Eigenvalues of Bethe vectors in the Gaudin model

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Abstract

A theorem of Feigin, Frenkel and Reshetikhin provides expressions for the eigenvalues of the higher Gaudin Hamiltonians on the Bethe vectors in terms of elements of the center of the affine vertex algebra at the critical level. In our recent work, explicit Harish-Chandra images of generators of the center were calculated in all classical types. We combine these results to calculate the eigenvalues of the higher Gaudin Hamiltonians on the Bethe vectors in an explicit form. The Harish-Chandra images can be interpreted as elements of classical $\mathcal{W}$-algebras. We provide a direct connection between the rings of q-characters and classical $\mathcal{W}$-algebras by calculating classical limits of the corresponding screening operators.

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1 Introduction

In their seminal paper [7], Feigin, Frenkel and Reshetikhin established a connection between the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of the affine vertex algebra at the critical level and the higher Gaudin Hamiltonians. They used the Wakimoto modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ to calculate the eigenvalues of the Hamiltonians on the Bethe vectors of the Gaudin model associated with an arbitrary simple Lie algebra $\mathfrak{g}$. The calculation depends on the choice of an element z of the center and the result is written in terms of the Harish-Chandra image of z; see also [8], [9] and [24] for a relationship with the opers and generalizations to non-homogeneous Hamiltonians.

The center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative associative algebra whose structure was described by a theorem of Feigin and Frenkel [6], which states that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables; see [10] for a detailed exposition of these results. Simple explicit formulas for generators of this algebra were found in [5] for type A and in [15] for types B, C and D; see also [4] and [17] for simpler arguments in type A and extensions to Lie superalgebras. The calculation of the Harish-Chandra images of the generators in type A is straightforward, whereas types B, C and D require a rather involved application of the q-characters; see [16]. Our goal in this paper is to apply these results to get the action of the higher Gaudin Hamiltonians on tensor products of representations of $\mathfrak{g}$ in an explicit form and calculate the corresponding eigenvalues of the Bethe vectors. In type A we thus reproduce the results of [19] obtained by a different method based on the Bethe ansatz.

We will begin with a brief exposition of some results of [7] and [9]. Our main focus will be on Theorem 6.7 from [9] expressing eigenvalues of a generalized Gaudin algebra on Bethe vectors in terms of opers associated with tensor products of Verma modules. Then we will apply this theorem to the classical Lie algebras to write explicit Gaudin operators and their eigenvalues on Bethe vectors.

A connection between the Yangian characters (or q-characters) and the Segal–Sugawara operators played an essential role in the calculation of the Harish-Chandra images in [16]. We will explore this connection further by constructing a map $\mathbf{gr}$ taking a character to an element of the associated classical $\mathcal{W}$-algebra. We will also establish multiplicativity and surjectivity properties of this map.

2 Feigin–Frenkel center and Bethe vectors

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ equipped with a standard symmetric invariant bilinear form $\langle\ ,\ \rangle$ defined as a normalized Killing form

$$\langle X, Y\rangle = \frac{1}{2h^\vee}\,\mathrm{tr}\,\big(\mathrm{ad}\,X\,\mathrm{ad}\,Y\big), \tag{2.1}$$

where $h^\vee$ is the *dual Coxeter number* for $\mathfrak{g}$. The corresponding *affine Kac–Moody algebra* $\widehat{\mathfrak{g}}$ is defined as the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \tag{2.2}$$

where $\mathfrak{g}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in $\mathfrak{g}$; see [14]. For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we set $X[r] = X\, t^r$. The commutation relations of the Lie algebra $\widehat{\mathfrak{g}}$ have the form

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\, \delta_{r,-s} \langle X, Y\rangle\, K, \qquad X, Y \in \mathfrak{g},$$

and the element K is central in $\widehat{\mathfrak{g}}$.

The universal enveloping algebra *at the critical level* $\mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K + h^\vee$. Let I denote the left ideal of $\mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$ generated by $\mathfrak{g}[t]$ and let $\mathrm{Norm}\,\mathrm{I}$ be its normalizer,

$$\mathrm{Norm}\,\mathrm{I} = \{v \in \mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}}) \mid \mathrm{I}\, v \subseteq \mathrm{I}\}.$$

The normalizer is a subalgebra of $\mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$, and I is a two-sided ideal of $\mathrm{Norm}\,\mathrm{I}$. The *Feigin–Frenkel center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the associative algebra defined as the quotient

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathrm{Norm}\,\mathrm{I}/\mathrm{I}. \tag{2.3}$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a *Segal–Sugawara vector*. The quotient

$$V_{\mathrm{cri}}(\mathfrak{g}) = \mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}})/\mathrm{I} \tag{2.4}$$

is the *vacuum module* at the critical level over $\widehat{\mathfrak{g}}$. It possesses a vertex algebra structure. As a vector space, $V_{\mathrm{cri}}(\mathfrak{g})$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}}_-)$, where $\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$. Hence, we have a vector space embedding

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow \mathrm{U}(\widehat{\mathfrak{g}}_-).$$

Since $\mathrm{U}(\widehat{\mathfrak{g}}_-)$ is a subalgebra of $\mathrm{U}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$, the embedding is an algebra homomorphism so that the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be regarded as a subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}_-)$. This subalgebra is commutative which is seen from its identification with the *center* of the vertex algebra $V_{\mathrm{cri}}(\mathfrak{g})$ by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V_{\mathrm{cri}}(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}. \tag{2.5}$$

As a vertex algebra, the vacuum module $V_{\mathrm{cri}}(\mathfrak{g})$ is equipped with the translation operator

$$T : V_{\mathrm{cri}}(\mathfrak{g}) \to V_{\mathrm{cri}}(\mathfrak{g}), \tag{2.6}$$

which is determined by the properties

$$T : 1 \mapsto 0 \qquad \text{and} \qquad \big[T, X[r]\big] = -r\, X[r-1], \quad X \in \mathfrak{g}.$$

We also regard T as a derivation of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}_-)$. Its subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is T-invariant. By the Feigin–Frenkel theorem, there exist elements $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$, where $n = \operatorname{rank} \mathfrak{g}$, such that all elements $T^r S_l$ are algebraically independent, and every Segal–Sugawara vector is a polynomial in these elements:

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \quad r \geqslant 0]. \tag{2.7}$$

We call such a family $S_1, \dots, S_n$ a *complete set of Segal–Sugawara vectors*.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Consider $\mathrm{U}(\widehat{\mathfrak{g}}_-)$ as the adjoint $\mathfrak{g}$-module by regarding $\mathfrak{g}$ as the span of the elements $X[0]$ with $X \in \mathfrak{g}$. Denote by $\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}}$ the subalgebra of $\mathfrak{h}$-invariants under this action. Consider the left ideal J of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}_-)$ generated by all elements $X[r]$ with $X \in \mathfrak{n}_-$ and $r < 0$. By the Poincaré–Birkhoff–Witt theorem, the intersection $\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}} \cap \mathrm{J}$ is a two-sided ideal of $\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}}$ and we have a direct sum decomposition

$$\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}} = \big(\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}} \cap \mathrm{J}\big) \oplus \mathrm{U}(\widehat{\mathfrak{h}}_-),$$

where $\widehat{\mathfrak{h}}_- = t^{-1}\mathfrak{h}[t^{-1}]$. The projection to the second summand is a homomorphism

$$\mathrm{U}(\widehat{\mathfrak{g}}_-)^{\mathfrak{h}} \to \mathrm{U}(\widehat{\mathfrak{h}}_-) \tag{2.8}$$

which is an affine version of the *Harish-Chandra homomorphism*. By the Feigin–Frenkel theorem, the restriction of the homomorphism (2.8) to the subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields an isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathcal{W}({}^L\mathfrak{g}), \tag{2.9}$$

where $\mathcal{W}({}^L\mathfrak{g})$ is the *classical* $\mathcal{W}$*-algebra* associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$; see [10] for a detailed exposition of these results. The $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ can be defined as a subalgebra of $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ which consists of the elements annihilated by the *screening operators*; see [10, Sec. 8.1] and also [16] for explicit formulas in the classical types.

Given any element $\chi \in \mathfrak{g}^*$ and a nonzero $z \in \mathbb{C}$, the mapping

$$\mathrm{U}(\widehat{\mathfrak{g}}_-) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto X\, z^r + \delta_{r,-1}\, \chi(X), \quad X \in \mathfrak{g}, \tag{2.10}$$

defines an algebra homomorphism. Using the coassociativity of the standard coproduct on $\mathrm{U}(\widehat{\mathfrak{g}}_-)$ defined by

$$\Delta : Y \mapsto Y \otimes 1 + 1 \otimes Y, \qquad Y \in \widehat{\mathfrak{g}}_-,$$

for any $\ell \geqslant 1$ we get the homomorphism

$$\mathrm{U}(\widehat{\mathfrak{g}}_-) \to \mathrm{U}(\widehat{\mathfrak{g}}_-)^{\otimes \ell} \tag{2.11}$$

as an iterated coproduct map. Now fix distinct complex numbers $z_1, \dots, z_\ell$ and let u be a complex parameter. Applying homomorphisms of the form (2.10) to the tensor factors in (2.11), we get another homomorphism

$$\Psi : \mathrm{U}(\widehat{\mathfrak{g}}_-) \to \mathrm{U}(\mathfrak{g})^{\otimes \ell}, \tag{2.12}$$

given by

$$\Psi : X[r] \mapsto \sum_{a=1}^{\ell} X_a (z_a - u)^r + \delta_{r,-1}\, \chi(X) \in \mathrm{U}(\mathfrak{g})^{\otimes \ell},$$

where $X_a = 1^{\otimes (a-1)} \otimes X \otimes 1^{\otimes (\ell - a)}$; see [24]. We will twist this homomorphism by the involutive anti-automorphism

$$\varsigma : \mathrm{U}(\widehat{\mathfrak{g}}_-) \to \mathrm{U}(\widehat{\mathfrak{g}}_-), \qquad X[r] \mapsto -X[r], \quad X \in \mathfrak{g}, \tag{2.13}$$

to get the anti-homomorphism

$$\Phi : \mathrm{U}(\widehat{\mathfrak{g}}_-) \to \mathrm{U}(\mathfrak{g})^{\otimes \ell}, \tag{2.14}$$

defined as the composition $\Phi = \Psi \circ \varsigma$. Since $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}_-)$, the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under Φ is a commutative subalgebra $\mathcal{A}(\mathfrak{g})_\chi$ of $\mathrm{U}(\mathfrak{g})^{\otimes \ell}$, depending on the chosen parameters $z_1, \dots, z_\ell$, but it does not depend on u [24]; see also [9, Sec. 2].

Introduce the standard Chevalley generators e_i, h_i, f_i with $i = 1, \dots, n$ of the simple Lie algebra $\mathfrak{g}$ of rank n. The generators h_i form a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, while the e_i and f_i generate the respective nilpotent subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Let $A = [a_{ij}]$ be the Cartan matrix of $\mathfrak{g}$ so that the defining relations of $\mathfrak{g}$ take the form

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, & [h_i, h_j] &= 0, \\ [h_i, e_j] &= a_{ij}\, e_j, & [h_i, f_j] &= -a_{ij}\, f_j, \end{aligned}$$

together with the Serre relations; see e.g. [14]. Given $\lambda \in \mathfrak{h}^*$, the Verma module M_λ is the quotient of $\mathrm{U}(\mathfrak{g})$ by the left ideal generated by $\mathfrak{n}_+$ and the elements $h_i - \lambda(h_i)$ with $i = 1, \dots, n$. We denote the image of 1 in M_λ by 1_λ.

For any weights $\lambda_1, \dots, \lambda_\ell \in \mathfrak{h}^*$ consider the tensor product of the Verma modules $M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$. We will now describe common eigenvectors for the commutative subalgebra $\mathcal{A}(\mathfrak{g})_\chi$ in this tensor product. For a set of distinct complex numbers $w_1, \dots, w_m$ with $w_i \neq z_j$ and a collection (multiset) of labels $i_1, \dots, i_m \in \{1, \dots, n\}$ introduce the *Bethe vector*

$$\phi(w_1^{i_1}, \dots, w_m^{i_m}) \in M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$$

by the following formula which originates in [25]; see [2] and also [7], [21] and references therein:

$$\phi(w_1^{i_1}, \dots, w_m^{i_m}) = \sum_{(I^1, \dots, I^\ell)} \bigotimes_{k=1}^{\ell} \prod_{s=1}^{a_k} \frac{1}{w_{j_s^k} - w_{j_{s+1}^k}} \prod_{r \in I^k} f_{i_r}\, 1_{\lambda_k}, \tag{2.15}$$

summed over all ordered partitions $I^1 \cup I^2 \cup \dots \cup I^\ell$ of the set $\{1, \dots, m\}$ into ordered subsets $I^k = \{j_1^k, j_2^k, \dots, j_{a_k}^k\}$ with the products taken from left to right, where $w_{j_{s+1}^k} := z_k$ for $s = a_k$.

Now suppose that $\chi \in \mathfrak{h}^*$. We regard χ as a functional on $\mathfrak{g}$ which vanishes on $\mathfrak{n}_+$ and $\mathfrak{n}_-$. The system of the *Bethe ansatz equations* takes the form

$$\sum_{i=1}^{\ell} \frac{\langle \check{\alpha}_{i_j}, \lambda_i \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_j}, \alpha_{i_s} \rangle}{w_j - w_s} = \langle \check{\alpha}_{i_j}, \chi \rangle, \qquad j = 1, \dots, m, \tag{2.16}$$

where the α_l and $\check{\alpha}_l$ denote the simple roots and coroots, respectively; see [14].

We are now in a position to describe the eigenvalues of the Gaudin Hamiltonians on the Bethe vectors. Given the above parameters, introduce the homomorphism from $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ to rational functions in u by the rule:

$$\varrho : H[-r-1] \mapsto \frac{\partial_u^r}{r!} \mathcal{H}(u), \qquad H \in \mathfrak{h}, \quad r \geqslant 0, \tag{2.17}$$

where

$$\mathcal{H}(u) = \sum_{a=1}^{\ell} \frac{\lambda_a(H)}{u - z_a} - \sum_{j=1}^{m} \frac{\alpha_{i_j}(H)}{u - w_j} - \chi(H).$$

Let $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$ be a Segal–Sugawara vector. The composition $\varrho \circ \mathfrak{f}$ of this homomorphism with the isomorphism (2.9) takes S to a rational function $\varrho\big(\mathfrak{f}(S)\big)$ in u. Furthermore, we regard the image $\Phi(S)$ of S under the anti-homomorphism (2.14) as an operator in the tensor product of Verma modules $M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$. The following is essentially a reformulation of Theorems 6.5 and 6.7 from [9]; in the case $\chi = 0$ the result goes back to [7, Theorem 3].

Theorem 2.1. *Suppose that the Bethe ansatz equations* (2.16) *are satisfied. If the Bethe vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is nonzero, then it is an eigenvector for the operator $\Phi(S)$ with the eigenvalue $\varrho\big(\mathfrak{f}(S)\big)$.* □

In what follows we will rely on the results of [4], [15] and [16] to give explicit formulas for the operators $\Phi(S_i)$ and their eigenvalues $\varrho\big(\mathfrak{f}(S_i)\big)$ on the Bethe vectors for complete sets of Segal–Sugawara vectors $S_1, \dots, S_n$ in all classical types.

3 Gaudin Hamiltonians and eigenvalues

We will use the extended Lie algebra $\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau$ where the element τ satisfies the commutation relations

$$\big[\tau, X[r]\big] = -r\, X[r-1], \qquad \big[\tau, K\big] = 0. \tag{3.1}$$

Consider the extension of (2.9) to the isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \otimes \mathbb{C}[\tau] \to \mathcal{W}({}^L\mathfrak{g}) \otimes \mathbb{C}[\tau], \tag{3.2}$$

which is identical on $\mathbb{C}[\tau]$.

For an arbitrary $N \times N$ matrix $A = [A_{ij}]$ with entries in a ring we define its *column-determinant* $\operatorname{cdet} A$ and *row-determinant* $\operatorname{rdet} A$ by the respective formulas

$$\operatorname{cdet} A = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \cdot A_{\sigma(1)1} \dots A_{\sigma(N)N} \tag{3.3}$$

and

$$\operatorname{rdet} A = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \cdot A_{1\sigma(1)} \dots A_{N\sigma(N)}, \tag{3.4}$$

where $\mathfrak{S}_N$ denotes the symmetric group.

3.1 Type A

We will work with the reductive Lie algebra $\mathfrak{gl}_N$ rather than the simple Lie algebra $\mathfrak{sl}_N$ of type A. We let E_{ij} with $i, j = 1, \dots, N$ be the standard basis of $\mathfrak{gl}_N$. Denote by $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the subalgebras of $\mathfrak{gl}_N$ spanned by the diagonal, upper-triangular and lower-triangular matrices, respectively, so that $E_{11}, \dots, E_{NN}$ is a basis of $\mathfrak{h}$.

We start by recalling the constructions of some complete sets of Segal–Sugawara vectors for $\mathfrak{gl}_N$. For each $a \in \{1, \dots, m\}$ introduce the element $E[r]_a$ of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \dots \otimes \operatorname{End} \mathbb{C}^N}_{m} \otimes \mathrm{U} \tag{3.5}$$

by

$$E[r]_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes E_{ij}[r], \tag{3.6}$$

where the e_{ij} are the standard matrix units and U stands for the universal enveloping algebra of $\widehat{\mathfrak{gl}}_N \oplus \mathbb{C}\tau$. Let $H^{(m)}$ and $A^{(m)}$ denote the respective images of the normalized symmetrizer and anti-symmetrizer in the group algebra for the symmetric group $\mathfrak{S}_m$ under its natural action on $(\mathbb{C}^N)^{\otimes m}$. In particular, $H^{(m)}$ and $A^{(m)}$ are idempotents and we identify them with the respective elements $H^{(m)} \otimes 1$ and $A^{(m)} \otimes 1$ of the algebra (3.5). Define the elements $\varphi_{ma}, \psi_{ma}, \theta_{ma} \in \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ by the expansions

$$\operatorname{tr} A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) = \varphi_{m\,0}\, \tau^m + \varphi_{m\,1}\, \tau^{m-1} + \dots + \varphi_{m\,m}, \tag{3.7}$$

$$\operatorname{tr} H^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) = \psi_{m\,0}\, \tau^m + \psi_{m\,1}\, \tau^{m-1} + \dots + \psi_{m\,m}, \tag{3.8}$$

where the traces are taken with respect to all m copies of $\operatorname{End} \mathbb{C}^N$ in (3.5), and

$$\operatorname{tr} \big(\tau + E[-1]\big)^m = \theta_{m\,0}\, \tau^m + \theta_{m\,1}\, \tau^{m-1} + \dots + \theta_{m\,m}. \tag{3.9}$$

Expressions like $\tau + E[-1]$ are understood as matrices, where τ is regarded as the scalar matrix τI. Furthermore, expand the column-determinant of this matrix as a polynomial in τ,

$$\operatorname{cdet}\big(\tau + E[-1]\big) = \tau^N + \varphi_1\,\tau^{N-1} + \dots + \varphi_N, \qquad \varphi_m \in \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big). \tag{3.10}$$

We have $\varphi_{mm} = \varphi_m$ for $m = 1, \dots, N$.

Theorem 3.1. *All elements $\varphi_{m\,a}$, $\psi_{m\,a}$ and $\theta_{m\,a}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$. Moreover, each of the families*

$$\varphi_1, \dots, \varphi_N, \qquad \psi_{11}, \dots, \psi_{NN} \qquad \textit{and} \qquad \theta_{11}, \dots, \theta_{NN}$$

is a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$. □

This theorem goes back to [5], where the elements φ_m were first discovered (in a slightly different form). A direct proof of the theorem for the coefficients of the polynomial (3.10) was given in [4]. The elements $\psi_{m\,a}$ are related to $\varphi_{m\,a}$ through the quantum MacMahon Master Theorem of [13], while a relationship between the $\varphi_{m\,a}$ and $\theta_{m\,a}$ is provided by a Newton-type identity given in [3, Theorem 15]. Note that super-versions of these relations between the families of Segal–Sugawara vectors for the Lie superalgebra $\mathfrak{gl}_{m|n}$ were given in the paper [17], which also provides simpler arguments in the purely even case.

We will calculate the images of the Segal–Sugawara vectors under the involution (2.13). We extend it to the algebra $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big) \otimes \mathbb{C}[\tau]$ with the action on $\mathbb{C}[\tau]$ as the identity map.

Lemma 3.2. *For the images with respect to the involution ς we have*

$$\operatorname{tr} A^{(m)}\big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) \mapsto \operatorname{tr} A^{(m)}\big(\tau - E[-1]_1\big) \dots \big(\tau - E[-1]_m\big), \tag{3.11}$$

$$\operatorname{tr} H^{(m)}\big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big) \mapsto \operatorname{tr} H^{(m)}\big(\tau - E[-1]_1\big) \dots \big(\tau - E[-1]_m\big), \tag{3.12}$$

$$\operatorname{tr}\big(\tau + E[-1]\big)^m \mapsto \operatorname{tr}\big(\tau - E^t[-1]\big)^m, \tag{3.13}$$

and

$$\operatorname{cdet}\big(\tau + E[-1]\big) \mapsto \operatorname{cdet}\big(\tau - E^t[-1]\big), \tag{3.14}$$

where t denotes the standard matrix transposition.

Proof. The left hand side of (3.11) equals a linear combination of expressions of the form

$$\operatorname{tr} A^{(m)} E[r_1]_{a_1} \dots E[r_p]_{a_p}\, \tau^k \tag{3.15}$$

with $1 \leqslant a_1 < \cdots < a_p \leqslant m$. However, such an expression remains unchanged under any permutation of the factors $E[r_i]_{a_i}$. This follows from the commutation relations

$$E[r]_a\, E[s]_b - E[s]_b\, E[r]_a = P_{a\,b} E[r+s]_b - E[r+s]_b P_{a\,b}$$

for $a < b$, where

$$P_{a\,b} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{j\,i} \otimes 1^{\otimes(m-b)} \tag{3.16}$$

is the permutation operator. We only need to observe that $A^{(m)} P_{a\,b} = P_{a\,b} A^{(m)} = -A^{(m)}$ and use the cyclic property of trace. Hence the image of (3.15) under ς equals

$$(-1)^p \operatorname{tr} A^{(m)} E[r_1]_{a_1} \dots E[r_p]_{a_p}\, \tau^k$$

which verifies (3.11). The same argument proves (3.12). Now (3.14) follows from the relation

$$\operatorname{cdet}\big(\tau + E[-1]\big) = \operatorname{tr} A^{(N)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_N\big) \tag{3.17}$$

which is implied by the fact that $\tau + E[-1]$ is a Manin matrix; see [3] for an extensive review on Manin matrices. Indeed, by (3.11) for the image of (3.17) under ς we get

$$\operatorname{tr} A^{(N)} \big(\tau - E[-1]_1\big) \dots \big(\tau - E[-1]_N\big) = \operatorname{tr} A^{(N)} \big(\tau - E^t[-1]_1\big) \dots \big(\tau - E^t[-1]_N\big),$$

where we have applied the transposition $t_1 \dots t_N$ with respect to all copies of $\operatorname{End} \mathbb{C}^N$ and used the invariance of $A^{(N)}$ under this transposition. Since $\tau - E^t[-1]$ is also a Manin matrix, the resulting expression coincides with $\operatorname{cdet}\big(\tau - E^t[-1]\big)$. Finally, (3.13) follows from the Newton-type formula connecting the coefficients of the polynomial in (3.9) with those of (3.10); see [4, (3.5)]. □

With the parameters chosen as in Sec. 2, suppose that χ vanishes on the subspace $\mathfrak{n}_- \oplus \mathfrak{n}_+$ of $\mathfrak{gl}_N$ so that we can regard χ as an element of $\mathfrak{h}^*$. Set

$$E_{ij}(u) = \sum_{a=1}^{\ell} \frac{(E_{ij})_a}{u - z_a} - \chi(E_{ij}) \in \mathrm{U}(\mathfrak{gl}_N)^{\otimes \ell}.$$

Consider the row-determinant $\operatorname{rdet}\big(\partial_u + E(u)\big)$ of the matrix $\partial_u + E(u) = \big[\delta_{ij}\, \partial_u + E_{ij}(u)\big]$ as a differential operator in ∂_u with coefficients in $\mathrm{U}(\mathfrak{gl}_N)^{\otimes \ell}$. Furthermore, in accordance with (2.17), set

$$\mathcal{E}_{ii}(u) = \sum_{a=1}^{\ell} \frac{\lambda_a(E_{ii})}{u - z_a} - \sum_{j=1}^{m} \frac{\alpha_{i_j}(E_{ii})}{u - w_j} - \chi(E_{ii}).$$

In all the following eigenvalue formulas for the Gaudin Hamiltonians we will assume that the Bethe ansatz equations (2.16) hold.

Theorem 3.3. *The eigenvalue of the operator* $\mathrm{rdet}\big(\partial_u+E(u)\big)$ *on the Bethe vector* (2.15) *is found by*

$$\mathrm{rdet}\big(\partial_u+E(u)\big)\,\phi(w_1^{i_1},\dots,w_m^{i_m})=\big(\partial_u+\mathcal{E}_{NN}(u)\big)\dots\big(\partial_u+\mathcal{E}_{11}(u)\big)\,\phi(w_1^{i_1},\dots,w_m^{i_m}).$$

Proof. To apply Theorem 2.1, we will find the image of the polynomial $\mathrm{cdet}\big(\tau+E[-1]\big)$ under the anti-homomorphism Φ. We regard Φ as the map

$$\Phi:\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)\otimes\mathbb{C}[\tau]\to\mathrm{U}(\mathfrak{gl}_N)^{\otimes\ell}\otimes\mathbb{C}[\partial_u]$$

such that $\tau\mapsto\partial_u$. Note that by definition of the homomorphism (2.12) we have

$$\Psi:E[-1]\mapsto -E(u)$$

and so, by (3.14),

$$\Phi:\mathrm{cdet}\,\big(\tau+E[-1]\big)\mapsto\mathrm{cdet}\big(\partial_u+E^t(u)\big)=\mathrm{rdet}\big(\partial_u+E(u)\big). \tag{3.18}$$

The images of the elements φ_i under the isomorphism (2.9) for $\mathfrak{g}=\mathfrak{gl}_N$ are easy to obtain from (3.10), they are found by

$$\mathfrak{f}:\mathrm{cdet}\big(\tau+E[-1]\big)\mapsto\big(\tau+E_{NN}[-1]\big)\dots\big(\tau+E_{11}[-1]\big). \tag{3.19}$$

Therefore,

$$\varrho\circ\mathfrak{f}:\mathrm{cdet}\big(\tau+E[-1]\big)\mapsto\big(\partial_u+\mathcal{E}_{NN}(u)\big)\dots\big(\partial_u+\mathcal{E}_{11}(u)\big)$$

completing the proof. □

Formula (3.19) can be generalized to get the Harish-Chandra images of the polynomials (3.7) and (3.8). We get

$$\mathfrak{f}:\mathrm{tr}\,A^{(m)}\big(\tau+E[-1]_1\big)\dots\big(\tau+E[-1]_m\big)\mapsto e_m\big(\tau+E_{11}[-1],\dots,\tau+E_{NN}[-1]\big),$$

$$\mathfrak{f}:\mathrm{tr}\,H^{(m)}\big(\tau+E[-1]_1\big)\dots\big(\tau+E[-1]_m\big)\mapsto h_m\big(\tau+E_{11}[-1],\dots,\tau+E_{NN}[-1]\big),$$

where we use standard noncommutative versions of the complete and elementary symmetric functions in the ordered variables $x_1,\dots,x_p$ defined by the respective formulas

$$h_m(x_1,\dots,x_p)=\sum_{i_1\leqslant\dots\leqslant i_m}x_{i_1}\dots x_{i_m}, \tag{3.20}$$

$$e_m(x_1,\dots,x_p)=\sum_{i_1>\dots>i_m}x_{i_1}\dots x_{i_m}. \tag{3.21}$$

The following corollaries can be derived from Theorem 3.3 or proved in a similar way with the use of Lemma 3.2.

Corollary 3.4. *The eigenvalues of the operators*

$$\operatorname{tr} A^{(m)}\big(\partial_u + E(u)_1\big)\dots\big(\partial_u + E(u)_m\big) \quad \textit{and} \quad \operatorname{tr} H^{(m)}\big(\partial_u + E(u)_1\big)\dots\big(\partial_u + E(u)_m\big)$$

on the Bethe vector (2.15) *are found by respective formulas*

$$e_m\big(\partial_u + \mathcal{E}_{11}(u),\dots,\partial_u + \mathcal{E}_{NN}(u)\big) \quad \textit{and} \quad h_m\big(\partial_u + \mathcal{E}_{11}(u),\dots,\partial_u + \mathcal{E}_{NN}(u)\big).$$

By [4, Corollary 6.4] we have

$$\mathfrak{f}: \sum_{k=0}^{\infty} z^k \operatorname{tr}\big(\tau + E[-1]\big)^k \mapsto \sum_{i=1}^{N} \Big(1 - z\big(\tau + E_{11}[-1]\big)\Big)^{-1} \cdots \Big(1 - z\big(\tau + E_{ii}[-1]\big)\Big)^{-1}$$
$$\times \Big(1 - z\big(\tau + E_{i-1\,i-1}[-1]\big)\Big) \cdots \Big(1 - z\big(\tau + E_{11}[-1]\big)\Big),$$

where z is an independent variable. So we get the following.

Corollary 3.5. *The eigenvalue of the series*

$$\sum_{k=0}^{\infty} z^k \operatorname{tr}\big(\partial_u + E^t(u)\big)^k$$

on the Bethe vector (2.15) *is found by the formula*

$$\sum_{i=1}^{N} \Big(1 - z\big(\partial_u + \mathcal{E}_{11}(u)\big)\Big)^{-1} \cdots \Big(1 - z\big(\partial_u + \mathcal{E}_{ii}(u)\big)\Big)^{-1}$$
$$\times \Big(1 - z\big(\partial_u + \mathcal{E}_{i-1\,i-1}(u)\big)\Big) \cdots \Big(1 - z\big(\partial_u + \mathcal{E}_{11}(u)\big)\Big).$$

3.2 Types B and D

Now turn to the orthogonal Lie algebras and let $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n$ or $N = 2n+1$. These are simple Lie algebras of types D_n and B_n, respectively. We use the involution on the set $\{1,\dots,N\}$ defined by $i' = N - i + 1$. The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij} = E_{ij} - E_{j'i'}$ with $i, j \in \{1,\dots,N\}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$.

Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{o}_N$ spanned by the basis elements $F_{11},\dots,F_{nn}$. We have the triangular decomposition $\mathfrak{o}_N = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ denote the subalgebras of $\mathfrak{o}_N$ spanned by the elements F_{ij} with $i > j$ and by the elements F_{ij} with $i < j$, respectively.

We will use the elements $F_{ij}[r] = F_{ij}t^r$ of the loop algebra $\mathfrak{o}_N[t, t^{-1}]$. Introduce the elements $F[r]_a$ of the algebra (3.5) by

$$F[r]_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ij}[r], \tag{3.22}$$

where U in (3.5) now stands for the universal enveloping algebra of $\widehat{\mathfrak{o}}_N \oplus \mathbb{C}\tau$.

For $1 \leqslant a < b \leqslant m$ consider the operators P_{ab} defined by (3.16) and introduce the operators

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

Set

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1}\Big), \tag{3.23}$$

where the product is taken in the lexicographic order on the pairs (a,b). The element (3.23) is the image of the symmetrizer in the Brauer algebra $\mathcal{B}_m(N)$ under its action on the vector space $(\mathbb{C}^N)^{\otimes m}$. In particular, for any $1 \leqslant a < b \leqslant m$ for the operator $S^{(m)}$ we have

$$S^{(m)} Q_{ab} = Q_{ab} S^{(m)} = 0 \qquad \text{and} \qquad S^{(m)} P_{ab} = P_{ab} S^{(m)} = S^{(m)}. \tag{3.24}$$

The symmetrizer admits a few other equivalent expressions which are reproduced in [15].

We will use the notation

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2} \tag{3.25}$$

and define the elements $\varphi_{m\,a} \in \mathrm{U}\big(t^{-1}\mathfrak{o}_N[t^{-1}]\big)$ by the expansion

$$\gamma_m(N)\,\mathrm{tr}\, S^{(m)} \big(\tau + F[-1]_1\big) \dots \big(\tau + F[-1]_m\big) = \varphi_{m\,0}\,\tau^m + \varphi_{m\,1}\,\tau^{m-1} + \dots + \varphi_{m\,m}, \tag{3.26}$$

where the trace is taken over all m copies of $\mathrm{End}\,\mathbb{C}^N$. By the main result of [15], all coefficients $\varphi_{m\,a}$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{o}}_N)$. In the even orthogonal case $\mathfrak{g} = \mathfrak{o}_{2n}$ there is an additional element $\varphi'_n = \mathrm{Pf}\,\widetilde{F}[-1]$ of the center defined as the (noncommutative) Pfaffian of the skew-symmetric matrix $\widetilde{F}[-1] = \big[\widetilde{F}_{ij}[-1]\big]$,

$$\mathrm{Pf}\,\widetilde{F}[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot \widetilde{F}_{\sigma(1)\,\sigma(2)}[-1] \dots \widetilde{F}_{\sigma(2n-1)\,\sigma(2n)}[-1], \tag{3.27}$$

where $\widetilde{F}_{ij}[-1] = F_{ij'}[-1]$. The family $\varphi_{2\,2}, \varphi_{4\,4}, \dots, \varphi_{2n\,2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$, whereas $\varphi_{2\,2}, \varphi_{4\,4}, \dots, \varphi_{2n-2\,2n-2}, \varphi'_n$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$.

We extend the involution (2.13) to the algebra $\mathrm{U}\big(t^{-1}\mathfrak{o}_N[t^{-1}]\big) \otimes \mathbb{C}[\tau]$ with the action on $\mathbb{C}[\tau]$ as the identity map.

Lemma 3.6. *The element* (3.26) *is stable under* ς. *Moreover, in type* D_n *we have*

$$\varsigma : \mathrm{Pf}\,\widetilde{F}[-1] \mapsto (-1)^n\,\mathrm{Pf}\,\widetilde{F}[-1]. \tag{3.28}$$

Proof. The same argument as in the proof of Lemma 3.2 shows that the image of (3.26) under the involution ς equals

$$\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(\tau - F[-1]_1\big)\dots\big(\tau - F[-1]_m\big). \tag{3.29}$$

Indeed, this is implied by (3.24) and the commutation relations

$$F[r]_a\,F[s]_b - F[s]_b\,F[r]_a = (P_{a\,b} - Q_{a\,b})\,F[r+s]_b - F[r+s]_b\,(P_{a\,b} - Q_{a\,b})$$

for $a < b$. By applying the simultaneous transpositions $e_{ij} \mapsto e_{j'i'}$ to all m copies of $\mathrm{End}\,\mathbb{C}^N$ we conclude that (3.29) coincides with (3.26) because this transformation takes each factor $\tau - F[-1]_a$ to $\tau + F[-1]_a$ whereas the operator $S^{(m)}$ stays invariant. Relation (3.28) is immediate from (3.27). $\square$

By the main results of [16], the image of the polynomial (3.26) under the isomorphism (3.2) is given by the formula:

$$h_m\big(\tau + F_{11}[-1],\dots,\tau + F_{nn}[-1],\tau - F_{nn}[-1],\dots\tau - F_{11}[-1]\big),$$

for type B_n and by

$$\tfrac{1}{2}\,h_m\big(\tau + F_{11}[-1],\dots,\tau + F_{n-1\,n-1}[-1],\tau - F_{nn}[-1],\dots\tau - F_{11}[-1]\big)$$
$$+\tfrac{1}{2}\,h_m\big(\tau+F_{11}[-1],\dots,\tau+F_{nn}[-1],\tau-F_{n-1\,n-1}[-1],\dots\tau-F_{11}[-1]\big),$$

for type D_n. The latter sum can also be written in the form

$$h_m\big(\tau + F_{11}[-1],\dots,\tau + F_{nn}[-1],\tau - F_{nn}[-1],\dots\tau - F_{11}[-1]\big)$$
$$-\sum_{k+l=m-1} h_k\big(\tau + F_{11}[-1],\dots,\tau + F_{nn}[-1]\big)\,\tau\,h_l\big(\tau - F_{nn}[-1],\dots,\tau - F_{11}[-1]\big).$$

Furthermore, the image of the element φ'_n in type D_n is given by

$$\big(F_{11}[-1] - \tau\big)\dots\big(F_{nn}[-1] - \tau\big)\,1, \tag{3.30}$$

where τ is understood as the differentiation operator so that $\tau\,1 = 0$; see also [23] for a direct calculation of the Harish-Chandra image of φ'_n.

Choose parameters as in Sec. 2 and suppose that χ vanishes on the subspace $\mathfrak{n}_- \oplus \mathfrak{n}_+$ of $\mathfrak{o}_N$ so that we can regard χ as an element of $\mathfrak{h}^*$. Set

$$F_{ij}(u) = \sum_{a=1}^{\ell} \frac{(F_{ij})_a}{u - z_a} - \chi(F_{ij}) \in \mathrm{U}(\mathfrak{o}_N)^{\otimes\,\ell}.$$

In accordance with (2.17) set

$$\mathcal{F}_{ii}(u) = \sum_{a=1}^{\ell} \frac{\lambda_a(F_{ii})}{u - z_a} - \sum_{j=1}^{m} \frac{\alpha_{i_j}(F_{ii})}{u - w_j} - \chi(F_{ii}).$$

In the case $\mathfrak{g} = \mathfrak{o}_{2n}$ define the operator

$$\operatorname{Pf} \widetilde{F}(u) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot \widetilde{F}_{\sigma(1)\,\sigma(2)}(u) \dots \widetilde{F}_{\sigma(2n-1)\,\sigma(2n)}(u), \tag{3.31}$$

where $\widetilde{F}_{ij}(u) = F_{ij'}(u)$. As before, we will assume that the Bethe ansatz equations (2.16) hold.

Theorem 3.7. *The eigenvalue of the operator*

$$\gamma_m(N) \operatorname{tr} S^{(m)} \big(\partial_u + F(u)_1\big) \dots \big(\partial_u + F(u)_m\big) \tag{3.32}$$

on the Bethe vector (2.15) *is found by*

$$h_m\big(\partial_u + \mathcal{F}_{11}(u), \dots, \partial_u + \mathcal{F}_{nn}(u), \partial_u - \mathcal{F}_{nn}(u), \dots \partial_u - \mathcal{F}_{11}(u)\big)$$

for type B_n, and by

$$\tfrac{1}{2}\, h_m\big(\partial_u + \mathcal{F}_{11}(u), \dots, \partial_u + \mathcal{F}_{n-1\,n-1}(u), \partial_u - \mathcal{F}_{nn}(u), \dots \partial_u - \mathcal{F}_{11}(u)\big)$$
$$+ \tfrac{1}{2}\, h_m\big(\partial_u + \mathcal{F}_{11}(u), \dots, \partial_u + \mathcal{F}_{nn}(u), \partial_u - \mathcal{F}_{n-1\,n-1}(u), \dots \partial_u - \mathcal{F}_{11}(u)\big)$$

for type D_n. Moreover, the eigenvalue of the operator $\operatorname{Pf} \widetilde{F}(u)$ *in type D_n is given by*

$$\big(\mathcal{F}_{11}(u) - \partial_u\big) \dots \big(\mathcal{F}_{nn}(u) - \partial_u\big)\, 1. \tag{3.33}$$

Proof. We apply Theorem 2.1 again and regard Φ as the map

$$\Phi : \mathrm{U}\big(t^{-1}\mathfrak{o}_N[t^{-1}]\big) \otimes \mathbb{C}[\tau] \to \mathrm{U}(\mathfrak{o}_N)^{\otimes \ell} \otimes \mathbb{C}[\partial_u]$$

such that $\tau \mapsto \partial_u$. By the definition of the homomorphism (2.12) we have

$$\Psi : F[-1] \mapsto -F(u).$$

Hence, using the equivalent formula (3.29) for the polynomial (3.26) we find that its image under Φ coincides with the operator (3.32). The proof of the first part of the theorem is completed by using the formulas for the images of (3.26) under the respective isomorphisms (3.2) recalled above. Finally, by Lemma 3.6, in type D_n,

$$\Phi : \operatorname{Pf} \widetilde{F}[-1] \mapsto \operatorname{Pf} \widetilde{F}(u)$$

so that the last claim follows by using formula (3.30) for the image of $\operatorname{Pf} \widetilde{F}[-1]$ under the isomorphism (2.9). □

Corollary 3.8. *The eigenvalue of the generating function*

$$\Big(\sum_{m=0}^{\infty}(-z)^m\,\gamma_m(N)\,\mathrm{tr}\,S^{(m)}\big(\partial_u+F(u)_1\big)\dots\big(\partial_u+F(u)_m\big)\Big)^{-1} \tag{3.34}$$

on the Bethe vector (2.15) *is found by*

$$\Big(1+\big(\partial_u-\mathcal{F}_{11}(u)\big)z\Big)\dots\Big(1+\big(\partial_u-\mathcal{F}_{nn}(u)\big)z\Big)\Big(1+\big(\partial_u+\mathcal{F}_{nn}(u)\big)z\Big)\dots\Big(1+\big(\partial_u+\mathcal{F}_{11}(u)\big)z\Big)$$

for type B_n and by

$$\Big(1+\big(\partial_u-\mathcal{F}_{11}(u)\big)z\Big)\dots\Big(1+\big(\partial_u-\mathcal{F}_{nn}(u)\big)z\Big)\Big(1+\partial_u z\Big)^{-1}$$
$$\times\Big(1+\big(\partial_u+\mathcal{F}_{nn}(u)\big)z\Big)\dots\Big(1+\big(\partial_u+\mathcal{F}_{11}(u)\big)z\Big)$$

for type D_n.

3.3 Type C

We identify the symplectic Lie algebra $\mathfrak{g}=\mathfrak{sp}_{2n}$ with the Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements $F_{ij}=E_{ij}-\varepsilon_i\varepsilon_j E_{j'i'}$ with $i,j\in\{1,\dots,2n\}$, where $i'=2n-i+1$ and $\varepsilon_i=1$ for $i=1,\dots,n$ and $\varepsilon_i=-1$ for $i=n+1,\dots,2n$.

Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{sp}_{2n}$ spanned by the basis elements $F_{11},\dots,F_{nn}$. We have the triangular decomposition $\mathfrak{sp}_{2n}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ denote the subalgebras of $\mathfrak{sp}_{2n}$ spanned by the elements F_{ij} with $i>j$ and by the elements F_{ij} with $i<j$, respectively.

We will use the elements $F_{ij}[r]=F_{ij}t^r$ of the loop algebra $\mathfrak{sp}_{2n}[t,t^{-1}]$. Introduce the elements $F[r]_a$ of the algebra (3.5) by

$$F[r]_a=\sum_{i,j=1}^{2n}1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(m-a)}\otimes F_{ij}[r], \tag{3.35}$$

where U in (3.5) now stands for the universal enveloping algebra of $\widehat{\mathfrak{sp}}_{2n}\oplus\mathbb{C}\tau$.

For $1\leqslant a<b\leqslant m$ consider the operators P_{ab} defined by (3.16) and introduce the operators

$$Q_{ab}=\sum_{i,j=1}^{2n}\varepsilon_i\varepsilon_j\,1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(b-a-1)}\otimes e_{i'j'}\otimes 1^{\otimes(m-b)}.$$

For $1\leqslant m\leqslant n$ set

$$S^{(m)}=\frac{1}{m!}\prod_{1\leqslant a<b\leqslant m}\Big(1-\frac{P_{ab}}{b-a}-\frac{Q_{ab}}{n-b+a+1}\Big), \tag{3.36}$$

where the product is taken in the lexicographic order on the pairs (a, b). The element (3.36) is the image of the symmetrizer in the Brauer algebra $\mathcal{B}_m(-2n)$ under its action on the vector space $(\mathbb{C}^{2n})^{\otimes m}$. Use the notation (3.25) to introduce the polynomial in τ by

$$\gamma_m(-2n)\,\mathrm{tr}\, S^{(m)}\big(\tau + F[-1]_1\big)\dots\big(\tau + F[-1]_m\big) = \varphi_{m\,0}\,\tau^m + \varphi_{m\,1}\,\tau^{m-1} + \dots + \varphi_{m\,m}, \quad (3.37)$$

where the trace is taken over all m copies of $\mathrm{End}\,\mathbb{C}^{2n}$. By the results of [15], the values of m in (3.37) can be extended to the range $1 \leqslant m \leqslant 2n$ (and, in fact, for $m = 2n + 1$ as well) to get a well-defined polynomial in τ. Moreover, the family $\varphi_{2\,2}, \varphi_{4\,4}, \dots, \varphi_{2n\,2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{sp}_{2n}$.

Extend the involution (2.13) to the algebra $\mathrm{U}\big(t^{-1}\mathfrak{sp}_{2n}[t^{-1}]\big) \otimes \mathbb{C}[\tau]$ with the action on $\mathbb{C}[\tau]$ as the identity map.

Lemma 3.9. *The element* (3.37) *is stable under* ς.

Proof. The proof is the same as for Lemma 3.6, which also provides an equivalent formula

$$\gamma_m(-2n)\,\mathrm{tr}\, S^{(m)}\big(\tau - F[-1]_1\big)\dots\big(\tau - F[-1]_m\big) \tag{3.38}$$

for the polynomial (3.37). □

By the main result of [16], the image of the polynomial (3.37) with $1 \leqslant m \leqslant 2n + 1$ under the isomorphism (3.2) is given by the formula:

$$e_m\big(\tau + F_{11}[-1], \dots, \tau + F_{nn}[-1], \tau, \tau - F_{nn}[-1], \dots \tau - F_{11}[-1]\big),$$

where we use notation (3.21).

With parameters chosen as in Sec. 2, suppose that χ vanishes on the subspace $\mathfrak{n}_- \oplus \mathfrak{n}_+$ of $\mathfrak{sp}_{2n}$ so that we can regard χ as an element of $\mathfrak{h}^*$. Set

$$F_{ij}(u) = \sum_{a=1}^{\ell} \frac{(F_{ij})_a}{u - z_a} - \chi(F_{ij}) \in \mathrm{U}(\mathfrak{sp}_{2n})^{\otimes \ell}.$$

In accordance with (2.17) set

$$\mathcal{F}_{ii}(u) = \sum_{a=1}^{\ell} \frac{\lambda_a(F_{ii})}{u - z_a} - \sum_{j=1}^{m} \frac{\alpha_{i_j}(F_{ii})}{u - w_j} - \chi(F_{ii}).$$

As before, we will assume that the Bethe ansatz equations (2.16) hold.

Theorem 3.10. *For any* $1 \leqslant m \leqslant 2n + 1$ *the eigenvalue of the operator*

$$\gamma_m(-2n)\,\mathrm{tr}\, S^{(m)}\big(\partial_u + F(u)_1\big)\dots\big(\partial_u + F(u)_m\big) \tag{3.39}$$

on the Bethe vector (2.15) *is found by*

$$e_m\big(\partial_u + \mathcal{F}_{11}(u), \dots, \partial_u + \mathcal{F}_{nn}(u), \partial_u, \partial_u - \mathcal{F}_{nn}(u), \dots \partial_u - \mathcal{F}_{11}(u)\big).$$

Proof. This is derived from Theorem 2.1 and Lemma 3.9 as in the proof of Theorem 3.7. □

3.4 Connection with the results of [19] and [20]

Theorem 3.3 was previously proved in [19] is a slightly different form; see Theorem 9.2 there. We will make a connection between these results by showing that one is obtained from the other by using an automorphism of the current algebra. The notation of [19] corresponds to ours (we used the settings of [7] and [9]) as follows. The highest weights $\Lambda_k = (\Lambda_k^1, \ldots, \Lambda_k^N)$ correspond to our λ_k so that $\Lambda_k^i = \lambda_k(E_{ii})$; the evaluation parameters z_i are the same. The diagonal matrix $K = \mathrm{diag}\,[K_1, \ldots, K_N]$ corresponds to our element $-\chi$ so that $K_i = -\chi(E_{ii})$. Finally, the collection of nonnegative integers $\xi = (\xi^1, \ldots, \xi^{N-1})$ gives rise to our multiset of simple roots α_{i_j} where $\alpha_l = \varepsilon_l - \varepsilon_{l+1}$ occurs ξ^l times for each $l = 1, \ldots, N-1$. The corresponding variables $t_1^1, \ldots, t_{\xi^1}^1, \ldots, t_1^{N-1}, \ldots, t_{\xi^{N-1}}^{N-1}$ are then respectively identified with our parameters $w_1, \ldots, w_m$ with $m = |\xi|$. The coroots $\check{\alpha}_l$ coincide with the elements $E_{ll} - E_{l+1\, l+1}$ so that the Bethe ansatz equations (9.3) in [19] turn into (2.16). Using this correspondence between the settings, we can now state [19, Theorem 9.2] in our notation as the relation

$$\mathrm{cdet}\big(\partial_u - E(u)\big)\,\phi(w_1^{i_1}, \ldots, w_m^{i_m}) = \big(\partial_u - \mathcal{E}_{11}(u)\big) \ldots \big(\partial_u - \mathcal{E}_{NN}(u)\big)\,\phi(w_1^{i_1}, \ldots, w_m^{i_m})$$

for the eigenvalue of the operator $\mathrm{cdet}\big(\partial_u - E(u)\big)$ on the Bethe vector (2.15). This relation is implied by Theorem 3.3 by twisting the action of $\mathrm{U}(\mathfrak{gl}_N)$ on each Verma module M_{λ_k} by the automorphism $E_{ij} \mapsto -E_{j'i'}$, where $i' = N - i + 1$. The automorphism takes $\mathrm{rdet}\big(\partial_u + E(u)\big)$ to $\mathrm{cdet}\big(\partial_u - E(u)\big)$ and $\mathcal{E}_{ii}(u)$ to $-\mathcal{E}_{i'i'}(u)$.

We also make a connection of Theorems 3.3, 3.7 and 3.10 with formulas for universal differential operators corresponding to populations of critical points of the master functions associated with flag varieties; see [20]. With the recalled above notation of [19], we follow [20] to introduce polynomials in type A,

$$T_a(u) = \prod_{k=1}^{\ell} (u - z_k)^{\Lambda_k^a}, \qquad a = 1, \ldots, N,$$

and

$$y_a(u) = \prod_{p=1}^{\xi^a} (u - t_p^a), \qquad a = 1, \ldots, N-1.$$

Then the eigenvalue of the Bethe vector in Theorem 3.3 with $\chi = 0$ coincides with the differential operator

$$\overleftarrow{\prod_{a=1,\ldots,N}} \left(\partial_u + \ln' \frac{T_a(u)\, y_{a-1}(u)}{y_a(u)} \right) \tag{3.40}$$

rewritten in our notation, where we set $y_0(u) = y_N(u) = 1$; see [20, Sec. 5.2].

Using a similar notation, in type B_n set

$$T_a^B(u) = \prod_{k=1}^{\ell} (u - z_k)^{\Lambda_k^a} \qquad \text{and} \qquad y_a^B(u) = \prod_{p=1}^{\xi^a} (u - t_p^a), \qquad a = 1, \dots, n. \tag{3.41}$$

Then the coefficient of z^{2n} in the eigenvalue in type B_n (see Corollary 3.8) coincides with (3.40), if we take $N = 2n$ and set

$$y_a(u) = y_{2n-a}(u) = y_a^B(u) \qquad \text{for} \quad a = 1, \dots, n$$

and

$$T_a(u) = T_{2n-a+1}(u)^{-1} = T_a^B(u) \qquad \text{for} \quad a = 1, \dots, n;$$

cf. [20, Sec. 7.1]. In type C_n, introducing $T_a^C(u)$ and $y_a^C(u)$ for $a = 1, \dots, n$ as in (3.41), we find that the eigenvalue of the operator with $m = 2n + 1$ in Theorem 3.10 is given by (3.40) with $N = 2n + 1$, where we set

$$y_a(u) = y_{2n-a+1}(u) = \begin{cases} y_a^C(u) & \text{for} \quad a = 1, \dots, n-1 \\ y_a^C(u)^2 & \text{for} \quad a = n \end{cases}$$

and

$$T_a(u) = T_{2n-a+2}(u)^{-1} = \begin{cases} T_a^C(u) & \text{for} \quad a = 1, \dots, n \\ 1 & \text{for} \quad a = n+1; \end{cases}$$

cf. [20, Sec. 7.2].

4 From q-characters to classical $\mathcal{W}$-algebras

The Harish-Chandra images of the Segal–Sugawara elements (3.26) and (3.37) in types B, C and D were calculated in [16] by taking a classical limit of certain Yangian characters (or q-characters). Our goal in this section is to prove general results providing a connection between the rings of q-characters and the corresponding classical $\mathcal{W}$-algebras. We will rely on the original work [12] for the basic definitions and properties of the q-characters; see also [11]. However, we will use an equivalent additive version of the character ring as in [22] and indicate the connection between the notation in Remarks 4.1 and 4.3 below. Although this version can be introduced independently via the Yangian representation theory, we will not make a direct use of the Yangians which will only appear in the notation $\operatorname{Rep} \mathrm{Y}(\mathfrak{g})$ for the ring of characters; cf. [16].

The screening operators for classical $\mathcal{W}$-algebras are constructed as limits of certain intertwiners between $\widehat{\mathfrak{g}}_\kappa$-modules at a level κ, as $\kappa \to -h^\vee$; see [10, Ch. 7]. They can also be obtained by applying a Chevalley-type theorem to the $\mathcal{W}$-algebras defined in the context of classical Hamiltonian reduction; see, e.g., [18]. It was conjectured in [12] and proved in [11], that the ring of characters can be defined as the intersection of the kernels of the screening operators. We will apply a classical limit procedure to derive the screening operators characterizing elements of the $\mathcal{W}$-algebra; cf. [12, Sec. 8]. The main result of [16] will play an important role in the proof of the surjectivity of the procedure.

4.1 Type A

Introduce the algebra of polynomials

$$\mathcal{L} = \mathbb{C}[\lambda_i(a) \,|\, i = 1, \dots, N,\ a \in \mathbb{C}]$$

in the variables $\lambda_i(a)$. For every $i \in \{1, \dots, N-1\}$ consider the free left $\mathcal{L}$-module $\widetilde{\mathcal{L}}_i$ with the generators $\sigma_i(a)$, where a runs over $\mathbb{C}$ and denote by $\mathcal{L}_i$ its quotient by the relations

$$\lambda_i(a)\, \sigma_i(a) = \lambda_{i+1}(a)\, \sigma_i(a+1), \qquad a \in \mathbb{C}. \tag{4.1}$$

Define the linear operator $\widetilde{S}_i : \mathcal{L} \to \widetilde{\mathcal{L}}_i$ by the formula

$$\widetilde{S}_i : \lambda_j(a) \mapsto \begin{cases} \lambda_i(a)\, \sigma_i(a) & \text{for} \quad j = i \\ -\lambda_{i+1}(a)\, \sigma_i(a+1) & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad j \neq i, i+1 \end{cases} \tag{4.2}$$

and the Leibniz rule

$$\widetilde{S}_i(AB) = B\, \widetilde{S}_i(A) + A\, \widetilde{S}_i(B). \tag{4.3}$$

Now the *i-th screening operator*

$$S_i : \mathcal{L} \to \mathcal{L}_i$$

is defined as the composition of $\widetilde{S}_i$ and the projection $\widetilde{\mathcal{L}}_i \to \mathcal{L}_i$.

In accordance with [11, Theorem 5.1], we can define the subalgebra $\operatorname{Rep} \mathrm{Y}(\mathfrak{gl}_N)$ of Yangian characters in $\mathcal{L}$ as the intersection of kernels of the screening operators:

$$\operatorname{Rep} \mathrm{Y}(\mathfrak{gl}_N) = \bigcap_{i=1}^{N-1} \ker S_i.$$

Remark 4.1. Our variables $\lambda_i(a)$ and $\sigma_i(a)$ correspond to $\Lambda_{i,q^{2a}}$ and $S_{i,q^{2a+i-1}}$ from [12], respectively; cf. [22]. $\square$

Now we recall the definition of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_N)$ via screening operators as in [10, Sec. 8.1]; see also [16] and [18]. With the notation as in Sec. 3.1, we will regard $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ as the algebra of polynomials in the variables $E_{ii}[r]$ with $i = 1, \dots, N$ and $r < 0$. The screening operators

$$V_i : \mathrm{U}(\widehat{\mathfrak{h}}_-) \to \mathrm{U}(\widehat{\mathfrak{h}}_-), \qquad i = 1, \dots, N-1$$

are defined by

$$V_i=\sum_{r=0}^{\infty}V_{i\,[r]}\left(\frac{\partial}{\partial E_{ii}[-r-1]}-\frac{\partial}{\partial E_{i+1\,i+1}[-r-1]}\right),$$

where the coefficients $V_{i\,[r]}$ are found from the expansion of a formal generating function in a variable z,

$$\sum_{r=0}^{\infty}V_{i\,[r]}\,z^r=\exp\sum_{m=1}^{\infty}\frac{E_{ii}[-m]-E_{i+1\,i+1}[-m]}{m}\,z^m.$$

The *classical* $\mathcal{W}$*-algebra* $\mathcal{W}(\mathfrak{gl}_N)$ is a subalgebra of $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ defined as the intersection of kernels of the screening operators:

$$\mathcal{W}(\mathfrak{gl}_N)=\bigcap_{i=1}^{N-1}\ker V_i.$$

We will now construct a map $\mathbf{gr}:\operatorname{Rep}\mathrm{Y}(\mathfrak{gl}_N)\to\mathcal{W}(\mathfrak{gl}_N)$ and describe its properties. First, embed $\mathcal{L}$ into the algebra of formal power series $\mathbb{C}[[\lambda_i^{(r)}]]$ in variables $\lambda_i^{(r)}$ with $i=1,\dots,N$ and $r=0,1,\dots$ by setting

$$\lambda_i(a)\mapsto\sum_{r=0}^{\infty}\frac{\lambda_i^{(r)}}{r!}\,a^r. \tag{4.4}$$

Identify the formal power series in the $\lambda_i^{(r)}$ with those in new variables $\mu_i^{(r)}$ defined by

$$\lambda_i^{(0)}=1+\mu_i^{(0)}\qquad\text{and}\qquad\lambda_i^{(r)}=\mu_i^{(r)}\quad\text{for}\quad r\geqslant 1. \tag{4.5}$$

Define the degrees of the new variables by $\deg\mu_i^{(r)}=-r-1$. Given $A\in\mathcal{L}$, consider the corresponding element $\mathbb{C}[[\mu_i^{(r)}]]$ and take its homogeneous component $\overline{A}$ of the maximum degree. This component is a polynomial in the variables $\mu_i^{(r)}$ and so we have a map

$$\mathbf{gr}:\mathcal{L}\to\mathbb{C}[\mu_i^{(r)}],\qquad A\mapsto\overline{A}. \tag{4.6}$$

Note its property which is immediate from the definition:

$$\mathbf{gr}(AB)=\mathbf{gr}(A)\,\mathbf{gr}(B). \tag{4.7}$$

In the following proposition we identify $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ with the algebra of polynomials $\mathbb{C}[\mu_i^{(r)}]$ via the isomorphism $E_{ii}[-r-1]\mapsto\mu_i^{(r)}/r!$.

Proposition 4.2. *The image of the restriction of the map* (4.6) *to the subalgebra of characters* $\operatorname{Rep}\mathrm{Y}(\mathfrak{gl}_N)$ *is contained in* $\mathcal{W}(\mathfrak{gl}_N)$ *and so it defines a map*

$$\mathbf{gr}:\operatorname{Rep}\mathrm{Y}(\mathfrak{gl}_N)\to\mathcal{W}(\mathfrak{gl}_N).$$

Moreover, any homogeneous element of $\mathcal{W}(\mathfrak{gl}_N)$ *is contained in the image of* $\mathbf{gr}$.

Proof. Similar to (4.4), introduce variables $\sigma_i^{(r)}$ by the expansion

$$\sigma_i(a) \mapsto \sum_{r=0}^{\infty} \frac{\sigma_i^{(r)}}{r!}\, a^r \tag{4.8}$$

and set $\deg \sigma_i^{(r)} = -r-1$. Regarding a as a formal variable in (4.1) and (4.2), write the screening operators in terms of the variables $\mu_i^{(r)}$. Explicitly, for $i = 1, \dots, N-1$ define operators

$$S_i^\circ : \mathbb{C}[[\mu_j^{(r)}]] \to \mathbb{C}[[\mu_j^{(r)}, \sigma_i^{(r)}]]/\sim, \tag{4.9}$$

where the target space is the quotient of $\mathbb{C}[[\mu_j^{(r)}, \sigma_i^{(r)}]]$ by the relations (4.1) written in terms of the $\mu_i^{(r)}$ with a understood as a variable. Set

$$S_i^\circ : \mu_j^{(0)} \mapsto \begin{cases} \big(1+\mu_i^{(0)}\big)\, \sigma_i^{(0)} & \text{for} \quad j = i \\ -\big(1+\mu_{i+1}^{(0)}\big) \displaystyle\sum_{k\geqslant 0} \frac{\sigma_i^{(k)}}{k!} & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad j \neq i, i+1 \end{cases}$$

and

$$S_i^\circ : \mu_j^{(r)} \mapsto \partial^r \Big(S_i^\circ \big(\mu_j^{(0)}\big)\Big), \qquad r \geqslant 1,$$

where the derivation ∂ acts on the variables by the rule

$$\partial : \mu_j^{(r)} \mapsto \mu_j^{(r+1)}, \qquad \sigma_j^{(r)} \mapsto \sigma_j^{(r+1)}, \qquad r \geqslant 0.$$

The action of S_i° then extends to the entire algebra $\mathbb{C}[[\mu_j^{(r)}]]$ via the Leibniz rule as in (4.3).

Now suppose that $A \in \operatorname{Rep} \mathrm{Y}(\mathfrak{gl}_N)$ so that $S_i A = 0$ for all $i = 1, \dots, N-1$. Denote by A° the corresponding element of $\mathbb{C}[[\mu_j^{(r)}]]$. By the definition of the operators S_i°, their restriction to the subalgebra $\mathcal{L}$ coincides with the action of the respective operators S_i. Therefore, $S_i^\circ A^\circ = 0$. Taking the top degree component $\overline{A}$ of A° we can write

$$S_i^\circ A^\circ = \overline{S}_i \overline{A} + \text{lower degree terms},$$

where the operator $\overline{S}_i$ is given by

$$\overline{S}_i : \mu_j^{(r)} \mapsto \begin{cases} \sigma_i^{(r)} & \text{for} \quad j = i \\ -\sigma_i^{(r)} & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad j \neq i, i+1. \end{cases} \tag{4.10}$$

On the other hand, relations (4.1) give

$$\mu_i(a)\, \sigma_i(a) = \big(1 + \mu_{i+1}(a)\big) \sum_{k=0}^{\infty} \frac{\sigma_i^{(k)}(a)}{k!} - \sigma_i(a), \tag{4.11}$$

where $\sigma_i^{(k)}(a)$ is defined as the k-th derivative over a from (4.8) and

$$\mu_j(a) = \sum_{r=0}^{\infty} \frac{\mu_j^{(r)}}{r!}\, a^r.$$

Regarding a as a variable, we get from (4.11) a sequence of relations by comparing the coefficients of the same powers of a. The top degree components in these relations are homogeneous relations which can be written in terms of generating functions in the form

$$\sigma_i'(z) = \big(\mu_i(z) - \mu_{i+1}(z)\big)\, \sigma_i(z)$$

so that for the images under $\overline{S}_i$ we have

$$\overline{S}_i : \mu_i(z) \mapsto \exp \int \big(\mu_i(z) - \mu_{i+1}(z)\big)\, dz, \qquad \mu_{i+1}(z) \mapsto -\exp \int \big(\mu_i(z) - \mu_{i+1}(z)\big)\, dz,$$

and $\overline{S}_i : \mu_j(z) \mapsto 0$ for $j \neq i, i+1$. However, this coincides with the action of the operator V_i on the series

$$\mu_k(z) = \sum_{r=0}^{\infty} E_{kk}[-r-1]\, z^r, \qquad k = 1, \dots, N.$$

Thus, we may conclude that if an element $A \in \mathcal{L}$ is annihilated by all operators S_i, then its image $\overline{A}$ under the map (4.6) is annihilated by all operators V_i completing the proof of the first part of the proposition.

The second part follows from [16], where generators of the algebra $\mathcal{W}(\mathfrak{gl}_N)$ were obtained as images of certain elements of $\mathcal{L}$ under the map $\mathbf{gr}$. $\square$

4.2 Types B, C and D

We let $\mathfrak{g}$ denote the orthogonal Lie algebra $\mathfrak{o}_N$ (with $N = 2n$ or $N = 2n+1$) or the symplectic Lie algebra $\mathfrak{sp}_N$ (with $N = 2n$). Introduce a parameter κ by $\kappa = N/2 - 1$ in the orthogonal case and $\kappa = N/2 + 1$ in the symplectic case. As before, we set $i' = N - i + 1$.

Consider the algebra of polynomials in variables $\lambda_i(a)$ with $i = 1, \dots, N$ and $a \in \mathbb{C}$ and denote by $\mathcal{L} = \mathcal{L}(\mathfrak{g})$ its quotient by the relations

$$\lambda_i(a + \kappa - i)\, \lambda_{i'}(a) = \lambda_{i+1}(a + \kappa - i)\, \lambda_{(i+1)'}(a), \qquad a \in \mathbb{C}, \tag{4.12}$$

for $i = 0, 1, \dots, n-1$ if $\mathfrak{g} = \mathfrak{o}_{2n}$ or $\mathfrak{sp}_{2n}$, and for $i = 0, 1, \dots, n$ if $\mathfrak{g} = \mathfrak{o}_{2n+1}$, where $\lambda_0(a) = \lambda_{0'}(a) = 1$.

For $i = 1, \dots, n$ consider the free left $\mathcal{L}$-module $\widetilde{\mathcal{L}}_i$ with the generators $\sigma_i(a)$, where a runs over $\mathbb{C}$ and denote by $\mathcal{L}_i$ its quotient by the relations

$$\lambda_i(a)\, \sigma_i(a) = \lambda_{i+1}(a)\, \sigma_i(a+1), \qquad i = 1, \dots, n-1, \quad a \in \mathbb{C}, \tag{4.13}$$

together with

$$\begin{aligned}
\lambda_n(a)\,\sigma_n(a) &= \lambda_{n+1}(a)\,\sigma_n(a+1/2), && \text{for} \quad \mathfrak{g}=\mathfrak{o}_{2n+1}\\
\lambda_n(a)\,\sigma_n(a) &= \lambda_{n+1}(a)\,\sigma_n(a+2), && \text{for} \quad \mathfrak{g}=\mathfrak{sp}_{2n}\\
\lambda_{n-1}(a)\,\sigma_n(a) &= \lambda_{n+1}(a)\,\sigma_n(a+1), && \text{for} \quad \mathfrak{g}=\mathfrak{o}_{2n}.
\end{aligned} \tag{4.14}$$

For every $i\in\{1,\dots,n\}$ define a linear operator $\widetilde{S}_i:\mathcal{L}\to\widetilde{\mathcal{L}}_i$ satisfying the Leibniz rule (4.3). For $i=1,\dots,n-1$ set

$$\widetilde{S}_i:\lambda_j(a)\mapsto\begin{cases}\lambda_i(a)\,\sigma_i(a) & \text{for} \quad j=i\\ -\lambda_{i+1}(a)\,\sigma_i(a+1) & \text{for} \quad j=i+1\\ -\lambda_{i'}(a)\,\sigma_i(a+\kappa-i+1) & \text{for} \quad j=i'\\ \lambda_{(i+1)'}(a)\,\sigma_i(a+\kappa-i) & \text{for} \quad j=(i+1)'\\ 0 & \text{for} \quad j\neq i,i',i+1,(i+1)'.\end{cases} \tag{4.15}$$

The action of $\widetilde{S}_n$ depends on the type and is given as follows.

Case $\mathfrak{g}=\mathfrak{o}_{2n+1}$: $\widetilde{S}_n:\lambda_j(a)\mapsto 0$ if $j<n$ or $j>n'$ and

$$\begin{aligned}
\lambda_n(a)&\mapsto\lambda_n(a)\,\big(\sigma_n(a)+\sigma_n(a-1/2)\big)\\
\lambda_{n+1}(a)&\mapsto\lambda_{n+1}(a)\,\big(\sigma_n(a-1/2)-\sigma_n(a+1/2)\big)\\
\lambda_{n'}(a)&\mapsto-\lambda_{n'}(a)\,\big(\sigma_n(a)+\sigma_n(a+1/2)\big).
\end{aligned}$$

Case $\mathfrak{g}=\mathfrak{sp}_{2n}$: $\widetilde{S}_n:\lambda_j(a)\mapsto 0$ if $j<n$ or $j>n'$ and

$$\begin{aligned}
\lambda_n(a)&\mapsto\lambda_n(a)\,\sigma_n(a)\\
\lambda_{n'}(a)&\mapsto-\lambda_{n'}(a)\,\sigma_n(a+2).
\end{aligned}$$

Case $\mathfrak{g}=\mathfrak{o}_{2n}$: $\widetilde{S}_n:\lambda_j(a)\mapsto 0$ if $j<n-1$ or $j>(n-1)'$ and

$$\begin{aligned}
\lambda_{n-1}(a)&\mapsto\lambda_{n-1}(a)\,\sigma_n(a)\\
\lambda_n(a)&\mapsto\lambda_n(a)\,\sigma_n(a)\\
\lambda_{n'}(a)&\mapsto-\lambda_{n'}(a)\,\sigma_n(a+1)\\
\lambda_{(n-1)'}(a)&\mapsto-\lambda_{(n-1)'}(a)\,\sigma_n(a+1).
\end{aligned}$$

The relations (4.12) are easily seen to be preserved by the action of the $\widetilde{S}_i$ so that the operators on $\mathcal{L}$ are well-defined. The *i-th screening operator*

$$S_i:\mathcal{L}\to\mathcal{L}_i$$

is now defined as the composition of $\widetilde{S}_i$ and the projection $\widetilde{\mathcal{L}}_i \to \mathcal{L}_i$.

Due to [11, Theorem 5.1], we can define the subalgebra $\operatorname{Rep} \mathrm{Y}(\mathfrak{g})$ of Yangian characters in $\mathcal{L}$ as the intersection of kernels of the screening operators:

$$\operatorname{Rep} \mathrm{Y}(\mathfrak{g}) = \bigcap_{i=1}^{n} \ker S_i.$$

Remark 4.3. The variables $\lambda_i(a)$ and $\sigma_i(a)$ are related to the corresponding elements used in [12] as follows: $\lambda_i(a) = \Lambda_{i,q^{4a}}$ for $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\lambda_i(a) = \Lambda_{i,q^{2a}}$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{g} = \mathfrak{o}_{2n}$. Moreover, for $\mathfrak{g} = \mathfrak{o}_{2n+1}$

$$\sigma_i(a) = S_{i,q^{4a+2i-2}} \qquad \text{for} \quad i = 1, \dots, n-1; \qquad \sigma_n(a) = S_{n,q^{4a+2n-1}},$$

while for $\mathfrak{g} = \mathfrak{sp}_{2n}$ we have

$$\sigma_i(a) = S_{i,q^{2a+i-1}} \qquad \text{for} \quad i = 1, \dots, n;$$

the latter relations with $i = 1, \dots, n-1$ hold for $\mathfrak{g} = \mathfrak{o}_{2n}$ as well, but $\sigma_n(a) = S_{n,q^{2a+n-2}}$; cf. [22]. Note also that relations (4.12) were obtained in [1, Prop. 5.2 and 5.14] as the conditions for the highest weight representations of the Yangian $\mathrm{Y}(\mathfrak{g})$ to be nontrivial, whereas (4.13) and (4.14) are consistent with the conditions on the representation to be finite-dimensional; cf. [1, Theorem 5.16]. □

We follow [10, Sec. 8.1] again to define the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$; see also [16] and [18]. We will regard $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ as the algebra of polynomials in the variables $F_{ii}[r]$ with $i = 1, \dots, n$ and $r < 0$. The screening operators

$$V_i : \mathrm{U}(\widehat{\mathfrak{h}}_-) \to \mathrm{U}(\widehat{\mathfrak{h}}_-), \qquad i = 1, \dots, n$$

are defined as follows. For $i = 1, \dots, n-1$ set

$$V_i = \sum_{r=0}^{\infty} V_{i\,[r]} \left(\frac{\partial}{\partial F_{ii}[-r-1]} - \frac{\partial}{\partial F_{i+1\,i+1}[-r-1]} \right),$$

where the coefficients $V_{i\,[r]}$ are found from the expansion of a formal generating function in a variable z,

$$\sum_{r=0}^{\infty} V_{i\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{F_{ii}[-m] - F_{i+1\,i+1}[-m]}{m}\, z^m.$$

For the action of V_n we have the following formulas.

Case $\mathfrak{g} = \mathfrak{o}_{2n+1}$:

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]} \frac{\partial}{\partial F_n[-r-1]},$$

where

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{F_n[-m]}{m}\, z^m.$$

Case $\mathfrak{g} = \mathfrak{sp}_{2n}$:

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]} \frac{\partial}{\partial F_n[-r-1]},$$

where

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{2\,F_n[-m]}{m}\, z^m.$$

Case $\mathfrak{g} = \mathfrak{o}_{2n}$:

$$V_n = \sum_{r=0}^{\infty} V_{n\,[r]} \left(\frac{\partial}{\partial F_{n-1}[-r-1]} + \frac{\partial}{\partial F_n[-r-1]} \right)$$

where

$$\sum_{r=0}^{\infty} V_{n\,[r]}\, z^r = \exp \sum_{m=1}^{\infty} \frac{F_{n-1}[-m] + F_n[-m]}{m}\, z^m.$$

The *classical $\mathcal{W}$-algebra* $\mathcal{W}(\mathfrak{g})$ is a subalgebra of $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ defined as the intersection of kernels of the screening operators:

$$\mathcal{W}(\mathfrak{g}) = \bigcap_{i=1}^{n} \ker V_i.$$

Now construct a map $\mathbf{gr} : \mathrm{Rep}\,\mathrm{Y}(\mathfrak{g}) \to \mathcal{W}({}^L\mathfrak{g})$ and describe its properties. First, embed $\mathcal{L}$ into the algebra of formal power series $\mathbb{C}\,[[\lambda_i^{(r)}]]$ in variables $\lambda_i^{(r)}$ with $i = 1, \dots, N$ and $r = 0, 1, \dots$ by using (4.4) and taking the quotient by the corresponding relations (4.12). Introduce new variables $\mu_i^{(r)}$ by (4.5) for $i = 1, \dots, n$ and define their degrees by $\deg \mu_i^{(r)} = -r-1$. Given $A \in \mathcal{L}$, consider the corresponding element $\mathbb{C}\,[[\mu_i^{(r)}]]$ and take its homogeneous component $\overline{A}$ of the maximum degree. This component is a polynomial in the variables $\mu_i^{(r)}$ and so we have a map

$$\mathbf{gr} : \mathcal{L} \to \mathbb{C}\,[\mu_i^{(r)}], \qquad A \mapsto \overline{A}. \tag{4.16}$$

Note its property (4.7). We will identify $\mathrm{U}(\widehat{\mathfrak{h}}_-)$ with the algebra of polynomials $\mathbb{C}\,[\mu_i^{(r)}]$ via the isomorphism $F_{ii}[-r-1] \mapsto \mu_i^{(r)}/r!$.

Proposition 4.4. *The image of the restriction of the map* (4.16) *to the subalgebra of characters* $\mathrm{Rep}\,\mathrm{Y}(\mathfrak{g})$ *is contained in* $\mathcal{W}({}^L\mathfrak{g})$ *and so it defines a map*

$$\mathbf{gr} : \mathrm{Rep}\,\mathrm{Y}(\mathfrak{g}) \to \mathcal{W}({}^L\mathfrak{g}).$$

Moreover, any homogeneous element of $\mathcal{W}({}^L\mathfrak{g})$ *is contained in the image of* $\mathbf{gr}$.

Proof. The proof is quite similar to that of Proposition 4.2 so we only point out the changes to be made. Introduce variables $\sigma_i^{(r)}$ by (4.8) and set $\deg \sigma_i^{(r)} = -r-1$. Define operators S_i° for $i = 1, \dots, n$ as in (4.9), where the quotient is now taken by the respective relations (4.13) and (4.14) written in terms of the $\mu_i^{(r)}$ with a understood as a variable. Since relations (4.13) are identical to (4.1), the argument for the operators S_i° with $i = 1, \dots, n-1$ follows the same steps as for type A. To complete the proof for the operator S_n°, consider the three cases separately.

Case $\mathfrak{g} = \mathfrak{o}_{2n+1}$. As with (4.10), the corresponding operator $\overline{S}_n$ is now given by

$$\overline{S}_n : \mu_j^{(r)} \mapsto \begin{cases} 2\sigma_n^{(r)} & \text{for} \quad j = n \\ 0 & \text{for} \quad j \neq n. \end{cases} \tag{4.17}$$

Note that

$$\lambda_{n+1}(a) = \frac{\lambda_1(a+n-1)\lambda_2(a+n-2)\dots\lambda_n(a)}{\lambda_1(a+n-1/2)\lambda_2(a+n-3/2)\dots\lambda_n(a+1/2)}$$

which is easy to derive from (4.12). Now use (4.14) and write $\lambda_i(a) = 1 + \mu_i(a)$ for $i = 1, \dots, n$ to get the corresponding analogue of (4.11). As a result, we get the equation

$$\sigma_n'(z) = 2\mu_n(z)\,\sigma_n(z)$$

so that for the images under $\overline{S}_n$ we have

$$\overline{S}_n : \mu_n(z) \mapsto 2\exp 2\int \mu_n(z)\, dz,$$

and $\overline{S}_n : \mu_j(z) \mapsto 0$ for $j \neq n$. This coincides with the action of the operator $2V_n$ associated with $\mathfrak{sp}_{2n}$ on the series

$$\mu_n(z) = \sum_{r=0}^{\infty} F_{nn}[-r-1]\, z^r.$$

Hence, if an element $A \in \mathcal{L}$ is annihilated by all operators S_i, then its image $\overline{A}$ under the map (4.6) is annihilated by all operators V_i associated with $\mathfrak{sp}_{2n}$ which is Langlands dual to $\mathfrak{o}_{2n+1}$.

Case $\mathfrak{g} = \mathfrak{sp}_{2n}$. Similar to (4.17), we have

$$\overline{S}_n : \mu_j^{(r)} \mapsto \begin{cases} \sigma_n^{(r)} & \text{for} \quad j = n \\ 0 & \text{for} \quad j \neq n. \end{cases} \tag{4.18}$$

Relations (4.12) now imply

$$\lambda_{n+1}(a) = \frac{\lambda_1(a+n)\lambda_2(a+n-1)\dots\lambda_{n-1}(a+2)}{\lambda_1(a+n+1)\lambda_2(a+n)\dots\lambda_n(a+2)}.$$

Write $\lambda_i(a) = 1 + \mu_i(a)$ for $i = 1, \dots, n$ and use (4.14) to get the equation

$$\sigma'_n(z) = \mu_n(z)\, \sigma_n(z).$$

Hence, for the images under $\overline{S}_n$ we have

$$\overline{S}_n : \mu_n(z) \mapsto \exp \int \mu_n(z)\, dz,$$

and $\overline{S}_n : \mu_j(z) \mapsto 0$ for $j \neq n$. This coincides with the action of the operator V_n associated with $\mathfrak{o}_{2n+1}$ on the series

$$\mu_n(z) = \sum_{r=0}^{\infty} F_{nn}[-r-1]\, z^r.$$

Therefore, if an element $A \in \mathcal{L}$ is annihilated by all operators S_i, then its image $\overline{A}$ under the map (4.6) is annihilated by all operators V_i associated with $\mathfrak{o}_{2n+1}$ which is Langlands dual to $\mathfrak{sp}_{2n}$.

Case $\mathfrak{g} = \mathfrak{o}_{2n}$. Similar to (4.10), we have

$$\overline{S}_n : \mu_j^{(r)} \mapsto \begin{cases} \sigma_n^{(r)} & \text{for} \quad j = n-1, n \\ 0 & \text{for} \quad j \neq n-1, n. \end{cases} \tag{4.19}$$

We derive from (4.12) that

$$\lambda_{n+1}(a) = \frac{\lambda_1(a+n-2)\lambda_2(a+n-3)\dots\lambda_{n-1}(a)}{\lambda_1(a+n-1)\lambda_2(a+n-2)\dots\lambda_n(a)}.$$

Write $\lambda_i(a) = 1 + \mu_i(a)$ for $i = 1, \dots, n$ and use (4.14) to get the equation

$$\sigma'_n(z) = \big(\mu_{n-1}(z) + \mu_n(z)\big)\, \sigma_n(z).$$

Hence, for the images under $\overline{S}_n$ we have

$$\overline{S}_n : \mu_{n-1}(z) \mapsto \exp \int \big(\mu_{n-1}(z) + \mu_n(z)\big)\, dz, \qquad \mu_n(z) \mapsto \exp \int \big(\mu_{n-1}(z) + \mu_n(z)\big)\, dz,$$

and $\overline{S}_n : \mu_j(z) \mapsto 0$ for $j \neq n-1, n$. This coincides with the action of the operator V_n associated with $\mathfrak{o}_{2n}$ on the series

$$\mu_i(z) = \sum_{r=0}^{\infty} F_{ii}[-r-1]\, z^r, \qquad i = n-1, n.$$

Thus, if an element $A \in \mathcal{L}$ is annihilated by all operators S_i, then its image $\overline{A}$ under the map (4.6) is annihilated by all operators V_i associated with $\mathfrak{o}_{2n}$ which is Langlands self-dual.

The last part of the proposition follows from [16], where generators of the classical $\mathcal{W}$-algebra were obtained as images of the Yangian characters under the map $\mathbf{gr}$. $\square$

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