A LIOUVILLE THEOREM FOR p-HARMONIC FUNCTIONS ON EXTERIOR DOMAINS

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ABSTRACT. We prove Liouville type theorems for p-harmonic functions on exterior domains of $\mathbb{R}^d$, where $1 < p < \infty$ and $d \geq 2$. We show that every positive p-harmonic function satisfying zero Dirichlet, Neumann or Robin boundary conditions and having zero limit as $|x|$ tends to infinity is identically zero. In the case of zero Neumann boundary conditions, we establish that any semi-bounded p-harmonic function is constant if $1 < p < d$. If $p \geq d$, then it is either constant or it behaves asymptotically like the fundamental solution of the homogeneous p-Laplace equation.

1. INTRODUCTION AND MAIN RESULTS

Assume that Ω is a general *exterior domain* of $\mathbb{R}^d$, that is, a connected open set such that $\Omega^c = \mathbb{R}^d \setminus \Omega$ is compact and nonempty. We assume that the boundary $\partial\Omega$ is the disjoint union of the sets Γ_1, Γ_2, where Γ_1 is closed. We denote by ν the outward pointing unit normal vector on $\partial\Omega$ and $\mathcal{H}$ the $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$. For $1 < p < \infty$ define the p-Laplace operator Δ_p by $\Delta_p v := \operatorname{div}(|\nabla v|^{p-2}\nabla v)$.

The aim of this paper is to establish a Liouville theorem for weak solutions of the elliptic boundary-value problem

$$\begin{aligned} -\Delta_p v &= 0 \quad \text{in } \Omega, \\ \mathcal{B}v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$\mathcal{B}v := \begin{cases} v|_{\Gamma_1} & \text{on } \Gamma_1 \text{ (Dirichlet b.c.)}, \\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} + h(x,v) & \text{on } \Gamma_2 \text{ (Robin/Neumann b.c.)}. \end{cases}$$

Here we assume that $h\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (see [23]) satisfying

$$h(\cdot,v) \in L^{p/(p-1)}(\Gamma_2) \quad \text{and} \quad h(x,v)v \geq 0 \text{ for } \mathcal{H}\text{-a.e. } x \in \Gamma_2, \tag{1.2}$$

for every $v \in L^p(\Gamma_2)$. Note that the first condition in (1.2) implicitly implies a growth condition on the function $v \mapsto h(\cdot,v)$; see [17]. As usual, a function $v \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ is said to be *p-harmonic* (or simply *harmonic* if $p = 2$) on Ω if $\Delta_p v = 0$ in Ω in the weak sense, that is,

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi \, dx = 0$$

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for every $\varphi \in C_c^\infty(\Omega)$; see [20]. Throughout, we call a p-harmonic function v *positive* if $v \geq 0$.

The classical Liouville theorem asserts that every harmonic function on the whole space $\mathbb{R}^d$ is constant if it is bounded from below or from above; see for instance [3, Theorem 3.1] or [19, p. 111]. The classical Liouville theorem was generalised to p-harmonic functions on the whole space $\mathbb{R}^d$ for $1 < p < \infty$; see [22, Theorem II] or [15, Corollary 6.11]. The result extends to d-harmonic function on $\mathbb{R}^d \setminus \{0\}$ for $d \geq 2$; see [3, Corollary 3.3] for $p = d = 2$ or [16, Corollary 2.2] for $p = d \geq 2$. In our investigation, the *fundamental solution*

$$\mu_p(x) := \begin{cases} |x|^{(p-d)/(p-1)} & \text{if } p \neq d, \\ \log|x| & \text{if } p = d. \end{cases} \tag{1.3}$$

on $\mathbb{R}^d \setminus \{0\}$ plays an important role. If $1 < p < d$, then μ_p provides an example of a non-constant p-harmonic function bounded from below. Another example valid for $1 < p < d$ is given by $v(x) := 1 - \mu_p(x)$ for every $x \in \mathbb{R}^d \setminus B_1$, where B_1 denotes the open unit ball. In this case v is a positive p-harmonic function on the exterior domain $\Omega := \mathbb{R}^d \setminus \overline{B}_1$ satisfying zero Dirichlet boundary conditions at $\partial\Omega$. Hence, in order to have a chance of proving a Liouville type theorem for exterior domains we need to make use of the boundary conditions and the behavior of a p-harmonic function near infinity.

First, we consider the case $1 < p < d$. Then by [21, Corollary, p.84] or [2, Theorem 2 & Theorem 3] and by rescaling if necessary, we know that for every positive p-harmonic function v on an exterior domain $\Omega \subseteq \mathbb{R}^d$, the limit

$$b := \lim_{|x|\to\infty} v(x) \quad \text{exists and} \quad |v(x) - b| \leq c_1\, \mu_p(x) \text{ whenever } |x| \geq 2 \tag{1.4}$$

where $c_1 > 0$. With this in mind, our first result is a kind of maximum principle for weak solution of (1.1) on an unbounded domain. A precise definition of weak solutions of (1.1) is given in Definition 3.5 below.

Theorem 1.1. *Let Ω be an exterior domain with Lipschitz boundary and let $1 < p < \infty$. Suppose that* (1.2) *is satisfied and that v is a positive weak solution of* (1.1) *such that $\lim_{|x|\to\infty} v(x) = 0$. Then $v \equiv 0$.*

If $p \geq d$, the conclusion of the theorem is valid without any restrictions on the boundary conditions or regularity of Ω due to a result in [12]; see Section 4. If $1 < p < d$, then under some additional assumptions on v we can remove the assumption that $\partial\Omega$ is Lipschitz. The condition is that v has a trace in some weak sense which is in $L^p(\Gamma_2)$. Such a condition is satisfied in the setting discussed in [6, 1, 10, 11].

The proof of Theorem 1.1 relies on the asymptotic decay estimates for positive p-harmonic functions on exterior domains as stated in (1.4). We give a simple alternative proof of such estimates in case $p = 2$ and $d \geq 3$ in Section 2.

If $p \geq d$, then there are two alternatives for a positive p-harmonic function v: Either v is bounded in a neighbourhood of infinity and has a limit as $|x| \to \infty$, or $v \sim \mu_p$ near infinity, that is,

$$\lim_{|x|\to\infty} \frac{v(x)}{\mu_p(x)} = c$$

for some constant $c > 0$; see [12, Theorem 2.3]. In the first case, if $v > 0$, then the limit is strictly positive; see [12, Lemma A.2]. See also the related work in [13]. As an example let $\Omega := \mathbb{R}^d \setminus \overline{B}_1$ and set $v_p := \mu_p + 1$ if $p = d$ and $v_p := \mu_p$ if $p > d$. Then v_p is a positive p-harmonic function on Ω satisfying zero Robin boundary conditions

$$|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} + |v|^{p-2}v = 0 \qquad \text{on } \partial\Omega.$$

Similarly, $w_p := \mu_p$ if $p = d$ and $w_p := \mu_p - 1$ if $p > d$ satisfies zero Dirichlet boundary conditions on $\partial\Omega$ and is a positive unbounded p-harmonic function on Ω.

Our second main result is a Liouville theorem for p-harmonic functions on exterior domains with zero Neumann boundary conditions, that is, the case $\Gamma_2 = \partial\Omega$ and $h \equiv 0$.

Theorem 1.2. *Let Ω be an exterior domain with no regularity assumption on $\partial\Omega$. Suppose that v is a weak solution of (1.1) on Ω that is bounded from below or from above. Moreover, assume that v satisfies homogeneous Neumann boundary conditions, that is, $h(x, v) \equiv 0$ and $\Gamma_2 = \partial\Omega$. If $1 < p < d$, then v is constant. If $p \geq d$, then v is either constant or $v \sim \pm\mu_p$ near infinity.*

The proofs of the theorems are based on a general criterion for Liouville type theorems established in Section 3. We fully prove the two Theorems in Section 4.

There is an intimate relationship between Liouville-type theorems and pointwise a priori estimates of solutions of boundary value problems. On the one hand, Liouville's theorem for some semi-linear equations on $\mathbb{R}^d$ can be seen as a corollary of pointwise a priori estimates; see [8, Lemma 1]. On the other hand, Liouville's theorem can be used to derive universal upper bounds for positive solutions on bounded domains. These connections were outlined in [22, p.82] and recently revisited in [18]. More precisely, it is shown in [18, p.556] that Liouville's theorem and universal boundedness theorems are equivalent for semi-linear equations and systems of Lane-Emden type; see also [16]. This relationship becomes again apparent in this paper. This article was motivated by application to domain perturbation problems for semi-linear elliptic boundary value problems on domains with shrinking holes; see [9].

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2. Estimates near infinity in the linear case

In this section we establish pointwise decay estimates for semi-bounded harmonic functions on an exterior domain $\Omega \subseteq \mathbb{R}^d$ when $d \geq 3$. The result is a special case of estimates proved in [2], but it seems appropriate to provide a much shorter proof in the linear case.

Proposition 2.1. *Let v be harmonic on the exterior domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$. Further assume that v is bounded from below or from above. Then $b := \lim_{|x|\to\infty} v(x)$ exists. Moreover, there exist positive constants r_0 and C_1 such that*

$$|v(x) - b| \leq C_1 \, |x|^{2-d} \tag{2.1}$$

for all $|x| \geq r_0$.

To prove the proposition let v be a harmonic function on the exterior domain Ω, and suppose that v is bounded form below or from above. Since by assumption $\overline{\Omega}^c$ is bounded, by translating and rescaling if necessary, we can assume without loss of generality that $\overline{\Omega}^c$ is contained in the unit ball B_1, and that $0 \in \overline{\Omega}^c$. Furthermore, without loss of generality we can consider non-negative harmonic functions on Ω. Indeed, if v is bounded from below we consider $v + \inf v \geq 0$, and if v is bounded from above we consider $-v + \sup v \geq 0$.

If v is positive and harmonic on Ω, then in particular v is positive and harmonic on $\overline{B}_1^c$. Hence, the Kelvin transform $K[v]$ of v given by $K[v](x) := |x|^{2-d} v(x/|x|^2)$ for $x \in B_1 \setminus \{0\}$ is positive and harmonic on $B_1 \setminus \{0\}$; see [3, Theorem 4.7]. By Bôcher's theorem there exist a harmonic function w on B_1 and a constant $b \geq 0$ such that

$$K[v](x) = w(x) + b\,|x|^{2-d} \qquad \text{or} \qquad K[v-b](x) = w(x)$$

for every $x \in B_1 \setminus \{0\}$ see [3, Theorem 3.9]. Applying the Kelvin transform again yields

$$v(x) - b = K[w](x) = |x|^{2-d}\, w(x/|x|^2) \tag{2.2}$$

for every $x \in \overline{B}_1^c$. Note that $w(x/|x|^2) \to w(0)$ as $|x| \to \infty$. Hence (2.2) implies the existence of constants $C_1, r_0 > 0$ such that (2.1) holds whenever $|x| > r_0$.

3. A GENERAL CRITERION FOR LIOUVILLE TYPE THEOREMS

The proofs of the main theorems are based on a general criterion showing that a function satisfying suitable integral conditions is constant. We generalise an idea from [5, Lemma 2.1]. Similar ideas were used for instance in [4, 22] or in [7, Theorem 19.8] for $p = 2$.

Proposition 3.1. *Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ on $\mathbb{R}^d$, $\varphi \equiv 1$ on $\overline{B}(0,1)$ and with support contained in $\overline{B}(0,2)$. For $x \in \mathbb{R}^d$ and $r > 0$ let $\varphi_r(x) := \varphi(x/r)$. Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and suppose that there exist constants $b \in \mathbb{R}$ and $C_0, C_1, r_0 > 0$ such that*

$$\begin{aligned}\int_{\Omega\cap B_{2r}} |\nabla v|^p \,\varphi_r^p \,\mathrm{d}x \\ \leq \frac{C_0}{r} \left(\int_{(\Omega\cap B_{2r})\setminus B_r} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x \right)^{(p-1)/p} \left(\int_{(\Omega\cap B_{2r})\setminus B_r} |v-b|^p \,\mathrm{d}x \right)^{1/p}\end{aligned} \tag{3.1}$$

and

$$\frac{1}{r^p} \int_{(\Omega\cap B_{2r})\setminus B_r} |v-b|^p \,\mathrm{d}x \leq C_1 \tag{3.2}$$

for all $r > r_0$. Then v is constant.

The above proposition is a direct consequence of the following stronger result. To prove Proposition 3.1, we set $C := C_0 C_1^{1/p}$ and $\delta = (p-1)/p$ in the lemma below. Then inequality (3.3) follows from (3.1) and (3.2).

Lemma 3.2. *Let φ and φ_r be as in Proposition 3.1. Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and suppose that there exist constants $C, r_0 > 0$ and $\delta \in (0,1)$ such that*

$$\int_{\Omega\cap B_{2r}} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x \le C \left(\int_{(\Omega\cap B_{2r})\setminus B_r} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x \right)^{\delta} \tag{3.3}$$

for all $r > r_0$. Then v is constant.

Proof. In a first step we show that $\nabla v \in L^p(\Omega)^d$. In a second step we then prove that $\nabla v \in L^p(\Omega)^d$ and (3.3) imply that $\nabla v = 0$. As Ω is assumed to be connected we can apply [14, Lemma 7.7] to conclude that v is constant on Ω.

(i) We first show that $\nabla v \in L^p(\Omega)^d$. If $\nabla v = 0$, then there is nothing to show, so assume that $\nabla v \ne 0$. By possibly increasing r_0 we can assume that

$$\int_{\Omega\cap B_{2r}} |\nabla v|^p \varphi_r^p \,\mathrm{d}r > 0$$

for all $r > r_0$. Rearranging inequality (3.3) and using that $\delta < 1$ yields

$$\int_{\Omega} |\nabla v|^p \varphi_r^p \,\mathrm{d}x = \int_{\Omega\cap B_{2r}} |\nabla v|^p \varphi_r^p \,\mathrm{d}x \le C^{1/(1-\delta)}$$

for all $r > r_0$. Note that $\varphi_r^p \to 1_{\mathbb{R}^d}$ pointwise and monotonically increasing as $r \to \infty$. Hence, the monotone convergence theorem implies that

$$\int_{\Omega} |\nabla v|^p \,\mathrm{d}x = \lim_{r\to\infty} \int_{\Omega\cap B_{2r}} |\nabla v|^p \varphi_r^p \,\mathrm{d}x \le C^{1/(1-\delta)} < \infty. \tag{3.4}$$

In particular $\nabla v \in L^p(\Omega)^d$ as claimed.

(ii) Assuming that $\nabla v \in L^p(\Omega)^d$ we now show that $\nabla v = 0$. We can rewrite (3.3) in the form

$$\int_{\Omega\cap B_{2r}} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x \le C \left(\int_{\Omega} |\nabla v|^p \,\mathrm{d}x - \int_{\Omega\cap B_r} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x \right)^{\delta}.$$

Letting $r \to \infty$, making use of (3.4) and the fact that $\delta > 0$, we deduce that $\|\nabla v\|_p \le 0$, that is, $\|\nabla v\|_p = 0$. ☐

Remark 3.3. Suppose that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies inequality (3.1), and that there exists $r_0 > 0$ such that $v \in L^\infty(\Omega \cap B^c_{r_0})$. Then, for every $b \in \mathbb{R}$

$$\frac{1}{r^p} \int_{B_{2r}\setminus B_r} |v-b|^p \,\mathrm{d}x \le \frac{\|v-b\|_\infty^p}{r^p} \int_{B_{2r}\setminus B_r} 1 \,\mathrm{d}x \le \frac{\omega_d}{d}(2^d - 1)\|v-b\|_\infty^p r^{d-p}$$

for all $r \ge r_0$, where $\|v-b\|_\infty := \|v-b\|_{L^\infty(B^c_{r_0})}$ and ω_d is the surface area of the unit sphere in $\mathbb{R}^d$. If $p \ge d$, then Proposition 3.1 implies that v is constant.

We next show that weak solutions of (1.1) satisfy (3.1). Before we can do that we want to state our precise assumptions and give a definition of weak solutions of boundary-value problem (1.1).

Assumption 3.4. By assumption, an exterior domain Ω as defined in the introduction is an open connected set such that Ω^c is compact. In particular $\partial\Omega$ is compact. Thus there exists $r_0 > 0$ such that $\partial\Omega \subseteq B_r := B(0,r)$ for all $r \geq r_0$. We consider solutions of (1.1) that lie in

$$V^{1,p}(\Omega) := \Big\{ v \in W^{1,p}_{\mathrm{loc}}(\Omega) \colon v \in W^{1,p}(\Omega \cap B_r) \text{ for all } r > r_0 \Big\}.$$

For simplicity we now assume that $\partial\Omega$ is Lipschitz. We assume that Γ_1, Γ_2 are disjoint subsets of $\partial\Omega$ such that Γ_1 is closed and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. We let $V^{1,p}_{\Gamma_1}(\Omega)$ be the closure of the vector space

$$\Big\{ v \in V^{1,p}(\Omega) \colon v = 0 \text{ in a neighbourhood of } \Gamma_1 \Big\}$$

in $V^{1,p}(\Omega)$. If $h(x,v) \equiv 0$ no regularity assumption on $\partial\Omega$ is needed.

We use the space $V^{1,p}(\Omega)$ because we do not want to assume that the solutions of (1.1) are in $L^p(\Omega)$.

Definition 3.5. We say that a function v is a *weak solution* of the boundary value problem (1.1) on Ω if $v \in V^{1,p}_{\Gamma_1}(\Omega)$ and

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi \, \mathrm{d}x + \int_{\Gamma_2} h(x,v)\, \varphi \, \mathrm{d}\mathcal{H} = 0 \tag{3.5}$$

for every $\varphi \in V^{1,p}_{\Gamma_1}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq B_r$.

The above definitions have to be modified in an obvious manner for non-smooth domains. In particular, when using the setting from [1, 10, 11] we require that v is in the Maz'ya space $W^1_{p,p}(\Omega \cap B_r, \partial\Omega)$ for all r large enough.

If Ω admits the divergence theorem and the solution v is smooth enough, then an integration by parts shows that v is a weak solution of (1.1) if and only if v satisfies (1.1) in a classical sense. We next show that positive solutions of (1.1) satisfy (3.1).

Proposition 3.6. *Let Assumption 3.4 be satisfied and let φ_r be as in Proposition 3.1, and $r_0 > 0$ such that $\Omega^c \subseteq B_{r_0}$. Suppose that (1.2) is satisfied and that v is a weak solution of (1.1). Then, inequality (3.1) holds with $b = 0$. In the case of homogeneous Neumann boundary conditions, that is, if $h(x,v) \equiv 0$ and $\Gamma_2 = \partial\Omega$, then every weak solution of (1.1) satisfies (3.1) for every $b \in \mathbb{R}$.*

Proof. Let $r \geq r_0$ and let φ_r be the same test-function as in Proposition 3.1. Then $v\varphi_r^p \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with support in B_{2r}. Moreover, by definition of φ_r we have $v\varphi^p = v$ on $\Omega \cap B_r$. Hence $v\varphi^p$ is a suitable test function to be used in

(3.5). Using that v is a weak solution of (1.1) gives

$$\begin{aligned} 0 &= \int_{\Omega\cap B_{2r}} |\nabla v|^{p-2}\nabla v \nabla (v\varphi_r^p)\,\mathrm{d}x + \int_{\Gamma_2} h(x,v)\, v\, \varphi_r^p \,\mathrm{d}\mathcal{H} \\ &= \int_{\Omega\cap B_{2r}} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x + \frac{p}{r}\int_{(\Omega\cap B_{2r})\setminus B_r} v\varphi_r^{p-1}|\nabla v|^{p-2}\nabla v\nabla\varphi(\cdot/r)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_2} h(x,v)\, v\, \varphi_r^p\,\mathrm{d}\mathcal{H}. \end{aligned}$$

Rearranging this equation we arrive at

$$\begin{aligned} \int_{\Omega\cap B_{2r}} |\nabla v|^p\, \varphi_r^p \,\mathrm{d}x &+ \int_{\Gamma_2} h(x,v)\, v\, \varphi_r^p\,\mathrm{d}\mathcal{H} \\ &= -\frac{p}{r}\int_{(\Omega\cap B_{2r})\setminus B_r} v\varphi_r^{p-1}|\nabla v|^{p-2}\nabla v\nabla\varphi(\cdot/r)\,\mathrm{d}x. \end{aligned}$$

By assumption (1.2) we have $h(x,v)v \geq 0$. Setting $C_0 := p\,\|\nabla\varphi\|_{L^\infty(B_2)}$ and applying Hölder's inequality we obtain

$$\begin{aligned} &\int_{\Omega\cap B_{2r}} |\nabla v|^p\, \varphi_r^p\,\mathrm{d}x \\ &\quad\leq \frac{C_0}{r}\left(\int_{(\Omega\cap B_{2r})\setminus B_r} |\nabla v|^p\, \varphi_r^p\,\mathrm{d}x\right)^{(p-1)/p} \left(\int_{(\Omega\cap B_{2r})\setminus B_r} |v|^p\,\mathrm{d}x\right)^{1/p}, \end{aligned}$$

which is (3.1) with $b = 0$. In the case of homogeneous Neumann boundary conditions, for every $b \in \mathbb{R}$, the function $v - b$ is another weak solution of (1.1). Hence we can replace v by $v - b$ in the above calculations to obtain (3.1). ☐

Remark 3.7. Note that the above proof only uses that

$$0 \leq \int_{\Gamma_2} h(x,v)v\varphi_r^p\,\mathrm{d}\mathcal{H} = \int_{\Gamma_2} h(x,v)v\,\mathrm{d}\mathcal{H} < \infty.$$

4. Proofs of the main theorems

This section is dedicated to the proofs of Theorems 1.1 and 1.2. By rescaling, we can assume without loss of generality that $\Omega^c \subseteq B_1$ and that v is p-harmonic on $\overline{B}_1^c$.

4.1. **Proof of Theorem 1.1.** Assume that $1 < p < d$, and that v is a positive weak solution of (1.1) satisfying $\lim_{|x|\to\infty} v(x) = 0$. We need to show that $v \equiv 0$. Due to Propositions 3.1 and 3.6 we only need to show that there exists $r_0 > 0$ such that v satisfies (3.2) with $b = 0$ for all $r \geq r_0$. By (1.4) or Proposition 2.1 if $p = 2$, there are constants $c_1, c_2 > 0$ such that

$$0 \leq v(x) \leq c_1\,|x|^{(p-d)/(p-1)}$$

for every $x \in \overline{B_2^c}$. Hence,

$$\begin{aligned} \frac{1}{r^p}\int_{B_{2r}\setminus B_r} |v|^p\,\mathrm{d}x &\leq \frac{c_1^p}{r^p}\int_{B_{2r}\setminus B_r} |x|^{p\,(p-d)/(p-1)}\,\mathrm{d}x \\ &= c_1^p\frac{\omega_d}{r^p}\int_r^{2r} s^{p\,(p-d)/(p-1)}s^{d-1}\,\mathrm{d}s = c_1^p\omega_d\, c_2\, r^{(p-d)/(p-1)} \end{aligned} \tag{4.1}$$

for all $r \geq r_0 := 2$, where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$ and $c_2 = \ln 2$ if $d = p^2$ and $c_2 = \frac{p-1}{d-p^2}(2^{(p^2-d)/(p-1)} - 1)$ if $d \neq p^2$. As $p < d$ we conclude that v satisfies (3.2) with $b = 0$ for every $r \geq 2$. As $\lim_{|x|\to\infty} v(x) = 0$ we conclude that $v \equiv 0$.

If $p \geq d$, then every non-trivial positive bounded solution of (1.1) has a strictly positive limit as $|x| \to \infty$; see [12, Lemma A.2]. Because we assume that the limit is zero, we must have $v \equiv 0$. Observe that these arguments do not make use of the boundary conditions. This completes the proof of Theorem 1.1.

4.2. **Proof of Theorem 1.2.** Let v be a semi-bounded weak solution of problem (1.1) with homogeneous Neumann boundary conditions, that is, $\Gamma_2 = \partial\Omega$ and $h(x, v) \equiv 0$. Recall also that no regularity assumptions on $\partial\Omega$ are needed.

Note that the p-Laplace operator Δ_p is an *odd* operator, that is, $\Delta_p(-v) = -\Delta_p v$. Hence, for every $c \in \mathbb{R}$, the function $c \pm v$ is another solution of problem (1.1). If v is bounded from below we can therefore replace v by $v - \inf_{x\in\Omega} v(x)$, and if v is bounded from above we can replace v by $\sup_{x\in\Omega} v(x) - v$. In either case we get a new solution $v \geq 0$ with $\inf_{x\in\Omega} v(x) = 0$. As before, we also assume that $\Omega^c \subseteq B_1$.

If $1 < p < d$, then by (1.4) the finite limit $b := \lim_{|x|\to\infty} v(x)$ exists. By Proposition 3.6 inequality (3.1) is satisfied. To show that v satisfies (3.2) with b just defined we repeat the calculation (4.1) with v replaced by $v - b$, using the decay estimate from (1.4). We can now apply Proposition 3.1 to conclude that v is constant.

It remains to deal with the case $p \geq d$. Recall that by [12, Theorem 2.3], every positive p-harmonic function v on Ω is either bounded in a neighbourhood of infinity and has a limit $b := \lim_{|x|\to\infty} v(x)$ or $v \sim \mu_p$ near infinity. In the second case the original solution considered is asymptotically equivalent to $\pm\mu_p$ near infinity. Assume now that v has a limit as $|x| \to \infty$. To show that v is constant we first note that by Proposition 3.6, v satisfies (3.1) with $b = 0$. As v is bounded in a neighbourhood of infinity and since $p \geq d$, Remark 3.3 implies that v satisfies (3.2). Hence by Proposition 3.1, v is constant. This completes the proof of Theorem 1.2.

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