ON BOHR SETS OF INTEGER-VALUED TRACELESS MATRICES

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Abstract. In this paper we show that any Bohr-zero non-periodic set B of traceless integer valued matrices, denoted by Λ, intersects non-trivially the conjugacy class of any matrix from Λ. As a corollary, we obtain that the family of characteristic polynomials of B contains all characteristic polynomials of matrices from Λ. The main ingredient used in this paper is an equidistribution result of Burgain-Furman-Lindenstrauss-Mozes [6].

1. Introduction

Let us denote by $\Lambda = Mat_d^0(\mathbb{Z})$, $d \geq 2$, the set of integer valued $d \times d$ matrices with zero trace, and by $\mathbb{T}^n$, $n \geq 1$, the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. Let G be a countable abelian group. A set $B \subset G$ is called a *non-periodic Bohr set* if there exist a homomorphism $\tau : G \to \mathbb{T}^n$, for some $n \geq 1$, with $\overline{\tau(G)} = \mathbb{T}^n$, and an open set $U \subset \mathbb{T}^n$ satisfying $B = \tau^{-1}(U)$. If the open set U contains the zero element of $\mathbb{T}^n$, then the set B is called a *Bohr-zero set*. We will also denote by $SL_d(\mathbb{Z})$ the group of $d \times d$ integer-valued matrices of determinant one.

The main result of this paper is the following.

Main Theorem. *Let $d \geq 2$, and $B \subset Mat_d^0(\mathbb{Z})$ be a Bohr-zero non-periodic set. Then for any matrix $C \in Mat_d^0(\mathbb{Z})$ there exists a matrix $A \in B$ and a matrix $g \in SL_d(\mathbb{Z})$ such that $C = g^{-1}Ag$.*

The same result has been also proved independently by Björklund and Bulinski [4]. They use the recent works of Benoist-Quint [2] and [3], instead of the work of Bourgain-Furman-Lindenstrauss-Mozes as the main ingredient in the proof.

Corollary 1.1. *Let $d \geq 2$, and $B \subset Mat_d^0(\mathbb{Z})$ be a Bohr-zero non-periodic set. The set of characteristic polynomials of the matrices in B coincides with the set of all characteristic polynomials of the matrices in $Mat_d^0(\mathbb{Z})$.*

The following number-theoretic statement conjectured by B. Green and T. Sanders is an immediate implication of Corollary 1.1.

Corollary 1.2. *Let $B \subset \mathbb{Z}$ be a Bohr-zero non-periodic set. Then the set of the discriminants over B defined by*

$$D := \{xy - z^2 \mid x, y, z \in B\}$$

satisfies that $D = \mathbb{Z}$.

At this point we will define Furstenberg's system corresponding to a set B of positive density in a countable abelian group G. Recall, we say that B has *positive density* if *upper Banach density* of B is positive:

$$d^*(B) = \sup_{\lambda \in \mathcal{F}} \lambda(1_B) > 0,$$

where $\mathcal{F}$ is the set of all Λ-invariant means on $\ell^\infty(G)$, i.e., non-negative normalised G-invariant linear functionals on $\ell^\infty(G)$. Since G is abelian, this implies that $\mathcal{F} \neq \emptyset$.

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Furstenberg in his seminal paper [8] constructed an (ergodic) G-measure-preserving system[1] (X, η, σ) and a clopen set $\tilde{B} \subset X$ such that

- $d^*(B \cap (B+h)) \geq \eta\left(\tilde{B} \cap \sigma(h)\tilde{B}\right)$, for any $h \in G$.
- $\eta(\tilde{B}) = d^*(B)$.

We will denote Furstenberg's system corresponding to B by $X_B = (X, \eta, \sigma, \tilde{B})$. Next, we will define the notion of the spectral measure corresponding to a set B of a countable abelian group G of positive density and its Furstenberg's system $X_B = (X, \eta, \sigma, \tilde{B})$. Denote by $1_{\tilde{B}}$ the indicator function of the set $\tilde{B}$. Then by Bochner's spectral theorem [7] there exists a non-negative finite Borel measure ν on $\widehat{G}$ (the dual of G) which satisfies:

$$\langle 1_{\tilde{B}}, \sigma(h)1_{\tilde{B}} \rangle = \int_{\widehat{G}} \chi(h) d\nu(\chi), \text{ for } h \in G.$$

The measure ν will be called the *spectral measure of the set B and its Furstenberg's system* X_B, and we will denote by $\widehat{\nu}(h)$ the right hand side of the last equation. We are at the position to state the main technical claim of the paper.

Theorem 1.1. *Let $d \geq 2$, and let $B \subset Mat_d^0(\mathbb{Z})$ be a set of positive density such that the spectral measure[2] of B has no atoms at non-trivial characters having finite torsion. Then for every $C \in Mat_d^0(\mathbb{Z})$ there exist $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.*

Theorem 1.1 is the strengthening of the following result that has been proved in [5] by use of the equidistribution result of Benoist-Quint [1].

Theorem 1.2. *Let $d \geq 2$, and let $B \in Mat_d^0(\mathbb{Z})$ be a set of positive density. Then there exists $k \geq 1$ such that for any matrix $C \in kMat_d^0(\mathbb{Z})$ there exists $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.*

We would like to finish the introduction by stating the piecewise version of Main Theorem. We recall that a set $B \subset \Lambda$ called *piecewise Bohr set* if there is a Bohr set $B_0 \subset \Lambda$ and a (thick) set $T \subset \Lambda$ of upper Banach density one, i.e., $d^*(T) = 1$ such that $B = B_0 \cap T$. Moreover, if the set B_0 is non-periodic Bohr-zero, then the set B will be called *piecewise Bohr-zero non-periodic*. Theorem 1.1 implies the following result.

Theorem 1.3. *Let $d \geq 2$, and let $B \subset Mat_d^0(\mathbb{Z})$ be a piecewise Bohr non-periodic set. Then for every $C \in Mat_d^0(\mathbb{Z})$ there exist $A \in B - B$ and $g \in SL_d(\mathbb{Z})$ with $C = g^{-1}Ag$.*

Let us show that Theorem 1.3 implies Main Theorem.

Proof of Main Theorem. Let $B \subset \Lambda$ be a Bohr-zero non-periodic set. Notice that there exists $B_0 \subset \Lambda$ a Bohr-zero non-periodic set with the property that

$$B_0 - B_0 \subset B.$$

Now, we apply Theorem 1.3 for the set B_0, and as a conclusion obtain the statement of the theorem. □

Organisation of the paper. In Section 2 we establish the consequences of the equidistribution result of Bourgain-Furman-Lindenstrauss-Mozes [6] related to the adjoint action of

[1] A triple (X, η, σ) is a *G-measure-preserving system*, if X is a compact metric space on which acts G by a measurable action denoted by σ, η is a Borel probability measure on X, and the action of G preserves η. A G-measure-preserving system is *ergodic* if any G-invariant measurable set has measure either zero or one.

[2] We assume the existence of some Furstenberg's system X_B corresponding to the set B, such that the associated spectral measure satisfies the requirement of the theorem.

$SL_d(\mathbb{Z})$ on $Mat_d^0(\mathbb{R})/Mat_d^0(\mathbb{Z})$. In Section 3 we prove Theorems 1.1, and 1.3. In Section 4 we prove main technical propositions stated in Sections 2 and 3.

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2. Consequences of the work of Bourgain-Furman–Lindenstrauss-Mozes

We start by recalling the property of strong irreducibility of an action of a discrete group. Let Γ be a countable group, and V be a finite dimensional real space. We say that an action $\rho : \Gamma \to End(V)$ is *strongly irreducible* if for every finite index subgroup H of Γ, the restriction of the action of ρ to H is irreducible. We also will be using the notion of a proximal element. An operator $T \in End(V)$ will be called *proximal*, if there is only one eigenvalue of the largest absolute value, and corresponding to it eigenspace is one-dimensional.

Assume that a countable group Γ acts on a compact Borel measure space (X, ν). Let μ be a probability measure on Γ. Then the convolution measure $\mu * \nu$ on X is defined by:

$$\int_X f d\,(\mu * \nu) = \int_X \left(\sum_{g\in\Gamma} f(gx)\mu(g) \right) d\nu(x), \text{ for any } f \in C(X).$$

We will denote the Dirac probability measure at a point $x \in X$ by δ_x. For every $k \geq 2$, we define the probability measure μ^{*k} on Γ by

$$\mu^{*k}(g) = \sum_{g_1\cdot\ldots\cdot g_k = g} \mu(g_1)\mu(g_2)\ldots\mu(g_k).$$

The main ingredient in the proofs of all our main results is the following seminal equidistribution statement due to Bourgain-Furman-Lindenstrauss-Mozes [6].

Theorem 2.1 (Corollary B in [6])**.** *Let $\Gamma < SL_n(\mathbb{Z})$ be a subgroup which acts totally irreducibly on $\mathbb{R}^n$, and having a proximal element. Let μ be a finite generating probability measure on Γ. Let $x \in \mathbb{T}^n$ be a non-rational point. Then the measures $\mu^{*k} * \delta_x$ converge in weak*-topology as $k \to \infty$ to Haar measure on $\mathbb{T}^n$.*

In this note, the acting group will be $\Gamma = SL_d(\mathbb{Z})$. The group Γ acts by the conjugation on the real vector space $V = Mat_d^0(\mathbb{R})$ of real valued $d \times d$ matrices with zero trace. So, an element $g \in SL_d(\mathbb{Z})$ acts on $v \in V$ by $Ad(g)v = g^{-1}vg$, and such action called the *adjoint action* of $SL_d(\mathbb{Z})$. Notice that V is isomorphic to $\mathbb{R}^{d^2-1}$. The next claim will allow us to apply Theorem 2.1 in our setting.

Proposition 2.1. *The adjoint action of $SL_d(\mathbb{Z})$ on $Mat_d^0(\mathbb{R})$ is strongly irreducible, and $SL_d(\mathbb{Z})$ contains an element which acts proximally.*

Let us denote by $A_d = V/\Lambda$. Notice that A_d is isomorphic to $\mathbb{T}^{d^2-1}$, and it is the dual group of Λ. The adjoint action of $SL_d(\mathbb{Z})$ leaves Λ invariant. Therefore, $SL_d(\mathbb{Z})$ also acts on A_d. Proposition 2.1 implies by Corollary B from [6] the following statement.

Proposition 2.2. *Let μ be a probability measure on $SL_d(\mathbb{Z})$ with finite generating support. Let $x \in A_d$ be a non-rational point. Then the measures $\mu^{*k} * \delta_x$ converge as $k \to \infty$ in the weak* topology to the normalised Haar measure on A_d.*

We will be using Proposition 2.2 to prove the following claim.

Proposition 2.3. *Let μ be a probability measure on $SL_d(\mathbb{Z})$ with finite generating support. Let ν be a probability measure on A_d with no atoms at rational points. Then the measures $\mu^{*k} * \nu$ converge as $k \to \infty$ in the weak* topology to the normalised Haar measure on A_d.*

3. Proofs of Theorems 1.1, and 1.3

Proof of Theorem 1.1. Recall that we denote by $\Lambda = Mat_d^0(\mathbb{Z})$, and by $\Gamma = SL_d(\mathbb{Z})$. Let $B \subset \Lambda$ be a set of positive density with Furstenberg's system $X_B = (X, \eta, \sigma, \tilde{B})$ and such that the spectral measure of B has no atoms at non-trivial characters. We make the identification of the dual space of Λ with the torus A_d by corresponding for every $x \in A_d$ the character χ_x on Λ given by:

$$\chi_x(h) = \exp(2\pi i x \cdot h), \text{ for } h \in \Lambda.$$

Notice that the trivial character on Λ corresponds to the zero element o_{A_d} of A_d, and characters having finite torsion correspond to the rational points of A_d. Denote by ν the spectral measure of B, i.e., for every $h \in \Lambda$ we have

$$\langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle = \int_{A_d} \exp(2\pi i x \cdot h) d\nu(x). \tag{1}$$

By the assumptions of the theorem, ν has no atoms at the rational points. Then Γ acts on Λ by the conjugation. We will show that for every $h \in \Lambda$ there exists $g \in \Gamma$ such that

$$\widehat{\nu}(g^{-1}hg) = \langle 1_{\tilde{B}}, \sigma(g^{-1}hg) 1_{\tilde{B}} \rangle > 0.$$

This will imply the claim of the theorem by the first property of Furstenberg's system X_B. Assume, that on the contrary, that there exists $h \in \Lambda$ such that for all $g \in \Gamma$ we have

$$\widehat{\nu}(g^{-1}hg) = 0. \tag{2}$$

It follows from the first property of the spectral measure listed below that the equation 2 holds for a non-zero $h \in \Lambda$. By the assumptions on B, we know that

- $\nu(\{o_{A_d}\}) = \eta(\tilde{B})^2 > 0.$
- For every rational non-zero point $x \in A_d$ we have $\nu(\{x\}) = 0$.

Indeed, the second property is given to us by the assumptions. To prove the first one, notice that for any Følner sequence[3] (F_n) in Λ:

$$\frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle = \tag{3}$$

$$\int_{A_d} \frac{1}{|F_n|} \sum_{h \in F_n} \exp(2\pi i x \cdot h) d\nu(x) \to \nu(\{o_{A_d}\}), \text{ as } N \to \infty.$$

In the last transition, we have used Lebesgue's dominated convergence theorem and the easy claim that for any non-trivial character χ on Λ, and a Følner sequence (F_n) in Λ we have:

$$\frac{1}{|F_n|} \sum_{h \in F_n} \chi(h) \to 0, \text{ as } n \to \infty.$$

By ergodicity of Furstenberg's system and von-Neumann's ergodic theorem it follows that the left hand side of (3) satisfies

$$\frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle \to \eta(\tilde{B})^2, \text{ as } n \to \infty.$$

This finishes the proof of the second property of the spectral measure ν.

[3] A sequence of finite sets (F_n) in Λ is called Følner if it is asymptotically Λ-invariant, i.e. for every $h \in \Lambda$ we have $\frac{|F_n \cap (F_n + h)|}{|F_n|} \to 1$, as $n \to \infty$. For any countable abelian group the family of Følner sequences is non-empty.

Let μ be a probability measure on $SL_d(\mathbb{Z})$ having a finite generating support. By Proposition 2.3 the measures $\mu^{*k} * \nu$ converge as $k \to \infty$ in weak* topology to

$$\eta(\tilde{B}) \left(1 - \eta(\tilde{B})\right) m_{A_d} + \eta(\tilde{B})^2 \delta_{o_{A_d}},$$

where m_{A_d} stands for the normalised Haar measure on A_d. Notice that Γ also acts on A_d by $g \cdot x = g^t x (g^t)^{-1}$, for $g \in \Gamma$. The action of Γ on A_d and the adjoint action of Γ on Λ are related by the following:

$$(g \cdot x) \cdot h = x \cdot Ad(g)h, \text{ for every } g \in \Gamma, h \in Lambda, x \in A_d.$$

Since

$$\widehat{\mu^{*k} * \nu}(h) = \int_{A_d} \exp\left(2\pi i x \cdot h\right) d\left(\mu^{*k} * \nu\right)(x) =$$

$$\int_{A_d} \left(\sum_{g \in \Gamma} \exp\left(2\pi i (g \cdot x) \cdot h\right) \mu^{*k}(g)\right) d\nu(x) =$$

$$\sum_{g \in \Gamma} \left(\int_{A_d} \exp\left(2\pi i x \cdot \left(g^{-1} h g\right)\right) d\nu(x)\right) \mu^{*k}(g) = \sum_{g \in \Gamma} \widehat{\nu}(g^{-1} h g) \mu^{*k}(g).$$

Recall, we assumed that there exists a non-zero $h \in \Lambda$ such that $\widehat{\nu}(g^{-1}hg) = 0$, for all $g \in \Gamma$. Therefore, we have $\widehat{\mu^{*k} * \nu}(h) = 0$, for all $k \geq 1$. On other hand, since $\widehat{m_{A_d}}(h) = 0$, and $\widehat{\delta_{o_{A_d}}}(h) = 1$, we have:

$$\widehat{\mu^{*k} * \nu}(h) \to \eta(\tilde{B})^2 > 0, \text{ as } k \to \infty.$$

Thus, we have a contradiction. This finishes the proof of the theorem. $\square$

Proof of Theorem 1.3. Theorem 1.3 follows immediately from Theorem 1.1 by use of the following statement which will be proved in the next section.

Proposition 3.1. *Let $B \subset Mat_d^0(\mathbb{Z})$ be a non-periodic piecewise Bohr set corresponding to a Jordan measurable*[4] *open set in a finite-dimensional torus. There exists a spectral measure associated with B that does not have atoms at non-zero rational points of A_d.*

Indeed, let $B \subset \Lambda$ be a piecewise non-periodic Bohr set given by $B = \tau^{-1}(U) \cap T$, where $\tau : \Lambda \to \mathbb{T}^n$ is a homomorphism with a dense image, $U \subset \mathbb{T}^n$ is an open set, and $T \subset \Lambda$ is a set with $d^*(T) = 1$. Then U contains an open ball U_o, and $m_{\mathbb{T}^n}(\partial U_o) = 0$, where $m_{\mathbb{T}^n}$ denotes the Haar normalised measure on $\mathbb{T}^n$. Denote by $B' = \tau^{-1}(U_o) \cap T \subset B$. Then the statement of Theorem 1.3 for the non-periodic piecewise Bohr set B' follows from Proposition 3.1. The latter implies the statement of the Theorem for the set B.

$\square$

4. Proofs of Propositions 2.1, 2.3, and 3.1

4.1. **Proof of Proposition 2.1.** It is proved in [5] [Corollary 5.4] that the adjoint action of $SL_d(\mathbb{Z})$ on $Mat_d^0(\mathbb{R})$ is strongly irreducible. Therefore, it is remained to prove that there is at least one element of $SL_d(\mathbb{Z})$ which acts on $Mat_d^0(\mathbb{R})$ proximaly. The next claim finishes the proof of Proposition 2.1.

[4] A set A in a topological space X equipped with a measure m_X is *Jordan measurable* if $m_X(\partial A) = 0$, where $\partial A = \overline{A} \setminus \mathring{A}$.

Proposition 4.1. *The matrix*

$$B_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

acts (by conjugation) on $Mat_2^0(\mathbb{R})$ proximally. For $d \geq 3$, the matrix

$$B_d = \left[\begin{array}{cc|c} 1 & -1 & \mathbf{0}_{2\times(d-2)} \\ -1 & 2 & \\ \hline \mathbf{0}_{(d-2)\times(d-2)} & & Id_{(d-2)\times(d-2)} \end{array}\right]$$

acts proximally on $Mat_d^0(\mathbb{R})$.

Proof. It is straithforward to check that the operator $B_2 : Mat_2^0(\mathbb{R}) \to Mat_2^0(\mathbb{R})$ can be written in the matrix form as[5]

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of this operator is $\chi_{B_2}(\lambda) = (1-\lambda)(\lambda^2 - \lambda - 1)$. Since all eigenvalues are distinct by their absolute value, it follows that the operator acts proximally.

In the case $d \geq 3$, notice that the action of B_d on $Mat_d^0(\mathbb{R})$ is decomposed into 4 orthogonal spaces. The actions on the 2×2 upper left corner, $2\times(d-2)$ upper right corner, $(d-2)\times 2$ bottom left corner, and the identity action on the bottom right $(d-2)\times(d-2)$ corner. Correspondingly, the dimensions of the spaces are $4, 2\cdot(d-2), (d-2)\cdot 2$, and $(d-2)^2 - 1$.

The 4-dimensional left upper corner part can be written in the matrix form as

$$\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 4 & -1 & 2 \\ 1 & -1 & 1 & -1 \\ -1 & 2 & -1 & 2 \end{bmatrix}.$$

Its characteristic polynomial is $(\lambda-1)^2(\lambda^2 - 7\lambda + 1)$. Therefore there is a unique highest eigenvalue by the absolute value equal to $\frac{7+3\sqrt{5}}{2}$, and it has multiplicity one.

The operator B_d acts on the upper right corner in the following way

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_{d-2} & y_{d-2} \end{bmatrix}^t \to \begin{bmatrix} 2x_1 + y_1 & x_1 + y_1 \\ 2x_2 + y_2 & x_2 + y_2 \\ \dots & \dots \\ 2x_{d-2} + y_{d-2} & x_{d-2} + y_{d-2} \end{bmatrix}^t.$$

It is clear that it has two eigenvalues with multiplicity $d-2$. These eigenvalues correspond to the eigenvalues of the matrix

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

These eigenvalues are the roots of the characteristic polynomial of the matrix C which are $\frac{3\pm\sqrt{5}}{2}$.

[5]We use the identification between $Mat_2^0(\mathbb{R})$ and $\mathbb{R}^3$, by

$$\begin{bmatrix} x & y \\ z & -x \end{bmatrix} \to [x, y, z]^t$$

The operator B_d acts on the bottom left corner in the following way:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_{d-2} & y_{d-2} \end{bmatrix} \to \begin{bmatrix} x_1 - y_1 & -x_1 + 2y_1 \\ x_2 - y_2 & -x_2 + 2y_2 \\ \dots & \dots \\ x_{d-2} - y_{d-2} & -x_{d-2} + 2y_{d-2} \end{bmatrix}.$$

Therefore it has two eigenvalues of the matrix C^{-1} each one having multiplicity $d-2$. It is immediate to check that C^{-1} has the same characteristic polynomial as C, therefore the eigenvalues of the operator B_d acting on the bottom left corner are $\frac{3\pm\sqrt{5}}{2}$, each one having multiplicity $d-2$.

As the conclusion of the previous considerations we find the the eigenvalues of the operator B_d are $\frac{7+3\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 1, \frac{7-3\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$ with corresponding multiplicities equal to $1, 2(d-2), [(d-2)^2-1]+2, 1, 2(d-2)$. This implies that B_d acts proximally on $Mat_d^0(\mathbb{R})$. $\square$

4.2. **Proof of Proposition 2.3.** Let ν be a probability measure on A_d with no atoms at rational points, and let μ be a probability measure on $\Gamma = SL_d(\mathbb{Z})$ with a finite generating support. By Proposition 2.2 for every $x \in supp(\nu)$ the measures $\mu^{*k} * \delta_x$ converge in weak*-topology as $k \to \infty$ to the Haar measure on A_d. Let f be a continuous function on A_d. Then for every $x \in supp(\nu)$ we have that $f_k(x) := \int f d\left(\mu^{*k} * \delta_x\right) \to \int f$. We have to show that

$$\int_{A_d} f d\left(\mu^{*k} * \nu\right) \to \int f.$$

By Egorov's theorem, for every $\varepsilon > 0$, there exists $X' \subset A_d$ with $\nu(X') \geq 1 - \varepsilon$ and $K(\varepsilon)$ with the property that for every $x \in X'$ and every $k \geq K(\varepsilon)$ we have

$$\left| f_k(x) - \int f \right| < \varepsilon.$$

Notice that

$$\int f d\mu^{*k} * \nu = \sum_{g\in\Gamma} \int f(gx)\mu^{*k}(g) d\nu(x) = \int \left(\sum_{g\in\Gamma} f(gx)\mu^{*k}(g) \right) d\nu(x)$$
$$= \int \left(\int f d\left(\mu^{*k} * \delta_x\right) \right) d\nu(x).$$

Let $\delta > 0$. Denote by $M = \|f\|_\infty$, and take $\varepsilon > 0$ so small that $\varepsilon M < 2\delta$, and $\varepsilon < \delta$. Then we have

$$\left| \int f d\left(\mu^{*k} * \nu\right) - \int f \right| < (1-\varepsilon)\varepsilon + \varepsilon M < 3\delta,$$

for $k \geq K(\varepsilon)$. Since δ can be chosen arbitrary small, we have shown that

$$\int f d\left(\mu^{*k} * \nu\right) \to \int f.$$

This finishes the proof because the function f was an arbitrary continuous function on A_d.

4.3. **Proof of Proposition 3.1.** Recall that $\Lambda = Mat_d^0(\mathbb{Z})$. We are given a piecewise Bohr non-periodic set $B \subset \Lambda$ corresponding to a Jordan measurable open set in a finite dimensional torus. This means that $B = B_o \cap T$, where $T \subset \Lambda$ with $d^*(T) = 1$, and $B_o \subset \Lambda$ given via a homomorphism $\tau : \Lambda \to \mathbb{T}^n$, for some $n \geq 1$ with the dense image, and an open Jordan measurable set $U_o \subset \mathbb{T}^n$ such that

$$B_o = \tau^{-1}(U_o).$$

We will construct an ergodic Furstenberg's Λ system $X_B = (X, \eta, \sigma, \tilde{B})$ corresponding to the set B, and will show that the spectral measure of the function $1_{\tilde{B}}$ has no atoms at the rational non-zero points of $A_d := \widehat{\Lambda}$.

Let $X = \mathbb{T}^n$, η be the Haar normalised measure on X, $\sigma_h(x) := x + \tau(h)$ for $x \in X, h \in \Lambda$, and $\tilde{B} = U_o$. We will denote by $X_B := (X, \eta, \sigma, \tilde{B})$. It remains to show that

- For every $h \in \Lambda$ we have $d^*\left(B \cap (B+h)\right) \geq \eta(\tilde{B} \cap \sigma_h(\tilde{B}))$.
- $\eta(\tilde{B}) = d^*(B)$.
- The spectral measure of $1_{\tilde{B}}$ has no atoms at non-zero rational points of A_d.

The first two properties will follow from the statement that for every $h \in \Lambda$:

$$d^*\left(B \cap (B+h)\right) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})).$$

First, notice that for every $h \in \Lambda$ the set $U_o \cap \sigma_h(U_o)$ is Jordan measurable. By unique ergodicity of X_B, for every Følner sequence (F_k) in Λ and any $h \in \Lambda$ we have

$$\frac{1}{|F_k|} \sum_{h \in F_k} \frac{|B_o \cap (B_o + h) \cap F_k|}{|F_k|} \to \int_X 1_{U_o \cap \sigma_h(U_o)}(x) d\eta(x) = \eta\left(\tilde{B} \cap \sigma_h(\tilde{B})\right), \text{ as } k \to \infty.$$

The latter will imply that for every $h \in \Lambda$

$$\eta\left(\tilde{B} \cap \sigma_h(\tilde{B})\right) \geq d^*(B \cap (B+h)).$$

On the other hand, for any Følner sequence (F_k) which lies inside the thick set T we will have for every $h \in \Lambda$ by a similar argument as before

$$\frac{1}{|F_k|} \sum_{h \in F_k} \frac{|B \cap (B + h) \cap F_k|}{|F_k|} \to \int_X 1_{U_o \cap \sigma_h(U_o)}(x) d\eta(x) = \eta\left(\tilde{B} \cap \sigma_h(\tilde{B})\right), \text{ as } k \to \infty.$$

This establishes that for every $h \in \Lambda$:

$$d^*\left(B \cap (B+h)\right) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})).$$

It remains to prove that the spectral measure corresponding to $1_{\tilde{B}}$ and the system X_B has no atoms at non-zero rational points of A_d. We will be abusing the notation and will also use T to denote the Koopman operator on $L^2(X)$ corresponding to σ. Let us list two important properties of the system X_B:

(1) X_B is *totally ergodic*, i.e., every subgroup $H < \Lambda$ of a finite index acts ergodically on X_B.

(2) For $f \geq 0, f \in L^2(X)$, the spectral measure μ_f of f defined by

$$\widehat{\mu_f}(h) := \int_{A_d} \exp(2\pi h \cdot x) d\mu_f(x) = \langle f, T_h f \rangle$$

is non-negative.

The first property follows from Lemma 4.2, while the second property is standard fact, see [7]. To prove Lemma 4.2 we will need the following result.

Lemma 4.1. *Let $H < \Lambda$ be a subgroup of a finite index. Then for every point $x \in X$, the H-orbit of x, i.e., $\{\sigma_h(x) \,|\, h \in H\}$, is dense in X.*

Proof. Lemma follows from two facts that utilise the connectivity of X, and Baire Category theorem:

- For every subgroup $H < \Lambda$ the closed subgroup $\overline{\tau(H)} < X$ nowhere dense.
- Finite union of nowhere dense sets cannot cover X.

□

Lemma 4.2. *Let $H < \Lambda$ be a subgroup of a finite index. The restriction of the Λ-action of X to H is uniquely ergodic.*

Proof. It follows from Lemma 4.1 that any H-invariant Borel probability measure on X is also X-invariant. The uniqueness of the Haar normalised measure on X implies the statement of the lemma. $\square$

Let $f \in L^2(X)$, then by the ergodicity of X_B (property (1)) it follows that for any Følner sequence $(F_k)_{k\geq 1}$ of finite sets in Λ we have

$$\frac{1}{|F_k|}\sum_{h\in F_k}\langle f, T_h f\rangle \to |\langle f, 1\rangle|^2\,,\ \text{as } k\to\infty.$$

On the other hand, it is a standard identity:

$$\frac{1}{|F_k|}\sum_{h\in F_k}\exp\left(2\pi i h\cdot x\right) \to \left[\begin{array}{ll} 1, & x = o_{A_d} \\ 0, & x \neq o_{A_d}. \end{array}\right.$$

It follows from Lebesgue's dominated convergence theorem that

$$|\langle f, 1\rangle|^2 = \mu_f(o_{A_d}). \tag{4}$$

Let $x_0 \in A_d$ be a non-zero rational point with the least common denominator equal to q. Then the stabiliser of x_0 in Λ is $H_{x_0} = q\Lambda$. Using the ergodicity of H_{x_0} action on X_B (property (1)), we obtain

$$\frac{1}{|F_k|}\sum_{h\in F_k}\langle f, T_{qh} f\rangle \to |\langle f, 1\rangle|^2\,,\ \text{as } k\to\infty.$$

On the other hand, we have

$$\frac{1}{|F_k|}\sum_{h\in F_k}\exp\left(2\pi i h\cdot (qx)\right) \to \left[\begin{array}{ll} 1, & qx = o_{A_d} \\ 0, & qx \neq o_{A_d}. \end{array}\right.$$

Therefore, by Lebesgue's dominated convergence theorem we obtain

$$|\langle f, 1\rangle|^2 = \sum_{qx = o_{A_d}} \mu_f(\{x\}). \tag{5}$$

If we know in addition that $f \geq 0$, then by property (2), the spectral measure μ_f is non-negative. Therefore, by use of equations (4) and (5) we get that for all non-zero points $x \in A_d$ with $qx = o_{A_d}$ we have

$$\mu_f(\{x\}) = 0.$$

In particular, we have that $\mu_f(\{x_0\}) = 0$. This finishes the proof of Proposition 2.3, if we choose $f = 1_{\tilde{B}}$.

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