THE $\mathbb{F}_2$-COHOMOLOGY RINGS OF $\mathbb{S}ol^3$-MANIFOLDS

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ABSTRACT. We compute $H^*(N;\mathbb{F}_2)$ for N a $\mathbb{S}ol^3$-manifold, and then determine the Borsuk-Ulam indices $BU(N,\phi)$ with $\phi \neq 0$ in $H^1(N;\mathbb{F}_2)$.

The Borsuk-Ulam Theorem states that any continuous function $f : S^n \to \mathbb{R}^n$ takes the same value at some antipodal pair of points. This may be put in a broader context as follows. Let N be an n-manifold and let N_ϕ be the double cover associated to an epimorphism $\phi : \pi \to Z/2Z$. Let t_ϕ be the covering involution. The *Borsuk-Ulam index* $BU(N,\phi)$ is the maximal value of k such that for all maps $f : N_\phi \to \mathbb{R}^k$ there is an $x \in N_\phi$ with $f(x) = f(t_\phi(x))$. Then the Borsuk-Ulam Theorem is equivalent to the assertion that $BU(RP^n,\alpha) = n$, where $\alpha : \pi_1(RP^n) \to Z/2Z$ is the canonical epimorphism.

In low dimensions this invariant may be determined cohomologically, and is known for many pairs (N,ϕ), with N a Seifert fibred 3-manifold, including all those with geometry $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2\times\mathbb{E}^1$, $\mathbb{N}il^3$ or $\mathbb{H}^2\times\mathbb{E}^1$ [1, 2]. Here we shall determine this invariant for all such pairs with N a closed $\mathbb{S}ol^3$-manifold. This follows easily once we know the mod-2 cohomology rings of such manifolds. We compute these using Poincaré duality and elementary properties of cup-product in the low-degree cohomology of groups. (Our approach can also be applied to $\mathbb{E}^3$- and $\mathbb{N}il^3$-manifolds.)

1. $\mathbb{S}ol^3$-MANIFOLDS AND THEIR GROUPS

Let M be a closed $\mathbb{S}ol^3$-manifold. Then $\pi = \pi_1(M)$ has an unique maximal abelian normal subgroup $\sqrt{\pi}$, which is free abelian of rank 2. (This subgroup is in fact the Hirsch-Plotkin radical [5] of π.) The quotient $\pi/\sqrt{\pi}$ is virtually $\mathbb{Z}$ (i.e., has two ends), and so is an extension of $\mathbb{Z}$ or $D_\infty = Z/2Z * Z/2Z$ by a finite normal subgroup. The preimage of this finite normal subgroup is torsion-free, and so is either $\mathbb{Z}^2$ or $\mathbb{Z} \rtimes_{-1} \mathbb{Z}$ (the Klein bottle group). Since $Out(\mathbb{Z} \rtimes_{-1} \mathbb{Z})$ is finite and π is not virtually abelian, this preimage must be $\sqrt{\pi}$. Hence $\pi/\sqrt{\pi} \cong \mathbb{Z}$ or D_∞.

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Suppose first that $\pi/\sqrt{\pi} \cong \mathbb{Z}$. Then M is the mapping torus of a self-homeomorphism of $T = S^1 \times S^1$, and $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, where $\Theta = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$. Thus π has a presentation

$$\langle t, x, y \mid txt^{-1} = x^a y^b,\ tyt^{-1} = x^c y^d,\ xy = yx\rangle.$$

Let $\varepsilon = det(\Theta) = \pm 1$ and $\tau = tr(\Theta) = a + d$. Then M is orientable if and only if $\varepsilon = 1$, in which case $|\tau| > 2$, since π is not virtually nilpotent. Let θ be a root of the characteristic polynomial $det(\Theta - XI_2) = X^2 - \tau X + \varepsilon$. Then θ is a unit in the quadratic number field $\mathbb{Q}[\theta]$, and $\sqrt{\pi}$ is isomorphic to an ideal I in the ring $\mathbb{Z}[\theta]$. (The latter may not be the full ring of integers in $\mathbb{Q}[\theta]$!)

Conversely, if α is a quadratic algebraic unit and I is an ideal in $\mathbb{Z}[\alpha]$ then I is free abelian of rank 2 as an abelian group, and $\pi = I \rtimes_\alpha \mathbb{Z}$ is the group of the mapping torus of a self-homeomorphism of T. If α is not a root of unity this mapping torus is a $\mathbb{S}ol^3$-manifold. If $(\alpha, [I])$ and $(\beta, [J])$ are two such pairs the corresponding groups are isomorphic if and only if either $\beta = \pm\alpha$ and $[I] = [J]$ or $\beta = \pm\alpha^{-1}$ and $[I] = \overline{[J]}$. (Here $[I]$ denotes the ideal class of I and the overbar denotes the Galois involution given by $\alpha \leftrightarrow \varepsilon\alpha^{-1}$.) Each such ring $\mathbb{Z}[\alpha]$ has finitely many ideal classes, by the Jordan-Zassenhaus Theorem.

If $\pi/\sqrt{\pi} \cong D_\infty$ then $\pi \cong B *_T C$, where B and C are torsion-free, $T \cong \mathbb{Z}^2$ and $[B : T] = [C : T] = 2$. Thus M is the union of two twisted I-bundles. Moreover, $\beta_1(\pi;\mathbb{Q}) = 0$. Hence M is orientable, since $\chi(M) = 0$, and so B and C must be copies of the Klein bottle group. Hence M is the union of two copies of the mapping cylinder of the double cover of the Klein bottle. The double cover of M corresponding to the preimage of $\sqrt{D_\infty}$ in π is a mapping torus.

In particular, π has a presentation

$$\langle u, v, y, z \mid uyu^{-1} = y^{-1},\ vzv^{-1} = z^{-1},\ yz = zy,\ v^2 = u^{2a}y^b,\ z = u^{2c}y^d\rangle,$$

where $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ corresponds to the identification of $\sqrt{C}$ with $T = \sqrt{B}$. This presentation simplifies immediately to

$$\langle u, v, y \mid uyu^{-1} = y^{-1},\ v^2 = u^{2a}y^b,\ vu^{2c}y^d v^{-1} = u^{-2c}y^{-d}\rangle.$$

Hence $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$ if b is odd, and $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ if b is even. Let $x = u^2$. Then conjugation by uv acts on $\langle x, y\rangle \cong \mathbb{Z}^2$ via $\Psi = \eta \left(\begin{smallmatrix} ad+bc & 2ac \\ 2bd & ad+bc \end{smallmatrix}\right)$, where $\eta = ad - bc = \pm 1$. We have $det(\Psi) = 1$, $tr(\Psi) \equiv 2\ mod\ (4)$, $\Psi \equiv I_2\ mod\ (2)$ and $abcd \neq 0$.

Conversely, any $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ with $abcd \neq 0$ gives rise to such a $\mathbb{S}ol^3$-manifold, for then $|tr(\Psi)| = 2|ad + bc| \geq 6$. Moreover, suppose $P = \left(\begin{smallmatrix} 2k+1 & 2m \\ 2n & 2k+1 \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, where $mn \neq 0$. Then $k(k+1) = mn$, and so we may write $m = m_1m_2$ and $n = n_1n_2$, with $k = m_1n_1$

and $k+1=m_2n_2$. The $\mathbb{S}ol^3$-rational homology sphere corresponding to $\left(\begin{smallmatrix} m_1 & -m_2 \\ -n_2 & n_1\end{smallmatrix}\right) \in GL(2,\mathbb{Z})$, is doubly covered by the mapping torus asociated to P.

Every quadratic unit α such that $\alpha+\bar{\alpha}\equiv 2 \ mod\ (4)$ and $\alpha\bar{\alpha}=1$ is realized in this way. Which ideal classes are realized? The ideal class must be invariant under the conjugation $\alpha\mapsto\alpha^{-1}$.

Every subgroup of finite index in π can be generated by three elements, while proper subgroups of infinite index need at most two generators. If a nontrivial normal subgroup N has infinite index in π then it has Hirsch length ≤ 2. Hence it is abelian, and so has finite index in $\sqrt{\pi}$. Thus proper quotients of a $\mathbb{S}ol^3$-group π either have two ends or are finite.

2. THE MOD-2 COHOMOLOGY RING

Martins has constructed an explicit free resolution $P_*\to\mathbb{Z}$ of the augmentation $\mathbb{Z}[\pi]$-module, and a partial diagonal approximation $\Delta : P_*\to P_*\otimes P_*$, which he used to compute the integral cohomology ring, for semirect products $\pi\cong\mathbb{Z}^2\rtimes_\Theta\mathbb{Z}$ with $\Theta\in GL(2,\mathbb{Z})$ [4]. His formulae should (in principle) apply with coefficients $\mathbb{F}_2$ also.

We shall take a somewhat different approach, first computing cup products into $H^2(\pi;\mathbb{F}_2)$ and then using Poincaré duality. Our strategy in determing relations in $H^2(\pi;\mathbb{F}_2)$ shall be to use restrictions to subgroups (such as $\sqrt{\pi}$) and epimorphisms to quotient groups (such as $\pi/\sqrt{\pi}$ or small finite 2-groups), with known cohomology rings.

We shall usually write $H_*(X)$ and $H^*(X)$ for the homology and cohomology of a space or group X, with coefficients $\mathbb{F}_2$, and denote the cup-product by juxtaposition. In each case considered below, the given generators for a group G represent a basis for $H_1(G)$, and we shall use the corresponding Kronecker dual bases for $H^1(G)=Hom(H_1(G),\mathbb{F}_2)$.

Lemma 1. *Let $w=w_1(\pi)$. Then $w\alpha\beta=\alpha^2\beta+\alpha\beta^2$, for all $\alpha,\beta\in H^1(\pi)$. In particular, if $w=0$ then $\alpha^2\beta=\alpha\beta^2$ and $(\alpha+\beta)^3=\alpha^3+\beta^3$.*

Proof. The first assertion follows from the Wu relation $Sq^1z=w\cup z$ for all $z\in H^{n-1}(X)$, which holds for any PD_n-complex X. The second follows easily. □

If G is a group let $X^n(G)=\langle g^n | g\in G\rangle$ be the subgroup generated by all n^{th} powers. The next lemma is a refinement of Theorem 2 of [3] (which is restated here as part (1) of the lemma).

Lemma 2. *Let G be a group, and $\rho,\phi,\psi\in H^1(G)$. Let $K=\mathrm{Ker}(\rho)$ and $L=K\cap\mathrm{Ker}(\phi)$. Then*

(1) *the kernel of cup product from the symmetric product* $\odot^2 H^1(G)$ *to* $H^2(G)$ *is the dual of* $X^2(G)/X^4(G)[G, X^2(G)]$*;*
(2) *the canonical projections induce isomorphisms* $H^1(G/X^2(K)) \cong H^1(G/X^2(L)) \cong H^1(G/X^4(G)) \cong H^1(G)$*;*
(3) $\rho\phi = 0$ *in* $H^2(G) \Leftrightarrow \rho\phi = 0$ *in* $H^2(G/X^2(K))$*;*
(4) $\phi^2 = \rho\phi + \rho\psi$ *in* $H^2(G) \Leftrightarrow \phi^2 = \rho\phi + \rho\psi$ *in* $H^2(G/X^2(L))$*.*

Proof. Part (1) is Theorem 2 of [3], while part (2) is clear.

If $\phi\psi = 0$ in $H^2(G)$ then there is a 1-cochain $F : G \to \mathbb{F}_2$ such that $\phi(g)\psi(h) = \delta F(g, h) = F(gh) + F(g) + F(h)$, for all $g, h \in G$. Part (3) follows easily, since F restricts to a homomorphism on K, and is constant on cosets of $X^2(K)$.

Part (4) is similar. □

In most of the cases considered here, the coefficients in the linear relations determining the kernel of cup product may be found by restricting to 2-generator subgroups.

Lemma 3. *Let* $\{T, Y\}$ *be the basis for* $H^1(D_8)$ *corresponding to the presentation* $D_8 = \langle t, y | t^2 = y^4 = 1,\ tyt^{-1} = y^{-1}\rangle$*. Then* $(T+Y)Y = 0$ *in* $H^2(D_8)$*.*

Proof. Let D_∞ have the presentation $\langle u, v | u^2 = v^2 = 1\rangle$, and let U, V be the dual basis for $H^1(D_\infty)$. Then $H^*(D_\infty) = \mathbb{F}_2[U, V]/(UV)$. Let $f : D_\infty \to D_8$ be the epimorphism given by $f(u) = t$ and $f(v) = ty$. Then f induces an isomorphism $D_\infty/X^4(D_\infty) \cong D_8$, so $H^2(f)$ is injective. Since $f^*U = T + Y$ and $f^*V = Y$, we see that $(T+Y)Y = 0$ in $H^2(D_8)$. □

Let E be the "almost extraspecial" 2-group with presentation

$$\langle t, u, v \mid t^2 = 1,\ u^2 = v^2,\ tut^{-1} = u^{-1},\ tv = vt,\ uv = vu\rangle.$$

Lemma 4. *Let* $\{T, U, V\}$ *be the basis for* $H^1(E)$ *corresponding to the above presentation. Then* $TU + U^2 + V^2 = 0$ *in* $H^2(E)$*.*

Proof. Since $X^2(E) \cong Z/2Z$, the kernel of cup product from $\odot^2 H^1(G)$ to $H^2(G)$ has dimension 1 [3]. Thus there is an unique nontrivial linear relation $aT^2 + bU^2 + cV^2 + dTU + eTV + fUV = 0$ in $H^2(E)$. The coefficients can be determined by restriction to the subgroups $\langle t\rangle \cong Z/4Z$, $\langle t, u\rangle \cong D_8$, $\langle t, v\rangle \cong Z/4Z \oplus Z/2Z$, and $\langle u, v\rangle \cong Z/4Z \oplus Z/2Z$. □

3. MAPPING TORI

Suppose that $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, where $\Theta = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$. Let $\varepsilon = ad - bc = \pm 1$ and $\tau = a + d$. Let $\Delta_1 = det(\Theta - I_2) = 1 - \tau + \varepsilon$ and

$\Delta_2 = (a-1, b, c, d-1)$ be the elementary divisors of $\Theta - I_2$. Then Δ_2^2 divides Δ_1, and

$$\pi^{ab} \cong \mathbb{Z} \oplus Z/(\Delta_1/\Delta_2)Z \oplus Z/\Delta_2 Z.$$

Let $\beta = \beta_1(\pi; \mathbb{F}_2)$. Then $1 \le \beta \le 3$, and $\beta_2(\pi; \mathbb{F}_2) = \beta$, by Poincaré duality. Let $\rho : \pi \to Z/2Z$ be the unique epimorphimorphism which factors through $\pi/\sqrt{\pi} \cong \mathbb{Z}$. If π is non-orientable then $\rho = w_1(M)$, and $K = \pi^+$, the maximal orientable subgroup of π.

1. If τ is odd then Δ_1 is odd and $\pi^{ab} \cong \mathbb{Z} \oplus odd$. In this case ρ is the unique epimorphism from π to $Z/2Z$, and

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \Xi]/(\rho^2, \Xi^2),$$

where Ξ has degree 2, by Poincaré duality.

2. If $\tau \equiv \varepsilon - 1 \ mod\,(4)$ then $\pi^{ab} \cong \mathbb{Z} \oplus Z/2Z \oplus odd$, and $\beta = 2$. Hence $H^1(\pi) = \langle \rho, \sigma \rangle$, where σ does not factor through $Z/4Z$. Moreover, if $G = \pi/X^4(\pi)$ then $X^2(G) \cong (Z/2Z)^2$ is central in G. Thus $\rho^2 = \rho\sigma = 0$, by Lemma 2, while $\sigma^2 \neq 0$. Hence $H^2(\pi) = \langle \sigma^2, \Xi \rangle$, for some Ξ of degree 2. Duality then implies that $\sigma^3 = \rho\Xi \neq 0$. We may assume also that $\sigma\Xi = 0$, and so

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \Xi]/(\rho^2, \rho\sigma, \sigma\Xi, \rho\Xi + \sigma^3, \Xi^2).$$

3. If $\tau \equiv \varepsilon + 1 \ mod\,(4)$ and Δ_2 is odd then $\pi^{ab} \cong \mathbb{Z} \oplus Z/2^k Z \oplus odd$, for some $k \ge 2$. Hence $H^1(\pi) = \langle \rho, \sigma \rangle$, where $\sigma^2 = \rho^2 = 0$. Since $\rho\sigma = 0$, by the nondegeneracy of Poincaré duality,

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \Xi, \Omega]/(\rho^2, \rho\sigma, \sigma^2, \rho\Omega, \sigma\Xi, \rho\Xi + \sigma\Omega, \Xi^2, \Omega^2, \Xi\Omega),$$

where Ξ and Ω have degree 2.

In all the remaining cases $\beta = 3$. For if $\tau \equiv \varepsilon + 1 \ mod\,(4)$ and Δ_2 is even then a and d are odd and b and c are even. Hence $\Delta_1 = 2^k q$ and $\Delta_2 = 2^\ell q'$, where $0 < \ell \le \frac{k}{2}$ and q, q' are odd. In this case $\pi^{ab} \cong \mathbb{Z} \oplus Z/2^{k-\ell}Z \oplus Z/2^\ell Z \oplus odd$, so the images of $\{t, x, y\}$ form a basis for $H_1(\pi)$. Let $\{\rho, \sigma, \psi\}$ be the dual basis, so that

$$\sigma(x) = \psi(y) = 1 \quad \text{and} \quad \sigma(t) = \sigma(y) = \psi(t) = \psi(x) = 0.$$

If $G = \pi/X^4(\pi)$ then $X^2(G) = \langle t^2, x^2, y^2 \rangle \cong (Z/2Z)^3$ is central in G, so the kernel of cup product from $\odot^2 H^1(\pi)$ to $H^2(\pi)$ has rank 3. It then follows from Poincaré duality that $H^*(\pi)$ is generated as a ring by $H^1(\pi)$. In each case, $\rho\sigma^2 = \rho\rho\sigma = 0$ and $\rho\psi^2 = \rho\rho\psi = 0$, by Lemma 1. Hence $\rho\sigma\psi \neq 0$, by the nondegeneracy of Poincaré duality. It then follows easily that $\rho\sigma$, $\rho\psi$ and $\sigma\psi$ are linearly independent, and so form a basis for $H^2(\pi)$. We may write

$$\sigma^2 = m\rho\sigma + n\rho\psi + p\sigma\psi \quad \text{and} \quad \psi^2 = q\rho\sigma + r\rho\psi + s\sigma\psi,$$

for some $m, \dots, s$. On restricting to $\sqrt{\pi}$, we see that $p = s = 0$, since $\sigma^2|_{\sqrt{\pi}} = \psi^2|_{\sqrt{\pi}} = 0$ and $\rho|_{\sqrt{\pi}} = 0$, while $\sigma\psi|_{\sqrt{\pi}} \neq 0$. Since $\rho\sigma^2 = \rho^2\sigma = \rho\psi^2 = \rho^2\psi = 0$, taking cup products with σ and ψ gives

$$\sigma^3 = n\rho\sigma\psi, \quad \sigma^2\psi = m\rho\sigma\psi, \quad \psi^3 = q\rho\sigma\psi \quad \text{and} \quad \sigma\psi^2 = r\rho\sigma\psi.$$

4. If $\ell \geq 2$ then $a \equiv d \equiv 1$ and $b, c \equiv 0$ *mod* (4), so $\varepsilon \equiv 1$ *mod* (4) also, i.e., π is orientable. In this case $\sigma^2 = \psi^2 = \rho^2 = 0$, and so

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \psi^2).$$

Suppose now that $\ell = 1$.

5. If π is orientable and $\Delta_1 \equiv 0$ *mod* (8) we may assume that one of σ, ψ or $\sigma + \psi$ factors through $Z/4Z$. Thus either $\sigma^2 = 0$, $\psi^2 = 0$ or $\sigma^2 = \psi^2$. We may assume that $\sigma^2 \neq 0$. Then $\rho\sigma^2 = \rho^2\sigma = 0$ and $\psi\sigma^2 = \psi^2\sigma = 0$, and so $\sigma^3 \neq 0$, by the nonsingularity of Poincaré duality. Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\psi + \sigma^2, \psi^2).$$

In this case we see that $\phi^3 = 0 \Leftrightarrow \phi^2 = 0$.

If π is orientable and $\Delta_1 \equiv 4$ *mod* (8) then $\tau \equiv 6$ *mod* (8) and a, d are odd, and so $a \equiv d$ *mod* (4). In this case $\psi^2 \neq 0$ and $(\sigma + \psi)^2 \neq 0$ also, and so $\sigma^2 = m\rho\sigma + n\rho\psi$ and $\psi^2 = q\rho\sigma + r\rho\psi$ are linearly independent. Hence $mr + nq = 1$ in $\mathbb{F}_2$. Since $w = 0$, $\sigma^2\psi = \sigma\psi^2$ and so $m = r$.

6. Suppose first that $a \equiv 1$ *mod* (4). Then $bc \equiv 4$ *mod* (8), and so $b \equiv c \equiv 2$ *mod* (4). Let $L_\phi = \mathrm{Ker}(\rho) \cap \mathrm{Ker}(\phi)$. Then $\pi/X^2(L_\phi)$ has a presentation

$$\langle t, x, y \mid t^4 = x^4 = y^2 = 1,\ tx = xt,\ tyt^{-1} = x^2y,\ xy = yx\rangle.$$

Let $J = \langle t, x\rangle \cong (Z/4Z)^2$. Then $\sigma^2|_J = \rho\psi|_J = 0$, while $\rho\sigma|_J \neq 0$. Applying part (3) of Lemma 2, we see that $m = 0$, and so $\sigma^2 = \rho\psi$ and $\psi^2 = \rho\sigma$. (Note, however, that Lemma 2 does *not* assert that the relation $\psi^2 = q\rho\sigma + r\rho\psi$ also holds in $\pi/X^2(L_\phi)$! For this, we could use $L_\psi = \mathrm{Ker}(\rho) \cap \mathrm{Ker}(\psi)$ instead.) Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\psi + \sigma^2, \rho\sigma + \psi^2).$$

In particular, $\sigma^3 = \psi^3 = (\rho + \sigma)^3 = (\rho + \psi)^3 \neq 0$.

If $a \equiv -1$ *mod* (4) then $bc \equiv 0$ *mod* (8). If, say, $b \equiv 2$ *mod* (4) (so $c \equiv 0$ *mod* (4)) then the change of basis $x' = x$, $y' = xy$ reduces this case to the one just considered. In terms of the given basis, we have

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\sigma + \sigma^2, \rho\psi + \sigma^2 + \psi^2).$$

In this case $\sigma^3 \neq 0$, but $\psi^3 = 0$. A similar result holds if $b \equiv 0$ *mod* (4) and $c \equiv 2$ *mod* (4).

7. If, however, $a \equiv -1$ *mod* (4) and $b \equiv c \equiv 0$ *mod* (4) then $\pi/X^4(\pi)$ has a presentation

$$\langle t, x, y \mid t^4 = x^4 = y^4 = 1,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1},\ xy = yx\rangle.$$

In this case $J = \langle t, x\rangle$ is non-abelian, and $\sigma^2|_J \neq 0$, while $\rho\psi|_J = 0$. Hence we must have $m = r = 1$. It is clear from the symmetry of the presentation for $\pi/X^4(\pi)$ that we must also have $n = q$ in this case, and so $n = q = 0$. Thus

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\sigma + \sigma^2, \rho\psi + \psi^2).$$

We now find that $\phi^3 = 0$ for all $\phi \in H^1(\pi)$.

If $\ell = 1$ and M is non-orientable then a and d are odd, and $\Delta_1 = -a - d \equiv 0$ *mod* (4). In this case $\rho = w_1(M)$, and so $\sigma^2\psi + \sigma\psi^2 = \rho\sigma\psi \neq 0$, by Lemma 1. After swapping x and y, if necessary, we may assume that $a \equiv 1$ *mod* (4).

8. If $bc \equiv 0$ *mod* (8) then, after a further change of basis of the form $x' = x, y' = xy$ or $x' = xy, y' = y$, if necessary, we may assume that $b \equiv c \equiv 0$ *mod* (4). Then $\sigma^2 = 0$, and $\pi/\langle\langle t^2, x, y^4\rangle\rangle \cong D_8$, so $(\rho + \psi)\psi = 0$ also. Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \rho\psi + \psi^2).$$

In particular, $(\sigma + \psi)^3 = (\rho + \sigma + \psi)^3 \neq 0$, and all other classes have cube 0. In terms of the given bases, the other cases are:

If $b \equiv 0$ and $c \equiv 2$ *mod* (4) then

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2 + \psi^2, \rho\psi + \psi^2, \sigma^2\psi).$$

Here $\sigma^3 = (\rho + \sigma)^3 \neq 0$ and all other classes have cube 0.

If $b \equiv 2$ and $c \equiv 0$ *mod* (4) then

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \psi^2 + \rho\sigma + \rho\psi).$$

Here $\psi^3 = (\rho + \psi)^3 \neq 0$ and all other classes have cube 0.

9. If $b \equiv c \equiv 2$ *mod* (4) then σ^2 and ψ^2 are linearly independent. There are three distinct epimorphisms from π to the almost extraspecial group E, given by $f(x) = u^{-1}v), f(y) = u$; $g(x) = v, g(y) = uv^{-1}$; and $h(x) = v, h(y) = u$. Using these epimorphisms to pull back the relation given in Lemma 3, we find that

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2 + \rho\psi, \psi^2 + \rho\sigma + \rho\psi).$$

In particular, every epimorphism $\phi \neq \rho$ has nonzero cube.

4. UNIONS OF TWISTED I-BUNDLES

Suppose that $\pi/\sqrt{\pi} \cong D_\infty$. Then π is orientable, and has a presentation

$$\langle u, v, y \mid uyu^{-1} = y^{-1},\ v^2 = u^{2a}y^b,\ vu^{2c}y^dv^{-1} = u^{-2c}y^{-d}\rangle,$$

where $ad - bc = \pm 1$ and $abcd \neq 0$. Let $B = \langle u, y\rangle$ and $C = \langle v, u^{2c}y^d\rangle$.

If b is odd then $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$, where the summands are generated by u and $u^{-a}v$, respectively. Let $U(u) = V(v) = 1$, $U(v) = a$ and $V(u) = 0$. Then

$$H^*(\pi) \cong \mathbb{F}_2[U, V, \Xi, \Omega]/(U^2, UV, V^2, U\Xi + V\Omega, \Xi^2, \Omega^2, \Xi\Omega),$$

where Ξ and Ω have degree 2.

If b is even then $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and the images of u, v and y represent a basis for $H_1(\pi)$. Let $\{U, V, Y\} \in H^1(\pi)$ be the dual basis. Then U^2, V^2 and Y^2 are all nonzero, but $W = U + V$ lifts to a homomorphism from π to $Z/4Z$, and so $W^2 = 0$. Hence $U^2 = V^2$. Since U and V are induced from classes in $H^1(D_\infty)$ we have $UV = 0$. We also have $UY|_B = Y^2|_B$ and $VY|_C = Y^2|_C$, while $U|_C, V|_B, U^2|_B$ and $V^2|_C$ are all 0.

Suppose that $pU^2+qY^2+rUY+sVY = 0$ in $H^2(\pi)$. On restricting to the subgroups B and C, we find that $q+r = q+s = 0$. Since $U^2 \neq 0$ we must have $q = r = s = 1$. Multiplying by U and V, we find that $UY^2 + U^2Y = 0$ and $VY^2+V^2Y = 0$. Poincaré duality for π now implies that $\{U^2, Y^2, UY\}$ is a basis for $H^2(\pi)$, while $UY^2 = U^2Y = VY^2$ generates $H^3(\pi)$. We see also that $U^3 = U^2V = UV^2 = V^3 = (U + V)^3 = 0$, while $(U + Y)^3 = (V + Y)^3 = (U + V + Y)^3 = Y^3$.

Suppose first that $b \equiv 0$ *mod* (4). Then $G = \pi/\langle\langle uv, u^2, y^4\rangle\rangle \cong D_8$. Hence $(U + V + Y)Y = 0$ in $H^3(\pi)$. It follows easily that $Y^3 = 0$, and so all cubes are 0 in $H^3(\pi)$.

If $b \equiv 2$ *mod* (4) then $\pi/\langle\langle u^2, (uv)^2, v^4, y^4\rangle\rangle$ has a presentation

$$\langle u, v, y \mid u^2 = (uv)^2 = v^4 = 1,\ uyu^{-1} = vyv^{-1} = y^{-1},\ v^2 = y^2\rangle$$

Hence there is an epimorphism $f : \pi \to E$, given by $f(u) = t$, $f(v) = u$ and $f(y) = u^{-1}t^{-1}v$. Since $f^*T = U + Y$, $f^*U = V + Y$, $f^*V = Y$ and $UV = 0$, it follows from Lemma 4 that $UY + VY + V^2 + Y^2 = 0$ in $H^2(\pi)$. Multiplying by Y, we find that $UY^2 + Y^3 = 0$ and so $Y^3 \neq 0$. In this case, only the cubes induced from $H^*(\pi/\sqrt{\pi})$ are zero.

5. THE BORSUK-ULAM INDEX

We may identify an epimorphism ϕ with a nonzero class in $H^1(N; \mathbb{F}_2)$. Then $BU(N, \phi) = 1 \Leftrightarrow \phi$ lifts to an integral class $\Phi \in H^1(N; \mathbb{Z})$, while $BU(N, \phi) = n \Leftrightarrow \phi^n \neq 0$ in $H^n(N; \mathbb{F}_2)$ In general, $1 \leq BU(N, \phi) \leq n$.

See [1]. When $n = 3$ the remaining possibility is that $BU(M, \phi) = 2 \Leftrightarrow \phi^2 = 0$ but ϕ is not the reduction of an integral class.

Suppose first that $\pi/\sqrt{\pi} \cong \mathbb{Z}$. Then the following results are immediate from §3.

1. If $\rho : \pi \to Z/2Z$ is the unique epimorphism which factors through $\pi/\sqrt{\pi} \cong \mathbb{Z}$ then $BU(M, \rho) = 1$.
2. If $\tau \equiv \varepsilon - 1$ *mod* (4) then $BU(M, \phi) = 3$ for all $\phi \neq \rho$.
3. If $\tau \equiv \varepsilon + 1$ *mod* (4) and either Δ_2 is odd or $a \equiv d \equiv 1$ *mod* (4) and b, c are divisible by 4, then $BU(M, \phi) = 2$ for all $\phi \neq \rho$.
4. If $\varepsilon = 1$, $\Delta_1 \equiv 0$ *mod* (8) and $\Delta_2 \equiv 2$ *mod* (4) then $BU(M, \phi) = 2$ for the two epimorphisms $\phi \neq \rho$ such that $\phi^2 = 0$ (i.e, that factor through $Z/4Z$) and $BU(M, \phi) = 3$ for the four such that $\phi^2 \neq 0$.
5. If $\varepsilon = 1$, $\Delta_1 \equiv 4$ *mod* (8) and $\Theta \equiv -I_2$ *mod* (4) then $BU(M, \phi) = 2$ for all $\phi \neq \rho$.
6. If $\varepsilon = 1$ and $\Delta_1 \equiv 4$ *mod* (8), but $\Theta \not\equiv -I_2$ *mod* (4), then $BU(M, \phi) = 2$ for the two epimorphisms $\phi \neq \rho$ such that $\phi^2 = 0$ and $BU(M, \phi) = 3$ for the four such that $\phi^2 \neq 0$.
7. If $\varepsilon = -1$, $\tau \equiv 0$ *mod* (4), $\Delta_2 \equiv 2$ *mod* (4) and $bc \equiv 0$ *mod* (8) then $BU(M, \phi) = 2$ for the four epimorphisms $\phi \neq \rho$ such that $\phi^3 = 0$ and $BU(M, \phi) = 3$ for the two such that $\phi^3 \neq 0$.
8. If $\varepsilon = -1$, $\tau \equiv 0$ *mod* (4), $\Delta_2 \equiv 2$ *mod* (4) and $bc \equiv 4$ *mod* (8) then $BU(M, \phi) = 3$ for all $\phi \neq \rho$.

Suppose now that $\pi/\sqrt{\pi} \cong D_\infty$. Then the following results are immediate from §4.

9. If $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$ then $BU(M, \phi) = 2$ for all ϕ.
10. If $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and $b \equiv 0$ *mod* (4) then $BU(M, \phi) = 2$ for all ϕ.
11. If $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and $b \equiv 2$ *mod* (4) then $BU(M, \phi) = 2$ for epimorphisms ϕ which factors through $\pi/\sqrt{\pi}$, while $BU(M, \phi) = 3$ otherwise.

6. OTHER GEOMETRIES

We remark finally that similar arguments may be used to determine the $\mathbb{F}_2$-cohomology rings and Borsuk-Ulam invariants for pairs (N, ϕ) with N a closed $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold. These manifolds are all Seifert fibred over flat 2-orbifolds. Since they have been covered in [2], we shall confine ourselves to some brief observations.

The ten closed flat 3-manifolds may be easily treated individually. The only one admitting a class ϕ with $\phi^3 \neq 0$ has group G_4, with holonomy $Z/4Z$ and abelianization $\mathbb{Z} \oplus Z/2Z$. Thus $H^1(\pi) = \langle T, X \rangle$, where $T^2 = 0$ and $X^2 \neq 0$. We may deduce that $TX = 0$ also, by mapping G_4 onto D_8. It follows easily that

$$H^*(G_4) \cong \mathbb{F}_2[T, X, \Omega]/(T^2, TX, X\Omega, T\Omega + X^3, \Omega^2),$$

where Ω has degree 2. (Thus $X^3 = (T + X)^3 \neq 0$. These classes correspond to the two epimorphisms without integral lifts.)

The possible Seifert bases B of closed $\mathbb{N}il^3$-manifolds are the seven flat 2-orbifolds with no reflector curves: $B = T$, Kb, $S(2, 2, 2, 2)$, $S(2, 4, 4)$, $S(2, 3, 6)$, $S(3, 3, 3)$ or $P(2, 2)$. Let $\beta = \pi_1^{orb}(B)$ be the orbifold fundamental group of the base. Then π^{ab} is an extension of β^{ab} by a finite cyclic group Z/qZ, if the base is orientable ($B \neq Kb$ or $P(2, 2)$), and by $Z/(2, q)Z$ otherwise. The ring $H^*(\pi)$ depends only on the base B and the residue of q *mod* (4). If $B = T$ or Kb then $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, for some $\Theta \in GL(2, \mathbb{Z})$. These are in fact the cases requiring most effort. In all other cases π^{ab} is finite, and the projection of π onto β induces an isomorphism $H_1(\pi) \cong H_1(\beta)$. When $B = S(2, 3, 6)$ or $S(3, 3, 3)$ this group is cyclic. (In particular, such $\mathbb{N}il^3$-manifolds are not unions of twisted I-bundles.) When $B = S(2, 4, 4)$ we have $\pi/X^4(\pi) \cong \beta/X^4(\beta) \cong G_4/X^4(G_4)$. The cases of $S(2, 2, 2, 2)$ and $P(2, 2)$ are related to those of the flat 3-manifolds G_2 and B_4, respectively.

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