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The Bound State S-matrix of the Deformed Hubbard Chain

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Abstract: In this work we use the q-oscillator formalism to construct the atypical (short) supersymmetric representations of the centrally extended $\mathcal{U}_q(\mathfrak{su}(2|2))$ algebra. We then determine the S-matrix describing the scattering of arbitrary bound states. The crucial ingredient in this derivation is the affine extension of the aforementioned algebra.

Contents

1 Introduction

Integrable systems constitute a special class of models in mathematics and physics. Their properties allow them to be solved exactly and thus they appear to be a very useful playground for studying various systems. One common feature shared by these models is that they are closely related to some underlying algebraic structures. Thus for most of the quantum integrable systems there is some sort of large and powerful symmetry hidden in the origins of it, for example a Yangian or a quantum affine algebra. A particularly interesting example is the Hubbard model.

The Hubbard model, which was named after John Hubbard, is the simplest model of interacting particles on a lattice, with only two terms in the Hamiltonian, the hopping term (kinetic energy) and the Coulomb potential [1]. The model describes an ensemble of particles in a periodic potential at sufficiently low temperatures such that all the particles may be considered to be in the lowest Bloch band and also any long-range interactions between the particles are considered to be weak enough and thus are ignored. It is based on the tight-binding approximation of the superconducting systems and the motion of electrons between the atoms of a crystalline solid. Despite its apparent simplicity, it is very rich applications and generalizations describing including phase shifts and a plethora of interesting phenomena. In the case when interactions between particles on different sites of the lattice can not be neglected and are included, the model is often referred to as the Extended Hubbard model. The particles can either be fermions, as in Hubbard's original work, or bosons, and the model is then referred as either the BoseHubbard model or the boson Hubbard model that can be used to study systems such as bosonic atoms on an optical lattice (for a decent overview of various generalizations see reprint volumes [2–4] and also a more recent book [5]).

A very specific class of models are those that share features of the one-dimensional Hubbard model and the supersymmetric t-J model [6]. The very interesting case being the Alcaraz and Bariev model [7] having an extra spin-spin interaction term in the Hamiltonian and showing some characteristics of superconductivity. This model can be viewed as a quantum deformation of the Hubbard model in much the same way as the Heisenberg XXZ model is a quantum deformation of the XXX model. This model has a specific R-matrix which can not be written as a function of the difference of two associated spectral parameters. This paradigm is related to the very interesting but at the same time complicated algebraic properties of the model.

In recent years there has been renewed interest in integrable models arising from the discovery of integrable structures in the context of the AdS/CFT correspondence. For a recent review see [8] and references therein. The worldsheet S-matrix encountered there is one of the central objects of research and it turns out to have a lot in common with the specific cases of the Hubbard model considered in [9, 10]. Interestingly, the S-matrix of such Hubbard model is obtained as a special limit of this worldsheet S-matrix [11].

The exact integrability of the one-dimensional Hubbard model was established by B. Shastry [12] where it was shown that the model exhibits $\mathcal{Y}(\mathfrak{su}(2)) \oplus \mathcal{Y}(\mathfrak{su}(2))$ Yangian symmetry [13]. However this symmetry is insufficient to constrain the R-matrix completely. Similarly, the worldsheet S-matrix for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring also turns out to have Yangian symmetry [14]. However the Yangian in this model is based on a larger Lie algebra, the centrally extended $\mathfrak{su}(2|2)$ Lie superalgebra. The underlying Lie superalgebra turns out to be powerful enough to constrain the S-matrix [15–17] (up to an overall phase, the so-called 'dressing factor' [18–20]) in the case where at least one of the representations is fundamental. However, Yangian symmetry (or equivalently the Yang-Baxter equation) is required in order to find the S-matrix describing the scattering of states that live in higher representations [21, 22]. This specifically concerns bound states in the system which transform in the supersymmetric short representations [23–26]. The bound state scattering matrix can be explicitly constructed with the help of the underlying Yangian symmetry [27].

Nevertheless, despite this success, there are still some problems concerning this infinite dimensional algebra due to some unusual features. The centrally extended $\mathfrak{su}(2|2)$ Lie superalgebra has a degenerate Cartan matrix which prohibits the direct application of the most of techniques related to the theory of Yangians. For the case at hand this has been partially circumvented in several ways: by enlarging the algebra by an $\mathfrak{sl}(2)$ automorphisms [14], by considering the $\alpha \to 0$ limits of the exceptional Lie algebra $\mathfrak{d}(2,1;\alpha)$ [28] or building the Drinfeld's second realization [29]. However this still proves to be an obstacle when, for example, one tries to construct the universal R-matrix [30, 31]. This object encodes all the scattering data in the theory of a purely algebraic form. Another issue that is not completely understood is the appearance of the so-called Secret symmetry [32]. This is an additional symmetry of the S-matrix that does not have a Lie algebra analogue. Resolving these issues could shed some light in understanding the complete underlying algebraic structures and put the model and the methods used to solve it on a more firm footing.

A possible route on attacking these issues was put forward in [10]. Here the quantum deformation $\mathcal{Q}$ of the extended $\mathfrak{su}(2|2)$ algebra was studied. This q-deformed algebra has a number of interesting features such as a rather symmetric realization of the different central elements. The algebra $\mathcal{Q}$ also seems to be better behaved when attacking the problems due to its rather simple and symmetric form. Just as in the non-deformed case, there is a link to Hubbard models, more specifically to the class of the deformed supersymmetric one-dimensional Hubbard models [10, 33]. The non-deformed model is revealed by taking a specific limit of the R-matrices that belong to this deformed model [34]. Moreover, by sending the quantum deformation parameter $q \to 1$, the R-matrix under the consideration reduces to $\mathrm{AdS}_5 \times \mathrm{S}^5$ string worldsheet S-matrix. As such, this matrix encompasses both different varieties of Hubbard models and the worldsheet S-matrices and seems to provide a clear algebraic framework for describing this class of models.

The q-deformed S-matrix in the fundamental representation is constrained up to an overall phase by requiring invariance under $\mathcal{Q}$ itself. However, in the light that both the $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the Hubbard model S-matrices are actually invariant under an infinite dimensional symmetry algebra, it should not be surprising that such structure is also present here. Indeed, the larger algebraic structure underlying this S-matrix is the quantum affine algebra $\widehat{\mathcal{Q}}$ [35]. This infinite dimensional algebra is obtained by adding an additional fermionic node to the Dynkin diagram. In the $q \to 1$ limit one can actually retrieve the Yangian generators of centrally extended $\mathfrak{su}(2|2)$ by considering the appropriate combinations of $\widehat{\mathcal{Q}}$ generators. This fuels the idea that $\widehat{\mathcal{Q}}$ plays a similar role as the Yangian in the undeformed case. More specifically, it is expected for the S-matrix in the higher representations to be uniquely defined (up to an overall phase) by the underlying quantum affine algebra $\widehat{\mathcal{Q}}$. This indeed turns out to be the case as we will show in this paper.

The class of representations we are considering in this work are the supersymmetric short representations. These representations are called short because the central elements are not independent; they satisfy the so-called shortening condition [23]. In order to construct these representations, we employ the formalism of quantum oscillators. It is the quantum version of the well-known harmonic oscillator algebra and is defined as

$$[N, \mathsf{a}] = -\mathsf{a}, \qquad [N, \mathsf{a}^\dagger] = \mathsf{a}^\dagger, \qquad \mathsf{a}\,\mathsf{a}^\dagger - q\,\mathsf{a}^\dagger\mathsf{a} = q^{-N}.$$

The use of quantum oscillators in the context of quantum groups was investigated some time ago in [36–38]. By employing the Fock space type modules, q-oscillators naturally give rise to the representations of quantum groups. This was first done for $\mathcal{U}_q(\mathfrak{sl}(2))$ but later extended to the simple Lie (super)algebras of more general type, see e.g. [39]. Since then the quantum oscillators become an important part of the theory of quantum deformed algebras.

Apart from being an interesting mathematical playground for studying $\widehat{\mathcal{Q}}$ and its S-matrix, there is also a more elaborate motivation for considering these representations and the corresponding S-matrix. Firstly, there might be some possible applications in the context of the deformed Hubbard model. Secondly, it turns out that bound state states transform exactly in these representations of q-oscillator algebra. It is important to study bound states due to many reasons. For example, bound states usually play a crucial role in the thermodynamics of the model. In the case of the non-deformed model in AdS/CFT, the thermodynamic Bethe ansatz (TBA) formalism is a key in describing the complete spectrum of the theory [40–43]. The bound state S-matrix then governs the large volume solutions of both the TBA equations and the Y-system. Thus this is one of the first steps towards the TBA and Y-system formalism for the q-deformed model. And, consequently, it might give some useful insights in these structures in the context of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. For example, there might be an interesting link to the recently constructed q-deformed Pohlmeyer reduced version of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring [44, 45] which seems to be closely related to the q-deformed model constructed in [10].

In this work we derive the general bound state S-matrix by employing the methods used in the context of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring [27], but rather than using the Yangian symmetry we make use of the underlying quantum affine algebra. Our approach is based on the identification of invariant subspaces in the scattering theory that are specified by their invariance properties under the Cartan elements of the algebra. Then we use the rest of the algebra generators to related the subspaces to each other in such way finding the explicit form of the corresponding S-matrix. Just as in [27] we find the S-matrix in a factorized form reminiscent of the Drinfeld twist [46].

The paper is organized as follows. In section 2 we discuss the quantum deformation $\mathcal{Q}$ of the extended $\mathcal{U}(\mathfrak{su}(2|2))$ algebra and its affine extension $\widehat{\mathcal{Q}}$. Then in section 3 we introduce the quantum oscillator formalism and construct the supersymmetric short representations of $\widehat{\mathcal{Q}}$. In section 4 we present the explicit derivation of the S-matrix for these representations. Subsequently, in section 5, we specify some explicit cases, we reproduce the fundamental R-matrix and also we give the precise form of the scattering matrix when one of the spaces forms a fundamental representation. We end with a brief discussion on the results and interesting directions for future research. The majority of the S-matrix coefficients and results of the intermediate steps of the performed calculations are spelled out in the appendices.

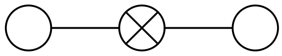

Figure 1. Dynkin diagram for the $\mathfrak{su}(2|2)$ algebra.

2 Quantum affine algebra of extended $\mathcal{U}_q(\mathfrak{su}(2|2))$

In this section we review the quantum deformation of the extended $\mathfrak{su}(2|2)$ algebra [10] and its affine extension [35].

2.1 Quantum deformation of extended $\mathfrak{su}(2|2)$

The quantum deformed extended $\mathfrak{su}(2|2)$ algebra $\mathcal{Q}$ was introduced in [10]. This algebra is generated by the three sets of Chevalley-Serre generators $\{E_j, K_j, F_j\}$ $(j = 1, 2, 3)$ where E_j and F_j are raising and lowering generators respectively and $K_j = q^{H_j}$ are the Cartan generators. We will consider the case when E_2 and F_2 are fermionic generators and the rest are bosonic. This corresponds to the $\mathfrak{su}(2|2)$ Dynkin diagram in Figure 1. In addition, this algebra has two central charges U and $V = q^C$ and two parameters: the deformation parameter q and the coupling constant g. There is also a third parameter α, which describes the relative scaling of E_2 and F_2. Even though it is possible absorb this parameter into the generators by a suitable redefinition, we will keep it unspecified.

Algebra. The commutation relations which include the mixed Chevalley-Serre generators are $(j, k = 1, 2, 3)$

$$K_j E_k = q^{+DA_{jk}} E_k K_j, \qquad K_j F_k = q^{-DA_{jk}} F_k K_j, \qquad [E_j, F_k\} = D_{jj}\delta_{jk}\frac{K_j - K_j^{-1}}{q - q^{-1}}, \tag{2.1}$$

where the associated Cartan matrix A and normalization matrix D are given by

$$A = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}, \qquad D = \mathrm{diag}(+1, -1, -1)\ . \tag{2.2}$$

There are also the unmixed commutation relations, called the Serre relations $(j = 1, 3)$,

$$\begin{aligned} [E_1, E_3] = \{E_2, E_2\} = \big[E_j, [E_j, E_2]\big] - (q - 2 + q^{-1})E_j E_2 E_j = 0, \\ [F_1, F_3] = \{F_2, F_2\} = \big[F_j, [F_j, F_2]\big] - (q - 2 + q^{-1})F_j F_2 F_j = 0. \end{aligned} \tag{2.3}$$

In addition, this algebra satisfies the extended Serre relations that give rise to two central elements U and V as follows,

$$\begin{aligned} g\alpha(1 - U^2V^2) = \big\{[E_2, E_1], [E_2, E_3]\big\} - (q - 2 + q^{-1})E_2E_1E_3E_2, \\ g\alpha^{-1}(V^{-2} - U^{-2}) = \big\{[F_2, F_1], [F_2, F_3]\big\} - (q - 2 + q^{-1})F_2F_1F_3F_2. \end{aligned} \tag{2.4}$$

The central element V is also related to the Cartan generators through

$$V^{-2} = K_1 K_2^2 K_3\ . \tag{2.5}$$

The conventional $\mathcal{U}_q(\mathfrak{su}(2|2))$ algebra is obtained in the limit $g \to 0$.

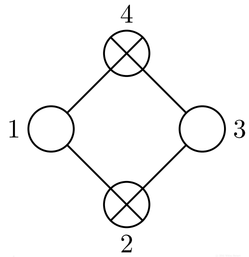

Figure 2. Dynkin diagram for the affine $\widehat{\mathfrak{su}}(2|2)$ algebra.

Coalgebra. The defining relations of $\mathcal{Q}$ are compatible with the following coalgebra structure. The coproduct of the group like elements $X \in \{U, V, K\}$ is $\Delta(X) = X \otimes X$ and the coproducts of the Chevalley-Serre generators E_j and F_j $(j = 1, 3)$ take the standard forms. However the coproducts of the fermionic generators E_2 and F_2 involve an additional braiding factor U, which is one of the central charges of the algebra alluded to in the previous paragraph

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U^{+\delta_{j,2}} \otimes E_j, \qquad \Delta(F_j) = F_j \otimes K_j + U^{-\delta_{j,2}} \otimes F_j. \tag{2.6}$$

The coalgebra can actually be extended to a Hopf algebra. We will give the relevant definitions of the antipode and counit later on.

2.2 Affine Extension

The infinite dimensional quantum affine algebra $\widehat{\mathcal{Q}}$ is the affine extension of $\mathcal{Q}$ introduced in [35]. The affine extension is obtained by adding an additional node into the Dynkin diagram as depicted in Figure 2. The remarkable property of this diagram is that the additional fermionic node is a copy of the second node. Therefore, we introduce the affine Chevalley-Serre generators $\{E_4, F_4, K_4\}$ as a copy of $\{E_2, F_2, K_2\}$ and assume that they satisfy the same commutation relations as are given in (2.1), (2.3) and (2.4) and also have the same coalgebra structure (2.6). Thus, we also introduce an additional set of the parameters g, α and central charges U, V. We distinguish these two sets by adhering subscripts to them arising from the generators to which they are associated,

$$g \to g_k, \qquad \alpha \to \alpha_k, \qquad U \to U_k, \qquad V \to V_k, \qquad \text{with} \qquad k = 2, 4. \tag{2.7}$$

Next, we need to determine the commutation relations $\{E_2, F_4\}$ and $\{E_4, F_2\}$ in such way that they would be compatible with the coalgebra structure

$$\Delta(\{E_2, F_4\}) = \{\Delta(E_2), \Delta(F_4)\} \qquad \text{and} \qquad \Delta(\{E_4, F_2\}) = \{\Delta(E_4), \Delta(F_2)\}\ . \tag{2.8}$$

Algebra. As a result, we obtain the quantum affine algebra $\widehat{\mathcal{Q}}$ [35]. The mixed commutation relations of it are given by $(i, j = 1, 3)$

$$\begin{aligned} K_i E_j &= q^{+DA_{ij}} E_j K_i, & K_i F_j &= q^{-DA_{ij}} F_j K_i, \\ \{E_2, F_4\} &= -\tilde{g}\tilde{\alpha}^{-1}(K_4 - U_2 U_4^{-1} K_2^{-1}), & \{E_4, F_2\} &= \tilde{g}\tilde{\alpha}(K_2 - U_4 U_2^{-1} K_4^{-1}), \\ [E_j, F_j\} &= D_{jj} \frac{K_j - K_j^{-1}}{q - q^{-1}} & [E_i, F_j\} &= 0, \quad \text{for} \ \ i \neq j, \ i + j \neq 6\ . \end{aligned} \tag{2.9}$$

with the two new constants $\tilde{g}$ and $\tilde{\alpha}$ and the associated supersymmetric Cartan matrix A and normalization matrix D given by

$$A = \begin{pmatrix} +2 & -1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -2 & +1 \\ -1 & 0 & +1 & 0 \end{pmatrix}, \qquad D = \mathrm{diag}(1, -1, -1, -1). \tag{2.10}$$

These are supplemented by the following Serre relations ($j = 1, 3$ and $k = 2, 4$)

$$\begin{aligned} &[E_1, E_3] = E_2 E_2 = E_4 E_4 = \{E_2, E_4\} = 0 \ , \\ &[F_1, F_3] = F_2 F_2 = F_4 F_4 = \{F_2, F_4\} = 0 \ , \\ &[E_j, [E_j, E_k]] - (q - 2 + q^{-1}) F_j F_k F_j = 0 \ , \\ &[F_j, [F_j, F_k]] - (q - 2 + q^{-1}) F_j F_k F_j = 0 \ . \end{aligned} \tag{2.11}$$

The central charges are related to the quartic Serre relations as ($k = 2, 4$)

$$\begin{aligned} g_k \alpha_k (1 - U_k^2 V_k^2) &= \big\{[E_k, E_1], [E_k, E_3]\big\} - (q - 2 + q^{-1}) E_k E_1 E_3 E_k \ , \\ g_k \alpha_k^{-1} (V_k^{-2} - U_k^{-2}) &= \big\{[F_k, F_1], [F_k, F_3]\big\} - (q - 2 + q^{-1}) F_k F_1 F_3 F_k \ . \end{aligned} \tag{2.12}$$

and the central charges V_k are related with Cartan charges through ($k = 2, 4$)

$$V_k^{-2} = K_1 K_k^2 K_3 \ . \tag{2.13}$$

Coalgebra. The group-like elements $X \in \{1, K_j, U_k, V_k\}$ ($j = 1, 2, 3, 4$ and $k = 2, 4$) have the coproduct Δ, the antipode S and the counit ε defined in the usual way,

$$\Delta(X) = X \otimes X, \qquad S(X) = X^{-1}, \qquad \varepsilon(X) = 1, \tag{2.14}$$

while the coproducts of the Chevalley-Serre generators are deformed by the central elements U_k as follows ($j = 1, 2, 3, 4$),

$$\begin{aligned} &\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} \otimes E_j, \quad \mathrm{S}(E_j) = -U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} K_j E_j, \quad \varepsilon(E_j) = 0, \\ &\Delta(F_j) = F_j \otimes K_j + U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} \otimes F_j, \qquad \mathrm{S}(F_j) = -U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} F_j K_j^{-1}, \quad \varepsilon(F_j) = 0. \end{aligned} \tag{2.15}$$

It is important to note that the above coproducts are compatible with all the defining relations, including the commutators $\{E_2, F_4\}$ and $\{E_4, F_2\}$ in (2.9). The opposite coproduct is defined as $\Delta^{op} = \mathcal{P}\,\Delta\,\mathcal{P}$ with $\mathcal{P}$ being the graded permutation operator.

Parameter constraints. In general, the quantum affine algebra $\widehat{\mathcal{Q}}$ has seven parameters $g_k, \alpha_k, \tilde{\alpha}, \tilde{g}, q$ ($k = 2, 4$). A suitable choice of them which lead to an interesting fundamental representation was performed in [35]:

$$g_2 = g_4 = g, \qquad \alpha_2 = \alpha_4\, \tilde{\alpha}^{-2} = \alpha, \qquad \tilde{g}^2 = \frac{g^2}{1 - g^2 (q - q^{-1})^2} \ . \tag{2.16}$$

This choice of parameters is also compatible with the bound state representations. Thus in this paper we only consider the quantum affine algebra $\widehat{\mathcal{Q}}$, parametrized by four independent parameters $g, \alpha, \tilde{\alpha}, q$ given in the relations above.

3 Quantum oscillators and representations

In this section we will provide all the necessary background for constructing the bound state S-matrix for the q-deformed Hubbard model. We will build the bound state representation by introducing q-oscillator formalism linking it to the aforementioned quantum affine algebra.

3.1 q-Oscillators

We first introduce the notion of q-oscillators and discuss how to obtain the representations of the quantum deformed algebras using q-oscillators. A concise overview of the q-oscillators and their relation to such representations may be found in [47, 48].

Definitions. The q-oscillator (q-Heisenberg-Weyl algebra) $\mathcal{U}_q(\mathfrak{h}_4)$ is the associative unital algebra consisting of the generators $\{\mathsf{a}^\dagger, \mathsf{a}, w, w^{-1}\}$ that satisfy the following relations

$$\begin{aligned} w\,\mathsf{a}^\dagger &= q\,\mathsf{a}^\dagger w, & qw\,\mathsf{a} &= \mathsf{a}\,w, \\ ww^{-1} &= w^{-1}w = 1, & \mathsf{a}\,\mathsf{a}^\dagger - q\,\mathsf{a}^\dagger\mathsf{a} &= w^{-1}. \end{aligned} \tag{3.1}$$

From the defining relations one can see that the element $w^{-1}(\mathsf{a}^\dagger\mathsf{a} - \frac{w-w^{-1}}{q-q^{-1}})$ is central. As such, we will set it to zero in the remainder. Then one easily obtains

$$\mathsf{a}^\dagger\mathsf{a} = \frac{w-w^{-1}}{q-q^{-1}}, \qquad \mathsf{a}\,\mathsf{a}^\dagger = \frac{qw-q^{-1}w^{-1}}{q-q^{-1}}. \tag{3.2}$$

We will also need to consider the fermionic version of the q-oscillator. The above notion is extended to include fermionic operators by adjusting the defining relations in the following way (we keep the same notation for bosonic and fermionic a, $\mathsf{a}^\dagger$ for now)

$$\begin{aligned} w\,\mathsf{a}^\dagger &= q\,\mathsf{a}^\dagger w, & qw\,\mathsf{a} &= \mathsf{a}\,w, \\ ww^{-1} &= w^{-1}w = 1, & \mathsf{a}\,\mathsf{a}^\dagger + q\,\mathsf{a}^\dagger\mathsf{a} &= w. \end{aligned} \tag{3.3}$$

In this case, the central element is $w(\mathsf{a}^\dagger\mathsf{a} - \frac{w-w^{-1}}{q-q^{-1}})$. Again we set this element to zero, resulting in the following identities

$$\mathsf{a}^\dagger\mathsf{a} = \frac{w-w^{-1}}{q-q^{-1}}, \qquad \mathsf{a}\,\mathsf{a}^\dagger = \frac{qw^{-1}-q^{-1}w}{q-q^{-1}}. \tag{3.4}$$

Of course in the fermionic case the operators $\mathsf{a}, \mathsf{a}^\dagger$ square to zero.

Fock space. The q-oscillator algebra can be used to define representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ in a very simple way. Let us first build the Fock representation of $\mathcal{U}_q(\mathfrak{h}_4)$. For this purpose consider a vacuum state $|0\rangle$ such that

$$\mathsf{a}|0\rangle = 0, \tag{3.5}$$

then the Fock vector space $\mathcal{F}$ generated by the states of the form

$$|n\rangle = (\mathsf{a}^\dagger)^n|0\rangle\,, \tag{3.6}$$

is an irreducible module of $\mathcal{U}_q(\mathfrak{h}_4)$. Let us first consider the bosonic q-oscillators. With the help of the defining relations (3.1) and (3.2) one finds that the action of the oscillator algebra generators on this module is

$$\mathsf{a}^\dagger|n\rangle = |n+1\rangle, \qquad \mathsf{a}|n\rangle = [n]_q|n-1\rangle, \qquad w|n\rangle = q^n|n\rangle. \tag{3.7}$$

This makes it natural to identify $w \equiv q^N$, where N is understood as a number operator. Analogously, fermionic generators are found to act as

$$\mathsf{a}^\dagger|n\rangle = |n+1\rangle, \qquad \mathsf{a}|n\rangle = [2-n]_q|n-1\rangle, \qquad w|n\rangle = q^n|n\rangle. \tag{3.8}$$

However, due to the fermionic nature, n can only take the values 0 and 1 and thus the identity $[2-n]_q = [n]_q$ holds.

Next consider two copies of bosonic q-oscillators $\mathsf{a}_i, \mathsf{a}_i^\dagger, w_i = q^{N_i}$ which mutually commute. Then the Fock space is naturally spanned by vectors of the form

$$|m,n\rangle = (\mathsf{a}_1^\dagger)^m(\mathsf{a}_2^\dagger)^n|0\rangle. \tag{3.9}$$

It is easy to see that under the identification

$$E = \mathsf{a}_2^\dagger\mathsf{a}_1, \qquad F = \mathsf{a}_1^\dagger\mathsf{a}_2, \qquad H = N_2 - N_1, \tag{3.10}$$

the Fock space forms an infinite dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-representation. Moreover, the subspace $\mathcal{F}_M = \mathrm{span}\{\, |m, M-m\rangle \mid m = 0,\dots,M\,\}$ is an irreducible $\mathcal{U}_q(\mathfrak{sl}(2))$-representation of dimension $M+1$. This can be straightforwardly generalized to $\mathfrak{sl}(n)$ and more generally, by including fermionic oscillators, this space is extended to the representations of $\mathfrak{sl}(n|m)$ [39].

Representations of centrally extended $\mathcal{U}_q(\mathfrak{su}(2|2))$. We will now construct the bound state representation for centrally extended $\mathcal{U}_q(\mathfrak{su}(2|2))$ in the q-oscillator language. We need to consider two copies of $\mathfrak{sl}(2)$, a bosonic and a fermionic one. Thus we need four sets of q-oscillators $\mathsf{a}_i, \mathsf{a}_i^\dagger, w_i = q^{N_i}$, where the index $i = 1,2$ denotes bosonic oscillators and $i = 3,4$ – fermionic ones. Using these we write

$$E_1 = \mathsf{a}_2^\dagger\mathsf{a}_1, \qquad F_1 = \mathsf{a}_1^\dagger\mathsf{a}_2, \qquad H_1 = N_2 - N_1, \tag{3.11}$$

$$E_2 = a\ \mathsf{a}_4^\dagger\mathsf{a}_2 + b\ \mathsf{a}_1^\dagger\mathsf{a}_3 \qquad F_2 = c\ \mathsf{a}_3^\dagger\mathsf{a}_1 + d\ \mathsf{a}_2^\dagger\mathsf{a}_4, \qquad H_2 = -C + \frac{N_1+N_3-N_2-N_4}{2}, \tag{3.12}$$

$$E_3 = \mathsf{a}_3^\dagger\mathsf{a}_4, \qquad F_3 = \mathsf{a}_4^\dagger\mathsf{a}_3, \qquad H_3 = N_4 - N_3, \tag{3.13}$$

where C is central. It is then straightforward to check that this set of generators forms a representation of $\mathcal{U}_q(\mathfrak{su}(2|2))$ on the Fock space when restricting to the subspace of total particle number M upon setting

$$ad = \frac{[C+\frac{M}{2}]_q}{[M]_q}, \qquad bc = \frac{[C-\frac{M}{2}]_q}{[M]_q}, \qquad ab = \frac{\mathfrak{P}}{[M]_q}, \qquad cd = \frac{\mathfrak{K}}{[M]_q}. \tag{3.14}$$

In the above $\mathfrak{K},\mathfrak{P}$ correspond to the right hand side of the Serre relations (2.12) following [10]. As a consequence, the central charges satisfy the shortening condition

$$[C]_q^2 - \mathfrak{P}\mathfrak{K} = \left[\tfrac{M}{2}\right]_q^2. \tag{3.15}$$

Here the q-numbers are defined as

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{3.16}$$

This way of constructing representations of the centrally extended algebra reminds us of the procedure used in, e.g. [30], where long representations were be obtained by twisting $\mathfrak{sl}(n|m)$ in a similar way.

In the $q \to 1$ limit the q-oscillators get reduced to the regular oscillators and the representations of them coincide with the superspace formalism introduced in [21]. The identification is as follows

$$\mathsf{a}_{1,2} \leftrightarrow \frac{\partial}{\partial w_{1,2}}, \qquad \mathsf{a}^\dagger_{1,2} \leftrightarrow w_{1,2}, \qquad \mathsf{a}_{3,4} \leftrightarrow \frac{\partial}{\partial \theta_{3,4}}, \qquad \mathsf{a}^\dagger_{3,4} \leftrightarrow \theta_{3,4}. \tag{3.17}$$

Parameterization and central elements. Introducing $V = q^C$ and U as in [35], we rewrite (3.14) as

$$\begin{aligned} ad &= \frac{q^{\frac{M}{2}}V - q^{-\frac{M}{2}}V^{-1}}{q^M - q^{-M}}, & bc &= \frac{q^{-\frac{M}{2}}V - q^{\frac{M}{2}}V^{-1}}{q^M - q^{-M}}, \\ ab &= \frac{g\alpha}{[M]_q}(1 - U^2V^2), & cd &= \frac{g\alpha^{-1}}{[M]_q}(V^{-2} - U^{-2}). \end{aligned} \tag{3.18}$$

which altogether lead to a constraint for U and V,

$$\frac{g^2}{[M]_q^2}(V^{-2} - U^{-2})(1 - U^2V^2) = \frac{(V - q^M V^{-1})(V - q^{-M}V^{-1})}{(q^M - q^{-M})^2}\,. \tag{3.19}$$

This constraint agrees with the one in [35] by identifying $q \to q^M$, $g \to g/[M]_q$. The explicit parametrization of the labels a, b, c, d shall be given a bit further.

3.2 Affine extension

Next we want to consider the affine extension introduced in [35]. Here we will show that our representation allows an affine extension. Analogously to [35] we make the ansatz that the affine charges act as copies of E_2, F_2, H_2. In other words, we set

$$E_4 = a_4\ \mathsf{a}_4^\dagger\mathsf{a}_2 + b_4\ \mathsf{a}_1^\dagger\mathsf{a}_3, \quad F_4 = c_4\ \mathsf{a}_3^\dagger\mathsf{a}_1 + d_4\ \mathsf{a}_2^\dagger\mathsf{a}_4, \quad H_4 = -C_4 + \frac{N_1 + N_3 - N_2 - N_4}{2}. \tag{3.20}$$

Checking all of the commutation relations is straightforward. Also, due to the defining relations (3.18), the equivalent expressions for the affine representation parameters are obtained

$$a_4 d_4 = \frac{q^{\frac{M}{2}} V_4 - q^{-\frac{M}{2}} V_4^{-1}}{q^M - q^{-M}}, \qquad b_4 c_4 = \frac{q^{-\frac{M}{2}} V_4 - q^{\frac{M}{2}} V_4^{-1}}{q^M - q^{-M}},$$
$$a_4 b_4 = \frac{g_4 \alpha_4}{[M]_q}(1 - U_4^2 V_4^2), \qquad c_4 d_4 = \frac{g_4 \alpha_4^{-1}}{[M]_q}(V_4^{-2} - U_4^{-2}). \tag{3.21}$$

However the commutators between the generators E_2 and E_4 and also between F_2 and F_4 induce relations between a_2, a_4, etc. These are found to be

$$a_2 d_4 = \frac{\tilde{g}\tilde{\alpha}^{-1}}{[M]_q}(q^{\frac{M}{2}} U_2 U_4^{-1} V_2 - q^{-\frac{M}{2}} V_4^{-1}), \quad b_2 c_4 = \frac{\tilde{g}\tilde{\alpha}^{-1}}{[M]_q}(q^{-\frac{M}{2}} U_2 U_4^{-1} V_2 - q^{\frac{M}{2}} V_4^{-1}),$$
$$c_2 b_4 = \frac{\tilde{g}\tilde{\alpha}}{[M]_q}(q^{\frac{M}{2}} V_2^{-1} - q^{-\frac{M}{2}} U_2^{-1} U_4 V_2), \quad d_2 a_4 = \frac{\tilde{g}\tilde{\alpha}}{[M]_q}(q^{-\frac{M}{2}} V_2^{-1} - q^{\frac{M}{2}} U_2^{-1} U_4 V_2), \tag{3.22}$$

and agree with [35] upon sending $q \to q^M$, $\tilde{g} \to \frac{\tilde{g}}{[M]_q}$, as in the non-affine case. The tilded $\tilde{g}$, $\tilde{\alpha}$ are not independent but constrained parameters; thus there are 12 constraints for 12 parameters $\{a_k, b_k, c_k, d_k, U_k, V_k\}$.

Hopf algebra and variables. The Hopf algebra structure is as discussed in Section 2. Here we will introduce Zhukowksy variables that will parameterize the representation labels $\{a_k, b_k, c_k, d_k\}$ and central elements U_k, V_k for the bound-state representation. Following [35] we choose

$$g_2 = g_4 = g, \qquad \alpha_2 = \alpha_4\, \tilde{\alpha}^{-2} = \alpha, \qquad \tilde{g}^2 = \frac{g^2}{1 - g^2(q - q^{-1})^2}. \tag{3.23}$$

Note that the powers of q in the expressions above are 1 and not M because $g^2(q - q^{-1})^2$ is invariant under the bound state map $(g, q) \mapsto (g/[M]_q, q^M)$, thus these equations are identical to the ones for the fundamental representation.

Also, there is a relation between the central elements of the algebra,

$$U_4 = \pm U_2^{-1}, \qquad V_4 = \pm V_2^{-1}, \tag{3.24}$$

that are called the two-parameter family of the representation [35]. We shall be using the *plus* relation in our calculations.

The mass-shell constraint (multiplet shortening condition) obtained from the expressions (3.18) and (3.21) reads as

$$(a_k d_k - q^M b_k c_k)(a_k d_k - q^{-M} b_k c_k) = 1, \tag{3.25}$$

and holds independently for $k = 2, 4$. In terms of the conventional $x^{\pm}$ parametrization it becomes

$$\frac{1}{q^M}\left(x^+ + \frac{1}{x^+}\right) - q^M\left(x^- + \frac{1}{x^-}\right) = \left(q^M - \frac{1}{q^M}\right)\left(\xi + \frac{1}{\xi}\right), \tag{3.26}$$

where $\xi = -i\tilde{g}(q - q^{-1})$. One can further introduce a function $\zeta(x)$

$$\zeta(x) = -\frac{x + 1/x + \xi + 1/\xi}{\xi - 1/\xi}, \tag{3.27}$$

in terms of which (3.26) becomes $q^{-M}\zeta(x^+) = q^M\zeta(x^-)$. This parametrization leads to the following expressions of the labels a_k, b_k, c_k, d_k of a 'canonical form':

$$
\begin{aligned}
a_k &= \sqrt{\frac{g}{[M]_q}}\gamma_k, & b_k &= \sqrt{\frac{g}{[M]_q}}\frac{\alpha_k}{\gamma_k}\frac{x_k^- - x_k^+}{x_k^-}, \\
c_k &= \sqrt{\frac{g}{[M]_q}}\frac{\gamma_k}{V_k\,\alpha_k}\frac{i\,q^{\frac{M}{2}}\tilde{g}}{g(x_k^+ + \xi)}, & d_k &= \sqrt{\frac{g}{[M]_q}}\frac{V_k\,\tilde{g}\,q^{\frac{M}{2}}}{i\,g\,\gamma_k}\frac{x_k^+ - x_k^-}{\xi x_k^+ + 1},
\end{aligned}
\tag{3.28}
$$

where the central charges are

$$
U_k^2 = \frac{1}{q^M}\frac{x_k^+ + \xi}{x_k^- + \xi} = q^M\frac{x_k^+}{x_k^-}\frac{\xi x_k^- + 1}{\xi x_k^+ + 1}, \qquad V_k^2 = \frac{1}{q^M}\frac{\xi x_k^+ + 1}{\xi x_k^- + 1} = q^M\frac{x_k^+}{x_k^-}\frac{x_k^- + \xi}{x_k^+ + \xi}, \tag{3.29}
$$

and the relations between $x_2^\pm$, γ_2 and $x_4^\pm$, γ_4 are constrained by (3.22) to be

$$
x_2^\pm = x^\pm, \qquad x_4^\pm = \frac{1}{x^\pm}, \qquad \gamma_2 = \gamma, \qquad \gamma_4 = \frac{i\tilde{\alpha}\gamma}{x^+}. \tag{3.30}
$$

The relation between normalization coefficients α_2 and α_4 was given in (3.23). Finally, the convenient multiplicative evaluation parameter z for the bound state representation is

$$
z = q^{-M}\zeta(x^+) = q^M\,\zeta(x^-). \tag{3.31}
$$

3.3 Summary

For the convenience of the reader we want to summarize all expressions that will be used in the consequent calculations of the bound state S-matrix. We will slightly change the notation for parameters related to the fermionic nodes. We rename the representation parameters and the central elements of the algebra as

$$
\begin{aligned}
(a_2, b_2, c_2, d_2, U_2, V_2) &\to (a, b, c, d, U, V), \\
(a_4, b_4, c_4, d_4, U_4, V_4) &\to (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \widetilde{U}, \widetilde{V}),
\end{aligned}
\tag{3.32}
$$

in order to reserve the subscript position for discriminating states living in different tensor spaces. We will also give some relations that we found to be very useful and handy to use.

Explicit representation. The bound state representation is defined as

$$
|m, n, k, l\rangle = (\mathsf{a}_3^\dagger)^m(\mathsf{a}_4^\dagger)^n(\mathsf{a}_1^\dagger)^k(\mathsf{a}_2^\dagger)^l\,|0\rangle. \tag{3.33}
$$

The total number of excitations is $k + l + m + n = M$. The triple corresponding to the bosonic $\mathfrak{sl}(2)$ is given by

$$
\begin{aligned}
&H_1|m, n, k, l\rangle = (l - k)|m, n, k, l\rangle, \\
&E_1|m, n, k, l\rangle = [k]_q\,|m, n, k-1, l+1\rangle, \qquad F_1|m, n, k, l\rangle = [l]_q\,|m, n, k+1, l-1\rangle.
\end{aligned}
\tag{3.34}
$$

The fermionic part is

$$
\begin{aligned}
&H_3|m, n, k, l\rangle = (n - m)|m, n, k, l\rangle, \\
&E_3|m, n, k, l\rangle = |m+1, n-1, k, l\rangle, \qquad F_3|m, n, k, l\rangle = |m-1, n+1, k, l\rangle.
\end{aligned}
\tag{3.35}
$$

The action of the supercharges is given by

$$\begin{aligned}
H_2|m,n,k,l\rangle &= -\left\{C - \frac{k-l+m-n}{2}\right\}|m,n,k,l\rangle, \\
E_2|m,n,k,l\rangle &= a\,(-1)^m[l]_q\,|m,n+1,k,l-1\rangle + b\,|m-1,n,k+1,l\rangle, \\
F_2|m,n,k,l\rangle &= c\,[k]_q\,|m+1,n,k-1,l\rangle + d\,(-1)^m\,|m,n-1,k,l+1\rangle.
\end{aligned} \tag{3.36}$$

The parameters a,b,c,d are related to the central charges via (3.14). The affine charges are defined exactly in the same way,

$$\begin{aligned}
H_4|m,n,k,l\rangle &= -\left\{\widetilde{C} - \frac{k-l+m-n}{2}\right\}|m,n,k,l\rangle, \\
E_4|m,n,k,l\rangle &= \tilde{a}\,(-1)^m[l]_q|m,n+1,k,l-1\rangle + \tilde{b}\,|m-1,n,k+1,l\rangle, \\
F_4|m,n,k,l\rangle &= \tilde{c}\,[k]_q\,|m+1,n,k-1,l\rangle + \tilde{d}\,(-1)^m\,|m,n-1,k,l+1\rangle.
\end{aligned} \tag{3.37}$$

The representation labels a,b,c,d are given by

$$\begin{aligned}
a &= \sqrt{\frac{g}{[M]_q}}\gamma, & b &= \sqrt{\frac{g}{[M]_q}}\frac{\alpha}{\gamma}\frac{x^- - x^+}{x^-}, \\
c &= \sqrt{\frac{g}{[M]_q}}\frac{\gamma}{\alpha\,V}\frac{i\,q^{\frac{M}{2}}\tilde{g}}{g(x^+ + \xi)}, & d &= \sqrt{\frac{g}{[M]_q}}\frac{\tilde{g}\,q^{\frac{M}{2}}V}{i\,g\,\gamma}\frac{x^+ - x^-}{\xi x^+ + 1},
\end{aligned} \tag{3.38}$$

and the affine parameters $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are acquired by replacing $V \to \widetilde{V} = V^{-1}$, $\gamma \to \frac{i\tilde{\alpha}\gamma}{x^+}$, $\alpha \to \alpha\,\tilde{\alpha}^2$ and $x^\pm \to \frac{1}{x^\pm}$; the corresponding central elements are given by $V = q^C$, $\widetilde{V} = q^{\widetilde{C}}$.

Useful relations. The evaluation parameter z may be expressed explicitly in terms of $x^\pm$ parametrization as

$$z\,(q-q^{-1})(\xi-\xi^{-1}) = -\frac{1}{[M]_q}\Big(x^+ - x^- + \frac{1}{x^+} - \frac{1}{x^-}\Big). \tag{3.39}$$

Then using the identity

$$\xi - \xi^{-1} = \frac{\tilde{g}}{i(q-q^{-1})g^2}, \tag{3.40}$$

one can further show that it is related to the representation labels (3.38) and their affine partners in a very nice way,

$$z = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}(a\tilde{b} - b\tilde{a}), \qquad \frac{1}{z} = \frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}(c\tilde{d} - d\tilde{c}), \tag{3.41}$$

while the consistency conditions (3.22) give

$$z = \frac{1-U^2V^2}{V^2-U^2} = \frac{1-\widetilde{U}^2\widetilde{V}^2}{\widetilde{V}^2-\widetilde{U}^2}\,. \tag{3.42}$$

Rational limit. The rational limit is usually obtained by substituting $q = 1 + h$ and then finding the $h \to 0$ limit. Thus by defining the evaluation parameter (3.31) as $z = q^{-2u}$ we can expand it in series of h as [35]

$$z = 1 - 2hu + \mathcal{O}(h^2), \qquad \text{where} \qquad u = \frac{ig}{2}(x^+ + x^-)(1 + 1/x^+x^-). \tag{3.43}$$

It is noted that the $x^{\pm}$ parameters in (3.43) satisfies the leading order of the following relation which is stemming from the mass-shell constraint (3.26) in the $h \to 0$ limit,

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iM}{g} + 2hMu + \mathcal{O}(h^2). \tag{3.44}$$

In fact, this is consistent with the rational constraint for $x^{\pm}$ parameters [27]. Finally, it would be important to see how the representation parameters reduce in the rational limit. The representation labels (3.38) in the $q \to 1$ limit reduce to the usual (undeformed) labels (a, b, c, d) of [27]. On the other hand, the affine parameters are related to the non-affine ones $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ through [35]

$$M\widetilde{T} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} T \begin{pmatrix} w^{-1} & 0 \\ 0 & wz \end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} 0 & \alpha\tilde{\alpha} \\ -\alpha^{-1}\tilde{\alpha}^{-1} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \tag{3.45}$$

where z is the evaluation parameter given in (3.31), (3.42) and w is defined by

$$w = \frac{\tilde{g}V}{gq^{1/2}} \frac{qU^2 - 1}{V^2U^2 - 1} = \frac{gq^{1/2}}{\tilde{g}V} \frac{U^2 - V^2}{U^2 - q}. \tag{3.46}$$

Since the central elements specialize to $(U, V) \to (\sqrt{\frac{x^+}{x^-}}, 1)$ in the limit $q \to 1$, it is easy to see that the matrix relation (3.45) reduces the following simple form,

$$M\widetilde{T} = T. \tag{3.47}$$

4 The S-matrix

We shall consider the bound state S-matrix which is an intertwining matrix of the tensor space furnished by the vectors

$$|m_1, n_1, k_1, l_1\rangle \otimes |m_2, n_2, k_2, l_2\rangle. \tag{4.1}$$

Here $0 \leq m_1, n_1, m_2, n_2 \leq 1$ and $k_1, l_1, k_2, l_2 \geq 0$ denote the numbers of fermionic and bosonic excitations respectively with the bound state number M_i being the total number of excitations $M_i = m_i + n_i + k_i + l_i$. Thus the S-matrix is the automorphism of the quantum deformed tensor space and is required to be invariant under the coproducts of the affine algebra $\widehat{\mathcal{Q}}$,

$$\mathbb{S}\,\Delta(J) = \Delta^{op}(J)\,\mathbb{S}, \qquad \text{for any} \qquad J \in \widehat{\mathcal{Q}}. \tag{4.2}$$

We normalize the S-matrix in such a way that the state $|0,0,0,M_1\rangle\otimes|0,0,0,M_2\rangle$ is invariant under the scattering. Therefore we will denote the state

$$|0\rangle = |0,0,0,M_1\rangle \otimes |0,0,0,M_2\rangle, \tag{4.3}$$

as the vacuum state.

The invariance under bosonic symmetries ΔH_1 and ΔH_3 requires the total number of fermions and the total number of fermions of one type[1]

$$\begin{aligned} N_f &= m_1+m_2+n_1+n_2+2l_1+2l_2, \\ N_{f_3} &= m_1+m_2+l_1+l_2. \end{aligned} \tag{4.4}$$

to be conserved. This conservation divides the space (4.1) into five types of invariant subspaces of the S-matrix:

I $\quad |0,1,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle$,

Ib $\quad |1,0,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle$,

II $\quad \{|0,0,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle,$
$|0,1,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |0,1,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle\}$,

IIb $\quad \{|0,0,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle,$
$|1,0,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |1,0,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle\}$,

III $\quad \{|0,0,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |0,0,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle,$
$|1,1,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle, |0,1,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle, |1,0,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle\}$.

Subspaces I, Ib and II, IIb are isomorphic, hence we need to find the S-matrix for one of the isomorphic subspaces only. In the following we will consider the scattering in the subspaces I, II and III only.

The invariant subspaces differ by the numbers N_{f,f_3}. By considering the action of the algebra charges it is easy to see that the different subspaces are related to each other in the way shown in figure 3.

Finally we want to give a remark on our choice of the basis. The q-oscillator basis we are considering is orthogonal, but not orthonormal,

$$\langle m',n',k',l'|m,n,k,l\rangle = \frac{1}{[k]!\,[l]!}\,\delta_{m,m'}\delta_{n,n'}\delta_{k,k'}\delta_{l,l'}, \tag{4.5}$$

where $[n]! = [n]_q[n-1]_q\cdots[1]_q$ is the quantum factorial. We shall choose the normalization for the *bra* vectors to be

$$\langle m,n,k,l| := \frac{1}{[k]!\,[l]!}\,|m,n,k,l\rangle^\dagger. \tag{4.6}$$

[1]Note that a bosonic excitation may be interpreted as a combined excitation of two fermionic ones of a different type.

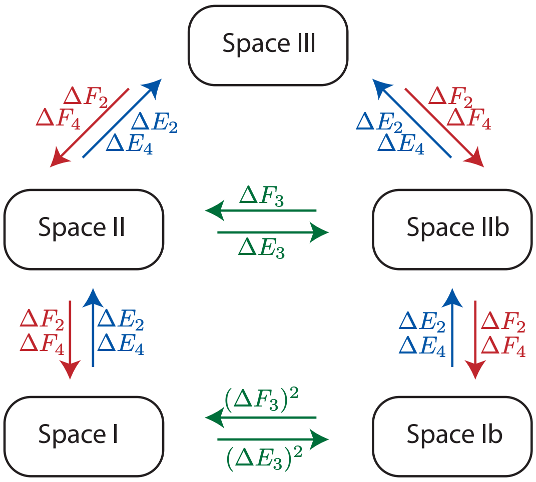

Figure 3. The invariant subspaces of the S-matrix and the algebraic relations between them.

which helps us to normalize the scalar product to unity and avoid the appearance of unpleasant numerical factors of the form $\big([k]![l]!\big)^{-1/2}$ in the derivations. The price we have to pay for this choice of the basis is that the S-matrix elements are not hermitian. However it is easy to obtain the hermitian ones,

$$S_{A\,B\;\text{Hermitian}}^{A'B'} = \left(\frac{[A']!\,[B']!}{[A]!\,[B]!}\right)^{1/2} S_{A\,B}^{A'B'}\,, \tag{4.7}$$

where $A = (m, n, k, l)$ and $[A]! = [k]!\,[l]!$ represents the set of quantum numbers describing the *ket* vector, while primed A' describe *bra* vector, and similarly for B, B'.

For further convenience we introduce these shorthands

$$\begin{aligned} &M = M_1 + M_2, \quad \delta M = M_1 - M_2, \quad K = k_1 + k_2, \quad \delta K = k_1 - k_2, \\ &\bar{k}_i = M_i - k_i - 1, \quad \delta k_i = \bar{k}_i - k_i = M_i - 2k_i - 1, \quad z_{12} = z_1/z_2, \quad \delta u = u_1 - u_2. \end{aligned} \tag{4.8}$$

4.1 Scattering in subspace I

The conserved fermionic numbers (4.4) for the subspace I are $N_f = 2K+2$ and $N_{f_3} = K+2$. Thus for the fixed K $(0 \leq K \leq M_1 + M_2 - 2)$ the dimension of the space is $K+1$ and the states in this space are defined as

$$|k_1, k_2\rangle^{\mathrm{I}} = |0, 1, k_1, M_1 - k_1 - 1\rangle \otimes |0, 1, k_2, M_2 - k_2 - 1\rangle. \tag{4.9}$$

We start by considering the highest weight state (the state with $k_1 = k_2 = 0$). The invariance under ΔH_1 and ΔH_3 requires it to be an eigenstate of the S-matrix,

$$\mathbb{S}\,|0,0\rangle^{\mathrm{I}} = \mathscr{D}\,|0,0\rangle^{\mathrm{I}}. \tag{4.10}$$

Let us compute $\mathscr{D}$. First, we construct the highest weight state by acting with the combination $\Delta E_2 \Delta E_4$ on the vacuum state (4.3) (we use the notation $a_i \equiv a(p_i)$ etc.)

$$\Delta E_2 \Delta E_4\,|0\rangle = q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,U_1V_1)\,|0,0\rangle^{\mathrm{I}}. \tag{4.11}$$

This construction let us to rewrite (4.10) as

$$\begin{aligned}\mathbb{S}\,|0,0\rangle^{\rm I} &= \frac{\mathbb{S}\,\Delta E_2\Delta E_4}{q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1)}\,|0\rangle \\ &= \frac{\Delta^{op}E_2\Delta^{op}E_4\,\mathbb{S}}{q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1)}\,|0\rangle \\ &= -q^{\frac{M_2-M_1}{2}}\,\frac{a_2\tilde{a}_1\,\widetilde{U}_2\widetilde{V}_2 - a_1\tilde{a}_2\,V_2U_2}{a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1}\,|0,0\rangle^{\rm I}, \end{aligned} \tag{4.12}$$

where we have used the invariance condition (4.2) when going from the first to the second line. Comparing (4.12) with (4.10) we find $\mathscr{D}$ to be

$$\mathscr{D} = -q^{\frac{M_2-M_1}{2}}\,\frac{a_2\tilde{a}_1\,\widetilde{U}_2\widetilde{V}_2 - a_1\tilde{a}_2\,V_2U_2}{a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1} = q^{-\delta M/2}\frac{U_2V_2}{U_1V_1}\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}. \tag{4.13}$$

In the $q \to 1$ limit this is an inverse of the result found in [27] due to the interchange of Δ and Δ^{op} with respect to the ones in [27].

Next we define the action of the S-matrix on the subspace I to be

$$\mathbb{S}\,|k_1,k_2\rangle^{\rm I} = \sum_{n=0}^{K}\mathscr{X}_n^{k_1,k_2}|n,K-n\rangle^{\rm I}. \tag{4.14}$$

The strategy for finding coefficients $\mathscr{X}_n^{k_1,k_2}$ will be based on building the generic state $|k_1,k_2\rangle^{\rm I}$ by starting from the highest weight state $|0,0\rangle^{\rm I}$. This will let us relate $\mathscr{X}_n^{k_1,k_2}$ with any k_1, k_2 and n to the already known coefficient $\mathscr{D}$. Thus we need to construct k_1- and k_2-raising operators. We start from inspecting the action of the coproduct of the bosonic charge F_1 giving

$$\Delta F_1|k_1,k_2\rangle^{\rm I} = [\bar{k}_1]_q\,q^{\delta k_2}|k_1+1,k_2\rangle^{\rm I} + [\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\rm I}, \tag{4.15}$$

and

$$\Delta^{op}F_1|k_1,k_2\rangle^{\rm I} = [\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\rm I} + [\bar{k}_2]_q\,q^{\delta k_1}|k_1,k_2+1\rangle^{\rm I}. \tag{4.16}$$

These coproducts do not have the desired properties we want, but are very close. However, with the help of E_2, E_3 and E_4 we can construct a new charge with a similar action:

$$\hat{F}_1 = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}\,\{E_2,[E_4,E_3]\}. \tag{4.17}$$

We call this new charge 'the affine partner' of the raising charge F_1. The action of $\hat{F}_1$ on the state of the form $|0,1,k,l\rangle$ is

$$\hat{F}_1|0,1,k,l\rangle = z\,[l]_q\,|0,1,k+1,l-1\rangle, \tag{4.18}$$

where we have used (3.39) implicitly[2]. Then it is straightforward to see that the new affine raising charge acts on generic states in subspace I as

$$\Delta\hat{F}_1\,|k_1,k_2\rangle^{\rm I} = z_1\,[\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\rm I} + z_2\,q^{\delta k_1}\,[\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\rm I}. \tag{4.20}$$

And the action of $\Delta^{op}\hat{F}_1$ is

$$\Delta^{op}\hat{F}_1\,|k_1,k_2\rangle^{\rm I} = z_1\,q^{\delta k_2}\,[\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\rm I} + z_2\,[\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\rm I}. \tag{4.21}$$

By combining $\Delta\hat{F}_1$ with ΔF_1 we obtain composite operators having the action of the desired form – raising k_1 and k_2 separately:

$$|k_1+1,k_2\rangle^{\rm I} = \frac{1}{[\bar{k}_1]_q}\frac{\Delta\hat{F}_1 - z_2\,q^{\delta k_1}\Delta F_1}{z_1 - z_2\,q^{\delta k_1+\delta k_2}}\,|k_1,k_2\rangle^{\rm I}, \tag{4.22}$$

$$|k_1,k_2+1\rangle^{\rm I} = \frac{1}{[\bar{k}_2]_q}\frac{z_1\,\Delta F_1 - q^{\delta k_2}\Delta\hat{F}_1}{z_1 - z_2\,q^{\delta k_1+\delta k_2}}\,|k_1,k_2\rangle^{\rm I}. \tag{4.23}$$

Then by induction we find that the generic state $|k_1,k_2\rangle^{\rm I}$ may be constructed as

$$|k_1,k_2\rangle^{\rm I} = \frac{\prod_{j_2=0}^{k_2-1}(z_1\,\Delta F_1 - q^{\delta j_2}\Delta\hat{F}_1)\prod_{i_1=0}^{k_1-1}(\Delta\hat{F}_1 - z_2\,q^{\delta i_1}\Delta F_1)}{\prod_{i=1}^{k_1}[M_1-i]_q\prod_{j=1}^{k_2}[M_2-j]_q\prod_{j=1}^{k_1+k_2}(z_1 - z_2 q^{M-2j})}\,|0,0\rangle^{\rm I}. \tag{4.24}$$

Finding $\mathscr{X}_n^{k_1,k_2}$ is then straightforward. We only need to act with the S-matrix on the expression above and sandwich with *bra*-vector as

$$\mathscr{X}_n^{k_1,k_2} = {}^{\rm I}\langle n,K-n|\,\mathbb{S}\,|k_1,k_2\rangle^{\rm I}. \tag{4.25}$$

Performing similar steps as we did in (4.12) and employing the relations

$$\begin{aligned}
&(\Delta^{op}\hat{F}_1 - z_2\,q^{\delta k_1}\Delta^{op}F_1)\,|n_1,n_2\rangle^{\rm I} \\
&\qquad = [\bar{n}_2]_q\,z_2\,(1-q^{\delta k_1+\delta n_1})\,|n_1,n_2+1\rangle^{\rm I} + [\bar{n}_1]_q\,(z_1\,q^{\delta n_2} - z_2\,q^{\delta k_1})\,|n_1+1,n_2\rangle^{\rm I}, && (4.26)\\
&(z_1\,\Delta^{op}F_1 - q^{\delta k_2}\Delta^{op}\hat{F}_1)\,|n_1,n_2\rangle^{\rm I} \\
&\qquad = [\bar{n}_1]_q\,z_1\,(1-q^{\delta n_2+\delta k_2})\,|n_1+1,n_2\rangle^{\rm I} + [\bar{n}_2]_q\,(z_1 q^{\delta n_1} - z_2 q^{\delta k_2})|n_1,n_2+1\rangle^{\rm I}, && (4.27)
\end{aligned}$$

[2]For the consistency of the algebra we also give a definition of the 'affine lowering charge' $\hat{E}_1$:

$$\hat{E}_1 = \frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}\,\{F_2,[F_4,F_3]\}, \qquad \hat{E}_1|0,1,k,l\rangle = \frac{[l]_q}{z}\,|0,1,k-1,l+1\rangle. \tag{4.19}$$

we find the coefficients of the S-matrix in the subspace I to be

$$\begin{aligned}
\mathscr{X}_n^{k_1,k_2} = \mathscr{D} \frac{\prod_{i=1}^{n}[M_1-i]_q \prod_{j=1}^{K-n}[M_2-j]_q}{\prod_{i=1}^{k_1}[M_1-i]_q \prod_{j=1}^{k_2}[M_2-j]_q} \frac{1}{\prod_{l=1}^{K}(z_{12}-q^{M-2l})} \\
\times \sum_{m=0}^{k_1} \Bigg(z_{12}^{n-m} q^{k_2(n-m)-k_1 m-k_2^2} \begin{bmatrix} k_1 \\ m \end{bmatrix}_q \begin{bmatrix} k_2 \\ n-m \end{bmatrix}_q \\
\times \prod_{p=0}^{m-1} (z_{12}\, q^{M_2+2p} - q^{M_1}) \prod_{p=1+m}^{k_1} (1-q^{2(M_1-p)}) \\
\times \prod_{p=1}^{n-m} (1-q^{2(M_2-K+n-p)}) \prod_{p=-m}^{k_2-n-1} (z_{12}\, q^{M_1+2p} - q^{M_2}) \Bigg),
\end{aligned} \tag{4.28}$$

where $z_{12} = \frac{z_1}{z_2}$ and the q-binomials are defined as

$$\begin{bmatrix} a \\ b \end{bmatrix}_q \equiv \frac{[a]_q!}{[b]_q![a-b]_q!}. \tag{4.29}$$

Apart from the prefactor $\mathscr{D}$, this expression only depends on the quotient z_{12} and on simple q-factors. The expression above has exactly the form that one would expect to obtain by an educated guess relying on the one given in [27].

Quantum $6j$-Symbol. The coefficients $\mathscr{X}_n^{k_1,k_2}$ of the bound state S-matrix may be regarded as the coefficients which arise in the fusion rule of the irreducible representations of $\mathcal{U}_q(su(2))$, thus it is expected that the expression (4.28) is related to the quantum $6j$-symbol, which is the q-deformation of $6j$-symbol and was first introduced in [49].

In order to see the relation with the quantum $6j$-symbol, we first rewrite (4.28) in terms of quantum factorials. This can be done by introducing the notation $z_{12} = q^{-2\delta u}$ and using the following identity several times,

$$\frac{q^A - q^B}{q-q^{-1}} = q^{\frac{A+B}{2}} \left[\frac{A-B}{2}\right]_q. \tag{4.30}$$

Secondary, we shift the index of summation m to $M_1 - 2 - m$. After some computation, we obtain the following form,

$$\begin{aligned}
\mathscr{X}_n^{k_1,k_2} = \mathscr{D}\, q^{(k_1-n)(k_2-n+\delta u+\frac{\delta M}{2})} \frac{[M_2-k_2-1]!}{[M_1-n-1]!} \frac{[\delta u + \frac{M}{2} - 1 - K]!}{[\delta u + \frac{M}{2} - 1]!} \\
\times [k_1]![k_2]![\delta u + \tfrac{\delta M}{2}]![\delta u - \tfrac{\delta M}{2} - k_2 + n + 1]! \\
\times \sum_{m\geq 0} [m+1]! \,([m-M_1+2+k_1]!\,[m-M_1+2+n]!\,[k_2-n+M_1-2-m]! \\
\times [m+\delta u - \tfrac{M}{2}+2]!\,[\delta u + \tfrac{M}{2} - 1 - m]!\,[M_1-2-m]!\,[M-K-3-m]!)^{-1}.
\end{aligned} \tag{4.31}$$

where the summation index m runs over the non-negative integers such that all arguments of the quantum factorials, which do not include δu, are non-negative. Finally, replacing the six variables $(M_1, M_2, k_1, k_2, n, \delta u)$ by the appropriate combinations of $(j_1, j_2, j_3, j_4, j_5, j_6)$ as (see also [27]),

$$\begin{aligned} j_1 &= \tfrac{1}{2}(K - n + \tfrac{\delta M}{2} + \delta u), & j_4 &= \tfrac{1}{2}(\tfrac{\delta M}{2} - 1 + k_2 - \delta u),\\ j_2 &= \tfrac{1}{2}(\tfrac{M}{2} - 2 - k_2 - \delta u), & j_5 &= \tfrac{1}{2}(\tfrac{M}{2} - 1 - K + n + \delta u),\\ j_3 &= \tfrac{1}{2}(M_1 - 2 - k_1 - n), & j_6 &= \tfrac{1}{2}(M_2 - 1), \end{aligned} \tag{4.32}$$

we have found that the expression (4.28) obtains a quite elegant form

$$\begin{aligned} \mathscr{X}_n^{k_1,k_2} &= \mathscr{D}\,(-1)^{j_1-j_3-j_4+2j_5+j_6} q^{(j_1-j_2+j_3)(j_1+j_2-j_4-j_5)} \frac{[j_1+j_2-j_3]!}{[1+j_1+j_2+j_3]!} \frac{[j_1+j_5-j_6]!}{[j_1+j_5+j_6]!} \\ &\times [j_3-j_4+j_5]!\,[j_3+j_4-j_5]!\,[j_2-j_4+j_6]!\,[-j_2+j_4+j_6]! \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}, \end{aligned} \tag{4.33}$$

where we have defined the rescaled quantum $6j$-symbol by

$$\begin{aligned} \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} &= \sum_{m\geq 0} (-1)^m [m+1]! \big([j_{1245}-m]!\,[j_{1346}-m]!\,[j_{2356}-m]! \\ &\qquad \times [m-j_{123}]!\,[m-j_{345}]!\,[m-j_{246}]!\,[m-j_{156}]!\big)^{-1}. \end{aligned} \tag{4.34}$$

Here we have used the bookkeeping notations $j_{abc} = j_a + j_b + j_c$ and $j_{abcd} = j_a + j_b + j_c + j_d$. The above expression is related with the quantum $6j$-symbol introduced in [49] as

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} &= \sqrt{2j_3-1}\sqrt{2j_6-1}\,(-1)^{-j_1-j_2+2j_3+j_4+j_5} \\ &\times \Delta(j_1,j_2,j_3)\Delta(j_1,j_5,j_6)\Delta(j_2,j_4,j_6)\Delta(j_3,j_4,j_5) \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}, \end{aligned} \tag{4.35}$$

where the triangle coefficient $\Delta(a, b, c)$ is defined to be

$$\Delta(a,b,c) = \left(\frac{[a+b-c]!\,[b+c-a]!\,[c+a-b]!}{[1+a+b+c]!} \right)^{1/2}. \tag{4.36}$$

Rational Limit. In order to find the rational limit of the matrix $\mathscr{X}$ (4.28) we first use the expansion (3.43) for the spectral parameter z. This leads to

$$\begin{aligned} \mathscr{X}_n^{k_1,k_2} &= \mathscr{D} \frac{\prod_{i=1}^{n}[M_1-i]_q \prod_{j=1}^{K-n}[M_2-j]_q}{\prod_{i=1}^{k_1}[M_1-i]_q \prod_{j=1}^{k_2}[M_2-j]_q} \frac{1}{\prod_{l=1}^{K}(z_{12}^{1/2}[\delta u]_q + q^{M/2-l}[\frac{M}{2}-l]_q)} \\ &\times \sum_{m=0}^{k_1} \Bigg(z_{12}^{n-m} q^{k_2(n-m)-k_1 m-k_2^2} \begin{bmatrix} k_1 \\ m \end{bmatrix}_q \begin{bmatrix} k_2 \\ n-m \end{bmatrix}_q \\ &\qquad \times \prod_{p=0}^{m-1} \left(z_{12}^{1/2}\, q^{M_2/2+p} \left[\delta u - \frac{M_2}{2} - p\right]_q + q^{M_1/2}\left[\frac{M_1}{2}\right]_q \right) \\ &\qquad \times \prod_{p=-m}^{k_2-n-1} \left(z_{12}^{1/2}\, q^{M_1/2+p} \left[\delta u - \frac{M_1}{2} - p\right]_q + q^{M_2/2}\left[\frac{M_2}{2}\right]_q \right) \\ &\qquad \times \prod_{p=1+m}^{k_1} q^{M_1-p}[M_1-p]_q \prod_{p=1}^{n-m} q^{M_2-K+n-p}[M_2-K+n-p]_q \Bigg), \end{aligned} \tag{4.37}$$

where $\delta u = u_1 - u_2$. Now we are ready to find $q \to 1$ limit. The q-numbers $[x]_q$ coalesce to x, thus (4.37) becomes

$$\mathscr{X}_n^{k_1,k_2} = \mathscr{D}\,\frac{\prod_{i=1}^{n}(M_1-i)\prod_{j=1}^{K-n}(M_2-j)}{\prod_{i=1}^{k_1}(M_1-i)\prod_{j=1}^{k_2}(M_2-j)}\,\frac{1}{\prod_{l=1}^{K}(\delta u+\frac{M}{2}-l)}$$
$$\times\sum_{m=0}^{k_1}\left(\binom{k_1}{m}\binom{k_2}{n-m}\prod_{p=0}^{m-1}\left(\delta u+\frac{\delta M}{2}-p\right)\prod_{p=-m}^{k_2-n-1}\left(\delta u-\frac{\delta M}{2}-p\right)\right.$$
$$\left.\times\prod_{p=1+m}^{k_1}(M_1-p)\prod_{p=1}^{n-m}(M_2-K+n-p)\right). \tag{4.38}$$

This result coincides exactly with the expression obtained in [27][3]

Classical Limit. It is also important to find the classical limit $g \to \infty$ of (4.28). This limit corresponds to the case 'T(h)' in the analysis of the classical algebra [34], where the deformation parameter q is expanded as

$$q = 1 + \frac{h}{2g} + \mathcal{O}(g^{-2}), \tag{4.39}$$

and the $x^{\pm}$ parameters become

$$x^{\pm} = x\left[1 \pm \frac{hM}{2g}\frac{(x+\tilde{h})(1+1/x\tilde{h})}{x-x^{-1}} + \mathcal{O}(g^{-2})\right], \quad \text{where} \quad \tilde{h} = -\frac{ih}{\sqrt{1-h^2}}\,. \tag{4.40}$$

The above expressions are compatible with the constraint (3.26) up to a given order. Since $\xi \to \tilde{h}$ and $x^{\pm} \to x$ in the classical limit, it is easy to see that the evaluation parameter z reduces to[4]

$$z = -\frac{(x+\tilde{h})(1+1/x\tilde{h})}{\tilde{h}-\tilde{h}^{-1}} = -\frac{C+D}{C-D}\,, \tag{4.41}$$

where elements C and D are the classical limits of $U = q^D$ and $V = q^C$ respectively, and are given by

$$D = \tfrac{1}{2}(z+1)\,\tilde{q}, \qquad C = \tfrac{1}{2}(z-1)\,\tilde{q}, \qquad \text{where} \qquad \tilde{q} = -M\frac{\tilde{h}-\tilde{h}^{-1}}{x-x^{-1}}. \tag{4.42}$$

[3] The normalization of the evaluation parameter is slightly different in here, $u_{\text{here}} = -2u_{[27]}$.

[4] The classical evaluation parameter given in [34] is related with ours as $z^{cl}_{[34]} = (z^{cl}_{\text{here}})^{-1}$ and the classical parameter is $x_{[34]} = -ih\tilde{h}^{-1}(x_{\text{here}} + \tilde{h})$.

With these preliminaries, we find the classical limit of (4.28) to be

$$\begin{aligned}\mathscr{X}_n^{k_1,k_2} \sim (1+\mathscr{D}_{cl}) & \frac{\prod_{i=1}^{n}(M_1-i)\prod_{j=1}^{K-n}(M_2-j)}{\prod_{i=1}^{k_1}(M_1-i)\prod_{j=1}^{k_2}(M_2-j)}\left(1+\frac{h}{g}\sum_{l=1}^{k_1+k_2}\frac{\frac{M}{2}-l}{z_{12}-1}\right)\\ &\times\sum_{m=0}^{k_1}\left[\left(-\frac{h}{g}\frac{1}{z_{12}-1}\right)^{k_1+n-2m} z_{12}^{n-m}\binom{k_1}{m}\binom{k_2}{n-m}\right.\\ &\times\left(1+\frac{h}{g}\sum_{p=0}^{m-1}\frac{z_{12}\left(\frac{M_2}{2}+p\right)-\frac{M_1}{2}}{z_{12}-1}+\frac{h}{g}\sum_{p=-m}^{k_2-n-1}\frac{z_{12}\left(\frac{M_1}{2}+p\right)-\frac{M_2}{2}}{z_{12}-1}\right.\\ &\left.\left.+\frac{h}{2g}(k_2(n-m)-k_1m-k_2^2)\right)\prod_{p=1+m}^{k_1}(M_1-p)\prod_{p=1}^{n-m}(M_2-K+n-p)\right],\end{aligned} \tag{4.43}$$

where $\mathscr{D}_{cl}$ is $\mathcal{O}(g^{-1})$ term of $\mathscr{D}$ in (4.28). Since the binomial coefficients force the index m to be $m \le \min\{k_1, n\}$, we will discuss the two possible cases separately. They are the $n \neq k_1$ case (off-diagonal sector) and the $n = k_1$ case (diagonal sector).

Off-diagonal sector. In the case when n is different from k_1, it is further classified by two more cases – if n is bigger or smaller than k_1. Firstly, in the $n > k_1$ case, the leading order of (4.43) is $\mathcal{O}(g^{-(n-k_1)})$ with $m = k_1$. Therefore the $\mathcal{O}(g^{-1})$ term, which contributes to the classical r-matrix, is obtained by setting $n = k_1 + 1$. In this situation, the classical limit of (4.43) turns out to be of a simple form,

$$\mathscr{X}_{k_1+1}^{k_1,k_2} \sim -\frac{h}{g}\frac{z_1}{z_1-z_2}k_2(M_1-k_1-1)\,. \tag{4.44}$$

Secondary, in the $n < k_1$ case, the leading order is $\mathcal{O}(g^{-(k_1-n)})$ with $m = n$. Therefore, the $\mathcal{O}(g^{-1})$ contribution is given by $n = k_1 - 1$. In this case the amplitude becomes

$$\mathscr{X}_{k_1-1}^{k_1,k_2} \sim -\frac{h}{g}\frac{z_2}{z_1-z_2}k_1(M_2-k_2-1)\,. \tag{4.45}$$

The other matrix elements do not contribute to the classical r-matrix.

Diagonal sector. This is the $n = k_1$ case and it needs a more elaborate treatment in comparison with the off-diagonal sector. In this case the leading order in (4.43) is $\mathcal{O}(1)$ with $m = k_1 = n$. Thus the classical limit turns out to be

$$\begin{aligned}\mathscr{X}_{k_1}^{k_1,k_2} \sim 1+\mathscr{D}_{cl}-\frac{h}{2g}(k_1^2+k_2^2)+\frac{h}{g}\frac{1}{z_1-z_2}&\left[\sum_{l=1}^{k_1+k_2} z_2\left(\frac{M}{2}-l\right)\right.\\ &\left.+\sum_{p=0}^{k_1-1}\left(\frac{z_1M_2-z_2M_1}{2}+z_1p\right)+\sum_{p=-k_1}^{k_2-k_1-1}\left(\frac{z_1M_1-z_2M_2}{2}+z_1p\right)\right].\end{aligned} \tag{4.46}$$

Full Rational Limit. It is noted that the classical limit still depends on the deformation parameter h. This allows us to take $h \to 0$ limit further, which corresponds to the case "R(full)" in the analysis of [34]. In this limit, the classical evaluation parameter (4.41) reads,

$$z \sim 1 - \frac{h}{g}u + \mathcal{O}(h^2), \qquad \text{with} \qquad u = x + \frac{1}{x}. \tag{4.47}$$

Then the off-diagonal elements of the classical r-matrix (4.44) and (4.45) turns out to be

$$\mathscr{X}^{k_1,k_2}_{k_1+1} \sim \frac{1}{\delta u} k_2(M_1 - k_1 - 1)\,, \qquad \mathscr{X}^{k_1,k_2}_{k_1-1} \sim \frac{1}{\delta u} k_1(M_2 - k_2 - 1)\,. \tag{4.48}$$

On the other hand, the diagonal elements (4.46) reduce to

$$\mathscr{X}^{k_1,k_2}_{k_1} \sim 1 + \mathscr{D}_{cl} - \frac{1}{\delta u}\left[\sum_{l=1}^{k_1+k_2}\left(\tfrac{M}{2} - l\right) + \sum_{p=0}^{k_1-1}\left(-\tfrac{\delta M}{2} + p\right) + \sum_{p=-k_1}^{k_2-k_1-1}\left(\tfrac{\delta M}{2} + p\right)\right]. \tag{4.49}$$

The above expressions (4.48) and (4.49) agree with the classical limits of rational case [27].

4.2 Scattering in subspace II

The S-matrix in the subspace II is defined to be

$$\mathbb{S}\,|k_1,k_2\rangle^{\rm II}_i = \sum_{n=0}^{K}\sum_{j=1}^{4} |n, K-n\rangle^{\rm II}_j \left(\mathscr{Y}^{k_1,k_2}_n\right)^j_i\,, \tag{4.50}$$

and the standard $4N+2$–dimensional basis is

$$\begin{aligned}
|k_1,k_2\rangle^{\rm II}_1 &= |0,1,k_1,M_1-k_1-1\rangle \otimes |0,0,k_2,M_2-k_2\rangle,\\
|k_1,k_2\rangle^{\rm II}_2 &= |0,0,k_1,M_1-k_1\rangle \otimes |0,1,k_2,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle^{\rm II}_3 &= |0,1,k_1,M_1-k_1-1\rangle \otimes |1,1,k_2-1,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle^{\rm II}_4 &= |1,1,k_1-1,M_1-k_1-1\rangle \otimes |0,1,k_2,M_2-k_2-1\rangle.
\end{aligned} \tag{4.51}$$

We shall express the coefficients $(\mathscr{Y}^{k_1,k_2}_n)^j_i$ in terms of already known $\mathscr{X}^{k_1,k_2}_n$ with the help of the charges ΔE_2 and ΔE_4 that relate the states in the subspace II to the states in subspace I:

$$\Delta E_2\,|k_1,k_2\rangle^{\rm II}_j = Q_j(k_1,k_2)\,|k_1,k_2\rangle^{\rm I}, \qquad \Delta E_4\,|k_1,k_2\rangle^{\rm II}_j = \widetilde{Q}_j(k_1,k_2)\,|k_1,k_2\rangle^{\rm I}. \tag{4.52}$$

The coefficients $Q_j(k_1,k_2)$, $\widetilde{Q}_j(k_1,k_2)$ and their partners for $\Delta^{op}E_2$ and $\Delta^{op}E_4$ are spelled out in the Appendix A.1.

The strategy of finding $\mathscr{X}_n^{k_1,k_2}$ is the following. We start by considering the matrix element

$$\begin{aligned}
{}^{\mathrm{I}}\langle n, K-n|\, \Delta^{op} E_2\, \mathbb{S}\, |k_1, k_2\rangle_i^{\mathrm{II}} &= \sum_{j=1}^{4}\sum_{m=0}^{K} {}^{\mathrm{I}}\langle n, K-n|\, \Delta^{op} E_2\, |m, K-m\rangle_j^{\mathrm{II}}\, \left(\mathscr{Y}_m^{k_1,k_2}\right)_i^j \\
&= \sum_{j=1}^{4}\sum_{m=0}^{K} {}^{\mathrm{I}}\langle n, K-n|m, K-m\rangle^{\mathrm{I}}\, Q_j^{op}(m, K-m)\, \left(\mathscr{Y}_m^{k_1,k_2}\right)_i^j \\
&= \sum_{j=1}^{4} Q_j^{op}(n, K-n)\, \left(\mathscr{Y}_n^{k_1,k_2}\right)_i^j .
\end{aligned} \tag{4.53}$$

Next, using the invariance of the S-matrix $\Delta^{op} E_2\, \mathbb{S} = \mathbb{S}\, \Delta E_2$, we could rewrite (4.53) as

$$\begin{aligned}
{}^{\mathrm{I}}\langle n, K-n|\, \mathbb{S}\, \Delta E_2\, |k_1, k_2\rangle_i^{\mathrm{II}} &= {}^{\mathrm{I}}\langle n, K-n|\, \mathbb{S}\, |k_1, k_2\rangle^{\mathrm{I}}\, Q_i(k_1, k_2) \\
&= \sum_{m=0}^{N} {}^{\mathrm{I}}\langle n, K-n|m, K-m\rangle^{\mathrm{I}}\, \mathscr{X}_m^{k_1,k_2}\, Q_i(k_1, k_2) \\
&= \mathscr{X}_n^{k_1,k_2}\, Q_i(k_1, k_2).
\end{aligned} \tag{4.54}$$

Also we get a similar set of relations by considering the charge E_4. These relations can be conveniently summarized in terms of matrix equation

$$\begin{aligned}
&\begin{pmatrix} Q_1^{op}(n, K-n) & Q_2^{op}(n, K-n) & Q_3^{op}(n, K-n) & Q_4^{op}(n, K-n) \\ \tilde{Q}_1^{op}(n, K-n) & \tilde{Q}_2^{op}(n, K-n) & \tilde{Q}_3^{op}(n, K-n) & \tilde{Q}_4^{op}(n, K-n) \end{pmatrix} \mathscr{Y}_n^{k_1,k_2} = \\
&\qquad = \mathscr{X}_n^{k_1,k_2} \begin{pmatrix} Q_1(k_1, k_2) & Q_2(k_1, k_2) & Q_3(k_1, k_2) & Q_4(k_1, k_2) \\ \tilde{Q}_1(k_1, k_2) & \tilde{Q}_2(k_1, k_2) & \tilde{Q}_3(k_1, k_2) & \tilde{Q}_4(k_1, k_2) \end{pmatrix},
\end{aligned} \tag{4.55}$$

giving a total number of 8 constraints. However, there is a further need of 8 more constraints. These can be obtained by considering a composite operator

$$\check{E}_2 = e_0\Big(e_1\, \hat{F}_1 F_3 F_2 + e_2\, F_1 F_3 F_2 + e_3\, F_3 F_2 F_1\Big), \tag{4.56}$$

where

$$\begin{aligned}
e_0 &= q^{1+K+\frac{M_1}{2}}(q^M z_1 - q^{2K+2} z_2)^{-1}, & e_1 &= (q - q^{-1}), \\
e_2 &= q^{M_2+2n}(q^{-2-2K} z_1 - q^{2-M} z_2), & e_3 &= -q^{M_2+2n}(q^{-1-2K} z_1 - q^{1-M} z_2),
\end{aligned} \tag{4.57}$$

and its affine partner $\check{E}_4$. These operators act on the states in the subspace II as

$$\begin{aligned}
\Delta\check{E}_2 |k_1, k_2\rangle_i^{\mathrm{II}} = & Z_i(k_1, k_2)|k_1, k_2\rangle^{\mathrm{I}} + Z_i^{+}(k_1, k_2)|k_1+1, k_2-1\rangle^{\mathrm{I}} \\
& + Z_i^{-}(k_1, k_2)|k_1-1, k_2+1\rangle^{\mathrm{I}},
\end{aligned} \tag{4.58}$$

giving

$$
\begin{aligned}
{}^{\mathrm{I}}\langle n, K-n|\ \Delta^{op}\check{E}_2\ \mathbb{S}\ |k_1,k_2\rangle_i^{\mathrm{II}} &= \sum_{j=1}^{4}\sum_{m=0}^{K}\ {}^{\mathrm{I}}\langle n, K-n|\ \Delta^{op}\check{E}_2\ |m, K-m\rangle_j^{\mathrm{II}}\ \left(\mathscr{Y}_m^{k_1,k_2}\right)_i^j \\
&= \sum_{j=1}^{4}\Big(Z_j^{op}(n,K-n)\ \left(\mathscr{Y}_n^{k_1,k_2}\right)_i^j + Z_j^{+,op}(n-1,K-n+1)\ \left(\mathscr{Y}_{n-1}^{k_1,k_2}\right)_i^j \\
&\qquad\qquad + Z_j^{-,op}(n+1,K-n-1)\ \left(\mathscr{Y}_{n+1}^{k_1,k_2}\right)_i^j\Big)\,.
\end{aligned} \tag{4.59}
$$

The coefficients (4.57) are chosen in a such way that the 'non-diagonal' part of this relation is vanishing, $Z_j^{+,op}(n-1,K-n+1) = Z_j^{-,op}(n+1,K-n-1) = 0$. Therefore the only surviving part of (4.59) is

$$
{}^{\mathrm{I}}\langle n, K-n|\ \Delta^{op}\check{E}_2\ \mathbb{S}\ |k_1,k_2\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4} Z_j^{op}(n,K-n)\ \left(\mathscr{Y}_n^{k_1,k_2}\right)_i^j\,. \tag{4.60}
$$

This results in the following matrix equation for $Z_j^{op}(n,K-n)$:

$$
\begin{aligned}
&\Big(Z_1^{op}(n,K-n)\ Z_2^{op}(n,K-n)\ Z_3^{op}(n,K-n)\ Z_4^{op}(n,K-n)\Big)\,\mathscr{Y}_n^{k_1,k_2} \\
&\quad= \Big(Z_1(k_1,k_2)\ Z_2(k_1,k_2)\ Z_3(k_1,k_2)\ Z_4(k_1,k_2)\Big)\,\mathscr{X}_n^{k_1,k_2} \\
&\qquad+ \Big(Z_1^{+}(k_1,k_2)\ 0\ Z_3^{-}(k_1,k_2)\ 0\Big)\,\mathscr{X}_n^{k_1+1,k_2-1} \\
&\qquad+ \Big(0\ Z_2^{-}(k_1,k_2)\ 0\ Z_4^{-}(k_1,k_2)\Big)\,\mathscr{X}_n^{k_1-1,k_2+1},
\end{aligned} \tag{4.61}
$$

plus a similar set of equations arising from the affine charge $\check{E}_4$. Both sets can further be united into a compact matrix form

$$
A\,\mathscr{Y}_n^{k_1,k_2} = B\,\mathscr{X}_n^{k_1,k_2} + B^{+}\,\mathscr{X}_n^{k_1+1,k_2-1} + B^{-}\,\mathscr{X}_n^{k_1-1,k_2+1}, \tag{4.62}
$$

which multiplied from the left by A^{-1} defines all coefficients of $\mathscr{Y}_n^{k_1,k_2}$ in terms of already known $\mathscr{X}_n^{k_1,k_2}$, $\mathscr{X}_n^{k_1\pm1,k_2\mp1}$. The explicit expressions of matrices A, A^{-1}, B, $B^{\pm}$, their $q\to1$ limit and the coefficients $Z_i(k_1,k_2)$, $Z_j^{op}(n,K-n)$ and their affine partners are spelled out the Appendix A.1.

To finalize we want to note that not all of the constraints in (4.61) are linearly independent. The set of independent constraints is chosen in such way that the inverse matrix A^{-1} would exist.

4.3 Scattering in subspace III

We will compute the S-matrix components in the subspace III in a very similar way as we did in the previous section for the scattering in subspace II. We start by defining the S-matrix for the subspace III as

$$
\mathbb{S}\,|k_1,k_2\rangle_i^{\mathrm{III}} = \sum_{n=0}^{K}\sum_{j=1}^{6} |n, K-n\rangle_j^{\mathrm{III}}\left(\mathscr{Z}_n^{k_1,k_2}\right)_i^j\,, \tag{4.63}
$$

where the standard basis for the $6N$-dimensional vector space is

$$\begin{aligned}
|k_1,k_2\rangle_1^{\rm III} &= |0,0,k_1,M_1-k_1\rangle\otimes|0,0,k_2,M_2-k_2\rangle,\\
|k_1,k_2\rangle_2^{\rm III} &= |0,0,k_1,M_1-k_1\rangle\otimes|1,1,k_2-1,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle_3^{\rm III} &= |1,1,k_1-1,M_1-k_1-1\rangle\otimes|0,0,k_2,M_2-k_2\rangle,\\
|k_1,k_2\rangle_4^{\rm III} &= |1,1,k_1-1,M_1-k_1-1\rangle\otimes|1,1,k_2-1,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle_5^{\rm III} &= |1,0,k_1-1,M_1-k_1\rangle\otimes|0,1,k_2,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle_6^{\rm III} &= |0,1,k_1,M_1-k_1-1\rangle\otimes|1,0,k_2-1,M_2-k_2\rangle.
\end{aligned}\tag{4.64}$$

Next we shall employ the same strategy as before. We perform the same steps as in (4.53) and (4.54) only with $\Delta^{op}E_2$, giving

$$\begin{aligned}
{}^{\rm II}_{i}\langle n,K-n|\,\Delta^{op}E_2\,\mathbb{S}\,|k_1,k_2\rangle_j^{\rm III} &= \sum_{l=1}^{6}\big(G^{op}(n,K-n)\big)^i_l\big(\mathscr{Z}_n^{k_1,k_2}\big)^l_j,\\
{}^{\rm II}_{i}\langle n,K-n|\,\mathbb{S}\,\Delta E_2\,|k_1,k_2\rangle_j^{\rm III} &= \sum_{m=1}^{4}\big(\mathscr{Y}_n^{k_1,k_2}\big)^i_m\big(G(k_1,k_2)\big)^m_j,
\end{aligned}\tag{4.65}$$

where $G^{(op)}$ are the matrix representations of the charges $\Delta^{(op)}E_2$. Once again these equations (together with the affine ones coming from E_4) do not provide enough constraints to define the matrix $\mathscr{Z}_n^{k_1,k_2}$ uniquely, and we need additional constraints. They are obtained with the help of $\Delta^{(op)}(F_3F_2)$, namely

$$\begin{aligned}
{}^{\rm II}_{i}\langle n-\theta_i,K-n+\theta_i-1|\,\Delta^{op}(F_3F_2)\,\mathbb{S}\,|k_1,k_2\rangle_j^{\rm III} &= \sum_{l=1}^{6}\big(H^{op}(n,n-K)\big)^i_l\big(\mathscr{Z}_n^{k_1,k_2}\big)^l_j,\\
{}^{\rm II}_{i}\langle n-\theta_i,K-n+\theta_i-1|\,\mathbb{S}\,\Delta(F_3F_2)\,|k_1,k_2\rangle_j^{\rm III} &= \sum_{m=1}^{4}\big(\overline{\mathscr{Y}}_n^{k_1,k_2}\big)^i_m\big(H(k_1,k_2)\big)^m_j,
\end{aligned}\tag{4.66}$$

where θ_i is defined by $\theta_i=(1-(-1)^i)/2$ and $H^{(op)}$ is the matrix representation of $\Delta^{(op)}(F_3F_2)$. Here we have also introduced $\overline{\mathscr{Y}}_n^{k_1,k_2}$ as

$$\big(\overline{\mathscr{Y}}_n^{k_1,k_2}\big)^i_j=\big(\mathscr{Y}_{n-\theta_i}^{k_1-\theta_j,k_2+\theta_j-1}\big)^i_j\,.\tag{4.67}$$

These equations may be written in a compact way using matrix notation

$$\begin{aligned}
G^{op}(n,K-n)\,\mathscr{Z}_n^{k_1,k_2} &= \mathscr{Y}_n^{k_1,k_2}\,G(k_1,k_2),\\
H^{op}(n,K-n)\,\mathscr{Z}_n^{k_1,k_2} &= \overline{\mathscr{Y}}_n^{k_1,k_2}\,H(k_1,k_2).
\end{aligned}\tag{4.68}$$

The explicit realization of the matrices in the expressions above are spelled out in the Appendix A.2.

Similarly as in the previous case, not all rows and columns of $G^{(op)}$ and $H^{(op)}$ are linearly independent, thus we have to select only the independent ones. Therefore by taking the following linear combinations

$$\overline{G}^{(op)}=q^{K-n-\frac{M_2}{2}}\big(\tilde a_2G^{(op)}-a_2\widetilde G^{(op)}\big)\quad\text{and}\quad\overline{H}^{(op)}=\tilde c_2V_1H^{(op)}-c_2V_1^{-1}\widetilde H^{(op)}\,,\tag{4.69}$$

where the tilded matrices are the affine counterparts and selecting the first three rows of each, we are able to combine them into the non-singular quadratic matrix A (6×6) and the rectangular matrix B (8×6) as follows $(j = 1, \cdots, 6)$,

$$(A)^i_j = \begin{cases} (\overline{G}^{op})^i_j\,, & i = 1,2,3\,, \\ (\overline{H}^{op})^{i-3}_j\,, & i = 4,5,6\,, \end{cases} \qquad \text{and} \qquad (B)^i_j = \begin{cases} (\overline{G})^i_j\,, & i = 1,2,3,4\,, \\ (\overline{H})^{i-4}_j\,, & i = 5,6,7,8\,. \end{cases} \tag{4.70}$$

This approach let us to rewrite the constraints (4.68) in terms of a single matrix relation

$$A\,\mathscr{X}^{k_1,k_2}_n = \check{\mathscr{Y}}^{k_1,k_2}_n B\,, \qquad \text{giving} \qquad \mathscr{X}^{k_1,k_2}_n = A^{-1}\,\check{\mathscr{Y}}^{k_1,k_2}_n B\,. \tag{4.71}$$

This relation let us to obtain any matrix element $(\mathscr{X}^{k_1,k_2}_n)^i_j$ of the scattering in the subspace III. Here we have also introduced the block diagonal matrix $\check{\mathscr{Y}}^{k,l}_n$ (6×8) as

$$(\check{\mathscr{Y}}^{k,l}_n)^i_j\,, = \begin{cases} \left(\mathscr{Y}^{k,l}_n\right)^i_j\,, & i = 1,2,3\,, \quad \text{and} \quad j = 1,2,3,4\,, \\ \left(\overline{\mathscr{Y}}^{k,l}_n\right)^{i-3}_{j-4}\,, & i = 4,5,6\,, \quad \text{and} \quad j = 5,6,7,8\,, \\ 0\,, & \text{the rest}\,. \end{cases} \tag{4.72}$$

The explicit form of matrices A, A^{-1}, B and their $q \to 1$ limit are given in Appendix A.2.

5 Special cases of the S-matrix

In this section we consider the reduction of the S-matrix in the case when one or both factors of the tensor space (4.1) are transforming in the fundamental representation.

5.1 Fundamental S-matrix

As a most simple case of the derivations presented in section 4, we want to compute the fundamental S-matrix found in [10]. The fundamental representation is defined by setting $M_1 = M_2 = 1$ and the corresponding S-matrix is 16×16 – dimensional. In order to make the comparison with [10] more explicit, let us denote

$$\mathsf{a}^\dagger_{1,2} = \phi^{1,2}, \qquad \text{and} \qquad \mathsf{a}^\dagger_{3,4} = \psi^{1,2}. \tag{5.1}$$

Then, starting with the subspaces I and Ib, we find

$$\mathbb{S}\,|\psi^\alpha\psi^\alpha\rangle = \mathscr{D}\,|\psi^\alpha\psi^\alpha\rangle, \tag{5.2}$$

where $\mathscr{D}$ is given by (4.13). Further, due to our normalization

$$\mathbb{S}\,|\phi^a\phi^a\rangle = |\phi^a\phi^a\rangle. \tag{5.3}$$

Here we would like to remark that our normalization differs from [10] where the S-matrix is normalized such that $\mathbb{S}\,|\psi^\alpha\psi^\alpha\rangle = -|\psi^\alpha\psi^\alpha\rangle$. In other words, the quantities given here need to be divided by an additional factor of $\mathscr{D}$.

Next we proceed to the subspaces II and IIb. For the subspace II (and analogously for IIb) the parameters k_1, k_2, n indexing the matrix $\mathscr{Y}$ can take the values 0 and 1, but fortunately, we find that $\mathscr{Y}$ is the same for both of these values. Next it is easy to observe that the matrices A (A.4) and B (A.5) get reduced to the upper left 2×2 blocks

$$A=\begin{pmatrix}-a_2 & q^{1/2}U_2V_2a_1\\ -\tilde{a}_2 & q^{1/2}\widetilde{U}_2\widetilde{V}_2\tilde{a}_1\end{pmatrix}, \qquad B=\begin{pmatrix}-a_2\sqrt{q}U_1V_1 & a_1\\ -\tilde{a}_2\sqrt{q}\widetilde{U}_1\widetilde{V}_1 & \tilde{a}_1\end{pmatrix}, \tag{5.4}$$

while the matrices B^+ and B^- do not contribute at all. This gives the following solution of (4.62)

$$\begin{aligned}\mathscr{Y}_0^{0,0}&=\mathscr{D}\begin{pmatrix}\frac{\sqrt{q}(a_2\tilde{a}_1U_1^2V_1^2-a_1\tilde{a}_2U_2^2V_2^2)}{U_1V_1(a_2\tilde{a}_1-a_1\tilde{a}_2U_2^2V_2^2)} & \frac{a_1\tilde{a}_1(1-U_2^2V_2^2)}{a_1\tilde{a}_2U_2^2V_2^2-a_2\tilde{a}_1}\\ \frac{a_2\tilde{a}_2U_2(U_1^2V_1^2-1)V_2}{U_1V_1(a_2\tilde{a}_1-a_1\tilde{a}_2U_2^2V_2^2)} & \frac{(a_2\tilde{a}_1-a_1\tilde{a}_2)U_2V_2}{\sqrt{q}(a_2\tilde{a}_1-a_1\tilde{a}_2U_2^2V_2^2)}\end{pmatrix}\\ &=\begin{pmatrix}q^{1/2}U_2V_2\frac{x_2^--x_1^-}{x_2^+-x_1^-} & \frac{\gamma_1}{\gamma_2}\frac{U_2V_2}{U_1V_1}\frac{x_2^+-x_2^-}{x_2^+-x_1^-}\\ \frac{\gamma_2}{\gamma_1}\frac{x_1^+-x_1^-}{x_2^+-x_1^-} & \frac{1}{q^{1/2}U_1V_1}\frac{x_2^+-x_1^+}{x_2^+-x_1^-}\end{pmatrix}.\end{aligned} \tag{5.5}$$

Then the corresponding explicit form of the fundamental S-matrix acting on the inequivalent states is

$$\begin{aligned}\mathbb{S}|\psi^\alpha\phi^b\rangle&=q^{1/2}U_2V_2\frac{x_2^--x_1^-}{x_2^+-x_1^-}|\psi^\alpha\phi^b\rangle+\frac{\gamma_2}{\gamma_1}\frac{x_1^+-x_1^-}{x_2^+-x_1^-}|\phi^b\psi^\alpha\rangle\,,\\ \mathbb{S}|\phi^a\psi^\beta\rangle&=\frac{\gamma_1}{\gamma_2}\frac{U_2V_2}{U_1V_1}\frac{x_2^+-x_2^-}{x_2^+-x_1^-}|\psi^\beta\phi^a\rangle+\frac{1}{q^{1/2}U_1V_1}\frac{x_2^+-x_1^+}{x_2^+-x_1^-}|\phi^a\psi^\beta\rangle\,.\end{aligned} \tag{5.6}$$

Finally we turn to the subspace III which is four dimensional in this case. Analogously to our strategy presented section 4.2, we inspect the action of ΔE_2 and ΔE_4 obtaining

$$\begin{aligned}\Delta E_2|1,0\rangle_1^{\rm III}&=\frac{U_1V_1}{\sqrt{q}}a_2|1,0\rangle_2^{\rm II}, & \Delta E_2|1,0\rangle_5^{\rm III}&=b_1|1,0\rangle_2^{\rm II},\\ \Delta E_2|0,1\rangle_1^{\rm III}&=a_1|0,0\rangle_1^{\rm II}, & \Delta E_2|0,1\rangle_6^{\rm III}&=-U_1V_1\sqrt{q}\,b_2|0,0\rangle_1^{\rm II},\end{aligned} \tag{5.7}$$

plus similar expressions for E_4. For completeness, let us spell out the opposite coproduct as well

$$\begin{aligned}\Delta^{op}E_2|1,0\rangle_1^{\rm III}&=a_2|1,0\rangle_2^{\rm II}, & \Delta^{op}E_2|1,0\rangle_5^{\rm III}&=b_1U_2V_2\sqrt{q}|1,0\rangle_2^{\rm II},\\ \Delta^{op}E_2|0,1\rangle_1^{\rm III}&=a_1\frac{U_2V_2}{\sqrt{q}}|0,0\rangle_1^{\rm II}, & \Delta^{op}E_2|0,1\rangle_6^{\rm III}&=-b_2|0,0\rangle_1^{\rm II}.\end{aligned} \tag{5.8}$$

The equation (4.71) in this case becomes

$$\begin{pmatrix}a_2 & b_1\sqrt{q}U_2V_2\\ \tilde{a}_2 & \tilde{b}_1\sqrt{q}\widetilde{U}_2\widetilde{V}_2\end{pmatrix}\begin{pmatrix}(\mathscr{X}_1^{1,0})_1^1 & (\mathscr{X}_1^{1,0})_1^5\\ (\mathscr{X}_1^{1,0})_1^5 & (\mathscr{X}_1^{1,0})_5^5\end{pmatrix}=\begin{pmatrix}\frac{U_1V_1}{\sqrt{q}}a_2 & b_1\\ \frac{\widetilde{U}_1\widetilde{V}_1}{\sqrt{q}}\tilde{a}_2 & \tilde{b}_1\end{pmatrix}(\mathscr{Y}_1^{1,0})_2^2\,, \tag{5.9}$$

the explicit solution of which is

$$\begin{pmatrix}(\mathscr{X}_1^{1,0})_1^1 & (\mathscr{X}_1^{1,0})_1^5\\ (\mathscr{X}_1^{1,0})_1^5 & (\mathscr{X}_1^{1,0})_5^5\end{pmatrix}=\begin{pmatrix}\frac{(1-x_2^-x_1^+)(x_1^+-x_2^+)}{(1-x_1^-x_2^-)(x_1^--x_2^+)}\frac{x_1^-}{qx_1^+} & \frac{\alpha(x_1^--x_1^+)(x_2^--x_2^+)(x_1^+-x_2^+)}{\sqrt{q}U_1V_1\gamma_1\gamma_2(x_1^-x_2^--1)(x_1^--x_2^+)}\\ \frac{\gamma_1\gamma_2(x_1^+-x_2^+)}{U_2V_2\alpha(1-x_1^-x_2^-)(x_2^+-x_1^-)}\frac{x_1^-}{q^{3/2}x_1^+} & \frac{(1-x_1^-x_2^+)(x_1^+-x_2^+)}{(1-x_1^-x_2^-)(x_1^--x_2^+)}\frac{U_2V_2}{U_1V_1}\frac{x_2^-}{qx_2^+}\end{pmatrix}. \tag{5.10}$$

The remaining matrix elements are then easily deduced from similar derivations. These results are in agreement with [10]. For a complete list of all the scattering elements we refer to the Appendix B.1.

5.2 The S-matrix $\mathbf{S}_{Q1}$

In this section we will derive the S-matrix describing the scattering of an arbitrary bound state with a fundamental one, $\mathbb{S}_{Q1}$. Once again, we will follow the derivations performed in section 4 step by step. First, by setting $M_2 = 1$, we find that the states in subspaces I and Ib scatter almost trivially

$$\mathbb{S}\,|k,0\rangle^{\mathrm{I}} = \mathscr{D}\,|k,0\rangle^{\mathrm{I}}. \tag{5.11}$$

However the scattering in the subspace II does not get simplified that much. Nevertheless, for fixed $k_1 + k_2$, the corresponding vector space gets restricted to

$$\{|k_1,0\rangle_1^{\mathrm{II}},\ |k_1-1,1\rangle_1^{\mathrm{II}},\ |k_1,0\rangle_2^{\mathrm{II}},\ |k_1,0\rangle_4^{\mathrm{II}}\}. \tag{5.12}$$

This is because the states $|k_1,k_2\rangle_3^{\mathrm{II}}$ have $M_2 \geq 2$ and thus they are not present. By reducing our general expressions to accommodate these 4 states, we are lead to 16 inequivalent scattering elements, however we found 2 of them to be vanishing. The rest may be casted in quite compact form as

$$\begin{aligned}
\mathbb{S}\,|k,0\rangle_1^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_1^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_1^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_1^2|k,0\rangle_2^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_1^4|k,0\rangle_4^{\mathrm{II}}\,,\\
\mathbb{S}\,|k-1,1\rangle_1^{\mathrm{II}} &= (\mathscr{Y}_0^{k-1,1})_1^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k-1,1})_1^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k-1,1})_1^2|k,0\rangle_2^{\mathrm{II}} + (\mathscr{Y}_0^{k-1,1})_1^4|k,0\rangle_4^{\mathrm{II}}\,,\\
\mathbb{S}\,|k,0\rangle_2^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_2^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_2^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_2^2|k,0\rangle_2^{\mathrm{II}}\,,\\
\mathbb{S}\,|k,0\rangle_4^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_4^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_4^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_4^4|k,0\rangle_4^{\mathrm{II}}\,.
\end{aligned} \tag{5.13}$$

The explicit expressions of the coefficients above are given in Appendix B.2. Upon setting $M_1 = 1$ the coefficients with indices 1 and 2 reduce to the ones of the fundamental S-matrix (5.6) derived previously.

The scattering in the subspace III simplifies considerably. It is easy to see, that the states $|k_1,k_2\rangle_{2,4}^{\mathrm{III}}$ need not to be considered. Thus we are led to the reduced case of our general expressions for subspace III that involve the states (5.12) and

$$\{|k,0\rangle_1^{\mathrm{III}},\ |k,0\rangle_3^{\mathrm{III}},\ |k,0\rangle_5^{\mathrm{III}},\ |k-1,1\rangle_1^{\mathrm{III}},\ |k-1,1\rangle_3^{\mathrm{III}},\ |k-1,1\rangle_6^{\mathrm{III}}\} \tag{5.14}$$

only. However, there is a more straightforward way to obtain the S-matrix in this particular case.

There are 36 scattering coefficients in subspace III that need to be determined, but not all of them are independent. Firstly we can relate the half of them to the other half by considering the identity

$$\Delta E_3|k-1,0\rangle^{\mathrm{I}} = |k,0\rangle_5^{\mathrm{III}} + q^{-1}|k-1,1\rangle_6^{\mathrm{III}}, \tag{5.15}$$

giving

$$\mathbb{S}\,|k-1,1\rangle_6^{\mathrm{III}} = \mathscr{D}\left(|k,0\rangle_5^{\mathrm{III}} + q\,|k-1,1\rangle_6^{\mathrm{III}}\right) - q\,\mathbb{S}\,|k,0\rangle_5^{\mathrm{III}}\,. \tag{5.16}$$

Subsequently we can express the states $|k-1,1\rangle_1^{\rm III}$, $|k-1,1\rangle_3^{\rm III}$ as follows

$$\begin{aligned}\frac{\Delta F_1 \Delta E_1 - q[k]_q[M-k+1]_q}{[k]_q}\,|k,0\rangle_1^{\rm III} &= |k-1,1\rangle_1^{\rm III},\\ \frac{\Delta F_1 \Delta E_1 - q[k-1]_q[M-k]_q}{[k-1]_q}\,|k,0\rangle_3^{\rm III} &= |k-1,1\rangle_3^{\rm III}.\end{aligned} \tag{5.17}$$

The explicit constraints that follow from these identities are listed in the Appendix B.2.

Then instead of reducing the general expression of the matrix $\mathscr{X}$, we follow its derivation path. By considering the action of the charges F_2 and F_4 on the subspace II states we are able to find simple expressions that relate subspaces III to subspace II as

$$\begin{aligned}|k,0\rangle_1^{\rm III} &= \frac{\tilde{c}_1 V_2 \Delta F_2 - c_1 \widetilde{V}_2 \Delta F_4}{\tilde{c}_1 d_2 \widetilde{U}_1 V_2 - c_1 \tilde{d}_2 U_1 \widetilde{V}_2}\,|k,0\rangle_2^{\rm II}, \qquad |k,0\rangle_3^{\rm III} = \frac{\tilde{d}_1 V_2 \Delta F_2 - d_1 \widetilde{V}_2 \Delta F_4}{\tilde{d}_1 d_2 \widetilde{U}_1 V_2 - \tilde{d}_2 d_1 U_1 \widetilde{V}_2}\,|k,0\rangle_4^{\rm II},\\ |k,0\rangle_5^{\rm III} &= \frac{\sqrt{q}}{[k]_q}\frac{\tilde{d}_2 U_1 \Delta F_2 - d_2 \widetilde{U}_1 \Delta F_4}{c_1 \tilde{d}_2 U_1 \widetilde{V}_2 - \tilde{c}_1 d_2 \widetilde{U}_1 V_2}\,|k,0\rangle_2^{\rm II}.\end{aligned} \tag{5.18}$$

This approach let us to find the expressions of the matrix elements of $\mathscr{X}$ in terms of the matrix elements of $\mathscr{Y}$ for this particular case in quite an easy way. The explicit expressions are once again given in the Appendix B.2.

6 Discussion and outlook

In this work we have constructed the supersymmetric short representations of the quantum affine algebra $\widehat{\mathcal{Q}}$ based on the centrally extended $\mathfrak{su}(2|2)$ by making use of quantum oscillator algebra. These representations are of great importance as they accommodate the bound states of the model. We found that the bound state representation of the affine extension shows a lot of similarities with the fundamental one constructed in [35]. In particular, we found that the affine central elements are inverse to their non-affine partners, exactly as for the fundamental representation. Moreover, the parameterization can be derived from the fundamental one simply by applying the map $(q,g) \to (q^M, g/[M]_q)$.

The affine extension plays a key role in the construction of the bound state S-matrix. Indeed, the affine generators E_4 and F_4 are crucial in constructing the elements $\mathscr{X}$ and $\mathscr{Y}$. In other words, the bound state S-matrix is uniquely fixed by requiring invariance under the affine algebra $\widehat{\mathcal{Q}}$.

We have also spelled out the explicit coefficients of the S-matrix when one of the spaces is the fundamental representation. And in particular, we have checked that our formalism correctly reproduces the fundamental S-matrix found in [10]. Furthermore, our results are in a very good agreement with those of [27], where a similar derivation based on the Yangian symmetry related to the same underlying Lie super algebra was performed. More precisely, the S-matrix we have obtained in the $q \to 1$ limit for the subspace I coalesce exactly to the one found in [27]. However we can not make a direct comparison for subspaces II and III as the intermediate expressions are different, due to the fact that the affine rather than Yangian generators are used. Nevertheless, the expressions we have obtained in this work are of more symmetric form than those of [27]. This is an expected result, as the deformed quantum affine algebra itself is of more symmetric form than its Yangian limit.

We have not checked the Yang-Baxter equation in full generality due to this being extremely challenging from the technical point of view. However, we have performed a series of checks for a wide variety of states using numerical computations and found that it was perfectly satisfied. This is to be expected as this S-matrix is uniquely defined by the algebra $\widehat{\mathcal{Q}}$.

In order to complete the investigations concerning the S-matrix it would be interesting to consider the crossing symmetry and the corresponding solutions for a q-deformed dressing phase.

A particularly interesting direction for future research would be to study representations and their S-matrices for q being a root of unity. It is well known that the representation theory for these values of q differs from the one for real q. Due to the bound state map being of the form $q \to q^M$, it is not difficult to see that there appears to be some intrinsic periodicity to these representations. One could hope, for example in the context of the thermodynamic Bethe ansatz, that this would result in a finite number of bound states. Thus such approach could lead to some useful insights.

A different topic related to this, would be to investigate the algebraic Bethe ansatz and the bound state transfer matrices. This could perhaps be used to find a q-deformed version of the T-system.

One more possible direction of investigations is to consider the boundary conditions and boundary scattering for the deformed Hubbard Chain. A good starting point for this approach would be to consider the boundary conditions equivalent to the ones of the $Y = 0$ and $Z = 0$ giant gravitons in the framework of $\mathrm{AdS}_5 \times \mathrm{S}^5$ correspondence [50]. We expect some sort of deformed (twisted) coideal subalgebra of $\widehat{\mathcal{Q}}$ to be governing the boundary scattering of the aforementioned type that in the rational limit would reproduce the twisted Yangian algebras constructed in [51–53].

Other, more open questions on the more algebraic side include algebraic R-matrix and a detailed investigation of the classical limit along the lines of [34]. It would also be interesting to extend the classical limit to the next order. For the undeformed case it was found that this order coincides with the square of the classical r-matrix [54].

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A Elements of the S-matrix

In this Appendix we have spelled out various coefficients and matrices that have been heavily used in the intermediate steps in deriving the final expressions of the S-matrix for the subspaces II and III.

A.1 Subspace II

The coefficients for the charge ΔE_2 in (4.52) are

$$\begin{aligned} Q_1(k_1,k_2) &= -q^{M_1/2-k_1}\, a_2\, U_1 V_1\, [\bar{k}_2+1]_q, & Q_2(k_1,k_2) &= a_1\, [\bar{k}_1+1]_q, \\ Q_3(k_1,k_2) &= -q^{M_1/2-k_1}\, b_2\, U_1 V_1, & Q_4(k_1,k_2) &= b_1. \end{aligned} \tag{A.1}$$

Similarly, the coefficients for the charge $\Delta^{op} E_2$ are

$$\begin{aligned} Q_1^{op}(k_1,k_2) &= -a_2\, [\bar{k}_2+1]_q, & Q_2^{op}(k_1,k_2) &= q^{M_2/2-k_2}\, a_1\, U_2 V_2\, [\bar{k}_1+1]_q, \\ Q_3^{op}(k_1,k_2) &= -b_2, & Q_4^{op}(k_1,k_2) &= q^{M_2/2-k_2}\, b_1\, U_2 V_2. \end{aligned} \tag{A.2}$$

By replacing $a, b \to \tilde{a}, \tilde{b}$ and $U, V \to \widetilde{U}, \widetilde{V}$, one obtains $\widetilde{Q}_i(k_1,k_2)$ and $\widetilde{Q}_i^{op}(k_1,k_2)$ related to the affine charge E_4.

The coefficients in (4.61) are

$$\begin{aligned} Z_1^{op}(n,K-n) &= c_2\, \widetilde{V}_1\, [M_2-K+n]_q, & Z_2^{op}(n,K-n) &= c_1\, \widetilde{U}_2\, [n-M_1]_q\, q^{n-K-\frac{M_1}{2}}, \\ Z_3^{op}(n,K-n) &= d_2\, \widetilde{V}_1\, q^{-M_2}, & Z_4^{op}(n,K-n) &= -d_1\, \widetilde{U}_2\, q^{n-K+\frac{M_1}{2}}. \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} Z_1(k_1,k_2) &= \frac{c_2 \widetilde{U}_1\, [\bar{k}_2+1]_q}{q^M z_{12} - q^{2(K+1)}} q^{M_1/2-k_1+M_2}\Big(q^{2n} z_{12} - q^{\delta M}\big(q^{2(n-\bar{k}_1)}-1\big) - q^{2k_2+\delta M}\Big), \\ Z_2(k_1,k_2) &= \frac{z_{12}\, c_1 \widetilde{V}_2\, [\bar{k}_1+1]_q}{q^M z_{12} - q^{2(K+1)}} q^{-\delta M/2+2}\Big(q^{2n} z_{21} - q^{\delta M}\big(q^{2(n+\bar{k}_2)}-q^{2K}\big) - q^{2k_2+\delta M}\Big), \\ Z_3(k_1,k_2) &= \frac{d_2 \widetilde{U}_1}{q^M z_{12} - q^{2(K+1)}} q^{M_1/2-k_1}\Big(q^{2n} z_{12} - q^{M}\big(q^{2(n-\bar{k}_1)}-1\big) - q^{2k_2+\delta M}\Big), \\ Z_4(k_1,k_2) &= \frac{z_{12}\, d_1 \widetilde{V}_2}{q^M z_{12} - q^{2(K+1)}} q^{M/2+2}\Big(q^{2n} z_{21} - q^{-M}\big(q^{2(n+\bar{k}_2)}-q^{2K}\big) - q^{2k_2+\delta M}\Big). \end{aligned}$$

The matrices in (4.62) are defined as

$$A = \begin{pmatrix} Q_1^{op}(n,K-n) & Q_2^{op}(n,K-n) & Q_3^{op}(n,K-n) & Q_4^{op}(n,K-n) \\ \tilde{Q}_1^{op}(n,K-n) & \tilde{Q}_2^{op}(n,K-n) & \tilde{Q}_3^{op}(n,K-n) & \tilde{Q}_4^{op}(n,K-n) \\ Z_1^{op}(n,K-n) & Z_2^{op}(n,K-n) & Z_3^{op}(n,K-n) & Z_4^{op}(n,K-n) \\ \tilde{Z}_1^{op}(n,K-n) & \tilde{Z}_2^{op}(n,K-n) & \tilde{Z}_3^{op}(n,K-n) & \tilde{Z}_4^{op}(n,K-n) \end{pmatrix}, \tag{A.4}$$

$$B = \begin{pmatrix} Q_1(k_1,k_2) & Q_2(k_1,k_2) & Q_3(k_1,k_2) & Q_4(k_1,k_2) \\ \tilde{Q}_1(k_1,k_2) & \tilde{Q}_2(k_1,k_2) & \tilde{Q}_3(k_1,k_2) & \tilde{Q}_4(k_1,k_2) \\ Z_1(k_1,k_2) & Z_2(k_1,k_2) & Z_3(k_1,k_2) & Z_4(k_1,k_2) \\ \tilde{Z}_1(k_1,k_2) & \tilde{Z}_2(k_1,k_2) & \tilde{Z}_3(k_1,k_2) & \tilde{Z}_4(k_1,k_2) \end{pmatrix}, \tag{A.5}$$

and

$$B^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z_1^+(k_1,k_2) & 0 & Z_3^+(k_1,k_2) & 0 \\ \tilde{Z}_1^+(k_1,k_2) & 0 & \tilde{Z}_3^+(k_1,k_2) & 0 \end{pmatrix}, \qquad B^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & Z_2^-(k_1,k_2) & 0 & Z_4^-(k_1,k_2) \\ 0 & \tilde{Z}_2^-(k_1,k_2) & 0 & \tilde{Z}_4^-(k_1,k_2) \end{pmatrix}. \tag{A.6}$$

The latter two have a quite compact explicit form

$$B^+ = [\bar{k}_1]_q \frac{q^{1+k_1-k_2-\frac{M_1}{2}}}{(q-q^{-1})^{-1}} \frac{q^{M_1+2k_2} z_{12} - q^{M_2+2(n+1)}}{q^M z_{12} - q^{2(K+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2 \widetilde{U}_1 [k_2]_q & 0 & d_2 \widetilde{U}_1 & 0 \\ -\tilde{c}_2 U_1 [k_2]_q & 0 & \tilde{d}_2 U_1 & 0 \end{pmatrix}, \tag{A.7}$$

$$B^- = [\bar{k}_2]_q \frac{q^{1-k_1+\frac{\delta M}{2}}}{(q-q^{-1})^{-1}} \frac{q^{M_2+2n} z_{12} - q^{M_1+2(k_2+1)}}{q^M z_{12} - q^{2(K+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -c_1 \widetilde{V}_2 [k_1]_q & 0 & d_1 \widetilde{V}_2 \\ 0 & -\tilde{c}_1 V_2 [k_1]_q & 0 & \tilde{d}_1 V_2 \end{pmatrix}, \tag{A.8}$$

The inverse of A has a very complex form, however it can be decomposed intro three quite compact matrices as $A^{-1} = CVD$, where

$$C = \begin{pmatrix} \frac{z_{12}\tilde{b}_2}{[M_2-K+n]_q} & 0 & \frac{z_{12}\tilde{\alpha} b_2}{[M_2-K+n]_q} & 0 \\ 0 & \frac{q^{K-\frac{M_2}{2}-n}\tilde{\alpha} b_1 U_2 V_2}{[n-M_1]_q} & 0 & \frac{q^{K-\frac{M_2}{2}-n}\tilde{b}_1}{[M_1-n]_q U_2 V_2} \\ -z_{12}\tilde{a}_2 & 0 & -z_{12}\tilde{\alpha} a_2 & 0 \\ 0 & q^{K-\frac{M_2}{2}-n}\tilde{\alpha} a_1 U_2 V_2 & 0 & -\frac{q^{K-\frac{M_2}{2}-n}\tilde{a}_1}{U_2 V_2} \end{pmatrix}, \tag{A.9}$$

$$D = \text{diag}\left(\frac{ig\xi}{\tilde{g}\alpha\tilde{\alpha} z_2}, \frac{ig\xi}{\tilde{g}\alpha\tilde{\alpha}^2 z_2}, \frac{q^{\frac{M_2}{2}}}{\widetilde{V}_1 \widetilde{V}_2 \tilde{\alpha}}, \frac{q^{\frac{M_2}{2}}}{V_1 V_2} \right), \tag{A.10}$$

$$V = \frac{1}{W} \begin{pmatrix} \frac{1}{i\xi}\left[U_z\xi^2 - V_z + \frac{\widetilde{V}_z V_z - \widetilde{U}_z U_z \xi^2}{z_{12}}\right] & V_z - U_z & i\xi U_z & -V_z \\ \widetilde{U}_z - \widetilde{V}_z & \frac{i}{\xi}\left(\widetilde{V}_z - \widetilde{U}_z\xi^2\right) & \widetilde{V}_z & i\widetilde{U}_z\xi \\ \widetilde{V}_z - \widetilde{U}_z & \frac{i}{\xi}\left[\widetilde{U}_z\xi^2 - \widetilde{V}_z + \frac{\widetilde{V}_z V_z - \widetilde{U}_z U_z \xi^2}{z_{12}}\right] & -\widetilde{V}_z & -i\widetilde{U}_z\xi \\ \frac{i}{\xi}\left(V_z - U_z\xi^2\right) & V_z - U_z & iU_z\xi & -V_z \end{pmatrix}, \tag{A.11}$$

here

$$W = \widetilde{V}_z V_z - \widetilde{U}_z U_z \xi^2, \qquad U_z = z_{12} - U_1^2 U_2^2, \qquad \widetilde{U}_z = z_{12} - \widetilde{U}_1^2 \widetilde{U}_2^2, \tag{A.12}$$

plus similar expressions for V_z.

Rational limit. The matrices B^+ (A.7) and B^- (A.8) in the $q \to 1 + h$ ($h \to 0$) limit become

$$B^+ = 2h\,\bar{k}_1 \frac{\delta u - \frac{\delta M}{2} - k_2 + n + 1}{\delta u - \frac{M}{2} + K + 1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2 c_2/U_1 & 0 & d_2/U_1 & 0 \\ k_2 a_2 U_1/\alpha\tilde{\alpha} & 0 & -b_2 U_1/\alpha\tilde{\alpha} & 0 \end{pmatrix}, \tag{A.13}$$

$$B^- = 2h\,\bar{k}_2 \frac{\delta u + \frac{\delta M}{2} + k_2 - n + 1}{\delta u - \frac{M}{2} + K + 1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_1 c_1 & 0 & d_1 \\ 0 & k_1 a_1/\alpha\tilde{\alpha} & 0 & -b_1/\alpha\tilde{\alpha} \end{pmatrix}. \tag{A.14}$$

The matrices A (A.4) and B (A.5) in the $q \to 1$ limit become

$$A = \begin{pmatrix} -(M_2-K+n)g_2\gamma_2 & (M_1-n)g_1U_2\gamma_1 & -\frac{\alpha g_2(x_2^- - x_2^+)}{\gamma_2 x_2^-} & \frac{\alpha g_1 U_2(x_1^- - x_1^+)}{\gamma_1 x_1^-} \\ -\frac{i(M_2-K+n)\tilde{\alpha} g_2\gamma_2}{x_2^+} & \frac{i(M_1-n)\tilde{\alpha} g_1\gamma_1}{U_2 x_1^+} & -\frac{i\alpha\tilde{\alpha} g_2(x_2^- - x_2^+)}{\gamma_2} & \frac{i\alpha\tilde{\alpha} g_1(x_1^- - x_1^+)}{U_2\gamma_1} \\ \frac{i(M_2-K+n)g_2\gamma_2}{\alpha x_2^+} & -\frac{i(M_1-n)g_1\gamma_1}{\alpha U_2 x_1^+} & \frac{i g_2(x_2^- - x_2^+)}{\gamma_2} & -\frac{i g_1(x_1^- - x_1^+)}{U_2\gamma_1} \\ -\frac{(M_2-K+n)g_2\gamma_2}{\alpha\tilde{\alpha}} & \frac{(M_1-n)g_1U_2\gamma_1}{\alpha\tilde{\alpha}} & -\frac{g_2(x_2^- - x_2^+)}{\tilde{\alpha}\gamma_2 x_2^-} & \frac{g_1U_2(x_1^- - x_1^+)}{\tilde{\alpha}\gamma_1 x_1^-} \end{pmatrix}, \tag{A.15}$$

$$B = \begin{pmatrix} -(M_2-k_2)g_2U_1\gamma_2 & (M_1-k_1)g_1\gamma_1 & -\frac{\alpha g_2 U_1(x_2^- - x_2^+)}{\gamma_2 x_2^-} & \frac{\alpha g_1(x_1^- - x_1^+)}{\gamma_1 x_1^-} \\ -\frac{i(M_2-k_2)\tilde{\alpha} g_2\gamma_2}{U_1 x_2^+} & \frac{i(M_1-k_1)\tilde{\alpha} g_1\gamma_1}{x_1^+} & -\frac{i\alpha\tilde{\alpha} g_2(x_2^- - x_2^+)}{U_1\gamma_2} & \frac{i\alpha\tilde{\alpha} g_1(x_1^- - x_1^+)}{\gamma_1} \\ \frac{i(M_2-k_2)g_2\gamma_2}{\alpha U_1 x_2^+} & -\frac{i(M_1-k_1)g_1\gamma_1}{\alpha x_1^+} & \frac{i g_2(x_2^- - x_2^+)}{U_1\gamma_2} & -\frac{i g_1(x_1^- - x_1^+)}{\gamma_1} \\ -\frac{(M_2-k_2)g_2U_1\gamma_2}{\alpha\tilde{\alpha}} & \frac{(M_1-k_1)g_1\gamma_1}{\alpha\tilde{\alpha}} & -\frac{g_2U_1(x_2^- - x_2^+)}{\tilde{\alpha}\gamma_2 x_2^-} & \frac{g_1(x_1^- - x_1^+)}{\tilde{\alpha}\gamma_1 x_1^-} \end{pmatrix}. \tag{A.16}$$

The notation used in here is $g_i = \sqrt{\frac{g}{M_i}}$ and $U_i = \sqrt{\frac{x_i^+}{x_i^-}}$.

It might seem that the matrices B^+ and B^- do not contribute in the $q \to 1$ limit as they are of order $\mathcal{O}(h)$, however the combinations $A^{-1}B^+$ and $A^{-1}B^-$ in (4.62) are of order $\mathcal{O}(1)$, thus are defined correctly. We do not spell out the explicit expression of A^{-1} in the $q \to 1$ limit as it is quite sizy and also not much illuminative.

A.2 Subspace III

The coefficients' matrices in the expressions (4.68)

$$G^{op}(n, K-n)\,\mathscr{Z}_n^{k_1,k_2} = \mathscr{Y}_n^{k_1,k_2}\,G(k_1,k_2),$$
$$H^{op}(n, K-n)\,\mathscr{Z}_n^{k_1,k_2} = \overline{\mathscr{Y}}_n^{k_1,k_2}\,H(k_1,k_2),$$

are

$$G^{op}=\begin{pmatrix} \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q\,a_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & 0 & -b_2 \\ [M_2-K+n]_q a_2 & b_2 & 0 & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 \\ 0 & \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q a_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & [M_2-K+n]_q a_2 \\ 0 & 0 & [M_2-K+n]_q a_2 & b_2 & \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q a_1}{-\widetilde{U}_2\widetilde{V}_2} & 0 \end{pmatrix}, \tag{A.17}$$

$$G=\begin{pmatrix} [M_1-k_1]_q a_1 & 0 & b_1 & 0 & 0 & \frac{q^{\frac{M_1}{2}-k_1}b_2}{-\widetilde{U}_1\widetilde{V}_1} \\ \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]_q a_2}{\widetilde{U}_1\widetilde{V}_1} & \frac{q^{\frac{M_1}{2}-k_1}b_2}{\widetilde{U}_1\widetilde{V}_1} & 0 & 0 & b_1 & 0 \\ 0 & [M_1-k_1]_q a_1 & 0 & b_1 & 0 & \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]_q a_2}{\widetilde{U}_1\widetilde{V}_1} \\ 0 & 0 & \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]_q a_2}{\widetilde{U}_1\widetilde{V}_1} & \frac{q^{\frac{M_1}{2}-k_1}b_2}{\widetilde{U}_1\widetilde{V}_1} & -[M_1-k_1]_q a_1 & 0 \end{pmatrix}, \tag{A.18}$$

and

$$H^{op}=\begin{pmatrix} \frac{[n]_q c_1}{U_2} & 0 & -\frac{d_1}{U_2} & 0 & -\frac{q^{n-\frac{M_1}{2}}d_2}{V_1} & 0 \\ \frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & \frac{q^{n-\frac{M_1}{2}}d_2}{-V_1} & 0 & 0 & 0 & \frac{d_1}{U_2} \\ 0 & \frac{[n]_q c_1}{U_2} & 0 & -\frac{d_1}{U_2} & -\frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & 0 \\ 0 & 0 & \frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & \frac{q^{n-\frac{M_1}{2}}d_2}{-V_1} & 0 & \frac{[n]_q c_1}{U_2} \end{pmatrix}, \tag{A.19}$$

$$H=\begin{pmatrix} \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} & 0 & \frac{q^{k_2-\frac{M_2}{2}}d_1}{-V_2} & 0 & -d_2\widetilde{U}_1 & 0 \\ [k_2]_q c_2\widetilde{U}_1 & -d_2\widetilde{U}_1 & 0 & 0 & 0 & \frac{q^{k_2-\frac{M_2}{2}}d_1}{V_2} \\ 0 & \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} & 0 & \frac{q^{k_2-\frac{M_2}{2}}d_1}{-V_2} & -[k_2]_q c_2\widetilde{U}_1 & 0 \\ 0 & 0 & [k_2]_q c_2\widetilde{U}_1 & -d_2\widetilde{U}_1 & 0 & \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} \end{pmatrix}. \tag{A.20}$$

Their affine counterparts $\widetilde{G}$, $\widetilde{G}^{op}$ and $\widetilde{H}$, $\widetilde{H}^{op}$ are obtained by the replacing non-affine (or affine) parameter to affine (or non-affine) ones. The matrix $\overline{\mathscr{Y}}_n^{k_1,k_2}$ is a slightly modified version of $\mathscr{Y}_n^{k_1,k_2}$,

$$\overline{\mathscr{Y}}_n^{k_1,k_2}\equiv\begin{pmatrix} (\mathscr{Y}_{n-1}^{k_1-1,k_2})_1^1 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_2^1 & (\mathscr{Y}_{n-1}^{k_1-1,k_2})_3^1 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_4^1 \\ (\mathscr{Y}_{n}^{k_1-1,k_2})_1^2 & (\mathscr{Y}_{n}^{k_1,k_2-1})_2^2 & (\mathscr{Y}_{n}^{k_1-1,k_2})_3^2 & (\mathscr{Y}_{n}^{k_1,k_2-1})_4^2 \\ (\mathscr{Y}_{n-1}^{k_1-1,k_2})_1^3 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_2^3 & (\mathscr{Y}_{n-1}^{k_1-1,k_2})_3^3 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_4^3 \\ (\mathscr{Y}_{n}^{k_1-1,k_2})_1^4 & (\mathscr{Y}_{n}^{k_1,k_2-1})_2^4 & (\mathscr{Y}_{n}^{k_1-1,k_2})_3^4 & (\mathscr{Y}_{n}^{k_1,k_2-1})_4^4 \end{pmatrix}. \tag{A.21}$$

The coefficient matrices in (4.71), $A\,\mathscr{Z}_n^{k_1,k_2} = \check{\mathscr{Y}}_n^{k,l} B$, are

$$A = \begin{pmatrix} -\frac{[M_1-n]\mathcal{A}_3}{U_2V_2} & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 & 0 & q_2\tilde{z}_2 \\ 0 & -q_2\tilde{z}_2 & 0 & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 \\ 0 & -\frac{[M_1-n]\mathcal{A}_3}{U_2V_2} & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 & 0 \\ -\frac{[n]_q\mathcal{A}_2}{U_2V_1} & 0 & -\frac{\mathcal{A}_4}{U_2V_1} & 0 & \frac{\tilde{g}^2q_1}{g^2\tilde{z}_2} & 0 \\ 0 & \frac{\tilde{g}^2q_1}{g^2\tilde{z}_2} & 0 & 0 & 0 & \frac{\mathcal{A}_4}{U_2V_1} \\ 0 & -\frac{[n]_q\mathcal{A}_2}{U_2V_1} & 0 & -\frac{\mathcal{A}_4}{U_2V_1} & 0 & 0 \end{pmatrix}, \tag{A.22}$$

$$A^{-1} = \begin{pmatrix} -\frac{U_2V_2}{\mathcal{A}_0\mathcal{A}_4^{-1}} & \frac{\tilde{g}^2q_1U_2^2V_1V_2}{g^2\mathcal{A}_0\tilde{z}_2} & 0 & -\frac{U_2V_1}{\mathcal{A}_0\mathcal{A}_1^{-1}} & \frac{q_2U_2^2V_1V_2\tilde{z}_2}{\mathcal{A}_0} & 0 \\ 0 & 0 & -\frac{U_2V_2}{\mathcal{A}_0\mathcal{A}_4^{-1}} & 0 & 0 & -\frac{U_2V_1}{\mathcal{A}_0\mathcal{A}_1^{-1}} \\ \frac{[n]_qU_2V_2}{\mathcal{A}_0\mathcal{A}_2^{-1}} & \frac{\tilde{g}^2[M_1-n]q_1U_2^2V_1V_2}{g^2\mathcal{A}_0\mathcal{A}_1\mathcal{A}_3^{-1}\tilde{z}_2} & \frac{\tilde{g}^2q_1q_2U_2^3V_1V_2^2}{-g^2\mathcal{A}_0\mathcal{A}_1} & \frac{[M_1-n]U_2V_1}{-\mathcal{A}_0\mathcal{A}_3^{-1}} & \frac{[n]_qq_2U_2^2V_1V_2\tilde{z}_2}{-\mathcal{A}_0\mathcal{A}_2^{-1}\mathcal{A}_4} & \frac{\tilde{g}^2q_1q_2U_2^3V_1^2V_2}{-g^2\mathcal{A}_0\mathcal{A}_4} \\ 0 & 0 & \frac{[n]_qU_2V_2}{\mathcal{A}_0\mathcal{A}_2^{-1}} & 0 & 0 & -\frac{[M_1-n]U_2V_1}{\mathcal{A}_0\mathcal{A}_3^{-1}} \\ 0 & \frac{U_2V_2}{\mathcal{A}_1} & -\frac{q_2U_2^2V_2^2\tilde{z}_2}{\mathcal{A}_0\mathcal{A}_1\mathcal{A}_4^{-1}} & 0 & 0 & -\frac{q_2U_2^2V_1V_2\tilde{z}_2}{\mathcal{A}_0} \\ 0 & 0 & \frac{\tilde{g}^2q_1U_2^2V_1V_2}{g^2\mathcal{A}_0\tilde{z}_2} & 0 & \frac{U_2V_1}{\mathcal{A}_4} & \frac{\tilde{g}^2q_1U_2^2V_1^2}{g^2\mathcal{A}_0\mathcal{A}_1^{-1}\mathcal{A}_4\tilde{z}_2} \end{pmatrix}, \tag{A.23}$$

here we have defined $\tilde{z}_i = \frac{\tilde{g}\alpha\tilde{\alpha}}{g} z_i$ and $\mathcal{A}_0 = [n]_q\,\mathcal{A}_1\mathcal{A}_2 + [M_1-n]_q\,\mathcal{A}_3\mathcal{A}_4$ where

$$\begin{aligned} \mathcal{A}_1 &= b_1\tilde{a}_2U_2^2V_2^2 - a_2\tilde{b}_1, & \mathcal{A}_2 &= c_2\tilde{c}_1U_2^2 - c_1V_1^2\tilde{c}_2, \\ \mathcal{A}_3 &= a_2\tilde{a}_1 - a_1\tilde{a}_2U_2^2V_2^2, & \mathcal{A}_4 &= d_1\tilde{c}_2V_1^2 - c_2\tilde{d}_1U_2^2. \end{aligned} \tag{A.24}$$

$$B = \begin{pmatrix} -[M_1-k_1]_qq_2\mathcal{B}_3 & 0 & q_2\mathcal{B}_2 & 0 & 0 & -\frac{q_3\mathcal{B}_1}{q_1U_1V_1} \\ -\frac{[M_2-k_2]_qq_3\mathcal{B}_7}{q_1U_1V_1} & \frac{q_3\mathcal{B}_1}{q_1U_1V_1} & 0 & 0 & q_2\mathcal{B}_2 & 0 \\ 0 & -[M_1-k_1]_qq_2\mathcal{B}_3 & 0 & q_2\mathcal{B}_2 & 0 & -\frac{[M_2-k_2]_qq_3\mathcal{B}_7}{q_1U_1V_1} \\ 0 & 0 & -\frac{[M_2-k_2]_qq_3\mathcal{B}_7}{q_1U_1V_1} & \frac{q_3\mathcal{B}_1}{q_1U_1V_1} & [M_1-k_1]_qq_2\mathcal{B}_3 & 0 \\ -\frac{[k_1]_qq_3\mathcal{B}_4}{V_1V_2} & 0 & -\frac{q_3\mathcal{B}_5}{V_1V_2} & 0 & -\frac{\mathcal{B}_6}{U_1V_1} & 0 \\ -\frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & -\frac{\mathcal{B}_6}{U_1V_1} & 0 & 0 & 0 & \frac{q_3\mathcal{B}_5}{V_1V_2} \\ 0 & -\frac{[k_1]_qq_3\mathcal{B}_4}{V_1V_2} & 0 & -\frac{q_3\mathcal{B}_5}{V_1V_2} & \frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & 0 \\ 0 & 0 & -\frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & -\frac{\mathcal{B}_6}{U_1V_1} & 0 & -\frac{[k_1]_qq_3\mathcal{B}_4}{V_1V_2} \end{pmatrix}, \tag{A.25}$$

here we are using the shorthand notation $q_1 = q^{n-\frac{M_1}{2}}$, $q_2 = q^{K-n-\frac{M_2}{2}}$, $q_3 = q^{k_2-\frac{M_2}{2}}$ and

$$\begin{aligned} \mathcal{B}_1 &= b_2\tilde{a}_2U_1^2V_1^2 - a_2\tilde{b}_2, & \mathcal{B}_2 &= b_1\tilde{a}_2 - a_2\tilde{b}_1, \\ \mathcal{B}_3 &= a_2\tilde{a}_1 - a_1\tilde{a}_2, & \mathcal{B}_4 &= c_2\tilde{c}_1V_2^2 - c_1\tilde{c}_2V_1^2, \\ \mathcal{B}_5 &= d_1\tilde{c}_2V_1^2 - c_2\tilde{d}_1V_2^2, & \mathcal{B}_6 &= d_2\tilde{c}_2V_1^2 - c_2\tilde{d}_2U_1^2, \\ \mathcal{B}_7 &= a_2\tilde{a}_2(1-U_1^2V_1^2), & \mathcal{B}_8 &= c_2\tilde{c}_2(U_1^2-V_1^2). \end{aligned} \tag{A.26}$$

The matrix $\breve{\mathscr{Y}}_n^{k_1,k_2}$ is defined as

$$\breve{\mathscr{Y}}_n^{k_1,k_2} = \begin{pmatrix} \mathscr{Y}_n^{k_1,k_2} & 0 \\ 0 & \overline{\mathscr{Y}}_n^{k_1,k_2} \end{pmatrix}, \tag{A.27}$$

where only first three rows of both $\mathscr{Y}_n^{k_1,k_2}$ and $\overline{\mathscr{Y}}_n^{k_1,k_2}$ are taken.

Rational limit. In the rational limit $q \to 1$ the coefficients (A.24) and (A.26) acquire quite compact expressions

$$\begin{aligned} \frac{\mathcal{A}_1}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{A}_4 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^- - x_1^+\right)\left(1 - x_1^- x_2^-\right)\gamma_2}{x_1^- x_2^- \gamma_1}, \\ \frac{\mathcal{A}_3}{\tilde{\alpha}} &= \tilde{\alpha}\,\mathcal{A}_2 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_2^- - x_1^+\right)\gamma_1\gamma_2}{x_2^- x_1^+}, \end{aligned} \tag{A.28}$$

giving

$$\mathcal{A}_0 = -\frac{g^2}{\alpha M_2}\frac{\left(1 - x_1^- x_2^-\right)\left(x_1^- - x_1^+\right)\left(x_2^- - x_1^+\right)\gamma_2^2}{x_1^-(x_2^-)^2 x_1^+}, \tag{A.29}$$

and also

$$\begin{aligned} \frac{\mathcal{B}_1}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{B}_6 = i\frac{g}{M_2}\frac{\left(x_2^- - x_2^+\right)\left(x_1^+ - x_1^- x_2^- x_2^+\right)}{x_1^- x_2^- x_2^+}, \\ \frac{\mathcal{B}_2}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{B}_5 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^- - x_1^+\right)\left(1 - x_1^- x_2^+\right)\gamma_2}{x_1^- x_2^+ \gamma_1}, \\ \frac{\mathcal{B}_3}{\tilde{\alpha}} &= \alpha^2\tilde{\alpha}\,\mathcal{B}_4 = -i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^+ - x_2^+\right)\gamma_1\gamma_2}{x_1^+ x_2^+}, \\ \frac{\mathcal{B}_7}{\tilde{\alpha}} &= \alpha^2\tilde{\alpha}\,\mathcal{B}_8 = i\frac{g}{M_2}\frac{\left(x_1^- - x_1^+\right)\gamma_2^2}{x_1^- x_2^+}. \end{aligned} \tag{A.30}$$

B Elements of the special cases of the S-matrix

B.1 Elements of the fundamental S-matrix

The fundamental S-matrix for the space III acquires the following form,

$$\begin{aligned} \mathbb{S}\,|\phi^1\phi^2\rangle &= (\mathscr{Z}_1^{1,0})_1^1|\phi^1\phi^2\rangle + (\mathscr{Z}_0^{1,0})_1^1|\phi^2\phi^1\rangle + (\mathscr{Z}_1^{1,0})_1^5|\psi^1\psi^2\rangle + (\mathscr{Z}_0^{1,0})_1^6|\psi^2\psi^1\rangle, \\ \mathbb{S}\,|\phi^2\phi^1\rangle &= (\mathscr{Z}_1^{0,1})_1^1|\phi^1\phi^2\rangle + (\mathscr{Z}_0^{0,1})_1^1|\phi^2\phi^1\rangle + (\mathscr{Z}_1^{0,1})_1^5|\psi^1\psi^2\rangle + (\mathscr{Z}_0^{0,1})_1^6|\psi^2\psi^1\rangle, \\ \mathbb{S}\,|\psi^1\psi^2\rangle &= (\mathscr{Z}_1^{1,0})_5^1|\phi^1\phi^2\rangle + (\mathscr{Z}_0^{1,0})_5^1|\phi^2\phi^1\rangle + (\mathscr{Z}_1^{1,0})_5^5|\psi^1\psi^2\rangle + (\mathscr{Z}_0^{1,0})_5^6|\psi^2\psi^1\rangle, \\ \mathbb{S}\,|\psi^2\psi^1\rangle &= (\mathscr{Z}_1^{0,1})_6^1|\phi^1\phi^2\rangle + (\mathscr{Z}_0^{0,1})_6^1|\phi^2\phi^1\rangle + (\mathscr{Z}_1^{0,1})_6^5|\psi^1\psi^2\rangle + (\mathscr{Z}_0^{0,1})_6^6|\psi^2\psi^1\rangle. \end{aligned} \tag{B.1}$$

In order to find these coefficients $\mathscr{Z}$ it is sufficient to consider the first relation of (4.68) and its affine counterpart only. In fact, the constraints read as follows,

$$\begin{aligned}
&\begin{pmatrix} (G^{op})^2_1 & (G^{op})^2_5 \\ (\widetilde{G}^{op})^2_1 & (\widetilde{G}^{op})^2_5 \end{pmatrix}(1,0) \begin{pmatrix} (\mathscr{Z}^{1,0}_1)^1_1 & (\mathscr{Z}^{1,0}_1)^1_5 \\ (\mathscr{Z}^{1,0}_1)^5_1 & (\mathscr{Z}^{1,0}_1)^5_5 \end{pmatrix} = (\mathscr{Y}^{1,0}_1)^2_2 \begin{pmatrix} (G)^2_1 & (G)^2_5 \\ (\widetilde{G})^2_1 & (\widetilde{G})^2_5 \end{pmatrix}(1,0)\,, \\
&\begin{pmatrix} (G^{op})^2_1 & (G^{op})^2_5 \\ (\widetilde{G}^{op})^2_1 & (\widetilde{G}^{op})^2_5 \end{pmatrix}(1,0) \begin{pmatrix} (\mathscr{Z}^{0,1}_1)^1_1 & (\mathscr{Z}^{0,1}_1)^1_6 \\ (\mathscr{Z}^{0,1}_1)^5_1 & (\mathscr{Z}^{0,1}_1)^5_6 \end{pmatrix} = (\mathscr{Y}^{0,1}_1)^2_1 \begin{pmatrix} (G)^1_1 & (G)^1_6 \\ (\widetilde{G})^1_1 & (\widetilde{G})^1_6 \end{pmatrix}(0,1)\,, \\
&\begin{pmatrix} (G^{op})^1_1 & (G^{op})^1_6 \\ (\widetilde{G}^{op})^1_1 & (\widetilde{G}^{op})^1_6 \end{pmatrix}(0,1) \begin{pmatrix} (\mathscr{Z}^{1,0}_0)^1_1 & (\mathscr{Z}^{1,0}_0)^1_5 \\ (\mathscr{Z}^{1,0}_0)^6_1 & (\mathscr{Z}^{1,0}_0)^6_5 \end{pmatrix} = (\mathscr{Y}^{1,0}_0)^1_2 \begin{pmatrix} (G)^2_1 & (G)^2_5 \\ (\widetilde{G})^2_1 & (\widetilde{G})^2_5 \end{pmatrix}(1,0)\,, \\
&\begin{pmatrix} (G^{op})^1_1 & (G^{op})^1_6 \\ (\widetilde{G}^{op})^1_1 & (\widetilde{G}^{op})^1_6 \end{pmatrix}(0,1) \begin{pmatrix} (\mathscr{Z}^{0,1}_0)^1_1 & (\mathscr{Z}^{0,1}_0)^1_6 \\ (\mathscr{Z}^{0,1}_0)^6_1 & (\mathscr{Z}^{0,1}_0)^6_6 \end{pmatrix} = (\mathscr{Y}^{0,1}_0)^1_1 \begin{pmatrix} (G)^1_1 & (G)^1_6 \\ (\widetilde{G})^1_1 & (\widetilde{G})^1_6 \end{pmatrix}(0,1)\,.
\end{aligned} \tag{B.2}$$

It is easy to solve these relations for $\mathscr{Z}$ and we find that they agree with [10]. For the completeness, we have listed the relations of our elements $\mathscr{Z}$ to those of [10][5]

$$\begin{aligned}
\begin{pmatrix} (\mathscr{Z}^{1,0}_1)^1_1 & (\mathscr{Z}^{1,0}_1)^1_5 \\ (\mathscr{Z}^{1,0}_1)^5_1 & (\mathscr{Z}^{1,0}_1)^5_5 \end{pmatrix} = \begin{pmatrix} (\mathscr{Z}^{0,1}_0)^1_1 & (\mathscr{Z}^{0,1}_0)^1_6 \\ (\mathscr{Z}^{0,1}_0)^6_1 & (\mathscr{Z}^{0,1}_0)^6_6 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{A_{12}-B_{12}}{q+q^{-1}} & -\frac{F_{12}}{q+q^{-1}} \\ \frac{C_{12}}{q+q^{-1}} & -\frac{D_{12}-E_{12}}{q+q^{-1}} \end{pmatrix}, \\
\begin{pmatrix} (\mathscr{Z}^{0,1}_1)^1_1 & (\mathscr{Z}^{0,1}_1)^1_6 \\ (\mathscr{Z}^{0,1}_1)^5_1 & (\mathscr{Z}^{0,1}_1)^5_6 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{q^{-1}A_{12}+qB_{12}}{q+q^{-1}} & \frac{qF_{12}}{q+q^{-1}} \\ -\frac{qC_{12}}{q+q^{-1}} & -\frac{q^{-1}D_{12}+qE_{12}}{q+q^{-1}} \end{pmatrix}, \\
\begin{pmatrix} (\mathscr{Z}^{0,1}_0)^1_1 & (\mathscr{Z}^{0,1}_0)^1_6 \\ (\mathscr{Z}^{0,1}_0)^6_1 & (\mathscr{Z}^{0,1}_0)^6_6 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{qA_{12}+q^{-1}B_{12}}{q+q^{-1}} & \frac{q^{-1}F_{12}}{q+q^{-1}} \\ -\frac{q^{-1}C_{12}}{q+q^{-1}} & -\frac{qD_{12}+q^{-1}E_{12}}{q+q^{-1}} \end{pmatrix}.
\end{aligned} \tag{B.3}$$

B.2 Elements of the S-matrix $\mathbf{S}_{Q1}$

Here we list the explicit forms of the coefficients of the matrix S_{Q1}.

Space II. First we give the coefficients of the matrix $\mathscr{Y}$ in the case of a bound state scattering with a fundamental particle. There are four different combinations of the parameters k_1, k_2, n that contribute. Thus we have to consider the case where $k_2 = 0$ and $k_1 = n = k$ leading to

$$\begin{aligned}
&(\mathscr{Y}^{k,0}_{k-1})^1_1 = q^{\frac{1}{2}+k} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{z_{12} - q^{Q-2k-1}}{z_{12} - q^{Q-1}}, && (\mathscr{Y}^{k,0}_{k-1})^2_2 = \frac{1}{q^{\frac{Q}{2}} U_1 V_1} \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}, \\
&(\mathscr{Y}^{k,0}_{k-1})^1_2 = q^{\frac{1-Q}{2}} \frac{[Q-k]_q}{\sqrt{[Q]_q}} \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \frac{U_2 V_2}{U_1 V_1} \frac{\gamma_1}{\gamma_2}, && (\mathscr{Y}^{k,0}_{k-1})^2_1 = \frac{1}{\sqrt{[Q]_q}} \frac{x_1^- - x_1^+}{x_1^- - x_2^+} \frac{\gamma_2}{\gamma_1}, \\
&(\mathscr{Y}^{k,0}_{k-1})^1_4 = \frac{q^{\frac{1-Q}{2}}\alpha}{\sqrt{[Q]_q}} \frac{U_2 V_2}{U_1 V_1} \frac{[x_1^- - x_1^+][x_2^- - x_2^+][x_2^- - x_1^+]}{(x_1^- - x_2^+)(x_1^- x_2^- - 1)\gamma_1\gamma_2}, && (\mathscr{Y}^{k,0}_{k-1})^4_2 = (\mathscr{Y}^{k,0}_{k-1})^2_4 = 0, \\
&(\mathscr{Y}^{k,0}_{k-1})^4_1 = \frac{q^{-Q}[k]_q}{\sqrt{[Q]_q}} \frac{x_1^+ - x_2^-}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)} \frac{x_1^-}{x_1^+} \frac{\gamma_1\gamma_2}{\alpha}, && (\mathscr{Y}^{k,0}_{k-1})^4_4 = \frac{q^{-\frac{Q}{2}}}{U_1 V_1} \frac{x_1^+ - x_1^-}{x_1^- - x_2^+} \frac{1 - x_1^- x_2^+}{1 - x_1^- x_2^-}.
\end{aligned} \tag{B.4}$$

[5] We remind that our $x^\pm$ parameterization is based on the one of [35] which are related to those of [10] by $x^\pm_{[35]} = g\tilde{g}^{-1}(x^\pm_{[10]} + \xi)$. This point must be taken into account when performing the concrete comparison.

Next we have three elements corresponding to $k_2 = 1$ and $k_1 + 1 = n = k$ giving

$$(\mathscr{Y}^{k,0}_{k-1})^1_1 = q^{\frac{1}{2}-Q} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{(q^{2(k+1)} - q^{2Q}) z_{12}}{z_{12} - q^{Q-1}}, \qquad (\mathscr{Y}^{k,0}_{k-1})^2_1 = \frac{q^{1+k-Q}}{\sqrt{[Q]_q}} \frac{x_1^- - x_1^+}{x_1^- - x_2^+} \frac{\gamma_2}{\gamma_1},$$
$$(\mathscr{Y}^{k,0}_{k-1})^4_1 = \frac{[Q-k-1]_q}{q^{Q-k-1}\sqrt{[Q]_q}} \frac{x_2^- - x_1^+}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)} \frac{x_1^-}{x_1^+} \frac{\gamma_1\gamma_2}{\alpha}. \tag{B.5}$$

Then we have another three scattering entries for $k_2 = 0$ and $k_1 = n + 1 = k$ contributing

$$(\mathscr{Y}^{k,0}_{k-1})^1_1 = q^{\frac{1}{2}+Q} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - q^{-2k}}{q^Q - q z_{12}}, \quad (\mathscr{Y}^{k,0}_{k-1})^1_2 = q^{\frac{1+Q-2k}{2}} \frac{[k]_q}{\sqrt{[Q]_q}} \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \frac{U_2 V_2}{U_1 V_1} \frac{\gamma_1}{\gamma_2},$$
$$(\mathscr{Y}^{k,0}_{k-1})^1_4 = -q^{-k} (\mathscr{Y}^{k,0}_{k})^1_4. \tag{B.6}$$

Finally, there is one element with $k_2 = 1$ and $k_1 = n = k - 1$ providing the last element

$$(\mathscr{Y}^{k-1,1}_{k-1})^1_1 = q^{\frac{1}{2}-k} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{q^{2k} - q^{1+Q} z_{12}}{q^Q - q z_{12}}. \tag{B.7}$$

Space III. There are 36 elements of the matrix $\mathscr{Z}$ that need be determined. As mentioned in Section 5, it follows that (5.16) becomes

$$\mathbb{S}\,|k-1,1\rangle^{\text{III}}_6 = \mathscr{D}\left(|k,0\rangle^{\text{III}}_5 + q\,|k-1,1\rangle^{\text{III}}_6\right) - q\,\mathbb{S}\,|k,0\rangle^{\text{III}}_5. \tag{B.8}$$

Acting with the S-matrix on both sides of the equations (5.17) and using its invariance property allows us to express the elements of the S-matrix of the left hand side to the ones on the right hand side. Explicitly we find

$$
\begin{aligned}
(\mathscr{Z}^{k-1,1}_{k})^1_1 =& (\mathscr{Z}^{k,0}_{k})^1_1 [Q-k+1]_q (q^{2k-Q-2} - q) + (\mathscr{Z}^{k,0}_{k-1})^1_1 \frac{[Q-k+1]_q}{[k]_q}, \\
(\mathscr{Z}^{k-1,1}_{k-1})^1_1 =& (\mathscr{Z}^{k,0}_{k})^1_1 + \frac{[k-1]_q [Q-k+2]_q (q^{2k-Q-4} - q) - [Q-2k+1]_q}{[k]_q} (\mathscr{Z}^{k,0}_{k-1})^1_1, \\
(\mathscr{Z}^{k-1,1}_{k})^3_1 =& (\mathscr{Z}^{k,0}_{k})^3_1 \frac{[k-1]_q [Q-k]_q q^{2k-Q-2} - q[k]_q [Q-k+1]_q}{[k]_q} + (\mathscr{Z}^{k,0}_{k-1})^3_1 \frac{[Q-k]_q}{[k]_q}, \\
(\mathscr{Z}^{k-1,1}_{k-1})^3_1 =& (\mathscr{Z}^{k,0}_{k-1})^3_1 \frac{[k-2]_q [Q-k+1]_q q^{2k-Q-4} + q[k-1]_q [k-Q-2]_q + [2k-Q-1]_q}{[k]_q} \\
&+ (\mathscr{Z}^{k,0}_{k})^3_1 \frac{[k-1]_q)}{[k]_q}, \\
(\mathscr{Z}^{k-1,1}_{k})^5_1 =& (\mathscr{Z}^{k,0}_{k})^5_1 \frac{([k-1]_q q^{2k-3-Q} - q[k]_q)[Q-k+1]_q}{[k]_q}, \\
(\mathscr{Z}^{k-1,1}_{k-1})^6_1 =& (\mathscr{Z}^{k,0}_{k-1})^6_1 \frac{([k-1]_q q^{2k-3-Q} - q[k]_q)[Q-k+1]_q}{[k]_q}, \\
(\mathscr{Z}^{k-1,1}_{k})^1_3 =& (\mathscr{Z}^{k,0}_{k})^1_3 \left[\frac{[k]_q [Q-k+1]_q q^{2k-Q-2}}{[k-1]_q} - q[Q-k]_q\right] + (\mathscr{Z}^{k,0}_{k-1})^1_3 \frac{[Q-k+1]_q}{[k-1]_q}, \\
(\mathscr{Z}^{k-1,1}_{k-1})^1_3 =& (\mathscr{Z}^{k,0}_{k-1})^1_3 \left[\frac{[Q-k+2]_q}{q^{Q+4-2k}} + [k-Q+1]_q - \frac{[k-2]_q q^{Q-k+1}}{[k-1]_q}\right] + (\mathscr{Z}^{k,0}_{k})^1_3 \frac{[k]_q}{[k-1]_q}, \\
(\mathscr{Z}^{k-1,1}_{k})^3_3 =& (\mathscr{Z}^{k,0}_{k})^3_3 [Q-k]_q (q^{2k-Q-2} - q) + (\mathscr{Z}^{k,0}_{k-1})^3_3 \frac{[Q-k]_q}{[k-1]_q},
\end{aligned}
$$

$$
\begin{aligned}
(\mathscr{Z}^{k-1,1}_{k-1})^3_3 =&(\mathscr{Z}^{k,0}_{k-1})^3_3\left[\frac{[k-2]_q[Q-k+1]_q}{q^{Q-2k+4}[k-1]_q}+\frac{q[k]_q[k-Q+1]_q-q^2[2k-Q-1]_q}{[k-1]_q}\right]+(\mathscr{Z}^{k,0}_{k})^3_3\,,\\
(\mathscr{Z}^{k-1,1}_{k})^5_3 =&(\mathscr{Z}^{k,0}_{k})^5_1([Q-k+1]_q q^{2k-Q-3}-q[Q-k])\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^6_3 =&(\mathscr{Z}^{k,0}_{k-1})^6_1([Q-k+1]_q q^{2k-Q-3}-q[Q-k])\,.
\end{aligned}
\tag{B.9}
$$

Finally, the remaining elements are

$$
\begin{aligned}
(\mathscr{Z}^{k,0}_{k})^1_5 =&\frac{\alpha}{U_1V_1}\frac{(x_1^--x_1^+)(x_2^--x_2^+)\left[\frac{(\xi x_1^++1)[Q-k]_q(q(\xi+x_2^-)-x_2^+-\xi)}{(\xi^2-1)q^Q}-[k]_q(x_1^+-x_2^+)\right]}{\gamma_1\gamma_2[k]_q\sqrt{[Q]}(1-x_1^-x_2^-)(x_1^--x_2^+)q^{\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^1_5 =&\frac{\alpha}{U_1V_1}\frac{(x_1^--x_1^+)(x_2^--x_2^+)[q(\xi+x_2^-)(\xi x_1^++1)-(\xi+x_1^+)(\xi x_2^++1)]}{\gamma_1\gamma_2(\xi^2-1)\sqrt{[Q]}(1-x_1^-x_2^-)(x_1^--x_2^+)q^{k+\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k})^3_5 =&\frac{\gamma_1}{\gamma_2 q^{\frac{Q}{2}}}\frac{[Q-k]_q}{\sqrt{[Q]_q}}\frac{(x_2^--x_2^+)[q(\xi+x_1^-)(\xi+x_2^-)-(\xi x_1^-+1)(\xi x_2^++1)]}{(\xi^2-1)(1-x_1^-x_2^-)U_1V_1(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^3_5 =&\frac{\gamma_1}{\gamma_2}\frac{[k-1]_q}{\sqrt{[Q]_q}}\frac{(x_2^--x_2^+)[q(\xi+x_1^-)(\xi+x_2^-)-(\xi x_1^-+1)(\xi x_2^++1)]}{(\xi^2-1)(1-x_1^-x_2^-)U_1V_1(x_1^--x_2^+)q^{k-\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k})^5_5 =&\frac{(x_1^+-x_2^+)[(\xi x_1^-+1)(\xi x_2^++1)-q(\xi+x_1^-)(\xi+x_2^-)]}{(\xi^2-1)(x_1^-x_2^--1)(x_1^--x_2^+)U_1V_1U_2V_2q^{\frac{Q+1}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^6_5 =&\frac{z_{12}(x_2^--x_2^+)(x_1^+(\xi x_2^-+1)(\xi x_2^++1)-V_1^4x_1^-(\xi+x_2^-)(\xi+x_2^+))}{(\xi^2-1)V_1^2x_2^+(x_1^-x_2^--1)(x_1^--x_2^+)q^{\frac{1+Q}{2}}}\frac{U_2V_2}{U_1V_1}\,,\\
(\mathscr{Z}^{k,0}_{k})^1_1 =&\frac{x_2^-(x_1^--x_1^+)[Q-k]_Q[(\xi x_1^-+1)(\xi x_1^++1)-V_2^2(\xi+x_1^-)(\xi+x_1^+)]}{(\xi^2-1)x_1^+z_{12}[Q]_q(x_1^-x_2^--1)(x_1^--x_2^+)q^Q}+\\
&+\frac{x_1^-(x_2^-x_1^+-1)(x_1^+-x_2^+)q^{k-2Q}}{x_1^+(x_1^-x_2^--1)(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^1_1 =&\frac{x_2^-[k]_qq^{-k}(x_1^--x_1^+)[(\xi x_1^-+1)(\xi x_1^++1)-V_2^2(\xi+x_1^-)(\xi+x_1^+)]}{(1-\xi^2)x_1^+z_{12}[Q]_q(1-x_1^-x_2^-)(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^3_1 =&\frac{\gamma_1^2x_1^-[k]_qq^{-Q-1}(\xi x_2^++1)[Q-k]_q[qx_2^+(\xi+x_2^-)-x_2^-(\xi+x_2^+)]}{\alpha(\xi^2-1)x_1^+x_2^+[Q]_q(1-x_1^-x_2^-)(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^3_1 =&\frac{\gamma_1^2x_1^-[k-1]_q[k]q^{-k-1}(\xi x_2^++1)[qx_2^+(\xi+x_2^-)-x_2^-(\xi+x_2^+)]}{\alpha(\xi^2-1)x_1^+x_2^+[Q]_q(1-x_1^-x_2^-)(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^5_1 =&\frac{\gamma_1\gamma_2(V_2^2-1)x_1^-x_2^-[k]_qq^{-Q-\frac{3}{2}}(x_1^+-x_2^+)(\xi+x_2^+)(\xi x_2^++1)}{\alpha(\xi^2-1)x_1^+x_2^+\sqrt{[Q]_q}(1-x_1^-x_2^-)(x_1^--x_2^+)(x_2^--x_2^+)U_2V_2}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^6_1 =&\frac{\gamma_1\gamma_2x_1^-[k]_qq^{-Q-\frac{3}{2}}(x_1^+-x_2^+)}{\alpha x_1^+\sqrt{[Q]_q}(x_1^-x_2^--1)(x_1^--x_2^+)U_2V_2}\,,\\
(\mathscr{Z}^{k,0}_{k})^1_3 =&\frac{\alpha q(1-V_2^2)(x_1^--x_1^+)^2(\xi+x_2^-)(\xi x_2^-+1)}{\gamma_1^2(\xi^2-1)[Q]_q(x_1^-x_2^--1)(x_1^--x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^1_3 =&\frac{\alpha(1-V_2^2)q^{k-3}(x_1^--x_1^+)^2(\xi+x_2^-)(\xi x_2^-+1)}{\gamma_1^2(\xi^2-1)[Q]_q(1-x_1^-x_2^-)(x_1^--x_2^+)}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\mathscr{Z}_k^{k,0})_3^3 =& \frac{q^k(x_1^- - x_2^-)(x_1^- x_2^+ - 1)}{(x_1^- x_2^- - 1)(x_1^- - x_2^+)} + \\
& - \frac{x_2^-[k]_q(x_1^- - x_1^+)[V_2^2(\xi x_1^- + 1)(\xi x_1^+ + 1) - (\xi + x_1^-)(\xi + x_1^+)]}{(\xi^2 - 1)x_1^+ z_{12}[Q]_q(x_1^- x_2^- - 1)(x_1^- - x_2^+)} , \\
(\mathscr{Z}_{k-1}^{k,0})_3^3 =& \frac{x_2^-[k-1]_q q^{-k}(x_1^- - x_1^+)[V_2^2(\xi x_1^- + 1)(\xi x_1^+ + 1) - (\xi + x_1^-)(\xi + x_1^+)]}{(\xi^2 - 1)x_1^+ z_{12}[Q]_q(x_1^- x_2^- - 1)(x_1^- - x_2^+)} , \\
(\mathscr{Z}_k^{k,0})_3^5 =& \sqrt{\frac{q}{[Q]_q}} \frac{(V_2^2 - 1)(x_1^- - x_1^+)(1 - x_1^- x_2^+)(\xi + x_2^-)(\xi x_2^- + 1)}{(\xi^2 - 1)(x_1^- x_2^- - 1)(x_1^- - x_2^+)(x_2^- - x_2^+)} , \\
(\mathscr{Z}_{k-1}^{k,0})_3^6 =& \frac{\gamma_2 q^{-Q-\frac{1}{2}}(x_1^- - x_1^+)(x_1^- x_2^+ - 1)}{\gamma_1 \sqrt{[Q]_q}(x_1^- x_2^- - 1)(x_1^- - x_2^+)U_2 V_2} .
\end{aligned}
\tag{B.10}
$$

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