Explicit generators in rectangular affine $\mathcal{W}$-algebras of type A

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Abstract

We produce in an explicit form free generators of the affine $\mathcal{W}$-algebra of type A associated with a nilpotent matrix whose Jordan blocks are of the same size. This includes the principal nilpotent case and we thus recover the quantum Miura transformation of Fateev and Lukyanov.

1 Main results

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ equipped with a symmetric invariant bilinear form κ and let f be a nilpotent element of $\mathfrak{g}$. The corresponding *affine $\mathcal{W}$-algebra* $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$ is defined by the generalized quantized Drinfeld–Sokolov reduction; see [5], [7] and [8].

In this note we take $\mathfrak{g} = \mathfrak{gl}_N$. The Jordan type of a nilpotent element $f \in \mathfrak{gl}_N$ is a partition of N. We will work with the elements f corresponding to partitions of the form (l^n) so that the associated Young diagram is the $n \times l$ rectangle with $nl = N$. Our main result is an explicit construction of free generators of the $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$. Moreover, we calculate the images of these generators with respect to the *Miura transformation*. In particular, if f is the principal nilpotent (i.e., $n = 1$) we thus reproduce the description of the $\mathcal{W}$-algebra due to Fateev and Lukyanov [4]. The results can be regarded as 'affine analogues' of the construction of the corresponding *finite $\mathcal{W}$-algebras* originated in [2], [10] and extended to arbitrary nilpotent elements f in [3].

To describe the results in more detail, identify $\mathfrak{g}$ with the tensor product of $\mathfrak{gl}_l$ and $\mathfrak{gl}_n$ via the isomorphism $\mathfrak{gl}_l \otimes \mathfrak{gl}_n \to \mathfrak{g}$ defined by

$$e_{ij} \otimes e_{rs} \mapsto e_{(i-1)n+r,\,(j-1)n+s}, \tag{1.1}$$

where the e_{ij} denote the standard basis elements of the corresponding general linear Lie algebras. Set

$$f_l = \sum_{i=1}^{l-1} e_{i+1\,i} \in \mathfrak{gl}_l$$

and

$$f = f_l \otimes I_n = \sum_{i=1}^{l-1} \sum_{j=1}^{n} e_{in+j,\,(i-1)n+j} \in \mathfrak{g},$$

where $I_n \in \mathfrak{gl}_n$ is the identity matrix. The matrix f is a nilpotent element of $\mathfrak{g}$ of Jordan type (l^n). Let

$$\mathfrak{gl}_l = \bigoplus_{p\in\mathbb{Z}} (\mathfrak{gl}_l)_p$$

be the standard principal grading of $\mathfrak{gl}_l$, obtained by defining the degree of e_{ij} to be equal to $j - i$. Set

$$\mathfrak{gl}_{l,\leqslant 0} = \bigoplus_{p\leqslant 0} (\mathfrak{gl}_l)_p \qquad \text{and} \qquad \mathfrak{gl}_{l,<0} = \bigoplus_{p<0} (\mathfrak{gl}_l)_p.$$

The isomorphism (1.1) then induces the $\mathbb{Z}$-grading on $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p, \qquad \mathfrak{g}_p = (\mathfrak{gl}_l)_p \otimes \mathfrak{gl}_n,$$

which is a *good grading* for f in the sense of [7]. We also set

$$\mathfrak{b} = \bigoplus_{p\leqslant 0} \mathfrak{g}_p = \mathfrak{gl}_{l,\leqslant 0} \otimes \mathfrak{gl}_n \qquad \text{and} \qquad \mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p = \mathfrak{gl}_{l,<0} \otimes \mathfrak{gl}_n. \tag{1.2}$$

For any $k \in \mathbb{C}$, we let κ be any symmetric invariant bilinear form on $\mathfrak{g}$ such that

$$\kappa(x, y) = k\,\mathrm{tr}(xy) \qquad \text{for} \quad x, y \in \mathfrak{sl}_N \subset \mathfrak{gl}_N. \tag{1.3}$$

For elements $x, y \in \mathfrak{b}$ set

$$\kappa_{\mathrm{b}}(x, y) = \kappa(x, y) + \frac{1}{2}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,x\,\mathrm{ad}\,y) - \frac{1}{2}\,\mathrm{tr}_{\mathfrak{g}_0}\,p_0(\mathrm{ad}\,x\,\mathrm{ad}\,y),$$

where p_0 denotes the restriction of the operator to $\mathfrak{g}_0$. Then κ_{b} defines a symmetric invariant bilinear form on $\mathfrak{b}$.

Example 1.1. Let

$$\kappa(x, y) = \frac{k}{2N}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,x\,\mathrm{ad}\,y) = k\,\big(\mathrm{tr}(xy) - \frac{1}{N}\,\mathrm{tr}(x)\,\mathrm{tr}(y)\big), \qquad x, y \in \mathfrak{g}.$$

Then for $i \geqslant i'$ and $j \geqslant j'$ we have

$$\begin{aligned} &\kappa_{\mathrm{b}}(e_{ii'} \otimes e_{pq},\, e_{jj'} \otimes e_{rs}) \\ &\qquad = \delta_{ii'}\delta_{jj'}\Big((k + nl)\big(\delta_{ij}\delta_{ps}\delta_{qr} - \frac{1}{nl}\delta_{pq}\delta_{rs}\big) - n\delta_{ij}\big(\delta_{ps}\delta_{qr} - \frac{1}{n}\delta_{pq}\delta_{rs}\big)\Big) \end{aligned}$$

with $N = nl$, as before. □

Let $\widehat{\mathfrak{b}} = \mathfrak{b}[t,t^{-1}] \oplus \mathbb{C}\mathbf{1}$ be the Kac–Moody affinization of $\mathfrak{b}$ with respect to the cocycle κ_{b}, and let $V^{\kappa_{\mathrm{b}}}(\mathfrak{b})$ be the universal affine vertex algebra associated with $\mathfrak{b}$ and κ_{b} [6]:

$$V^{\kappa_{\mathrm{b}}}(\mathfrak{b}) = \mathrm{U}(\widehat{\mathfrak{b}}) \otimes_{\mathrm{U}(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C},$$

where $\mathbb{C}$ is regarded as the one-dimensional representation of $\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{b}[t]$ acts trivially and $\mathbf{1}$ acts as 1. Note that by the Poincaré–Birkhoff–Witt theorem, $V^{\kappa_{\mathrm{b}}}(\mathfrak{b})$ is isomorphic to $\mathrm{U}(\mathfrak{b}[t^{-1}]t^{-1})$ as a vector space.

Due to [8, 9], the $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$ can be realized as a vertex subalgebra of $V^{\kappa_{\mathrm{b}}}(\mathfrak{b})$. Our aim is to give explicit description of the generators of $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$ inside $V^{\kappa_{\mathrm{b}}}(\mathfrak{b})$. We will use the identification

$$\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n \cong \mathfrak{b}[t^{-1}]t^{-1},$$

defined by

$$e_{ji}[-m] \otimes e_{pq} \mapsto e_{(j-1)n+p,\,(i-1)n+q}[-m], \qquad m \geqslant 1,$$

for $1 \leqslant i \leqslant j \leqslant l$ and $1 \leqslant p, q \leqslant n$, where we write $x[r] = x\,t^r$ for any $r \in \mathbb{Z}$.

By analogy with [3, Sec. 12], consider the tensor algebra $\mathrm{T}(\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1})$ of the vector space $\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1}$ and let M_n denote the matrix algebra with the basis formed by the matrix units e_{ij}, $1 \leqslant i, j \leqslant n$. Define the algebra homomorphism

$$\mathcal{T} : \mathrm{T}(\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1}) \to M_n \otimes \mathrm{U}(\mathfrak{b}[t^{-1}]t^{-1}), \qquad x \mapsto \mathcal{T}(x) = \sum_{i,j=1}^{n} e_{ij} \otimes \mathcal{T}_{ij}(x)$$

by setting

$$\mathcal{T}_{ij}(x) = x \otimes e_{ji} \in \mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n = \mathfrak{b}[t^{-1}]t^{-1}$$

for $x \in \mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}$. By definition, for any $x, y \in \mathrm{T}(\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1})$ we have

$$\mathcal{T}_{ij}(xy) = \sum_{r=1}^{n} \mathcal{T}_{ir}(x)\mathcal{T}_{rj}(y) = \sum_{r=1}^{n} (x \otimes e_{ri})(y \otimes e_{jr}).$$

Let us equip the tensor product space $\mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ with an associative algebra structure in such a way that the natural embeddings

$$\mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}) \hookrightarrow \mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \qquad \text{and} \qquad \mathbb{C}[\tau] \hookrightarrow \mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$$

are algebra homomorphisms and the generator τ satisfies the relations

$$\big[\tau, x[-m]\big] = m x[-m-1] \qquad \text{for} \quad x \in \mathfrak{g}_{l,\leqslant 0} \quad \text{and} \quad m \in \mathbb{Z}.$$

Furthermore, the tensor product space $\mathrm{U}(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ will also be considered as an associative algebra in a similar way. We will extend $\mathcal{T}$ to the algebra homomorphism

$$\mathcal{T} : \mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \to M_n \otimes \mathrm{U}(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$$

by setting $\mathcal{T}_{ij}(uS) = \mathcal{T}_{ij}(u)S$ for $u \in \mathrm{T}(\mathfrak{g}_{l,\leqslant 0}[t^{-1}]t^{-1})$ and any polynomial $S \in \mathbb{C}[\tau]$.

Set $\alpha = k + n(l-1)$ and consider the matrix

$$B = \begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \dots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ e_{l-1\,1}[-1] & e_{l-2\,2}[-1] & \dots & \alpha\tau + e_{l-1\,l-1}[-1] & -1 \\ e_{l1}[-1] & e_{l2}[-1] & \dots & \dots & \alpha\tau + e_{ll}[-1] \end{bmatrix}$$

with entries in $\mathrm{T}(\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$. Its *column-determinant* $\operatorname{cdet} B$ is defined as the usual alternating sum of the products of the entries taken in the order determined by the column numbers of the entries.[1] So $\operatorname{cdet} B$ is an element of $\mathrm{T}(\mathfrak{gl}_{l,\leqslant 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ and we can write

$$\mathcal{T}_{ij}(\operatorname{cdet} B) = \sum_{r=0}^{l} W_{ij}^{(r)} (\alpha\tau)^{l-r}$$

for certain coefficients $W_{ij}^{(r)}$ which are elements of $\mathrm{U}(\mathfrak{b}[t^{-1}]t^{-1})$, and we can also regard them as elements of $V^{\kappa_{\mathfrak{b}}}(\mathfrak{b})$. The following is our main result.

Theorem 1.2. *All coefficients $W_{ij}^{(r)}$ belong to the $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$. Moreover, the $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f) \subset V^{\kappa_{\mathfrak{b}}}(\mathfrak{b})$ is freely generated by the elements $W_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant n$ and $r = 1, 2, \dots, l$.*

Set $\mathfrak{l} = (\mathfrak{gl}_l)_0 \otimes \mathfrak{gl}_n \subset \mathfrak{gl}_N$. Then the projection $\mathfrak{b} \to \mathfrak{l}$ induces the vertex algebra homomorphism $V^{\kappa_{\mathfrak{b}}}(\mathfrak{b}) \to V^{\kappa_{\mathfrak{b}}}(\mathfrak{l})$, which restricts to the map

$$\nu : \mathcal{W}^{\kappa}(\mathfrak{g}, f) \to V^{\kappa_{\mathfrak{b}}}(\mathfrak{l}),$$

called the (*quantum*) *Miura transformation.* This is an injective vertex algebra homomorphism. The following formula for the images of the elements $W_{ij}^{(r)}$ under the Miura transformation is an immediate consequence of Theorem 1.2.

Theorem 1.3. *We have*

$$\sum_{r=0}^{l} \nu(W_{ij}^{(r)})(\alpha\tau)^{l-r} = \mathcal{T}_{ij}\Big(\big(\alpha\tau + e_{11}[-1]\big) \dots \big(\alpha\tau + e_{ll}[-1]\big)\Big).$$

[1]It is easy to verify that $\operatorname{cdet} B$ coincides with the *row-determinant* of B defined in a similar way.

Note that the principal $\mathcal{W}$-algebra of type A corresponds to the case $n = 1$. The elements $W^{(r)}$ are defined via the expansion of $\operatorname{cdet} B$,

$$\operatorname{cdet} B = \sum_{r=0}^{l} W^{(r)} (\alpha\tau)^{l-r}.$$

By applying the Miura transformation we recover the formula of Fateev and Lukyanov [4].

Corollary 1.4. *The principal $\mathcal{W}$-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$ is freely generated by the elements $W^{(1)}, \dots, W^{(l)}$. Moreover, we have*

$$\sum_{r=0}^{l} \nu(W^{(r)})(\alpha\tau)^{l-r} = \big(\alpha\tau + e_{11}[-1]\big) \dots \big(\alpha\tau + e_{ll}[-1]\big).$$

Example 1.5. Take $n = l = 2$ so that $N = 4$. We have

$$\operatorname{cdet} B = (\alpha\tau)^2 + \big(e_{11}[-1] + e_{22}[-1]\big)(\alpha\tau) + e_{11}[-1]e_{22}[-1] + e_{21}[-1] + \alpha e_{22}[-2].$$

Hence

$$\begin{aligned}
W_{11}^{(1)} &= e_{11}[-1] + e_{33}[-1], \qquad W_{22}^{(1)} = e_{22}[-1] + e_{44}[-1], \\
W_{21}^{(1)} &= e_{12}[-1] + e_{34}[-1], \qquad W_{12}^{(1)} = e_{21}[-1] + e_{43}[-1], \\
W_{11}^{(2)} &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + e_{31}[-1] + \alpha\, e_{33}[-2], \\
W_{22}^{(1)} &= e_{12}[-1]e_{43}[-1] + e_{22}[-1]e_{44}[-1] + e_{42}[-1] + \alpha\, e_{44}[-2], \\
W_{21}^{(1)} &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + e_{32}[-1] + \alpha\, e_{34}[-2], \\
W_{12}^{(1)} &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + e_{41}[-1] + \alpha\, e_{43}[-2].
\end{aligned}$$

For the images under the Miura transformation we have

$$\begin{aligned}
\nu(W_{11}^{(1)}) &= e_{11}[-1] + e_{33}[-1], \qquad \nu(W_{22}^{(1)}) = e_{22}[-1] + e_{44}[-1], \\
\nu(W_{21}^{(1)}) &= e_{12}[-1] + e_{34}[-1], \qquad \nu(W_{12}^{(1)}) = e_{21}[-1] + e_{43}[-1], \\
\nu(W_{11}^{(2)}) &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + \alpha\, e_{33}[-2], \\
\nu(W_{22}^{(1)}) &= e_{12}[-1]e_{43}[-1] + e_{22}[-1]e_{44}[-1] + \alpha\, e_{44}[-2], \\
\nu(W_{21}^{(1)}) &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + \alpha\, e_{34}[-2], \\
\nu(W_{12}^{(1)}) &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + \alpha\, e_{43}[-2].
\end{aligned}$$

Let the form κ_{b} be as in Example 1.1. The values $\kappa_{\mathrm{b}}(x, y)$ are then given in the following table, where the columns and rows correspond to the x and y variables, respectively:

	e_{11}	e_{22}	e_{33}	e_{44}	e_{12}	e_{21}	e_{34}	e_{43}
e_{11}	$\frac{3k+8}{4}$	$-\frac{k}{4}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	0	0	0	0
e_{22}	$-\frac{k}{4}$	$\frac{3k+8}{4}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	0	0	0	0
e_{33}	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	$\frac{3k+8}{4}$	$-\frac{k}{4}$	0	0	0	0
e_{44}	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	$-\frac{k}{4}$	$\frac{3k+8}{4}$	0	0	0	0
e_{12}	0	0	0	0	0	$k+2$	0	0
e_{21}	0	0	0	0	$k+2$	0	0	0
e_{34}	0	0	0	0	0	0	0	$k+2$
e_{34}	0	0	0	0	0	0	$k+2$	0

These values can be used to calculate the operator product expansion formulas for the generators of $\mathcal{W}^\kappa(\mathfrak{g}, f)$. In particular, set

$$\begin{aligned} L =& \frac{1}{2(k+4)}\Big(-2(W_{11}^{(2)} + W_{22}^{(2)}) + W_{12}^{(1)}W_{21}^{(1)} + \frac{3}{4}(W_{11}^{(1)}W_{11}^{(1)} + W_{22}^{(1)}W_{22}^{(1)}) \\ & - \frac{1}{2}W_{11}^{(1)}W_{22}^{(1)} - (k+2)(W_{11}^{(1)} + W_{22}^{(1)})' - (W_{11}^{(1)} - W_{22}^{(1)})'\Big), \end{aligned}$$

where the primes indicate the action of $\operatorname{ad}\tau$ taking $e_{ij}[-1]$ to $e_{ij}[-2]$. Then L is the conformal vector of $\mathcal{W}^\kappa(\mathfrak{g}, f)$:

$$L(z)L(w) \sim -\frac{12k^2+41k+32}{2(k+4)^2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{z-w}\partial L(w).$$

2 Proof of Theorem 1.2

Recall the notation (1.2) and let $\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_0 \oplus \widehat{\mathfrak{a}}_1$ be the Lie superalgebra such that $\widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{b}}$ and $\widehat{\mathfrak{a}}_1 = \mathfrak{m}[t, t^{-1}]$, where $\mathfrak{m}[t, t^{-1}]$ is regarded as the supercommutative Lie superalgebra, while

$$[x, y] = \operatorname{ad} x(y) \qquad \text{for} \quad x \in \widehat{\mathfrak{a}}_0 \quad \text{and} \quad y \in \widehat{\mathfrak{a}}_1.$$

We will write $\psi_{ji}[-m] \otimes e_{pq}$ for the element

$$e_{ji}[-m] \otimes e_{pq} \in \mathfrak{gl}_{l,<0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n = \mathfrak{m}[t^{-1}]t^{-1}$$

with $m \geqslant 1$, when it is considered as an element of $\widehat{\mathfrak{a}}_1$.

Let $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ be the representation of $\widehat{\mathfrak{a}}$ induced from the one-dimensional representation of $(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}) \oplus \mathfrak{m}[t]t$ on which $\mathfrak{b}[t] \subset \widehat{\mathfrak{a}}_0$ and $\mathfrak{m}[t]t \subset \widehat{\mathfrak{a}}_1$ act trivially and $\mathbf{1}$ acts as 1. Then $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ is naturally a vertex algebra which contains $V^{\kappa_{\mathfrak{b}}}(\mathfrak{b})$ as its vertex subalgebra. We will regard $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ as a (non-associative) algebra with repsect to the (-1)-product

$$V^{\kappa_{\mathfrak{b}}}(\mathfrak{a}) \otimes V^{\kappa_{\mathfrak{b}}}(\mathfrak{a}) \to V^{\kappa_{\mathfrak{b}}}(\mathfrak{a}), \qquad a \otimes b \mapsto a_{(-1)}b,$$

where the Fourier coefficients $a_{(n)}$ are defined in the usual way from the state-field correspondence map,

$$Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}\qquad \text{for}\quad a\in V^{\kappa_{\mathfrak{b}}}(\mathfrak{a}).$$

By [9] the $\mathcal{W}$-algebra is given by

$$\mathcal{W}^{\kappa}(\mathfrak{g},f)=\{v\in V^{\kappa_{\mathfrak{b}}}(\mathfrak{b})\mid Q\,v=0\},$$

where $Q: V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})\to V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ is the derivation of the non-associative algebra $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ defined by the following properties. First, Q commutes with the translation operator D of the vertex algebra $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$, that is, $[Q,D]=0$. Moreover, we have the commutation relations

$$[Q,e_{j\,i}\otimes e_{pq}]=\sum_{a=i}^{j-1}\sum_{r=1}^{n}(e_{a i}\otimes e_{rq})(\psi_{j\,a}\otimes e_{pr})-\sum_{a=i+1}^{j}\sum_{r=1}^{n}(\psi_{a i}\otimes e_{rq})(e_{j\,a}\otimes e_{pr})$$
$$+\alpha\,\psi'_{j\,i}\otimes e_{pq}+e_{i\,j+1}\otimes e_{pq}-e_{i-1\,j}\otimes e_{pq}$$

and

$$[Q,\psi_{j\,i}\otimes e_{pq}]=\frac{1}{2}\sum_{i<r<j,\,1\leqslant s\leqslant n}(\psi_{j\,r}\otimes e_{q\,s})(\psi_{r\,i}\otimes e_{ps})-\frac{1}{2}\sum_{i<r<j,\,1\leqslant s\leqslant n}(\psi_{r\,i}\otimes e_{sp})(\psi_{j\,r}\otimes e_{q\,s}),$$

where we used the abbreviations

$$e_{ij}\otimes e_{pq}=(e_{ij}\otimes e_{pq})[-1]\,\mathbf{1},\qquad \psi_{ij}\otimes e_{pq}=(\psi_{ij}\otimes e_{pq})[-1]\,\mathbf{1},$$
$$\psi'_{ij}\otimes e_{pq}=D(\psi_{ij}\otimes e_{pq})[-1]\,\mathbf{1}=(\psi_{ij}\otimes e_{pq})[-2]\,\mathbf{1},$$

and set $\psi'_{ii}=0$. Also, we used the fact that

$$\mathrm{tr}_{\mathfrak{m}}\,p_+\big(\mathrm{ad}\,(e_{j\,i}\otimes e_{pq})\,\mathrm{ad}\,(e_{ij}\otimes e_{pq})\big)=n\,(l+i-j-1)$$

for $1\leqslant i<j\leqslant l$ and $1\leqslant p,q\leqslant n$, where p_+ denotes the restriction of the operator to $\mathfrak{m}$.

Our goal now is to reduce the calculations to the principal nilpotent case. To this end, when $n=1$, we will write $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$ respectively, instead of $\mathfrak{a}$ and $\mathfrak{b}$, and replace k with $k+(n-1)(l-1)$ in (1.3). Consequently, $V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})$ will denote the vertex algebra $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ with $n=1$ (and k replaced by $k+(n-1)(l-1)$). We let $\overline{Q}$ denote the operator Q for $V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})$. We have

$$[\overline{Q},e_{j\,i}]=\sum_{a=i}^{j-1}e_{a i}\,\psi_{j\,a}-\sum_{a=i+1}^{j}\psi_{a i}\,e_{j\,a}+\alpha\,\psi'_{j\,i}+\psi_{j+1\,i}-\psi_{j\,i-1},$$
$$[\overline{Q},\psi_{j\,i}]=\frac{1}{2}\sum_{i<r<j}(\psi_{j\,r}\,\psi_{r\,i}-\psi_{r\,i}\,\psi_{j\,r}),$$

where we used the notation $e_{ji} = e_{ji}[-1]$, $\psi_{ij} = \psi_{ij}[-1]$, $\psi'_{ij} = \psi_{ij}[-2]$, and we set $\psi'_{ii} = 0$.

We will regard $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})\otimes\mathbb{C}[\tau]$ as a non-associative algebra with the natural subalgebras $V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$ and $\mathbb{C}[\tau]$ together with the relation $[\tau, u] = D\,u$ for $u \in V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})$. Similarly, the tensor product $V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})\otimes\mathbb{C}[\tau]$ will be regarded as a non-associative algebra with the relation $[\tau, u] = \overline{D}\,u$ for $u \in V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})$, where $\overline{D}$ denotes the translation operator of the vertex algebra $V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})$. Define the non-associative algebra homomorphism

$$\widetilde{\mathcal{T}} : V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})\otimes\mathbb{C}[\tau] \to M_n\otimes V^{\kappa_{\mathfrak{b}}}(\mathfrak{a})\otimes\mathbb{C}[\tau], \qquad x \mapsto \widetilde{\mathcal{T}}(x) = \sum_{p,q=1}^{n} e_{pq}\otimes\widetilde{\mathcal{T}}_{pq}(x)$$

by

$$\widetilde{\mathcal{T}}_{pq}\big(e_{ji}[-m]\big) = e_{ji}[-m]\otimes e_{qp}, \quad \widetilde{\mathcal{T}}_{pq}\big(\psi_{ji}[-m]\big) = \psi_{ji}[-m]\otimes e_{qp} \quad \text{and} \quad \widetilde{\mathcal{T}}_{pq}(\tau) = \tau.$$

We extend the definition of the column-determinant to matrices $A = [a_{ij}]$ with entries in a non-associative algebra by using right-normalized products,

$$\widetilde{\text{cdet}}\, A = \sum_{\sigma\in\mathfrak{S}_l} \operatorname{sgn}\sigma\cdot a_{\sigma(1)\,1}\big(a_{\sigma(2)\,2}\big(a_{\sigma(3)\,3}\big(\dots\big(a_{\sigma(l-1)\,l-1}\,a_{\sigma(l)\,l}\big)\big)\big)\big). \tag{2.4}$$

Note the relation

$$\widetilde{\mathcal{T}}(\widetilde{\text{cdet}}\, B) = \mathcal{T}(\text{cdet}\, B),$$

where $\widetilde{\text{cdet}}\, B$ is regarded as an element of $V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})\otimes\mathbb{C}[\tau]$. The first part of Theorem 1.2 will now be implied by the following two propositions.

Proposition 2.1. *For any $a \in V^{\kappa_{\bar{\mathfrak{b}}}}(\bar{\mathfrak{a}})\otimes\mathbb{C}[\tau]$ and $1 \leqslant p, q \leqslant n$ we have the relations*

$$[Q, \widetilde{\mathcal{T}}_{pq}(a)] = \widetilde{\mathcal{T}}_{pq}\big([\overline{Q}, a]\big).$$

Proof. This follows immediately from the definitions of the operators Q and $\overline{Q}$. □

Proposition 2.2. *We have the relation*

$$[\overline{Q}, \widetilde{\text{cdet}}\, B] = 0.$$

Proof. We use induction on l. For any $0 \leqslant s \leqslant l$ consider the submatrix $B^{(s)}$ of B corresponding to its last s rows and columns, which is given by

$$\begin{bmatrix} \alpha\tau + e_{l-s+1\,l-s+1}[-1] & -1 & 0 & \dots & 0 \\ e_{l-s+2\,l-s+1}[-1] & \alpha\tau + e_{l-s+2\,l-s+2}[-1] & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ e_{l-1\,l-s+1}[-1] & e_{l-1\,l-s+2}[-1] & \dots & \alpha\tau + e_{l-1\,l-1}[-1] & -1 \\ e_{l\,l-s+1}[-1] & e_{l\,l-s+2}[-1] & \dots & \dots & \alpha\tau + e_{ll}[-1] \end{bmatrix}.$$

Using the definition (2.4), set $D^{(s)} = \widetilde{\text{cdet}}\, B^{(s)}$.

Lemma 2.3. *We have the column expansion formula*

$$D^{(s)} = \sum_{i=1}^{s} B_{i\,1}^{(s)} D^{(s-i)},$$

where $B_{ij}^{(s)}$ denotes the (i,j) entry of $B^{(s)}$. $\square$

Suppose that $s < l$. By the induction hypothesis, the commutator $[\overline{Q}, D^{(s)}]$ equals

$$\sum_{i=1}^{s} \widetilde{\mathrm{cdet}} \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & \alpha\,\tau + e_{l-s+2\,l-s+2}[-1] & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\psi_{l-i+1\,l-s}[-1] & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & e_{l-1\,l-s+2}[-1] & \dots & \alpha\,\tau + e_{l-1\,l-1}[-1] & -1 \\ 0 & e_{l\,l-s+2}[-1] & \dots & \dots & \alpha\,\tau + e_{l\,l}[-1] \end{bmatrix}$$

so that

$$[\overline{Q}, D^{(s)}] = -\sum_{i=1}^{s} \psi_{l-i+1\,l-s}[-1]\, D^{(i-1)}. \tag{2.5}$$

Hence, by Lemma 2.3, we have

$$[\overline{Q}, D^{(l)}] = \sum_{i=1}^{l} [\overline{Q}, B_{i\,1}^{(l)}]\, D^{(l-i)} + \sum_{i=1}^{l} B_{i\,1}^{(l)}\, [\overline{Q}, D^{(l-i)}].$$

Now we use the definition of $\overline{Q}$ and relation (2.5) to write this expression as

$$\sum_{i=1}^{l} \Big(\sum_{a=1}^{i-1} e_{a1}\psi_{ia} - \sum_{a=2}^{i} \psi_{a1}e_{ia} + \alpha\,\psi'_{i\,1} + \psi_{i+1\,1} \Big) D^{(l-i)} - \sum_{i=1}^{l} B_{i\,1}^{(l)} \sum_{a=1}^{l-i} (\psi_{l-a+1\,i} D^{(a-1)}),$$

where, as before, we write $e_{ij} = e_{ij}[-1]$, $\psi_{ij} = \psi_{ij}[-1]$ and $\psi'_{ij} = \psi_{ij}[-2]$ for brevity. Thus,

$$\begin{aligned} [\overline{Q}, D^{(l)}] = & \sum_{i=1}^{l}\sum_{a=1}^{i-1} e_{a1}(\psi_{ia}D^{(l-i)}) - \sum_{i=1}^{l}\sum_{a=2}^{i} \psi_{a1}(e_{ia}D^{(l-i)}) + \sum_{i=1}^{l} (\alpha\,\psi'_{i\,1} + \psi_{i+1\,1}) D^{(l-i)} \\ & - \sum_{i=1}^{l} B_{i\,1}^{(l)} \sum_{a=1}^{l-i} (\psi_{l-a+1\,i} D^{(a-1)}) \end{aligned}$$

which equals

$$
\begin{aligned}
&-\alpha\sum_{a=2}^{l}\psi_{a\,1}\tau D^{(l-a)}-\sum_{a=2}^{l}\sum_{i=a}^{l}\psi_{a,1}(e_{i\,a}D^{(l-i)})+\sum_{i=1}^{l}\psi_{i+1\,1}D^{(l-i)}\\
&=-\sum_{a=1}^{l}\psi_{a\,1}\Big(\sum_{a=i}^{l}(\delta_{i\,a}\alpha\,\tau+e_{i\,a})D^{(l-i)}-D^{(l-a+1)}\Big)\\
&=\sum_{a=1}^{l}\Big(D^{(l-a+1)}-\sum_{i=1}^{l-a+1}B_{i\,a}^{(l-a+1)}D^{(l-a+1-i)}\Big)=0,
\end{aligned}
$$

where the last equality holds by Lemma 2.3. Here we used the relations

$$
\begin{aligned}
\big(e_{j\,i}[-m]\,\psi_{pq}[-n]\big)\,u&=e_{j\,i}[-m]\big(\psi_{pq}[-n]\,u\big),\\
\big(\psi_{pq}[-n]\,e_{j\,i}[-m]\big)\,u&=\psi_{pq}[-n]\big(e_{j\,i}[-m]\,u\big),
\end{aligned}
$$

which hold under the assumption

$$
u\in\text{span of }\{e_{i'j'}[-m'],\,\psi_{p'q'}[-n']\mid j'>j\quad\text{and}\quad q'>p\}.
$$

This completes the proof of the proposition. □

To see the second part of Theorem 1.2, consider the grading of $V^{\kappa_{\mathfrak{b}}}(\mathfrak{b})$ induced by the grading of $\mathfrak{b}$. One has

$$
W_{ij}^{(r)}=\mathcal{T}_{ij}\Big(\sum_{s=1}^{l-r+1}e_{r+s-1\,s}[-1]\Big)+\text{(terms of higher degree)}.
$$

Now the elements $\sum_{s=1}^{l-r+1}e_{r+s-1\,s}$ with $r=1,\dots,l$ form a basis of $\mathfrak{gl}_l^{f_l}$ and the elements

$$
\sum_{s=1}^{l-r+1}e_{r+s-1\,s}\otimes e_{j\,i},\qquad r=1,\dots,l\quad\text{and}\quad i,j=1,\dots,n,
$$

form a basis of $\mathfrak{g}^f$. Hence the claim follows from [8, Theorem 4.1] (cf. [1, Theorem 5.5.1]) thus completing the argument.

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