for all $a \in [1,n-1]$. Furthermore, $\mathrm{col}(t',n) = h = \mathrm{col}(t,n)$. So the columns of t' are the same as the columns of t, and since t' and t are both squashed it follows that $t = t'$. So t is maximal, contrary to hypothesis.

It remains to consider the possibility that $q = 1$, which means that

$$t_1 = \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & \cdots & n-1 & n \\ \hline \end{array}.$$

It will follow from the reasoning below that t_1 does not have an empty first column (to the left of the 2), but in any case we can say that the partition θ has only one part, $\theta_1 = \xi_1 + n - 1$, and ξ has at most one part. Moreover, $\mathrm{col}(t,a) = \mathrm{col}(t_1,a) = \xi_1 + a - 1$ for all $a \in [2,n]$.

The partition ζ has two parts, namely $\zeta_1 = g = \mathrm{col}(t,1)$ and $\zeta_2 = \theta_1 - 1 = \xi_1 + n - 2$. So $g \geqslant \xi_1 + n - 2$. We can see now that $\xi_1 = 0$, since t has no empty columns and $\mathrm{col}(t,a) > \xi_1$ for all $a \in [1,n]$. It is also clear that $g \leqslant n$, since t has at most n columns. So we have $\mathrm{col}(t,a) = \mathrm{col}(t_1,a) = a - 1$ for all $a \in [2,n]$, and $\mathrm{col}(t,1) \in \{n-2, n-1, n\}$. Note that since t is squashed, it is uniquely determined by its columns.

If $\mathrm{col}(t,1) = n$ then the columns (from left to right) are $\{2\}, \{3\}, \ldots, \{n\}$ and $\{1\}$, and so $t = \tau^{\lambda/\mu}$ with $\lambda/\mu = (n,n-1)/(n-1)$, contradicting the fact that t is not a maximal tableau. If $\mathrm{col}(t,1) = n-1$ then the columns of t are $\{2\}, \{3\}, \ldots, \{n-1\}$ and $\{1,n\}$, from which it follows that $t = \tau^{\lambda/\mu}$ with $\lambda/\mu = (n-1,n-1)/(n-2)$, which again contradicts the fact that t is not a maximal tableau. Finally, if $\mathrm{col}(t,1) = n-2$ then the columns of t are $\{2\}, \{3\}, \ldots, \{n-2\}, \{1,n-1\}$ and $\{n\}$, and it follows that t is the tableau u of Lemma 4.15, contradicting the assumption that t is a cell ideal generating tableau.

Case 2.

Suppose that 1 is not the unique entry in the first row of t_n and n is the unique entry in the last row of t_1. Thus $\eta_1 < \zeta_1 - 1$ and $\xi_q = \theta_q - 1$. This time case (i) of Corollary 4.8 applies to the left hand side of Eq. (4.1), and case (iv) applies to the right hand side. The partition ζ has $q-1$ parts, $\zeta_i = \theta_i$ and $\eta_i = \xi_i$ for $2 \leqslant i \leqslant q-1$, while $\zeta_1 = \theta_1$ and $g = \eta_1 + 1 = \xi_1$.

Suppose first that $q - 1 \geqslant 2$. Then $\eta_1 \geqslant \eta_2 = \xi_2$, and hence we can define a partition ξ' by putting $\xi'_i = \xi_i$ for $i \geqslant 2$ and $\xi'_1 = \eta_1 = \xi_1 - 1$. Note that $\xi'_i \leqslant \theta_i$ for all i, and that $[\theta/\xi'] = [\theta/\xi] \cup \{(1,g)\}$. Defining t' to be the maximal tableau of shape θ/ξ', we see that $1 + t_1 = 1 + \tau^{\theta/\xi}$ is the restriction of t' to $[\theta/\xi]$, and hence $\mathrm{col}(t',a) = \mathrm{col}(t_1,a) = \mathrm{col}(t,a)$ for all $a \in [2,n]$. Furthermore, $\mathrm{col}(t',1) = g = \mathrm{col}(t,1)$. So the columns of t' are the same as the columns of t, and since t' and t are both squashed it follows that $t = t'$. So t is maximal, contrary to hypothesis.

It remains to consider the possibility that $q - 1 = 1$, so that $t_n = \tau^{\zeta/\eta}$ has only one row. Thus $\zeta_1 = g + n - 2$ and $\eta_1 = g - 1$, and $\mathrm{col}(t,a) = \mathrm{col}(t_n,a) = g + a - 1$ for all $a \in [1,n-1]$. Note that $g \leqslant 2$, since t has at most n columns. If $g = 2$ then t has n columns, all with exactly one entry, and it follows that the columns of t must be $\{n\}, \{1\}, \{2\}, \ldots, \{n-1\}$. Hence $t = \tau^{\lambda/\mu}$ with $\lambda/\mu = (n,1)/(1)$, contradicting the fact that t is not maximal. If $g = 1$ then $h = \mathrm{col}(t,n)$ must be 1 or 2, since n is in the first nonempty column of t_1. If $h = 2$ then the columns of t are $\{1\}, \{2,n\}, \{3\}, \ldots, \{n-1\}$, which implies that t is the t of Lemma 4.15, contradicting to the fact that t is a cell ideal generating tableau. If $h = 1$ then the columns of t are $\{1,n\}, \{2\}, \{3\}, \ldots, \{n-1\}$, from which it follows that $t = \tau^{\lambda/\mu}$ where $\lambda = (n,1)$ and μ is empty. Again this contradicts the fact that t is not maximal.

Case 3.

Suppose that 1 is the unique entry in the first row of t_n and n is the unique entry in the last row of t_1. This time case (ii) of Corollary 4.8 applies to the left hand side of Eq. (4.1), and

case (iv) applies to the right hand side. Both t_1 and t_n have q rows, and the tableau in Eq. (4.1) has $q-1$ rows.

We have $\zeta_i = \theta_{i-1}$ and $\eta_i = \xi_{i-1}$ for $2 \leqslant i \leqslant q$, and it follows that $\zeta_q = \theta_{q-1} \geqslant \theta_q$ and $\eta_q = \xi_{q-1} \geqslant \xi_q$. So there is a $q+1$ row skew diagram $[\zeta'/\eta']$ with $\zeta'_{q+1} = \theta_q$ and $\eta'_{q+1} = \xi_q$, and $\zeta'_i = \zeta_i$ and $\eta'_i = \eta_i$ for $i \leqslant q$. Clearly $[\zeta'/\eta'] = [\zeta/\eta] \cup \{(q+1,h)\}$. Defining t' to be the maximal tableau of shape ζ'/η', we see that $t_n = \tau^{\zeta/\eta}$ is the restriction of t' to $[\zeta/\eta]$. Thus $\mathrm{col}(t',a) = \mathrm{col}(t_n,a) = \mathrm{col}(t,a)$ for all $a \in [1,n-1]$, and $\mathrm{col}(t',n) = h = \mathrm{col}(t,n)$. So $t = t'$, contradicting the fact that t is not maximal.

Case 4.

The only remaining possibility is that 1 is not the unique entry in the first row of t_n and n is not the unique entry in the last row of t_1. In this case t_n and t_1 both have q rows, and the tableau in Eq. (4.1) also has q rows. Case (i) of Corollary 4.8 applies to the left hand side of Eq. (4.1), and case (iii) applies to the right hand side. We find that $\theta_q = \zeta_q + 1$ and $\theta_i = \zeta_i$ for $i < q$, while $\eta_1 = \xi_1 - 1$ and $\eta_i = \xi_i$ for $i > 1$. Thus $\eta_i \leqslant \xi_i \leqslant \zeta_i \leqslant \theta_i$ for all i, and θ/η is a skew partition. Furthermore, $[\theta/\eta] = [\zeta/\eta] \cup \{(q,h)\}$. As in the previous cases it follows that $\mathrm{col}(t,a) = \mathrm{col}(\tau^{\theta/\eta},a)$ for all $a \in [1,n]$, so that $t = \tau^{\theta/\eta}$, contradicting the fact that t is not maximal. This final contradiction completes the proof. □

To conclude we state the following Corollary, whose proof was explained in Remark 4.13.

Corollary 4.17 *Let n be a positive integer and $(W,S) = (W_n,S_n)$, a Coxeter group of type A_{n-1}. If $w \in W$ and $J \subseteq S$ then $\{ x \in W \mid w_J \leqslant_{\mathrm{L}} x \leqslant_{\mathrm{L}} ww_J \}$ is a nonempty union of Kazhdan–Lusztig left cells if and only if $w = \mathrm{perm}(\tau^{\lambda/\mu})$ and $J = J_{\lambda/\mu}$ for some $\lambda/\mu \vdash n$, and then $x \in W$ satisfies $w_J \leqslant_{\mathrm{L}} x \leqslant_{\mathrm{L}} ww_J$ if and only if $xw_J\tau_{\lambda/\mu} \in \mathrm{Std}(\lambda/\mu)$.*

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