Implied volatility: small time-to-expiry asymptotics in exponential Lévy models

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Abstract

In this paper, we examine the small time-to-expiry behaviour of implied volatility in models of exponential Lévy type. In the at-the-money case, it turns out that the implied volatility converges, as time-to-expiry goes to zero, to the square root of the Gaussian member of the driving Lévy process' characteristic triplet. In particular, the limit is zero if the Lévy process has no Gaussian part. In the not at-the-money case, there are a number of possible behaviours. In most cases of interest, however, the implied volatility goes to infinity as time-to-expiry goes to zero. It is also shown that there are exponential Lévy models in which the implied volatility converges to zero as time-to-expiry goes to zero.

Implied volatility at strike K and time-to-expiry τ is the unique volatility parameter that when plugged into the Black-Scholes formula recovers the quoted call option price at strike K and and time-to-expiry τ. Hence implied volatility is another way of quoting call option prices. In practice, it is more popular than quoting actual call option prices. Hence the interest in implied volatility.

In this paper, we present first order asymptotics for implied volatility as time-to-expiry goes to zero. There has recently been much work on small time-to-expiry asymptotics of call options in stochastic volatility models, see, for example, Forde and Jacquier (2009) and references therein. In this paper, however, we exclusively examine small time-to-expiry asymptotics of implied volatility in models of exponential Lévy type. Despite the popularity of Lévy processes in mathematical finance, see Cont and Tankov (2004) and references therein, the literature treating the small time-to-expiry behaviour of implied volatility in this class of models is small. The method we use is to establish small time-to-expiry asymptotics for the call option price and then to use the results of Roper and Rutkowski (2009) to relate the call option price asymptotics to the implied volatility asymptotics.

It appears that the not at-the-money asymptotics in exponential Lévy models were first rigorously analysed in Roper (2008) and the at-the-money case was first rigorously analysed in Roper (2009). Note though that the results of Durrleman (2008) are applicable to some exponential Lévy models and when they are they agree with the results of this paper. Since then, there have been a number of works devoted to small time-to-expiry asymptotics of implied volatility in models of exponential Lévy type. The contributions that we make here are two-fold. Firstly, in models of exponential

Lévy type and in the not at-the-money case it is found that implied volatility may have a number of different behaviours close to expiry. In particular, it may go to zero but, in most cases of interest, it goes to infinity. Secondly, it is shown that in all exponential Lévy models implied volatility converges, as time-to-expiry goes to zero, to the square root of the Gaussian member of the driving Lévy process' characteristic triplet. In particular, the limit is zero if the Lévy process has no Gaussian part.

We now compare our results to the existing literature. Levendorskiǐ (see Levendorskiǐ (2008)) calculates small time-to-expiry asymptotics for the European put and call in exponential Lévy models. Attention is restricted to the not at-the-money case. Similar results are in Levendorskiǐ (2004). His results require some regularity conditions on the driving Lévy process that we are able to do without. Our approach to the small time-to-expiry asymptotics of European calls and puts is different to that of Levendorskiǐ. In addition, Levendorskiǐ does not apply his results to implied volatility asymptotics.

Let S be a non-negative martingale, not necessarily of exponential Lévy type. Under some assumptions on S, Carr and Wu (see Carr and Wu (2003)) claim that

$$\mathbb{E}\left((S_{t+\tau}-K)^{+}|\mathscr{F}_t\right)-(S_t-K)^{+}=\mathrm{O}\left(\tau\right), \quad \text{as } \tau\to 0^{+}, \tag{0.1}$$

in the case that $S_t \neq K$. The left-hand side of Equation (0.1) is termed the time value of a European call.

In the case of S being an exponential Lévy process, Carr and Wu (see Carr and Wu (2003)) argue for exact asymptotics for the time value, for example for the out of the money call, in terms of $\tau\int(S_0e^x-K)^{+}\nu(dx)$, where ν is the Lévy measure of the driving Lévy process. We establish rigorously this claim. They also claim that the time value of at-the-money European calls in the pure jump models that they consider decay at the rate $O(\tau^p)$ as $\tau\to 0^{+}$ for some p in $(0,1]$. For stochastic volatility plus jumps models (i.e. noise with non-vanishing Brownian part as well as a jump process part) they claim that the decay rate is $O(\tau^p)$ as $\tau\to 0^{+}$ for some p in $(0,1/2]$. The decay rate for out-of-the money calls is claimed to be $O(\tau)$ in the pure jump models and stochastic volatility plus jumps cases. In the purely continuous case, it is claimed that the decay rate is $O(e^{-c/\tau})$ as $\tau\to 0^{+}$ for some $c>0$.

Carr and Wu (see Carr and Wu (2003)) claim that in the not at-the-money case implied volatility explodes as time-to-expiry goes to zero once jumps are included in their stochastic volatility with jumps model. We show that this is not necessarily the case even in the simpler exponential Lévy models that we consider here.

The behaviour of at-the-money implied volatility in stochastic volatility models with bounded (spot) volatility and a finite variation jump component has been considered by Durrleman (see Durrleman (2008)). The limiting value of the implied volatility turns out to be the "instantaneous spot volatility" of the model. This agrees with the results we obtain in this paper.

In a recent article (Figueroa-Lopéz and Forde (2010)), the authors obtain second-order asymptotics for not at-the-money implied volatility in exponential Lévy models. Their results require some reasonably mild extra conditions on the Lévy process that we are able to do without, but it should be noted that we only give first order asymptotics. We show that not at-the-money implied volatility goes to infinity in models of exponential Lévy type under weak conditions. In the not at-the-money case, Tankov (2009) establishes the rate at which the implied volatility goes to infinity. This is obtained under basically the same weak conditions that we require in this paper. As for the at-the-money case, Tankov obtains the rate at which the at-the-money implied volatility goes to the square root of the Gaussian member of the process' characteristic triplet. A number

of different conditions on the driving Lévy process are required. Tankov (2009) does not establish that at-the-money implied volatility always goes to to the square root of the Gaussian member of the driving Lévy process' characteristic triplet. In contrast, we show that, in general, the at-the-money implied volatility in exponential Lévy process models goes to the square root of the Gaussian member of the process' characteristic triplet. Note, though, that we only get the limiting value of implied volatility, while Tankov (2009) obtains the rate at which the at-the-money implied volatility goes to its limiting value in the cases he considers.
We proceed as follows. Section 1 gives the background necessary for the formulation and proof of our main results and supporting lemmas. Supporting lemmas are given in Section 2 and the main results are given in Section 3. Section 4 gives examples of our main results. Section 5 concludes our study. Proofs of the various lemmas are given in the Appendix.

1 Background

1.1 Setup

We first recall the definition of a Lévy process. We are assuming that we are working on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.

Definition 1.1 (Lévy process)**.** Let X be a real-valued process with $X_0 = 0$ $\mathbb{P}$-a.s. and

(1) X has increments independent of the past; that is $X_t - X_s$ is independent of $\mathscr{F}_s$, $0 \leq s < t < \infty$; and

(2) X has stationary increments; that is $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$; and

(3) X is continuous in probability; that is, $\lim_{t\to s} X_t = X_s$, where the limit is taken in probability.

See Protter (2004), p. 20 for this definition.

Remark 1.2. Note Theorem 30 on p.20 of Protter (2004). It gives that if X is a Lèvy process, then there exists a unique modification Y of X which is càdlàg and also a Lévy process. We will work throughout with this modification. To avoid the introduction of further notation, we will use X to denote the càdlàg modification.

We recall that a Lévy process is described by its characteristic triplet (b, σ^2, ν) where $\sigma, b \in \mathbb{R}$, $\sigma \geq 0$, and ν is a non-negative Radon measure satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \int_{\mathbb{R}} (1 \wedge y^2) \nu(\,\mathrm{d}y) < \infty,$$

see Bertoin (1996), p. 3. We model the stock as the exponential of a Lévy process, that is

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0, \tag{1.1}$$

where X is a Lévy process and $S_0 > 0$ is some finite constant. So that our stock price process is a martingale we require that

$$\int_{|y|\geq 1} e^y\, \nu(\mathrm{d}y) < \infty \text{ and } b = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{|y|\leq 1})\, \nu(\mathrm{d}y), \tag{1.2}$$

see p. 354 of Cont and Tankov (2004).
For simplicity, we assume the interest rate and dividend yield are both zero. The stock is a martingale under $\mathbb{P}$ and the driving Lévy process satisfies the constraints set out in (1.2). The model is presented under the pricing measure, $\mathbb{P}$, chosen by the market so that we price options as expectations under $\mathbb{P}$ of their payoff. Since S is a time-homogeneous Markov process, there is no loss of generality in assuming that we are at time zero.
As is standard, the price of a K strike, τ time-to-expiry, European call option is given by $\mathbb{E}\left((S_\tau - K)^+\right)$.
We now need to define the Black-Scholes formula, see Musiela and Rutkowski (2005).

Definition 1.3. The Black-Scholes price of a European call option with strike $K > 0$, time-to-expiry $\tau \geq 0$, stock price $S_0 > 0$, and volatility $\sigma > 0$ is

$$BS(K, \tau, S_0, \sigma) = S_0 \Phi\left(\frac{\ln(S_0/K)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right) - K\Phi\left(\frac{\ln(S_0/K)}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right),$$

where $BS(K, \tau, S_0, 0) = (S_0 - K)^+$ and $BS(K, \tau, S_0, \infty) = S_0$.

Definition 1.4. The implied volatility of a K strike, τ time-to-expiry, and stock price S_0 is the unique $\Sigma(K, \tau)$ that satisfies

$$BS(K, \tau, S_0, \Sigma(K, \tau)) = \mathbb{E}\left((S_\tau - K)^+\right).$$

1.2 Auxiliary Facts

We need some definitions and results from the literature to prove our claims.

Definition 1.5. A non-negative locally bounded function $k : \mathbb{R} \to \mathbb{R}$ is submultiplicative if there exists a constant $\alpha > 0$ such that $k(x+y) \leq \alpha k(x)k(y)$ for all $x, y \in \mathbb{R}$ (see Figueroa-Lopéz (2008)).

Definition 1.6. A non-negative locally bounded function $q : \mathbb{R} \to \mathbb{R}$ is subadditive provided that there exists a constant $\beta > 0$ such that $q(x+y) \leq \beta(q(x) + q(y))$ for all $x, y \in \mathbb{R}$ (see Figueroa-Lopéz (2008)).

We will use the following class of "dominating functions".

Definition 1.7 (The class $\mathscr{S}(\nu)$)**.** Suppose that ν is a Lévy measure. A function $u : \mathbb{R} \to \mathbb{R}$ is of class $\mathscr{S}(\nu)$ if

(1) $u(x) = q(x)k(x)$ for some functions q and k, where q is subadditive and k is submultiplicative; and

(2) $\int_{|x|>1} |u(x)|\, \nu(\mathrm{d}x) < \infty$,

see Figueroa-Lopéz (2008).

We use the following result a number of times in our proofs.

Theorem 1.8 (Figueroa-López, Theorem 1.1 in Figueroa-Lopéz (2008), abbreviated)**.** *Let X be a Lévy process with characteristic triplet (b, σ^2, ν). Let $w : \mathbb{R} \to \mathbb{R}$ satisfy*

(1) $w(x) = o\left(x^2\right)$ as $x \to 0$;

(2) w is locally bounded;

(3) w is ν-a.e. continuous; and

(4) there exists a function $u \in \mathscr{S}(\nu)$ for which

$$\limsup_{|x|\to\infty} \frac{|w(x)|}{u(x)} < \infty.$$

Then

$$\tau^{-1}\mathbb{E}\left(w(X_\tau)\right) \to \int_{\mathbb{R}} w(x)\,\nu(\mathrm{d}x), \quad \text{as } \tau \to 0^+.$$

If conditions (2)-(4) hold, but (1) is replaced with

(1') $w(x) \sim x^2$ as $x \to 0$,

then

$$\tau^{-1}\mathbb{E}\left(w(X_\tau)\right) \to \sigma^2 + \int_{\mathbb{R}} w(x)\,\nu(\mathrm{d}x) \quad \text{as } \tau \to 0^+.$$

We now recall Sato's classification of Lévy processes.

Definition 1.9 (Sato, p. 65 of Sato (1999))**.** Let X be a Lévy process on $\mathbb{R}$ with characteristic triplet (b, σ^2, ν). Then

1. if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, then X is of type A;
2. if $\sigma = 0$, $\nu(\mathbb{R}) = \infty$, and $\int_{|x|\le 1} |x|\ \nu(\,\mathrm{d}x) < \infty$, then X is of type B;
3. if $\sigma \neq 0$ or $\int_{|x|\le 1} |x|\,\nu(\,\mathrm{d}x) = \infty$, then X is of type C.

In order to obtain Lévy processes with zero implied volatility we need the following result.

Theorem 1.10 (Sato, Theorem 24.10 in Sato (1999, abbreviated)**.** *Suppose that X is a Lévy process with characteristic triplet $(b, 0, \nu)$. Suppose that 0 is in the support of ν. Further, assume that X is either of Type A or Type B, see Definition 1.9, then, if the support of ν is a subset of $[0, \infty)$, we have $\mathbb{P}\left(X_\tau \in [b\tau, \infty)\right) = 1$. If the support of ν is a subset of $(-\infty, 0]$, then $\mathbb{P}\left(X_\tau \in (-\infty, b\tau]\right) = 1$.*

We will use the following result to obtain the limiting implied volatilities by first obtaining the small time-to-expiry call option prices.

Theorem 1.11 (Roper and Rutkowski, Corollary 5.1 in Roper and Rutkowski (2009))**.** *Suppose that for a fixed $K > 0$*

1. *$(S_0 - K)^+ \le \mathbb{E}\left((S_\tau - K)^+\right) \le S_0$, forall $\tau \ge 0$.*
2. *$\tau \mapsto \mathbb{E}\left((S_\tau - K)^+\right)$ is right-continuous on $[0, \infty)$.*
3. *$\tau \mapsto \mathbb{E}\left((S_\tau - K)^+\right)$ is non-decreasing.*

If there exists a constant $\delta > 0$ such that for every $\tau \in (0, \delta)$,

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+,$$

then

$$\lim_{\tau \to 0^+} \Sigma K, \tau = \begin{cases} \lim_{\tau \to 0^+} \dfrac{\sqrt{2\pi}\mathbb{E}\left((S_\tau - K)^+\right)}{S_0\sqrt{\tau}}, & \text{if } S_0 = K, \\ \lim_{\tau \to 0^+} \dfrac{|\ln(K/S_0)|}{\sqrt{-2\tau\ln(\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+)}}, & \text{if } S_0 \neq K, \end{cases}$$

in the sense that the left-hand side limit exists (is infinite, respectively) if and only if the right hand-side limit exists (is infinite, respectively) and then they are equal.

2 Lemmas

In this section, we present some definitions and lemmas that are used to prove the main results of this paper. The proofs of the lemmas are relegated to the Appendix.
So as to handle at-the-money implied volatilities we need the following result of Jacod.

Lemma 2.1 (Jacod, Lemma 4.1, p. 181 of Jacod (2007); abbreviated)**.** *Let $\widetilde{X}$ be a Lévy process with no Gaussian part, then $\tau^{-1/2}\widetilde{X}_\tau \xrightarrow{\mathbb{P}} 0$ as $\tau \to 0^+$.*

Proof. An original and I believe simpler proof of this result is given in the Appendix. □

We will frequently use the following call and put functions.

Definition 2.2. For a fixed $K > 0$ and $S_0 > 0$, let

$$\begin{aligned} C : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (S_0 e^x - K)^+, \end{aligned}$$

which is the call function.

Definition 2.3. For a fixed $K > 0$ and $S_0 > 0$, let

$$\begin{aligned} P : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (K - S_0 e^x)^+, \end{aligned}$$

which is the put function.

Definition 2.4. A Lévy process with characteristic triplet $(0, 0, 0)$ is termed a trivial process.

In order to use the results of Roper and Rutkowski (2009), we need some results about the conditional expectation of the call payoff.

Lemma 2.5. *Let*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where X is a Lévy process satisfying the constraints in (1.2) *and $S_0 > 0$ is some finite constant. Then, for each fixed $K > 0$,*

1. $(S_0 - K)^+ \leq \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right) < S_0, \forall \tau \geq 0$;
2. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is right-continuous on* $[0, \infty)$; *and*
3. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is non-decreasing.*
4. *If* $S_0 > K$, *then* $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ *if and only if* $\mathbb{P}\left(S_\tau < K\right) = \mathbb{P}\left(X_\tau < \ln(K/S_0)\right) = 0, \forall \tau > 0$.

Lemma 2.6. *Let S be defined by Equation* (1.1) *and assume that the driving Lévy process satisfies the constraints set out in* (1.2). *Consider the functions*

(1) P, with the additional restriction that $0 < K < S_0$; *and*

(2) C, with the additional restriction that $K > S_0 > 0$.

Then conditions (1)-(4) of Theorem 1.8 are satisfied by P and C under the respective stated conditions on S_0 and K.

Lemma 2.7. *Suppose that U is a non-negative process with representation*

$$U_\tau = U_0 e^{b\tau + \sigma W_\tau + Y_\tau}, \quad \tau \geq 0,$$

where $b \in \mathbb{R}$, $\sigma \geq 0$, $U_0 > 0$ are finite constants, W is a Brownian motion and Y is a compound Poisson process with finite, constant intensity $\lambda > 0$ and a finite exponential mean. That is we assume that $\mathbb{E}\left(e^{Y_\tau}\right) < \infty$ for all $\tau \geq 0$. In addition, suppose that the processes W_τ and Y_τ are independent. Then

$$\lim_{\tau \to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(U_0 - U_0 e^{b\tau + \sigma W_\tau + Y_\tau}\right)^+\right) = \frac{\sigma U_0}{\sqrt{2\pi}}.$$

3 Main Results

In this section, we present the main results of this paper. We begin by establishing small time-to-expiry asymptotics of the call/put option. We then invoke the results of Roper and Rutkowski (2009) to derive the small time-to-expiry asymptotics of implied volatility in exponential Lévy models.

3.1 Call Option Asymptotics

We begin by examining the call option asymptotics in the at-the-money case.

Theorem 3.1. *Let S be defined by Equation* (1.1) *and assume that the driving Lévy process, X, satisfies the constraints set out in* (1.2). *Then*

$$\lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_\tau - S_0)^+\right)}{\tau^{1/2}} = \lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_0 - S_\tau)^+\right)}{\tau^{1/2}} = \frac{\sigma S_0}{\sqrt{2\pi}}, \tag{3.1}$$

and, in particular, if $\sigma = 0$, then

$$\lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_\tau - S_0)^+\right)}{\tau^{1/2}} = \lim_{\tau \to 0^+} \frac{\mathbb{E}\left((S_0 - S_\tau)^+\right)}{\tau^{1/2}} = 0. \tag{3.2}$$

Proof. Since S is a martingale, the first equality in both Equations (3.1) and (3.2) follows from put-call parity. It remains to show the second equality of Equation (3.1) for which it is clearly enough to suppose that $S_0 = 1$. The final equality of Equation (3.2) will then be clear as we will nowhere in the proof use that $\sigma \neq 0$.
X is a Lévy process, with characteristic triplet (b, σ^2, ν), satisfying the constraints set out in (1.2). By the Lévy-Itô decomposition we have that there exists a probability space on which X is the sum of four independent Lévy processes

$$X_\tau = b\tau + \sigma W_\tau + Y_\tau + \widetilde{Y}_\tau, \quad \tau \geq 0,$$

where W is a Wiener process, Y is a compound Poisson process, and $\widetilde{Y}$ is a square-integrable pure jump martingale.
We are interested in the function

$$\begin{aligned} &\widetilde{P} : \mathbb{R} \to \mathbb{R} \\ &x \mapsto (1 - \exp(x))^+. \end{aligned}$$

Since $\widetilde{P}$ is globally Lipschitz with Lipschitz constant 1, we have that

$$\begin{aligned} \mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) &\leq \mathbb{E}\left(\left|X_\tau - (b\tau + \sigma W_\tau + Y_\tau)\right|\right) \\ &= \mathbb{E}\left(\left|\widetilde{Y}_\tau\right|\right). \end{aligned} \tag{3.3}$$

We now proceed to show uniform integrability of $\left(\tau^{-1/2}\widetilde{Y}_\tau\right)_{\tau \in (0,\epsilon)}$ (for some $\epsilon > 0$) using Theorem 1.8. Then we will be able to use Lemma 2.1 to get that

$$\lim_{\tau \to 0^+} \tau^{-1/2}\mathbb{E}\left(\left|\widetilde{Y}_\tau\right|\right) = 0. \tag{3.4}$$

The final step will be to approximate $\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(X_\tau)\right)$ by $\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)$ for which we have an explicit limiting expression.
We now proceed to show that Equation (3.4) holds. We begin by showing that

$$\lim_{\tau \to 0^+} \tau^{-1}\mathbb{E}\left(\widetilde{Y}_\tau^2\right)$$

exists and is finite. This is done by an application of Theorem 1.8. Denote the Lévy measure of $\widetilde{Y}$ by $\widetilde{\nu}$; it may be null. We now check the conditions of Theorem 1.8 . Write $w(y) = y^2$ for $y \in \mathbb{R}$. Trivially, $w(y) \sim y^2$ as $y \to 0$, which is condition (1') of Theorem 1.8. Local boundedness and $\widetilde{\nu}$-a.e. continuity of w, which are conditions (2) and (3) of Theorem 1.8, are obvious. For condition (4) of Theorem 1.8 we take $u(x) = 1 \vee x^2$. Clearly, u is the product of a submultiplicative function ($x \mapsto 1 \vee x^2$) and a subadditive function ($x \mapsto 1$). We therefore have that $u \in \mathscr{S}(\widetilde{\nu})$, since

$$\int_{|y|>1} (1 \vee y^2)\widetilde{\nu}(dy) = \int_{|y|>1} y^2 \widetilde{\nu}(dy) < \infty. \tag{3.5}$$

Where Equation (3.5) holds because of the square-integrability of $\widetilde{Y}$. Indeed, $\mathbb{E}\left(\widetilde{Y}_\tau^2\right) < \infty$ for each $\tau > 0$ if and only if

$$\int_{|y|>1} y^2 \widetilde{\nu}(y), \tag{3.6}$$

by Example 25.12, p. 163 of Sato (1999). Finally, it is clear that

$$\limsup_{|y|\to\infty} \frac{w(y)}{1 \vee y^2} < \infty.$$

We have shown that the conditions of Theorem 1.8 are satisfied and hence, by Theorem 1.8,

$$\lim_{\tau\to 0^+} \tau^{-1}\mathbb{E}\left(\widetilde{Y}_\tau^2\right) = \int_{\mathbb{R}} y^2\, \widetilde{\nu}(\mathrm{d}y) < \infty, \tag{3.7}$$

where we used that $\widetilde{Y}$ has no Gaussian part. We know that this limit is finite since by Equation (3.6) and the definition of a Lévy measure we have that

$$\int_{\mathbb{R}} y^2\, \widetilde{\nu}(\mathrm{d}y) = \int_{|y|>1} y^2\, \widetilde{\nu}(\mathrm{d}y) + \int_{|y|\leq 1} y^2\, \widetilde{\nu}(\mathrm{d}y) < \infty.$$

It follows from (3.7) that there exists an $\epsilon > 0$ such that $\left(\frac{\left|\widetilde{Y}_\tau\right|}{\sqrt{\tau}}\right)_{\tau\in(0,\epsilon)}$ is uniformly integrable and by Lemma 2.1

$$\left|\tau^{-1/2}\widetilde{Y}_\tau\right| \xrightarrow{\mathbb{P}} 0, \text{ as } \tau \to 0^+,$$

from Lemma 2.1. It is therefore clearly the case that

$$\lim_{\tau\to 0^+} \mathbb{E}\left(\left|\tau^{-1/2}\widetilde{Y}_\tau\right|\right) = 0, \tag{3.8}$$

as was claimed as Equation (3.4).
From Equations (3.3) and (3.8) we therefore have that

$$\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) = 0. \tag{3.9}$$

From Lemma 2.7 we have

$$\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) = \frac{\sigma}{\sqrt{2\pi}}. \tag{3.10}$$

Recalling that $S_0 = 1$ by assumption, it follows from both Equations (3.9) and (3.10) that the last equality of Equation (3.1) holds. To see this, we first note that Equation (3.9) implies that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\tau \in (0, \delta)$

$$\tau^{-1/2}\mathbb{E}\left(\left|\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right|\right) < \epsilon.$$

But, $\mathbb{E}\left(\widetilde{P}(X_\tau) - \widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)$ exists for every $\tau > 0$ since $\widetilde{P}$ is bounded above and below. Therefore $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall\tau \in (0, \delta)$

$$\tau^{-1/2}\left|\mathbb{E}\left(\widetilde{P}(X_\tau)\right) - \mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right)\right| < \epsilon$$

from which, $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall \tau \in (0, \delta)$,

$$\tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) - \epsilon < \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(X_\tau)\right) < \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(b\tau + \sigma W_\tau + Y_\tau)\right) + \epsilon,$$

which implies that

$$\lim_{\tau \to 0^+} \tau^{-1/2}\mathbb{E}\left(\widetilde{P}(X_\tau)\right) = \frac{\sigma S_0}{\sqrt{2\pi}},$$

using Equation (3.10) and recalling the definition of $\widetilde{P}$ and that we assumed that $S_0 = 1$. □

We now turn to the not at-the-money case. This case is much simpler than the at-the-money case, although we are unable to reach the same level of generality. We begin with a result of general interest.
Recall that $C(x) = (S_0 e^x - K)^+$ and $P(x) = (K - S_0 e^x)^+$ for $x \in \mathbb{R}$.

Theorem 3.2 (Small time-to-expiry asymptotics of not at-the-money European calls: Part I)**.** *Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying* (1.2) *and $S_0 > 0$. Fix $K > 0$ with $K \neq S_0$.*
Then, for every $K > 0$ where $K \neq S_0$

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ = O\left(\tau\right), \quad \tau \to 0^+.$$

Proof. By Lemma 2.6, we may apply Theorem 1.8 to get that

$$\lim_{\tau \to 0^+} \frac{1}{\tau}\mathbb{E}\left((S_\tau - K)^+\right) = \int_{\mathbb{R}} C(x)\,\nu(\mathrm{d}x) < \infty, \quad (S_0 < K), \tag{3.11}$$

and

$$\lim_{\tau \to 0^+} \frac{1}{\tau}\mathbb{E}\left((K - S_\tau)^+\right) = \int_{\mathbb{R}} P(x)\,\nu(\mathrm{d}x) < \infty, \quad (S_0 > K). \tag{3.12}$$

Now use put-call parity in (3.12). The claim is now clear: the right hand sides of Equations (3.11) and (3.12) lie in $[0, \infty)$. For example, if X has only positive jumps, then the right hand side of (3.12) will be zero. □

We now sharpen the result of the previous theorem to make it applicable to implied volatility asymptotics.

Theorem 3.3 (Small time-to-expiry asymptotics of not at-the-money European calls: Part II)**.** *Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying* (1.2) *and $S_0 > 0$. Fix $K > 0$. Recall that $C(x) = (S_0 e^x - K)^+$ and $P(x) = (K - S_0 e^x)^+$.*
Then, for all $S_0, K > 0$ with $K \neq S_0$,

(i) If X has characteristic triplet $(-\sigma^2/2, \sigma^2, 0)$ with $\sigma \neq 0$, then

$$\mathbb{E}\left((S_\tau - K)^+\right) = S_0 \Phi\left(\frac{-\ln(K/S_0)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right) - K\Phi\left(\frac{-\ln(K/S_0)}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right).$$

(ii) If $S_0 < K$, and $\int C(x)\nu(\mathrm{d}x) > 0$, then

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ \sim \tau \int C(x)\nu(\mathrm{d}x), \quad \tau \to 0^+.$$

(iii) If $S_0 > K$, and $\int_{\mathbb{R}} P(x)\nu(\mathrm{d}x) > 0$, then

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ \sim \tau \int P(x)\nu(\mathrm{d}x), \quad \tau \to 0^+.$$

(iv) Otherwise

$$\mathbb{E}\left((S_\tau - K)^+\right) - (S_0 - K)^+ = o\left(\tau\right), \quad \tau \to 0^+.$$

Proof.

(i) This is nothing but the Black-Scholes model (with zero interest rates and dividend yield). See, for example, Musiela and Rutkowski (2005).

(ii) See the proof of Theorem 3.2.

(iii) See the proof of Theorem 3.2.

(iv) See the proof of Theorem 3.2.

□

3.2 Implied Volatility

We now apply the results of the previous subsection on call option asymptotics to our primary area of interest: small time-to-expiry asymptotics of implied volatility. This is made easy by Theorem 1.11.

Theorem 3.4. *Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying* (1.2) *and $S_0 > 0$. Then, for every $K, \tau > 0$, the implied volatility of the European call for every expiry τ and strike K, i.e. $\Sigma(K,\tau)$, satisfies*

$$0 \leq \Sigma(K,\tau) < \infty. \tag{3.13}$$

Moreover, there exists a non-trivial Lévy process X such that the lower bound in Equation (3.13) *is obtained for some $K > 0$ and all τ small enough.*

Proof. From Lemma 2.5 , $\mathbb{E}\left((S_\tau - K)^+\right) < S_0$ for all τ and $K > 0$. We then have $\Sigma(K,\tau) < \infty$ for $\tau, K > 0$.
From Lemma 2.5, $\mathbb{E}\left((S_\tau - K)^+\right) \geq (S_0 - K)^+$ for all τ and $K > 0$. Therefore, $\Sigma(K,\tau) \geq 0$ for all $K, \tau > 0$.
To show that the lower bound is attained by a non-trivial Lévy process, we describe a non-trivial Lévy process, $\hat{X}$, for which $\hat{S}_0 \exp(\hat{X})$ has $\Sigma(\hat{K},\tau) = 0$ for all τ small enough and some chosen $\hat{K}$. Assume that $\hat{S}_0 = 1$.
Let $\hat{X}$ be a Lévy process of Type A or B with characteristic triplet $(\hat{b}, 0, \hat{\nu})$. Suppose that $\hat{\nu}$ is not null and $\hat{b} \neq 0$. Moreover suppose that the support of $\hat{\nu}$ is a subset of $[0,\infty)$ and 0 is in the support of $\hat{\nu}$.

With $\hat{S}_0 > \hat{K}$, we have $\mathbb{E}\left((\hat{S}_\tau - \hat{K})^+\right) = (\hat{S}_0 - \hat{K})^+$ if and only if $\mathbb{P}\left(\hat{S}_\tau < \hat{K}\right) = \mathbb{P}\left(\hat{X}_\tau < \ln(\hat{K})\right) = 0$, see Lemma 2.5. Of course, $\mathbb{E}\left((\hat{S}_\tau - \hat{K})^+\right) = (\hat{S}_0 - \hat{K})^+$ implies that $\Sigma(\hat{K}, \tau) = 0$.
Since $\hat{X}$ has only positive jumps, we have by Theorem 1.10 that $\mathbb{P}\left(\hat{X}_\tau < \hat{b}\tau\right) = 0$ where $\hat{b}(< 0)$ is the drift of $\hat{X}$. We need to choose a $\hat{K}$ such that $\mathbb{P}\left(\hat{X}_\tau < \ln \hat{K}\right) = 0$ for all τ smaller than some constant time, but with $\hat{K}$ fixed. Choose $\ln \hat{K} = \hat{b}$, so that $\hat{K} = \exp(\hat{b})$. Obviously, $\hat{K} < \hat{S}_0 = 1$ since $\hat{b} < 0$. What is more $\mathbb{P}\left(\hat{X}_\tau < \ln \hat{K}\right) = \mathbb{P}\left(\hat{X}_\tau < \hat{b}\right) = 0$ for all $0 < \tau < 1$. But then $\mathbb{E}\left((\hat{S}_\tau - \hat{K})^+\right) = (\hat{S}_0 - \hat{K})^+$ for $\tau \in (0,1)$, from which $\Sigma(\hat{K}, \tau) = 0$ for all $\tau \in (0,1)$. □

Theorem 3.5 (Limiting implied volatility)**.**
Suppose that $S_\tau = S_0 e^{X_\tau}$, for all $\tau \geq 0$, where X is a Lévy process satisfying (1.2) *and $S_0 > 0$. Fix $K > 0$. Assume that X is not the trivial process. Also, to avoid trivialities we assume that $\mathbb{E}\left((S_\delta - K)^+\right) > (S_0 - K)^+$ for every $\tau \in (0, \delta)$ $(\exists \delta > 0)$*

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+.$$

With X having characteristic triplet (b, σ^2, ν), the at-the-money implied volatility satisfies

$$\lim_{\tau \to 0^+} \Sigma(S_0, \tau) = \sigma,$$

and if, in particular, $\sigma = 0$, then

$$\lim_{\tau \to 0^+} \Sigma(S_0, \tau) = 0.$$

The not at-the-money implied volatility of a $K(> 0), K \neq S_0$ strike European call satisfies the following. Assume that $K \neq S_0$ and both are strictly positive.

(i) If X has characteristic triplet $(-\sigma^2/2, \sigma^2, 0)$ with $\sigma \in (0, \infty)$, then, for every $K, \tau > 0$, $\Sigma(K, \tau) = \sigma$ so that

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \sigma$$

(ii) If $S_0 < K$, and $\int_{\mathbb{R}} (S_0 e^x - K)^+ \nu(\mathrm{d}x) > 0$, then

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \infty.$$

(iii) If $S_0 > K$, and $\int_{\mathbb{R}} (K - S_0 e^x)^+ \nu(\mathrm{d}x) > 0$, then

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \infty.$$

Proof. We have $S_0 > 0$ by assumption. Also, we assumed that $\mathbb{E}\left((S_\delta - K)^+\right) > (S_0 - K)^+$ for every $\tau \in (0, \delta)$ $(\exists \delta > 0)$

$$\mathbb{E}\left((S_\tau - K)^+\right) > (S_0 - K)^+.$$

We can therefore apply Lemma 2.5 and use Theorem 1.11 to obtain the implied volatility limit from the call option limit obtained in Theorem 3.1.
We now consider the not at-the-money case.
Statements (i) is trivial. The statements (ii) and (iii) follow from Theorem 3.3 (ii) and (iii). and the fact that $\tau \ln(A\tau) \to 0$ as $\tau \to 0^+$ for $A > 0$. In the model constructed in Theorem 3.4, it is trivially the case that there exists a K^* such that $\lim_{\tau \to 0^+} \Sigma(K^*, \tau) = 0$. The considered Lévy process is not trivial. □

4 Examples

Let X be either a Generalised Hyperbolic, Variance Gamma, Normal Inverse Gaussian, CGMY, or Meixner process. Then the Lévy measure of X has a density that is typically positive under most parameter specifications of interest in finance at each point of $\mathbb{R}$ except zero. Recall that X has no Brownian component. (See Cont and Tankov (2004) and Schoutens (2003)). Defining a stock price model as $S = S_0 e^X$ (with $S_0 > 0$) for any of these processes such that the Lévy measure of X has a density that is positive at each point of $\mathbb{R}$ except zero, we find that

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \begin{cases} \infty, & \text{if } K \neq S_0 \\ 0, & \text{if } K = S_0. \end{cases}$$

Let Y be Merton's model or Kou's model (see Cont and Tankov (2004)). Then the Lévy measure has a density that is positive at each point of $\mathbb{R}$ outside of zero, and it contains a Brownian component. For the model $S = S_0 e^Y$, with $S_0 > 0$ and Y either Merton's or Kou's models we have that

$$\lim_{\tau \to 0^+} \Sigma(K, \tau) = \begin{cases} \infty, & \text{if } K \neq S_0 \\ \sigma, & \text{if } K = S_0, \end{cases}$$

where σ is the square root of the Brownian component of Y's characteristic triplet (see Cont and Tankov (2004)).

5 Summary of Results

In this paper, we presented a study of the small time-to-expiry asymptotics of implied volatility in models of exponential Lévy type that are of interest in mathematical finance. We found that:

(1) Implied volatility is restricted to $[0, \infty)$ where there is a non-trivial Lévy process X such that e^X attains the lower bound for some K and expiries small enough.

(2) In *all* well-defined exponential Lévy models, the at-the-money implied volatility converges to zero if the driving Lévy process has no Gaussian part and σ if it has Gaussian part σ^2.

(3) Not at-the-money implied volatility converges to infinity in most examples of interest in mathematical finance because they have a Lévy density which is positive for all $x \in \mathbb{R} \setminus \{0\}$. There exist non-trivial examples where the limiting implied volatility is zero, at least for some strikes.

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6 Appendix

Lemma 2.1
Let $\widetilde{X}$ be a Lévy process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and with no Gaussian part, then

$$\tau^{-1/2}\widetilde{X}_\tau \xrightarrow{\mathbb{P}} 0 \text{ as } \tau \to 0^+. \tag{6.1}$$

Proof. The result is due to Jacod (see Jacod (2007), Lemma 4.1, p. 181). However, this proof is original.
We prove the convergence in distribution to zero, i.e.

$$\widetilde{X}_\tau/\sqrt{\tau} \xrightarrow{d} 0, \quad \text{as } \tau \to 0^+,$$

which of course implies (6.1).
We write $\widetilde{\Psi}$ for the characteristic exponent of $\widetilde{X}$. The characteristic function of $\widetilde{X}_\tau/\sqrt{\tau}$, is given by

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) = \phi_{\widetilde{X}_\tau}(\lambda/\sqrt{\tau}) = \exp\left(-\tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right)\right).$$

To prove the claim we must show that

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) \to 1 \text{ as } \tau \to 0^+ \text{ for each } \lambda \in \mathbb{R}.$$

But Bertoin (1996), Proposition 2 (i), p. 16), gives that

$$\lim_{|\lambda|\to\infty} \lambda^{-2}\widetilde{\Psi}(\lambda) = 0, \tag{6.2}$$

since $\widetilde{X}$ has no Gaussian part.
We prove the claim by showing that

$$\lim_{\tau\to 0^+} \tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right) = 0,$$

for each $\lambda \in \mathbb{R}$. Fix $\lambda \in \mathbb{R}$. If $\lambda = 0$, then

$$\phi_{\widetilde{X}_\tau/\sqrt{\tau}}(\lambda) = \phi_{\widetilde{X}_\tau}\left(\lambda/\sqrt{\tau}\right) = \exp\left(-\tau\widetilde{\Psi}\left(0\right)\right) = \exp(-\tau \cdot 0) = 1, \quad \forall\tau > 0.$$

For non-zero λ we perform a change of variables: $\widetilde{\lambda} = \widetilde{\lambda}(\tau) = \lambda/\sqrt{\tau}$. We only consider strictly positive τ. From Equation (6.2), we have

$$\begin{aligned}
0 &= \lim_{|\widetilde{\lambda}|\to\infty} \widetilde{\lambda}^{-2}\widetilde{\Psi}\left(\widetilde{\lambda}\right) \\
&= \lim_{\tau\to 0^+} \left(\frac{\lambda}{\sqrt{\tau}}\right)^{-2} \widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right) \\
&= \frac{1}{\lambda^2} \lim_{\tau\to 0^+} \tau\widetilde{\Psi}\left(\frac{\lambda}{\sqrt{\tau}}\right).
\end{aligned}$$

□

Lemma 2.5 *Let*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where X is a Lévy process satisfying the constraints in Equation (1.2) *and $S_0 > 0$ is some finite constant. Then, for each fixed $K > 0$,*

1. $(S_0 - K)^+ \leq \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right) < S_0$, $\forall \tau \geq 0$;
2. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is right-continuous on* $[0, \infty)$; *and*
3. $\tau \mapsto \mathbb{E}\left((S_0 e^{X_\tau} - K)^+\right)$ *is non-decreasing.*
4. *If $S_0 > K$, then $\mathbb{E}\left((S_\tau - K)^+\right) = (S_0 - K)^+$ if and only if* $\mathbb{P}(S_\tau < K) = \mathbb{P}(X_\tau < \ln(K/S_0)) = 0$, *for $\tau > 0$.*

Proof. Elementary. □

Lemma 2.6 *Introduce*

$$S_\tau = S_0 e^{X_\tau}, \quad \forall \tau \geq 0,$$

where $S_0 > 0$ is a constant and X is a Lévy process satisfying (1.2), *i.e.*

$$\int_{|y|\geq 1} e^y\, \nu(\mathrm{d}y) < \infty \text{ and } b = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{|y|\leq 1})\, \nu(\mathrm{d}y). \tag{6.3}$$

Then consider the functions

(1) $P(\cdot)$, defined in Definition 2.3 as $P(x) = (K - S_0 e^x)^+$, with the additional restriction that $0 < K < S_0$; and

(2) $C(\cdot)$, defined in Definition 2.2 as $C(x) = (S_0 e^x - K)^+$, with the additional restriction that $K > S_0 > 0$.

Then conditions (1)-(4) of Theorem 1.8 are satisfied by P and C under the respective stated conditions on S_0 and K.

Proof. Clearly P and C are locally bounded and ν continuous; hence conditions (2)-(3) of Theorem 1.8 are satisfied.
With $K > S_0$, C vanishes in a neighbourhood of the origin so that certainly $C(x) = \mathrm{o}\left(x^2\right)$ as $x \to 0$ and condition (1) of Theorem 1.8 is satisfied.
For $K < S_0$, P vanishes in a neighbourhood of the origin so that $P(x) = \mathrm{o}\left(x^2\right)$ as $x \to 0$ and condition (1) of Theorem 1.8 is again satisfied.
It remains to check condition (4) for C and P. Without further comment, we note that $x \mapsto 1$ is both subadditive and submultiplicative. Now, for C, consider $x \mapsto 1 \cdot e^x$. By Equation (6.3), the function $x \mapsto 1 \cdot e^x$ is in $\mathcal{S}(\nu)$: it satisfies

$$\limsup_{|x|\to\infty} \frac{(S_0 e^x - K)^+}{1 \cdot e^x} < \infty.$$

For P, consider $x \mapsto K$. Clearly $x \mapsto K \cdot 1$ is in $\mathcal{S}(\nu)$: it satisfies

$$\limsup_{|x|\to\infty} \frac{(K - S_0 e^x)^+}{1 \cdot K} < \infty$$

and

$$\int_{|x|>1} K \, \nu(\mathrm{d}x) < \infty,$$

since ν is a Lévy measure. $\square$

Lemma 2.7 *Suppose that U is a non-negative process with representation*

$$U_\tau = U_0 e^{b\tau + \sigma W_\tau + Y_\tau}, \quad \tau \geq 0,$$

where $b \in \mathbb{R}, \sigma \geq 0$, and $U_0 > 0$ are finite constants, W is a Brownian motion and Y is a compound Poisson process with finite, constant intensity $\lambda > 0$ and a finite exponential mean. That is we assume that $\mathbb{E}\left(e^{Y_\tau}\right) < \infty$ for all $\tau \geq 0$, also that the processes W and Y are independent. Then

$$\lim_{\tau \to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(U_0 - U_0 e^{b\tau + \sigma W_\tau + Y_\tau}\right)^+\right) = \frac{\sigma U_0}{\sqrt{2\pi}}.$$

Proof.
It is clearly enough to prove the claim for $U_0 = 1$.
Case 1: No compound Poisson part

First suppose that U has representation

$$U_\tau = \exp(b\tau + \sigma W_\tau),$$

where $b, \sigma \in \mathbb{R}$, $\sigma \geq 0$, and W is a standard Wiener process.
We claim that

$$\lim_{\tau \to 0^+} \tau^{-1/2} \mathbb{E}\left(\left(1 - e^{b\tau + \sigma W_\tau}\right)^+\right) = \frac{\sigma}{\sqrt{2\pi}}.$$

If $\sigma = 0$, then it is trivially the case that $\lim_{t\to 0^+} \tau^{-1/2}\mathbb{E}\left((1-e^{b\tau})^+\right) = 0$. Observe that b can be any real number and this same limit holds.
Suppose now that $\sigma > 0$ and continue to let $b \in \mathbb{R}$.
We will use that for $\theta \in \mathbb{R}$,

$$\operatorname{erf}(\theta\sqrt{\tau}) \sim \frac{2\theta\sqrt{\tau}}{\sqrt{\pi}}, \quad \tau \to 0^+,$$

and $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ for $x \in \mathbb{R}$, and

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

See Abramowitz (1984) or Olver (1997) for the definition and facts about the error function.
It is well known that

$$\mathrm{d}U_\tau = rU_\tau\,\mathrm{d}\tau + \sigma U_\tau\,\mathrm{d}W_\tau, \quad U_0 = 1$$

has solution

$$U_\tau = \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right), \quad \tau \geq 0.$$

Indeed, this is just the Black-Scholes model with r the risk-neutral interest rate. Using the well-known formula for the price of an at-the-money put option under this model (see, for example, Fouque *et al.* (2000)) we obtain that

$$\begin{aligned} & e^{-r\tau}\mathbb{E}\left(\left(1 - \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right)\right)^+\right) \\ = \quad & e^{-r\tau}\Phi\left(-\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma}\right) - \Phi\left(-\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma}\right), \end{aligned}$$

from which

$$
\begin{aligned}
&\mathbb{E}\left(\left(1-\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma W_\tau\right)\right)^+\right) \\
&=\Phi\left(-\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma}\right)-e^{r\tau}\Phi\left(-\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma}\right) \\
&=\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)-\frac{e^{r\tau}}{2}+\frac{e^{r\tau}}{2}\operatorname{erf}\left(\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right) \\
&=\frac{1}{2}(1-e^{r\tau})-\frac{1}{2}\left(\operatorname{erf}\left(\frac{(r-\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)-e^{r\tau}\operatorname{erf}\left(\frac{(r+\sigma^2/2)\sqrt{\tau}}{\sigma\sqrt{2}}\right)\right) \\
&=\mathrm{O}(\tau)-\frac{1}{2}\left(\left(\frac{(r-\sigma^2/2)\sqrt{2\tau}}{\sqrt{\pi}\sigma}+\mathrm{O}(\tau)\right)-(1+\mathrm{O}(\tau))\left(\frac{(r+\sigma^2/2)\sqrt{2\tau}}{\sqrt{\pi}\sigma}+\mathrm{O}(\tau)\right)\right) \\
&=\mathrm{O}(\tau)-\frac{1}{2}\frac{(r-\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}-\frac{1}{2}\mathrm{O}(\tau)+\frac{1}{2}\frac{(r+\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau} \\
&\qquad+\frac{1}{2}\mathrm{O}(\tau)+\frac{1}{2}\mathrm{O}\left(\tau^{3/2}\right)+\frac{1}{2}\mathrm{O}\left(\tau^2\right) \\
&=-\frac{1}{2}\frac{(r-\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}+\frac{1}{2}\frac{(r+\sigma^2/2)\sqrt{2}}{\sigma\sqrt{\pi}}\sqrt{\tau}+\mathrm{O}(\tau) \\
&=\frac{\sqrt{2}}{2\sigma\sqrt{\pi}}\left(-(r-\sigma^2/2)+(r+\sigma^2/2)\right)\sqrt{\tau}+\mathrm{O}(\tau) \\
&=\frac{\sigma}{\sqrt{2\pi}}\sqrt{\tau}+\mathrm{O}(\tau),
\end{aligned}
$$

all as $\tau \to 0^+$.
Observe that we could have chosen r in such a way that $b = r - \sigma^2/2$ and there would be no difference in the final result. We therefore have that

$$\lim_{\tau\to 0^+}\tau^{-1/2}\mathbb{E}\left((1-\exp(b\tau+\sigma W_\tau))^+\right)=\frac{\sigma}{\sqrt{2\pi}},$$

for all $b \in \mathbb{R}$ and $\sigma \geq 0$.
Case 2: Compound Poisson part included

Suppose now that U has representation

$$U_\tau = \exp(b\tau + \sigma W_\tau + Y_\tau), \quad \tau \geq 0,$$

where W is a standard Wiener process and Y is a compound Poisson process which is such that each of the random variables comprising Y have an exponential moment. Then

$$\begin{aligned}
\mathbb{E}\left((1-U_\tau)^+\right) &= \sum_{n=0}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&= \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = 0\right) \mathbb{P}\left(N_\tau = 0\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&=: A_\tau^1 + A_\tau^2.
\end{aligned}$$

For the first term we can just apply the first part (Case 1) of this proof to get

$$\begin{aligned}
\lim_{\tau\to 0^+} \tau^{-1/2}\mathbb{E}\left((1-e^{b\tau+\sigma W_\tau})^+\right) \mathbb{P}\left(N_\tau = 0\right) &= \lim_{\tau\to 0^+} e^{-\lambda\tau}\tau^{-1/2}\mathbb{E}\left((1-e^{b\tau+\sigma W_\tau})^+\right) \\
&= \frac{\sigma}{\sqrt{2\pi}},
\end{aligned}$$

so

$$\lim_{\tau\to 0^+} \tau^{-1/2} A_\tau^1 = \frac{\sigma}{\sqrt{2\pi}}.$$

For the second,

$$\begin{aligned}
A_\tau^2 &= \sum_{n=1}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right) \\
&\le \sum_{n=1}^{\infty} \mathbb{P}\left(N_\tau = n\right) \\
&= 1 - \exp(-\lambda\tau) \\
&= \mathrm{O}\left(\tau\right)
\end{aligned}$$

as $\tau \to 0^+$. Since $A_\tau^2 = \mathrm{O}\left(\tau\right)$ as $\tau \to 0^+$, we have

$$\lim_{\tau\to 0^+} \tau^{-1/2} A_\tau^2 = 0.$$

That is

$$\begin{aligned}
\tau^{-1/2}\mathbb{E}\left((1-U_\tau)^+\right) &= \tau^{-1/2}\mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = 0\right) \mathbb{P}\left(N_\tau = 0\right) \\
&\quad + \tau^{-1/2} \sum_{n=1}^{\infty} \mathbb{E}\left((1-U_\tau)^+ \middle| N_\tau = n\right) \mathbb{P}\left(N_\tau = n\right)] \\
&= \tau^{-1/2}(A_\tau^1 + A_\tau^2) \\
&\to \frac{\sigma}{\sqrt{2\pi}} \text{ as } \tau \to 0^+.
\end{aligned}$$

$\square$