COCOMPACT LATTICES OF MINIMAL COVOLUME IN RANK 2 KAC–MOODY GROUPS, PART I: EDGE-TRANSITIVE LATTICES

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ABSTRACT. Let G be a topological Kac–Moody group of rank 2 with symmetric Cartan matrix, defined over a finite field. An example is $G = SL_2(K)$, where K is the field of formal Laurent series over $\mathbb{F}_q$. The group G acts on its Bruhat–Tits building X, a regular tree, with quotient a single edge. We classify the cocompact lattices in G which act transitively on the edges of X. Using this, for many such G we find the minimum covolume among cocompact lattices in G, by proving that the lattice which realises this minimum is edge-transitive. Our proofs use covering theory for graphs of groups, the dynamics of the G–action on X, the Levi decomposition for the parabolic subgroups of G, and finite group theory.

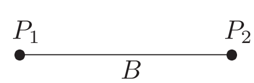
INTRODUCTION

A classical theorem of Siegel [23] states that the minimum covolume among lattices in $G = SL_2(\mathbb{R})$ is $\frac{\pi}{21}$, and determines the lattice which realises this minimum. In the nonarchimedean setting, Lubotzky [17] constructed the lattice of minimal covolume in $G = SL_2(K)$, where K is the field $\mathbb{F}_q((t^{-1}))$ of formal Laurent series over $\mathbb{F}_q$.

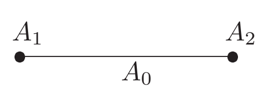
The group $G = SL_2(\mathbb{F}_q((t^{-1})))$ has, in recent developments, been viewed as the first example of a complete Kac–Moody group of rank 2 over a finite field. Such Kac–Moody groups are locally compact, totally disconnected topological groups, which may be thought of as infinite-dimensional analogues of semisimple algebraic groups (see Section 1.4 below for definitions). In this paper, we determine the cocompact lattice of minimal covolume in many such G, by classifying those lattices of G which act transitively on the edges of the associated Bruhat–Tits tree, and then showing that a cocompact lattice of minimal covolume is edge-transitive. Our main results are Theorems 1, 2 and 3 below, which give precise statements.

It is interesting that there exist any cocompact lattices in the groups G we consider, since starting with $n = 3$, most Kac–Moody groups of rank n do not possess any uniform lattices (with the possible exception of those whose root systems contain a subsystem of type $\tilde{A}_n$ – see Remark 4.4 of [6]). For rank 2, the only previous examples of cocompact lattices in complete Kac–Moody groups G are the free Schottky groups constructed by Carbone–Garland in [9].

The Kac–Moody groups G that we consider have a refined Tits system, and so have Bruhat–Tits building a regular tree X (see [19]). The action of G on X induces an edge of groups

$\mathbb{G} =$ 

where P_1 and P_2 are the standard parabolic/parahoric subgroups of G, and $B = P_1 \cap P_2$ is the standard Borel/Iwahori subgroup. Now let m, n be integers ≥ 2. An (m, n)*–amalgam* is a free product with amalgamation $A_1 *_{A_0} A_2$, where the group A_0 has index m in A_1 and index n in A_2. The amalgam is *faithful* if A_0, A_1 and A_2 have no common normal subgroup. In Bass–Serre theory (see Section 1.2), an (m, n)–amalgam is the fundamental group Γ of an edge of groups

$\mathbb{A} =$ 

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with universal cover the (m,n)–biregular tree, and this amalgam is faithful if and only if $\Gamma = \pi_1(\mathbb{A}) \cong A_1 *_{A_0} A_2$ acts faithfully on the universal cover.

The question of classifying amalgams is, in general, difficult. A deep theorem of Goldschmidt [15] established that there are only 15 faithful $(3,3)$–amalgams of finite groups, and classified such amalgams. Goldschmidt and Sims conjectured that when both m and n are prime, there are only finitely many faithful (m,n)–amalgams of finite groups (see [3, 13, 15]). This conjecture remains open, except for the case $(m,n) = (2,3)$, which was established by Djoković–Miller [11], and the work of Fan [13], who proved the conjecture when the edge group A_0 is a p–group, with p a prime distinct from both m and n. On the other hand, Bass–Kulkarni [3] showed that if either m or n is composite, there are infinitely many faithful (m,n)–amalgams of finite groups.

Now let Γ be a cocompact lattice in the complete Kac–Moody group G which acts transitively on the edges of the Bruhat–Tits tree X. As we explain in Section 1.4 below, Γ is the fundamental group of an edge of groups $\mathbb{A}$ as above, with moreover A_0, A_1 and A_2 finite groups. Hence to classify the edge-transitive cocompact lattices in G, we classify the amalgams $A_1 *_{A_0} A_2$ which embed in G. We note that, since the action of G on X is not in general faithful, an amalgam Γ may embed as a cocompact edge-transitive lattice in G even though it is not faithful.

We now state our first main result, Theorem 1. There are some exceptions for small values of p and q, which are stated separately below in Theorem 2. In Section 3 below, we state Theorems 1 and 2 for the special case $G = SL_2(\mathbb{F}_q((t^{-1})))$. The group G in our results is a *topological* Kac–Moody group, meaning that it is the completion of a minimal Kac–Moody group Λ with respect to some topology. We use the completion in the 'building topology', which is discussed in, for example, [8].

Our notation is as follows. We write C_n for the cyclic group of order n and S_n for the symmetric group on n letters. Since for a finite field $\mathbb{F}_q$ and the root system A_1 there exist at most two corresponding finite groups of Lie type (one isomorphic to $SL_2(\mathbb{F}_q)$, and the other to $PSL_2(\mathbb{F}_q)$), to avoid complications we use Lie-theoretic notation, and write $A_1(q)$ which stands for both of these groups. We will discuss this ambiguity whenever necessary. (Notice that as $PSL_2(\mathbb{F}_q) \cong SL_2(\mathbb{F}_q)/Z(SL_2(\mathbb{F}_q))$, if q is odd then $PSL_2(\mathbb{F}_q) \cong SL_2(\mathbb{F}_q)/\langle -I \rangle$, while if q is even, $SL_2(\mathbb{F}_q) = PSL_2(\mathbb{F}_q)$.) We denote by T a fixed maximal split torus of G with $T \leq P_1 \cap P_2$. The centre $Z(G)$ of G is then contained in T, and T is isomorphic to a quotient of $\mathbb{F}_q^* \times \mathbb{F}_q^*$ (the particular quotient depending upon G).

We say that two edge-transitive cocompact lattices $\Gamma = A_1 *_{A_0} A_2$ and $\Gamma' = A_1' *_{A_0'} A_2'$ in G are *isomorphic* if $A_i \cong A_i'$ for $i = 0,1,2$ and the obvious diagram commutes; our classification of edge-transitive lattices is up to isomorphism. In particular, this means that we assume $A_i \leq P_i$ for $i = 1,2$.

Theorem 1. *Let G be a topological Kac–Moody group of rank 2 defined over a finite field $\mathbb{F}_q$ of order $q = p^a$ where p is prime, with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Then G has edge-transitive cocompact lattices Γ of each of the following isomorphism types, and every edge-transitive cocompact lattice Γ in G is isomorphic to one of the following amalgams.*

(1) *If $p = 2$ then $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$:*
  (a) *$A_i = A_0 \times H_i$ with $H_i \cong C_{q+1}$; and*
  (b) *A_0 is a cyclic subgroup of T with $|A_0|$ dividing $(q-1)$.*
(2) *If p is odd and $q \equiv 1 \pmod 4$, then G does not contain any edge-transitive cocompact lattices (with finitely many exceptions, listed in Theorem 2 below).*
(3) *If p is odd and $q \equiv 3 \pmod 4$, then (with finitely many exceptions listed in Theorem 2 below) $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$:*
  (a) *$A_i = A_0 H_i$, with H_i isomorphic to the normaliser of a non-split torus in $A_1(q)$; and*
  (b) *A_0 is a subgroup of the normaliser $N_T(H_i)$ of H_i in T.*

We now give the finitely many exceptions to the statements in Theorem 1.

Theorem 2. *Let G be as in Theorem 1 above. The edge-transitive lattices for p odd and $q \equiv 1 \pmod 4$ are:*

(1) *$q = 5$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$, $A_i = A_0 H_i$ where $H_i \cong A_1(3)$, $A_0 \leq N_T(H_i)$, and $|H_i : H_i \cap A_0| = 6$; and*

(2) *$q = 29$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0H_i$ where $H_i \cong A_1(5)$, $A_0 \leq N_T(H_i)$, and $|H_i : H_i \cap A_0| = 30$.*

The exceptional edge-transitive lattices for p odd, $q \equiv 3 \pmod 4$ are:

(1) *If $q = 7$ or 23, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0H_i$, $H_i \cong S_4$ or $2S_4$, $A_0 \leq N_T(H_i)$ and $|H_i : H_i \cap A_0| = q + 1$ where $H_i \cap A_0$ is cyclic.*
(2) *If $q = 11$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0H_i$, and $A_0 \leq N_T(H_i)$ with $|H_i : H_i \cap A_0| = 12$, $H_i \cap A_0$ being cyclic, and one of the following holds:*
   (a) *$H_1 \cong H_2 \cong A_1(3)$, or*
   (b) *$H_1 \cong H_2 \cong A_1(5)$.*
(3) *If $q = 19$ or 59, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0H_i$, $H_i \cong A_1(5)$, $A_0 \leq N_T(H_i)$ and $|H_i : H_i \cap A_0| = q + 1$ with $H_i \cap A_0$ being cyclic.*

We now state our main result on covolumes, Theorem 3. We note in Section 1.4 below that the Haar measure μ on G may be normalised so that the covolume $\mu(\Gamma\backslash G)$ of an edge-transitive cocompact lattice $\Gamma = A_1 *_{A_0} A_2$ is equal to $|A_1|^{-1} + |A_2|^{-1}$. Using this normalisation, we obtain the following.

Theorem 3. *Let G be as in Theorem 1 above. If $p = 2$ then*

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \text{ a cocompact lattice in } G\} = \frac{2}{(q+1)|Z(G)|}.$$

If p is odd and $q \equiv 3 \pmod 4$, suppose also that $q \geq 300$. Then

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \text{ a cocompact lattice in } G\} = \frac{2}{2(q+1)|Z(G)|\delta}$$

where $\delta \in \{1, 2\}$ (depending upon the particular group G).

Moreover, in these cases, the cocompact lattice of minimal covolume in G is edge-transitive.

Even more precise statements of Theorems 1 and 2 above are obtained in Section 5 below, where we also prove Theorem 3. We plan to consider covolumes for the case $q \equiv 1 \pmod 4$, in which G does not generally admit any edge-transitive lattices, in Part II of this paper.

Theorem 3 above generalises Theorem 2 of Lubotzky [17], which found the lower bound on covolumes of cocompact lattices in $G = SL_2(\mathbb{F}_q((t^{-1})))$ by explicitly constructing the cocompact lattices of minimal covolume. Since many such lattices are edge-transitive, Lubotzky's constructions appear in our list above when $G = SL_2(\mathbb{F}_q((t^{-1})))$. In the special case $q = 2$, L. Carbone has informed us that she obtained such examples independently. Although our theorems in the case $G = SL_2(\mathbb{F}_q((t^{-1})))$ essentially follow from Lubotzky's work, in order to show where the difficulty in the general case lies, and to illustrate different techniques of proof, we prove Theorems 1 and 2 for $G = SL_2(\mathbb{F}_q((t^{-1})))$ in Section 3 below. We then present the general proof in Section 4.

Our main methods for determining whether or not a given amalgam is a cocompact lattice in G are described in Section 2 below. The first method is Bass' covering theory for graphs of groups [2], which is used in the proof for $G = SL_2(\mathbb{F}_q((t^{-1})))$, together with elementary matrix computations (which cannot be carried out in the general case). As we explain in Section 2.1 below, an amalgam $\Gamma = A_1 *_{A_0} A_2$ embeds as an edge-transitive cocompact lattice in G if and only if there is a covering of graphs of groups $\mathbb{A} \to \mathbb{G}$, where $\mathbb{A}$ and $\mathbb{G}$ are the edges of groups sketched above.

For the general proof in Section 4, an important tool is Lemma 4 below, which generalises Lemma 3.1 of Lubotzky [17]. Lubotzky's result gave sufficient conditions for an amalgam to embed in $G = SL_2(\mathbb{F}_q((t^{-1})))$. Our result, proved in Section 2.2, gives necessary and sufficient conditions, and applies to more general locally compact groups G acting on trees.

Lemma 4. *Let q_1 and q_2 be positive integers and let X be the (q_1+1, q_2+1)–biregular tree. Let G be a locally compact group of automorphisms of X, which acts on X with compact open stabilisers and with fundamental domain an edge (x_1, x_2), where for $i = 1, 2$ the vertex x_i of X has valence $q_i + 1$.*

Suppose for $i = 1, 2$ that A_i is a finite subgroup of the stabiliser G_{x_i} such that:

(1) *A_i acts transitively on the set of $q_i + 1$ neighbours of x_i in X; and*

(2) $\mathrm{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$.

Then $\Gamma = \langle A_1, A_2 \rangle$*, the group generated by* A_1 *and* A_2*, is a cocompact lattice in* G*, with fundamental domain the edge* (x_1, x_2)*. Moreover,* Γ *is isomorphic to the free product with amalgamation* $\Gamma \cong A_1 *_{A_1 \cap A_2} A_2$*, and* $\Gamma_{x_i} = A_i$*.*

Conversely, suppose Γ *is a cocompact lattice in* G *with fundamental domain the edge* (x_1, x_2)*. Let* $A_i = \Gamma_{x_i}$*. Then* $\Gamma \cong A_1 *_{A_1 \cap A_2} A_2$*, and* A_i *is a finite subgroup of* G_{x_i} *such that* (1) *and* (2) *hold.*

The other key result for the general proof is Proposition 5 below. This is in fact the statement that takes some work to prove, and is a nice result in its own right.

Proposition 5. *Let* G *be as in Theorem 1 above. If* Γ *is a cocompact lattice in* G*, then* Γ *does not contain* p*–elements.*

We apply Proposition 5 to restrict the possible finite groups A_0, A_1 and A_2 in a lattice amalgam $\Gamma = A_1 *_{A_0} A_2$. Our proof of Proposition 5 in Section 4 below was suggested by the Property (FPRS) in recent work of Caprace–Rémy [8], and makes use of the dynamics of the G–action on X, including some results of Carbone–Garland [9].

Our proofs in Sections 3, 4 and 5 below also use the Levi decompositions of the parabolic subgroups P_1 and P_2 of G, which we recall in Section 1.4, and classical results of finite group theory, which are stated in Section 1.5 below.

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1. Preliminaries

We recall some definitions and results concerning trees in Section 1.1, sketch the theory of graphs of groups in Section 1.2, and give some definitions and important properties for cocompact lattices in Section 1.3. In Section 1.4 we outline those parts of the theory of Kac–Moody groups that we will need. The required results of finite group theory are stated in Section 1.5.

1.1. **Trees.** Let X be a simplicial tree. We define *combinatorial balls* in X inductively as follows. Given a vertex v of X, the combinatorial ball $\mathrm{Ball}(v, 0)$ consists of the vertex v, and for integers $n \geq 1$, the combinatorial ball $\mathrm{Ball}(v, n)$ consists of all closed edges in X which meet $\mathrm{Ball}(v, n-1)$. Similarly, given an edge e of X, $\mathrm{Ball}(e, 0)$ consists of the (closed) edge e, and for $n \geq 1$, $\mathrm{Ball}(e, n)$ consists of all closed edges in X which meet $\mathrm{Ball}(e, n-1)$.

We may now define the *distance* $d(e, e')$ between edges e and e' of X to be 0 if $e = e'$, and to be $n \geq 1$ if $e' \in \mathrm{Ball}(e, n) - \mathrm{Ball}(e, n-1)$.

Two geodesic rays (that is, half-lines) α and α' in the tree X are said to be *equivalent* if their intersection is infinite. The set of *ends* of X is then the collection of equivalence classes of geodesic rays in X, under this relation. We say that an end is *determined* by a half-line α if α represents this end.

The following result of Serre will be very useful for us. A group A is said to act *without inversions* on a tree X if for all $g \in A$ and all edges $e \in EX$, if g preserves e then g fixes e pointwise.

Proposition 6 (Serre, Proposition 19, Section I.4.3 [21])**.** *Let* A *be a finite group acting without inversions on a tree* X*. Then there is a vertex of* X *which is fixed by* A*.*

1.2. **Bass–Serre theory.** Let A be a connected graph, with sets VA of vertices and EA of oriented edges. The initial and terminal vertices of $e \in EA$ are denoted by $\partial_0 e$ and $\partial_1 e$ respectively. The map $e \mapsto \overline{e}$ is orientation reversal, with $\overline{\overline{e}} = e$ and $\partial_{1-j}\overline{e} = \partial_j e$ for $j = 0, 1$ and all $e \in EA$.

A *graph of groups* $\mathbb{A} = (A, \mathcal{A})$ over a connected graph A consists of an assignment of vertex groups $\mathcal{A}_a$ for each $a \in VA$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in EA$, together with monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for each $e \in EA$. See for example [2] for the definitions of the *fundamental group* $\pi_1(\mathbb{A}, a_0)$ and the *universal cover* $X = \widetilde{(A, a_0)}$ of a graph of groups $\mathbb{A} = (A, \mathcal{A})$, with respect to a basepoint $a_0 \in VA$. The universal cover X is a tree, on which $\pi_1(\mathbb{A}, a_0)$ acts by isometries inducing a graph of groups isomorphic to $\mathbb{A}$. A graph of groups is *faithful* if its fundamental group acts faithfully on its universal cover.

In the special case that $\mathbb{A}$ is a graph of groups over an underlying graph A which is a single edge e, we say that $\mathbb{A}$ is an *edge of groups*. Suppose $\partial_0 e = a_1$ and $\partial_1 e = a_2$. Write A_0 for the edge group $\mathcal{A}_e$, and for $i = 1, 2$ let A_i be the vertex group $\mathcal{A}_{a_i}$. The fundamental group $\pi_1(\mathbb{A}, a_1)$ is then isomorphic to the free product with amalgamation $A_1 *_{A_0} A_2$, and the universal cover $X = \widetilde{(A, a_1)}$ is an (m, n)–biregular tree, where $m = [A_1 : A_0]$ and $n = [A_2 : A_0]$. Moreover, it follows from Proposition 1.23 of [2] that $\mathbb{A}$ is faithful if and only if for any normal subgroup N of $\mathcal{A}_e$, if $\alpha_e N$ is normal in $\mathcal{A}_{a_1}$ and $\alpha_{\overline{e}} N$ is normal in $\mathcal{A}_{a_2}$, then N is trivial.

1.3. **Cocompact lattices.** We recall some basic definitions and properties. Let G be a locally compact topological group with left-invariant Haar measure μ. A discrete subgroup $\Gamma \leq G$ is a *lattice* if $\Gamma\backslash G$ carries a finite G–invariant measure, and is *cocompact* if $\Gamma\backslash G$ is compact.

A well-known property of cocompact lattices that we will use is the following.

Theorem 7 (Gelfand–Graev–Piatetsky-Shapiro [14])**.** *Let G be a locally compact topological group, and Γ a cocompact lattice in G. If $u \in \Gamma$, then $u^G = \{gug^{-1} \mid g \in G\}$ is a closed subset of G.*

Proof. This is a statement on p. 10 of [14]. □

We will also use the following normalisation of Haar measure. In Section 1.4 below we will apply this result to the Kac–Moody groups G that we consider.

Proposition 8 (Serre, [22])**.** *Let G be a locally compact topological group acting on a set S with compact open stabilisers and a finite quotient $G\backslash S$. Then there is a normalisation of the Haar measure μ, depending only on the choice of G–set S, such that for each discrete subgroup Γ of G we have*

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash S) := \sum_{s \in \Gamma\backslash S} \frac{1}{|\Gamma_s|} \leq \infty.$$

Moreover, Γ is cocompact in G if and only if $\Gamma\backslash S$ is finite.

Note that a subgroup $\Gamma \leq G$ is discrete if and only if the stabilisers Γ_s, $s \in S$, are finite groups.

1.4. **Kac–Moody groups.** We first in Section 1.4.1 explain how one may associate, to a generalised Cartan matrix A and an arbitrary field, a Kac–Moody group Λ, the so-called minimal or incomplete Kac–Moody group. In Section 1.4.2 we specialise to rank 2 Kac–Moody groups over finite fields. Section 1.4.3 describes the completion G of Λ that appears in the statement of Theorem 1 above, and Section 1.4.4 discusses cocompact lattices in G. Our treatment of Kac–Moody groups is brief and combinatorial, and partly follows Appendix TKM of Dymara–Januszkiewicz [12]. For a more sophisticated and general approach, using the notion of a "twin root datum", we refer the reader to, for example, Caprace–Rémy [8].

1.4.1. *Incomplete Kac–Moody groups.* Let I be a finite set. A *generalised Cartan matrix* $A = (A_{ij})_{i,j \in I}$ is a matrix with integer entries, such that $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$, and $A_{ij} = 0$ if and only if $A_{ji} = 0$. (If A is positive definite, then A is the Cartan matrix of some finite-dimensional semisimple Lie algebra.) A *Kac–Moody datum* is a 5–tuple $(I, \mathfrak{h}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, A)$ where $\mathfrak{h}$ is a finitely generated free abelian group, $\alpha_i \in \mathfrak{h}$, $h_i \in \mathrm{Hom}(\mathfrak{h}, \mathbb{Z})$, and $A_{ij} = h_j(\alpha_i)$. The set $\Pi = \{\alpha_i\}_{i \in I}$ is called the set of *simple roots*.

Given a generalised Cartan matrix A as above, we define a *Coxeter matrix* $M = (m_{ij})_{i,j\in I}$ as follows: $m_{ii} = 1$, and if $i \neq j$ then $m_{ij} = 2, 3, 4, 6$ or ∞ as $A_{ij}A_{ji} = 0, 1, 2, 3$ or is ≥ 4. The associated *Weyl group* W is then the Coxeter group with presentation determined by M:

$$W = \langle \{w_i\}_{i\in I} \mid (w_i w_j)^{m_{ij}} \text{for } m_{ij} \neq \infty\rangle.$$

The Weyl group acts on $\mathfrak{h}$ via $w_i : \beta \mapsto \beta - h_i(\beta)\alpha_i$ for each $\beta \in \mathfrak{h}$ and each $i \in I$. In particular, $w_i(\alpha_i) = -\alpha_i$ for each simple root α_i. The set Φ of *real roots* is defined by $\Phi = W \cdot \Pi$. In general, the set of real roots is infinite.

We will, not by coincidence, use the same terminology and notation for simple roots and real roots which are defined in the following combinatorial fashion. Let ℓ be the word length on the Weyl group W, that is, $\ell(w)$ is the minimal length of a word in the letters $\{w_i\}_{i\in I}$ representing w. The *simple roots* $\Pi = \{\alpha_i\}_{i\in I}$ are then defined by

$$\alpha_i = \{w \in W \mid \ell(w_i w) > \ell(w)\}.$$

The set Φ of *real roots* is $\Phi = W \cdot \Pi = \{w\alpha_i \mid w \in W, \alpha_i \in \Pi\}$, and W acts naturally on Φ. The set Φ_+ of *positive roots* is $\Phi_+ = \{\alpha \in \Phi \mid 1_W \in \alpha\}$, and the set of *negative roots* Φ_- is $\Phi\backslash\Phi_+$. The complement of a root α in W, denoted $-\alpha$, is also a root. As before, $w_i(\alpha_i) = -\alpha_i$ for each simple root α_i.

We now define the split Kac–Moody group Λ associated to a Kac–Moody datum as above, over an arbitrary field k. The group Λ may be given by a presentation, which is essentially due to Tits (see [25]), and which appears in Carter [10]. For simplicity, we state this presentation only for the simply-connected group Λ_u and then discuss the general case. Let $(I, \mathfrak{h}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I}, A)$ be a Kac–Moody datum and k a field. The associated *simply-connected Kac–Moody group* Λ_u over k is generated by *root subgroups* $U_\alpha = U_\alpha(k) = \langle x_\alpha(t) \mid t \in k\rangle$, one for each real root $\alpha \in \Phi$. We write $x_i(u) = x_{\alpha_i}(u)$ and $x_{-i}(u) = x_{-\alpha_i}(u)$ for each $u \in k$ and $i \in I$, and put $\tilde{w}_i(u) = x_i(u)x_{-i}(u^{-1})x_i(u)$, $\tilde{w}_i = \tilde{w}_i(1)$, and $h_i(u) = \tilde{w}_i(u)\tilde{w}_i^{-1}$ for each $u \in k^*$ and $i \in I$. A set of defining relations for the simply-connected Kac-Moody group Λ_u is then:

(1) $x_\alpha(t)x_\alpha(u) = x_\alpha(t + u)$, for all roots $\alpha \in \Phi$ and all $t, u \in k$.
(2) If $\alpha, \beta \in \Phi$ is a prenilpotent pair of roots, that is, there exist $w, w' \in W$ such that $w(\alpha) \in \Phi_+$, $w(\beta) \in \Phi_+$, $w'(\alpha) \in \Phi_-$ and $w'(\beta) \in \Phi_-$, then for all $t, u \in k$:

$$[x_\alpha(t), x_\beta(u)] = \prod_{\substack{i,j \in \mathbb{N} \\ i\alpha + j\beta \in \Phi}} x_{i\alpha+j\beta}(C_{ij\alpha\beta}t^i u^j)$$

where the integers $C_{ij\alpha\beta}$ are uniquely determined by i, j, α, β, Φ, and the ordering of the terms on the right-hand side.
(3) $h_i(t)h_i(u) = h_i(tu)$ for all $t, u \in k^*$ and all $i \in I$.
(4) $[h_i(t), h_j(u)] = 1$ for all $t, u \in k^*$ and $i, j \in I$.
(5) $h_j(u)x_i(t)h_j(u)^{-1} = x_i(u^{A_{ij}}t)$ for all $t \in k$, $u \in k^*$ and $i, j \in I$.
(6) $\tilde{w}_i h_j(u)\tilde{w}_i^{-1} = h_j(u)h_i(u^{-A_{ij}})$ for all $u \in k^*$ and $i, j \in I$.
(7) $\tilde{w}_i x_\alpha(u)\tilde{w}_i^{-1} = x_{w_i(\alpha)}(\epsilon u)$ where $\epsilon \in \{\pm 1\}$, for all $u \in k$.

By a result of P.-E. Caprace (cf. 3.5(2) of [5]), any two split Kac–Moody groups of the same type defined over the same field are isogenic. That is, if Λ is any split Kac–Moody group associated to the same generalised Cartan matrix A as Λ_u, and defined over the same field k, then there exists a surjective homomorphism $i : \Lambda_u \to \Lambda$ with $\ker(i) \leq Z(\Lambda_u)$. The Kac–Moody group Λ so constructed is sometimes called the *incomplete* Kac–Moody group (for completions of Λ, see Section 1.4.3 below).

A first example of an incomplete Kac–Moody group Λ over a finite field is $\Lambda = SL_n(\mathbb{F}_q[t, t^{-1}])$, which is over the field $\mathbb{F}_q$, and is not simply-connected.

Again, for a complete and proper definition of *incomplete* Kac–Moody groups we encourage the reader to consult various papers of P.-E. Caprace and B. Rémy (cf. [19], [7]).

We now discuss several important subgroups of the Kac–Moody group Λ. For any version (simply-connected or not), the *unipotent* subgroup of Λ is

$$U = U_+ = \langle U_\alpha \mid \alpha \in \Phi_+\rangle.$$

For Λ_u simply-connected, the *torus*

$$T = \langle h_i(u) \mid i \in I, u \in k^* \rangle$$

is isomorphic to the direct product of $|I|$ copies of k^*. In general, the torus T of Λ is a homomorphic image of the direct product of $|I|$ copies of k^*. For all Λ, we define N to be the subgroup of Λ generated by the torus T and by the elements $\{\tilde{w}_i\}_{i\in I}$ (where, in general as in the simply-connected case, $\tilde{w}_i = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$ for all $i \in I$). The *standard Borel subgroup* $B = B_+$ of Λ is defined by

$$B = \langle T, U_+ \rangle = \langle T, U \rangle.$$

The group B has decomposition $B = T \ltimes U_+ = T \ltimes U$ (see [19]).

The subgroups B and N of Λ form a BN–pair (also known as a Tits system) with Weyl group W, and hence Λ has a Bruhat–Tits building X. (In fact, the group Λ has isomorphic twin buildings, associated to twin BN–pairs (B_+, N) and (B_-, N), but we need only concern ourselves with the positive pair.) The chambers of X correspond to the cosets of B in Λ, hence Λ acts naturally on X with quotient a single chamber. For each apartment Σ of X, the chambers in Σ are in bijection with the elements of the Weyl group W. Each root $\alpha \subset W$ corresponds to a "half-apartment". The construction of the building X for Λ of rank 2 is explained further in Section 1.4.2 below.

1.4.2. *Rank* 2. We now specialise to the cases considered in Theorem 1 above. Let A be a generalised Cartan matrix of the form $A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, with $m \geq 2$. For $m > 2$ such an A has rank 2. If $m = 2$ then A is *affine*, meaning that A is positive semidefinite but not positive definite. For all such A (affine and non-affine) the associated Weyl group W is

$$W = \langle w_1, w_2 \mid w_1^2, w_2^2 \rangle.$$

That is, W is the infinite dihedral group. Let ℓ be the word length on W. The simple roots $\Pi = \{\alpha_1, \alpha_2\}$ are then given by, for $i = 1, 2$,

$$\alpha_i = \{w \in W \mid \ell(w_i w) > \ell(w)\} = \{1, w_{3-i}, w_{3-i}w_i, w_{3-i}w_i w_{3-i}, \ldots\}.$$

The set Φ of real roots is $\Phi = \{w\alpha_i \mid w \in W, i = 1, 2\}$.

Now let Λ be an incomplete Kac–Moody group with generalised Cartan matrix A, defined over a finite field $\mathbb{F}_q$, where $q = p^a$ with p prime. As Λ is a group with BN-pair, as described above, for $i = 1, 2$, the *parabolic subgroup* P_i of Λ is defined by

$$P_i = B \sqcup B\tilde{w}_i B.$$

Since $J_i = \{\alpha_i\}$ is a root system of type A_1, and thus is of finite type, now [19, 6.2] applies. Hence, the group P_i has a *Levi decomposition* $P_i = L_i \ltimes U_i$. Here $U_i = U \cap U^{w_i}$ is called a *unipotent radical* of P_i, and the group L_i is called a *Levi complement* of P_i. The Levi complement factors as $L_i = TM_i$, where T is the torus of Λ, and $M_i = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle$, that is, $A_1(q) \cong M_i \lhd L_i$.

To describe the building X for Λ, we first describe its apartments. Let Σ be the *Coxeter complex* for the Weyl group W (the infinite dihedral group). That is, Σ is the one-dimensional simplicial complex homeomorphic to the line, with vertices the cosets in W of the subgroups $\langle w_i \rangle$, for $i = 1, 2$. Two vertices $w\langle w_1 \rangle$ and $w'\langle w_2 \rangle$ of Σ are adjacent if and only if $w^{-1}w' = w_i$ for $i = 1$ or 2. Observe that the set of real roots Φ, described above, is in bijection with the set of half-lines in Σ. The apartments of the building X are copies of the Coxeter complex Σ for W, and so X is a simplicial tree, with the roots corresponding to "half-apartments". The chambers of X are the edges of this tree. Since Λ has symmetric generalised Cartan matrix A and is defined over the finite field $\mathbb{F}_q$, the building X is a $(q+1)$–regular tree.

1.4.3. *Completions of* Λ. We are finally ready to describe the main object of our study: the locally compact topological Kac–Moody groups. In order to do this we will have to define a topological completion of the incomplete Kac–Moody group Λ. It turns out that there are several completions appearing in the literature. For example, Carbone–Garland [9] defined a representation-theoretic completion of Λ using the 'weight topology'. A different approach by Rémy and Ronan, appearing for instance in [20], is to use the action of Λ on the building X, as follows. The kernel K of the Λ–action on X is the centre $Z(\Lambda)$, which is a finite group when Λ is over a finite field (Rémy [19]). The closure of Λ/K in the automorphism group of X is then a completion of Λ. For example, when $\Lambda = SL_n(\mathbb{F}_q[t,t^{-1}])$, the centre $Z(\Lambda)$ is the finite group $\mu_n(\mathbb{F}_q)$ of nth roots of unity in $\mathbb{F}_q$, and the completion in this topology is $SL_n(\mathbb{F}_q((t^{-1})))/\mu_n(\mathbb{F}_q) \cong PSL_n(\mathbb{F}_q((t^{-1})))$. To avoid dealing with representation-theoretic constructions or with quotients, we are going to follow the completion in the building topology, defined by Caprace and Rémy in [8].

So, let Λ be an incomplete Kac–Moody group over a finite field, as defined in Section 1.4.1 above. We now describe the completion G of Λ which appears in Theorem 1 (for Λ with generalised Cartan matrix A as in Section 1.4.2 above).

Let $c_+ = B_+$ be the chamber of the Bruhat–Tits building X for Λ which is fixed by $B = B_+$. For each $n \in \mathbb{N}$, we define

$$U_{+,n} = \{g \in U_+ \mid g.c = c \text{ for each chamber } c \text{ such that } d(c, c_+) \leq n \ \}.$$

That is, $U_{+,n}$ is the kernel of the action of $U_+ = U$ on $\mathrm{Ball}(c_+, n)$. We now define a function $\mathrm{dist}_+ : \Lambda \times \Lambda \to \mathbb{R}_+$ by $\mathrm{dist}_+(g,h) = 2$ if $h^{-1}g \notin U_+$, and $\mathrm{dist}_+(g,h) = 2^{-n}$ if $g^{-1}h \in U_+$ and $n = \max\{k \in \mathbb{N} \mid g^{-1}h \in U_{+,k}\}$. It is not hard to see that dist_+ is a left-invariant metric on Λ (see [8]). Let G be the completion of Λ with respect to this metric. The group G is called the *completion of* Λ *in the building topology*, and we will refer to G as a *topological Kac–Moody group*. For example, when $\Lambda = SL_n(\mathbb{F}_q[t,t^{-1}])$, the topological Kac–Moody group G is $G = SL_n(\mathbb{F}_q((t^{-1})))$.

Some properties of topological Kac–Moody groups that we will need are gathered in Proposition 9 below. We state these results only for G as in Theorem 1 above, although they hold more generally.

Proposition 9. *Let G be a topological Kac–Moody group as in Theorem 1 above, with G being the completion in the building topology of an incomplete Kac–Moody group Λ.*

(1) *G is a locally compact, totally disconnected topological group.*
(2) *Let $\hat{B}$, $\hat{U}$, $\hat{P}_i$ and $\hat{U}_i$ be the closures in G of the subgroups $B = B_+$, $U = U_+$, P_i and U_i respectively of Λ. Then $\hat{B} \cong T \ltimes \hat{U}$ and $\hat{P}_i \cong L_i \ltimes \hat{U}_i$.*
(3) *$(\hat{B}, N)$ is a BN–pair of G. The corresponding building is canonically isomorphic to X, and so by abuse of notation we will denote it by X as well. The kernel of the action of G on X is the centre $Z(G)$, and $Z(G) = Z(\Lambda)$.*

Items (1) and (3) are established by Caprace–Rémy in [8], and item (2) in [8] and [7].

We will refer to $\hat{B}$ as the *(standard) Borel subgroup* of G, and to $\hat{P}_1$ and $\hat{P}_2$ as the *(maximal or standard) parabolic subgroups* of G. Alternatively, we may say that $\hat{B}$ is the *Iwahori* subgroup of G, and $\hat{P}_1$ and $\hat{P}_2$ are the *parahoric* subgroups of G, by analogy with terminology for $G = SL_2(\mathbb{F}_q((t^{-1})))$. To simplify notation, when the context is clear we will write B, P_1 and P_2 for the Borel and maximal parabolic subgroups of the topological Kac–Moody group G, rather than respectively $\hat{B}$, $\hat{P}_1$ and $\hat{P}_2$.

1.4.4. *Cocompact lattices in G.* Let G be as in Theorem 1 above, with Bruhat–Tits building the tree X. By definition, the vertices of X may be described by $VX = G/P_1 \sqcup G/P_2$, and the edges of X by G/B (here, we are abusing notation to write B, P_1 and P_2 for the standard Borel/Iwahori and parabolic/parahoric subgroups of the completed group G). It follows that in the G–action on X, the stabiliser of each vertex of X is a conjugate of either P_1 or P_2, and the stabiliser of each edge of X is a conjugate of B.

The action of G on the vertex set VX thus satisfies the hypotheses of Proposition 8 above. Hence $\Gamma < G$ is discrete if and only if Γ acts on X with finite vertex stabilisers, and $\Gamma < G$ discrete is a cocompact lattice in G if and only if $\Gamma\backslash X$ is a finite graph. Thus $\Gamma < G$ is an edge-transitive lattice if and only if Γ is the fundamental group of an edge of groups $\mathbb{A}$ as in the introduction, with A_0, A_1 and A_2 finite groups. Moreover, the covolume of such a Γ is the sum

$$\mu(\Gamma\backslash G) = \frac{1}{|A_1|} + \frac{1}{|A_2|}.$$

In particular, if Γ' is an edge-transitive lattice in G of minimal covolume (such as the lattices in $G = SL_2(\mathbb{F}_q((t^{-1})))$ constructed by Lubotzky in [17]), and $\Gamma = A_1 *_{A_0} A_2$ is another edge-transitive cocompact lattice in G, then $|A_i| \leq |A_i'|$ for $i = 1, 2$.

Note that, by construction, G acts without inversions on its Bruhat–Tits tree X. It follows from Proposition 6 above that if A is a finite subgroup of G, then A is contained in (a conjugate of) a standard parabolic/parahoric subgroup P_i of G.

1.5. **Finite groups.** In our quest for the cocompact lattices of Kac–Moody groups, we will need to look at the finite subgroups of G. The following celebrated result of L.E. Dickson and its corollary will be especially useful for us.

Theorem 10 (Dickson, 6.5.1 of [16]). *Let $K = PSL_2(q)$, where $q = p^a \geq 5$ and p is a prime. Set $d = (2, q-1)$. Then K has subgroups of the following isomorphism types (in the indicated cases), and every subgroup of K is isomorphic to a subgroup of one of the following groups:*

(1) *Borel subgroups of K, which are Frobenius groups of order $q(q-1)/d$;*
(2) *Dihedral groups of orders $2(q-1)/d$ and $2(q+1)/d$;*
(3) *The groups $PGL_2(p^b)$ (if $2b \mid a$) and $PSL_2(p^b)$ (if b is a proper divisor of a);*
(4) *The alternating group A_5, if 5 divides $|K|$;*
(5) *The symmetric group S_4 , if 8 divides $|K|$; and*
(6) *The alternating group A_4.*

Corollary 11. *Let $K = SL_2(q)$, where $q = p^a$ with p a prime, and suppose A is a proper subgroup of K.*

If $p = 2$ and $q+1$ divides $|A|$, then either $A \cong C_{q+1}$, a cyclic group of order $q+1$, or $A \cong D_{2(q+1)}$, a dihedral group of order $2(q+1)$.

If p is odd and the image of A in $K/Z(K) \cong PSL_2(q)$ has order divisible by $q+1$, then $Z(K) = \langle -I \rangle \leq A$. Moreover, either A is a subgroup of K of order $2(q+1)$ such that $A/Z(K) \cong D_{q+1}$, a dihedral group of order $q+1$, or one of the following conditions hold:

(1) *$q = 5$, $A \cong SL_2(3)$,*
(2) *$q = 7$, $A \cong 2S_4$,*
(3) *$q = 9$, $A \cong SL_2(5)$,*
(4) *$q = 11$, $A \cong SL_2(3)$ or $A \cong SL_2(5)$,*
(5) *$q = 19$, $A \cong SL_2(5)$,*
(6) *$q = 23$, $A \cong 2S_4$,*
(7) *$q = 29$, $A \cong SL_2(5)$,*
(8) *$q = 59$, $A \cong SL_2(5)$.*

Proof. Suppose that $p = 2$. Then $d = 1$ and $SL_2(q) = PSL_2(q)$. Assume first that $q \geq 5$. Then if $q+1$ divides $|A|$, Dickson's Theorem asserts that both C_{q+1} and $D_{2(q+1)}$ are the obvious candidates for the role of A. If not, A would be one of the following groups: A_4, S_4 or A_5. Then $q+1$ would divide 12, 24 or 60. Since q is a power of 2 and $q \geq 5$, this is not possible, proving the result. Otherwise $q \in \{2, 4\}$, and the result follows immediately from the structure of $K = SL_2(2) \cong S_3$, and $K = SL_2(4) \cong A_5$.

Suppose now that p is odd. This time $d = 2$ and the image of A in $PSL_2(q)$ is a group of order divisible by $q+1$. Since $|A|$ is even while K contains a unique involution $-I$, $\langle -I \rangle = Z(K) \leq A$. If $q \geq 5$, using the same argument as above, we obtain the desired conclusion. Otherwise $q = 3$ and $K = SL_2(3) \cong Q_8C_3$, and the result follows immediately. □

2. Embedding amalgams in G

Let $\Gamma = A_1 *_{A_0} A_2$ be an amalgam of finite groups. In this section we describe two methods that we will use to determine whether Γ embeds in a Kac–Moody group G as in Theorem 1 above as an edge-transitive cocompact lattice. In Section 2.1 we present a special case of Bass' covering theory for graphs of groups (see [2]), and in Section 2.2 we prove Lemma 4 of the introduction, which generalises Lemma 3.1 of Lubotzky [17] on embedding amalgams into the group $G = SL_2(\mathbb{F}_q((t^{-1})))$.

2.1. **Coverings of graphs of groups.** Lemma 12 below is a special case of Bass' covering theory for graphs of groups [2]. Coverings Φ of graphs of groups are defined in Section 2.6 of [2]. The notion of covering that we use in the statement of Lemma 12 below is a simplification of this definition, and is equivalent to the covering $\partial\Phi$ defined in Section 2.9 of [2]. As explained in Section 2.9 of [2], $\partial\Phi$ is a covering if and only if Φ is a covering (in the original sense of Section 2.6 of [2]). Moreover, given a (simplified) covering as below, it is not hard to construct a covering in the original sense. Hence we may work with this less complicated definition.

Lemma 12. *Let*

$$\mathbb{A} = \quad \overset{A_1}{\bullet}\underset{A_0}{\rule{4cm}{0.4pt}}\overset{A_2}{\bullet}$$

be a graph of finite groups, defined with respect to monomorphisms $\alpha_i : A_0 \to A_i$ for $i = 1, 2$. Let G be as in Theorem 1 above and let $\mathbb{G}$ be the graph of groups

$$\mathbb{G} = \quad \overset{P_1}{\bullet}\underset{B}{\rule{4cm}{0.4pt}}\overset{P_2}{\bullet}$$

induced by the action of G on its Bruhat–Tits tree X, where for $i = 1, 2$, the monomorphism $\phi_i : B \to P_i$ is inclusion. The following are equivalent.

1. *The amalgam $\Gamma = A_1 *_{A_0} A_2$ embeds as a cocompact edge-transitive lattice in G.*
2. *There is a covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$. That is, there are monomorphisms*

$$\rho_0 : A_0 \hookrightarrow B \quad \text{and} \quad \rho_i : A_i \hookrightarrow P_i \quad \text{for } i = 1, 2$$

such that:

(a) *for some $\delta_1 \in P_1$ and $\delta_2 \in P_2$, the following diagram commutes:*

$$\begin{array}{ccccc} A_1 & \xleftarrow{\alpha_1} & A_0 & \xrightarrow{\alpha_2} & A_2 \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho_2} \\ P_1 & \xleftarrow{\mathrm{ad}(\delta_1)\circ\phi_1} & B & \xrightarrow{\mathrm{ad}(\delta_2)\circ\phi_2} & P_2 \end{array}$$

where for $i = 1, 2$ and $g \in P_i$, $\mathrm{ad}(\delta_i)(g) = \delta_i g \delta_i^{-1}$; and

(b) *for $i = 1, 2$ the map of cosets*

$$A_i/\alpha_i(A_0) \quad \longrightarrow \quad P_i/\phi_i(B)$$

induced by

$$g \mapsto \rho_i(g)\delta_i$$

is a bijection.

Proof. A covering of graphs of groups induces a monomorphism of fundamental groups and an isomorphism of universal covers (see Proposition 2.7 of [2]). The equivalence between (1) and (2) in Lemma 12 then follows by Proposition 8 above applied to the action of G on X (see Section 1.4 above). □

2.2. **Generalisation of a method of Lubotzky.** In [17], Lubotzky studied the lattices of $SL_2(K)$, for K a nonarchimedean local field. An important tool in his work is Lemma 3.1 of [17], in which he gives a sufficient condition for an amalgam of two finite subgroups of $SL_2(K)$ to be a cocompact lattice in G. In Lemma 4, stated in the introduction, we generalise this lemma, and also prove the converse of this generalisation. Our proof differs in some details from that of Lubotzky. For convenience we restate Lemma 4 here.

Lemma 13. *Let q_1 and q_2 be positive integers and let X be the $(q_1 + 1, q_2 + 1)$–biregular tree. Let G be a locally compact group of automorphisms of X, which acts on X with compact open stabilisers and with fundamental domain an edge (x_1, x_2), where for $i = 1, 2$ the vertex x_i of X has valence $q_i + 1$.*

Suppose for $i = 1, 2$ that A_i is a finite subgroup of the stabiliser G_{x_i} such that:

1. *A_i acts transitively on the set of $q_i + 1$ neighbours of x_i in X; and*

(2) $\mathrm{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$.

Then $\Gamma = \langle A_1, A_2\rangle$, the group generated by A_1 and A_2, is a cocompact lattice in G, with fundamental domain the edge (x_1, x_2). Moreover, Γ is isomorphic to the free product with amalgamation $\Gamma \cong A_1 *_{A_1\cap A_2} A_2$, and $\Gamma_{x_i} = A_i$.

Conversely, suppose Γ is a cocompact lattice in G with fundamental domain the edge (x_1, x_2). Let $A_i = \Gamma_{x_i}$. Then $\Gamma \cong A_1 *_{A_1\cap A_2} A_2$, and A_i is a finite subgroup of G_{x_i} such that (1) and (2) hold.

Proof. Let Δ be the abstract free product with amalgamation $\Delta = A_1 *_{A_1\cap A_2} A_2$ and let $\varphi : \Delta \to \Gamma$ be the homomorphism onto Γ. Let $a_1, \ldots, a_{q_1}$ (respectively, $b_1, \ldots, b_{q_2}$) be representatives of the nontrivial cosets of $A_1 \cap A_2$ in A_1 (respectively, A_2). A word $w \in \Delta$ then has normal form $w = a_{i_1} b_{j_1} \cdots a_{i_t} b_{j_t} c$ where $c \in A_1 \cap A_2$, and possibly $a_{i_1} = 1$ or $b_{i_t} = 1$.

Let e be the edge (x_1, x_2). We claim that $d(\varphi(w)e, e) \geq t$ for all $w \in \Delta$ with normal form as above. For $t = 0$ we have $\varphi(w) = \varphi(c) \in A_1 \cap A_2$, hence $\varphi(w)$ fixes e, and so $d(\varphi(w)e, e) = d(e, e) = 0$. For $t = 1$, if $a_{i_1} = 1$ (respectively, $b_{i_1} = 1$) then $d(\varphi(w) \cdot e, e) = 1$ since the edge $\varphi(w) \cdot e$ will share the vertex x_2 (respectively, x_1) with e. Otherwise, if neither a_{i_1} nor b_{i_1} is trivial, we have $d(\varphi(w) \cdot e, e) = 2 \geq 1$.

For $t \geq 2$, assume inductively that for $w' = a_{i_2} b_{i_2} \cdots a_{i_t} b_{i_t} c$, the distance $d(\varphi(w')e, e) \geq t - 1$. Note that the edge path from e to $\varphi(w')e$ has the vertex x_1 in its interior since $a_{i_2} \neq 1$. Applying the element $\varphi(b_{i_1})$, which fixes x_2 and does not fix e, we obtain $d(\varphi(b_{i_1})\varphi(w')e, e) \geq (t - 1) + 1 \geq t$, and hence (whether or not a_{i_1} is trivial) we conclude that $d(\varphi(w)e, e) \geq t$ as claimed.

In particular, we have shown that $\varphi(w)e \neq e$ unless $w \in A_1 \cap A_2$. Suppose now that $\varphi(w) = 1$. Then $d(\varphi(w)e, e) = 0$ and so $w \in A_1 \cap A_2$, but the map φ is injective on $A_1 \cap A_2$, and thus $w = 1$. Hence Δ is isomorphic to Γ, and we have that Γ is discrete. Suppose $g \in \Gamma_{x_i}$. If g fixes e then $g \in A_1 \cap A_2$, and if g does not fix e then g is contained in some coset of $A_1 \cap A_2$ in A_i. Hence $\Gamma_{x_i} = A_i$.

We claim that Γ acts transitively on the edges of X. By induction, every edge of X at distance $s \geq 1$ from e can be written as $g \cdot e$, where $g \in \Gamma$ is a word of length s with letters alternating between a_i and b_j. Hence Γ acts transitively on the edges of X. This completes the proof of one direction of the lemma.

For the converse, the isomorphism $\Gamma \cong A_1 *_{A_1\cap A_2} A_2$ is a standard result of Bass–Serre theory. The inclusion $A_i \leq G_{x_i}$ holds because $\Gamma \leq G$. Properties (1) and (2) hold since Γ acts transitively on the set of edges of X, with fundamental domain the edge (x_1, x_2). □

3. Proof of Theorems 1 and 2 for $SL_2(\mathbb{F}_q((t^{-1})))$

We are now ready to proceed with a proof of our main results. In this section, we provide a proof of Theorems 1 and 2 above for the case $G = SL_2(K)$, where $K = \mathbb{F}_q((t^{-1}))$ with $q = p^a$, p a prime. We first restate Theorems 1 and 2 for this case. Then in Proposition 14 below, we use the Levi decomposition of the parahoric subgroups P_i of G, together with finite group theory, to restrict the possible vertex groups A_1 and A_2 in a cocompact edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G. The remaining argument is subdivided into two cases: $p = 2$, where we apply Lemma 12 above, and p odd, where we use Lemma 13.

Theorem 1′. *Let $G = SL_2(K)$ with $K = \mathbb{F}_q((t^{-1}))$ where $q = p^a$ with p a prime.*

(1) *If $p = 2$, then (up to isomorphism) there is only one edge-transitive cocompact lattice Γ in G, namely, the amalgam of cyclic groups $\Gamma = C_{q+1} * C_{q+1}$.*
(2) *If p is odd and $q \equiv 1 \pmod 4$, then G does not contain any edge-transitive cocompact lattices unless $q = p \in \{5, 29\}$, in which case $\Gamma = A_1 *_{A_0} A_2$ where*
  (a) *if $q = 5$, $A_1 \cong A_2 \cong SL_2(3)$ and $A_0 \cong C_4$; and*
  (b) *if $q = 29$, $A_1 \cong A_2 \cong SL_2(5)$ and $A_0 \cong C_4$.*
(3) *If p is odd and $q \equiv 3 \pmod 4$, the following are the only edge-transitive cocompact lattices Γ in G:*
  (a) *for all such q, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, A_i is a subgroup of order $2(q + 1)$ isomorphic to the normaliser of a non-split torus in $SL_2(q)$, and $A_0 \cong C_2$.*
  (b) *If $q = 7$, $\Gamma = A_1 *_{A_0} A_2$ where $A_1 \cong A_2 \cong 2S_4$ and $A_0 \cong C_6$.*
  (c) *If $q = 11$, $\Gamma = A_1 *_{A_0} A_2$ and one of the following holds:*
    (i) *$A_1 \cong A_2 \cong SL_2(3)$ and $A_0 \cong C_2$,*

(ii) $A_1 \cong A_2 \cong SL_2(5)$ *and* $A_0 \cong C_{10}$.
(d) *If* $q = 19$, $\Gamma = A_1 *_{A_0} A_2$ *where* $A_1 \cong A_2 \cong SL_2(5)$ *and* $A_0 \cong C_6$.
(e) *If* $q = 23$, $\Gamma = A_1 *_{A_0} A_2$ *where* $A_1 \cong A_2 \cong 2S_4$ *and* $A_0 \cong C_2$.
(f) *If* $q = 59$, $\Gamma = A_1 *_{A_0} A_2$ *where* $A_1 \cong A_2 \cong SL_2(5)$ *and* $A_0 \cong C_2$.

Proof. Suppose the amalgam of finite groups $\Gamma = A_1 *_{A_0} A_2$ is a cocompact edge-transitive lattice in G. Since the Bruhat–Tits building X for G is a $(q+1)$–regular tree, it follows that the edge group A_0 has index $q+1$ in both of the vertex groups A_1 and A_2. By Lemma 12 or Lemma 13 above, for $i = 1, 2$ there are injective group homomorphisms $A_i \hookrightarrow P_i$, where P_i is a standard parahoric subgroup of G. Hence A_1 and A_2 are finite subgroups of G of order divisible by $(q+1)$. Since the action of G on its Bruhat–Tits tree X is not faithful if p is odd (it is faithful when $p = 2$), we must take the kernel of the action $Z(G)$ into consideration. Thus what we are really looking for are finite subgroups A_i of G for $i = 0, 1, 2$, such that when we look at their images $\overline{A}_i$ in $G/Z(G) \cong PSL_2(K)$, $\overline{A}_0$ has index $(q+1)$ in both $\overline{A}_1$ and $\overline{A}_2$, and the images $\overline{A}_1$ and $\overline{A}_2$ in $G/Z(G)$ have orders divisible by $(q+1)$ as well.

Proposition 14. *Let* $G = SL_2(\mathbb{F}_q((t^{-1})))$. *If* X *is a finite subgroup of* G *such that* $|XZ(G)/Z(G)|$ *is divisible by* $(q+1)$, *then* $X \geq Z(G)$ *and* X *is isomorphic to a subgroup* A *of* $SL_2(q)$ *listed in the conclusions to Corollary 11 above.*

Proof. By Proposition 6 above, each finite subgroup of G sits inside a standard parahoric subgroup of G, which is isomorphic to $P = SL_2(\mathbb{F}_q[[t^{-1}]])$. Notice that $Z(G) = \langle -I \rangle$ is contained inside any standard parahoric subgroup of G.

The group P may be written as the semi-direct product, with respect to the conjugation action, of

$$L \cong SL_2(q) \text{ and } U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod t^{-1} \right\}.$$

Here, U is the principal congruence subgroup of P and is an infinite pro-p group. It contains a natural chain of subgroups

$$U = U_1 > U_2 > \cdots > U_i > \cdots$$

where

$$U_i = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod t^{-i} \right\}.$$

We will need the following well-known facts.

Lemma 15. *The chain of subgroups* U_i *has the following properties:*

(1) $\bigcap_i U_i = \{1\}$.
(2) *Each* U_i *is a normal subgroup of* P. *In particular, each* U_i *contains* U_{i+1} *as a normal subgroup, and each* U_i *is invariant under the conjugation action of* L.
(3) *For each* i, *the quotient group* U_i/U_{i+1} *is an elementary abelian* p*–group.*
(4) *The quotient* U_i/U_{i+1} *has the structure of an* L*–module, induced by the conjugation action of* L *on each* U_i.

Lemma 16. *Let* $g \in G$ *be an element of order* p. *Then* g *is* G*–conjugate to* $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, *for some* $b \in \mathbb{F}_q((t^{-1}))$ *with* $b \neq 0$, *and its centraliser* $C_G(g)$ *is an elementary abelian* p*–group.*

Proof. As is well known, as an algebraic group $G = SL_2(\mathbb{F}_q((t^{-1})))$ is a group with a BN–pair of rank 1. Its standard Borel subgroup is $\mathcal{B}$, the group of upper triangular matrices in G, which in turn has a unipotent radical $\mathcal{U}$, the group of strictly upper triangular matrices.

If $g \in G$ is an element of order p, $\langle g \rangle$ is a closed unipotent subgroup of G. We may now apply the Borel–Tits Theorem [4], to conclude that $\langle g \rangle \leq \mathcal{B}^h$ for some $h \in G$ (in fact, $g \in \mathcal{U}^h \leq \mathcal{B}^h$), and $C_G(g) \leq N_G(\langle g \rangle) \leq \mathcal{B}^h$. Now simple matrix calculation finishes the proof. $\square$

We now use Lemmas 15 and 16 above, together with Corollary 11 above, to prove Proposition 14. Assume there is a finite subgroup X of P such that $(q+1)$ divides $|XZ(G)/Z(G)|$. Then in particular, $|X|$ is divisible by $(q+1)$. If $X \cap U = 1$, then X is isomorphic to a subgroup of $L \cong SL_2(q)$. Using Corollary 11 and the fact that $Z(G) = Z(L) = \langle -I \rangle$ we obtain the desired conclusion.

We therefore assume by contradiction that $X \cap U \neq 1$, and so since $U = U_1$, we have that $X \cap U_1 \neq 1$. By part (1) of Lemma 15, it follows that $X \cap U_n = 1$ for some $n > 1$. Choose the smallest such n. The group X is then isomorphic to a subgroup Y of $\overline{P} := P/U_n$, with U_n a proper subgroup of U. Now $\overline{P} = \overline{L}\,\overline{U}$ where $\overline{L} \cong SL_2(q)$ and $\overline{U} = U/U_n$ is a (nontrivial) p–group. Since $Y \leq \overline{P}$ and $(q+1) \mid |Y|$, and since $Y \cap \overline{U}$ is a p–group, we have that $Y/Y \cap \overline{U} \cong Y\overline{U}/\overline{U}$ is isomorphic to a finite subgroup of $SL_2(q)$ of order divisible by $(q+1)$.

Using Corollary 11 together with the Schur–Zassenhaus Theorem (cf. [1]) we obtain that Y must contain a subgroup Z such that either $Z \cong C_{q+1}$, or Z is non-abelian of order coprime to p and divisible by $q+1$ (if $q = 9$, take $Z \cong C_{10} \leq SL_2(5) \leq SL_2(9)$). Also, $Z \cap \overline{U} = 1$. Without loss of generality we may assume that $Z \leq \overline{L}$.

By the minimality of n, we have that $H := X \cap U_{n-1} \neq 1$. Hence, H is a non-trivial normal subgroup of X. Since $HU_n \leq U_{n-1}$ and $H \cap U_n = 1$, we have $H \cong HU_n/U_n \leq U_{n-1}/U_n$. Now Lemma 15 (4) implies that H is a non-trivial elementary abelian p-subgroup of X normalised by Z' where $Z' \cong Z$ is the preimage of Z in X.

Let $h \in H$ be any nontrivial element. Then h is a genuine element of order p of G. By Lemma 16 above, we may assume without loss of generality that $h \in \mathcal{U}$ (the group of strictly upper triangular matrices in $G = SL_2(\mathbb{F}_q((t^{-1}))))$. Moreover $C_G(h) = \mathcal{U}$. But $H < C_G(h)$ since H is abelian, that is, $H < \mathcal{U}$. Hence $C_G(H) = \mathcal{U}$. Consider the normaliser $N_G(H)$. Since $C_G(H)$ is normal in $N_G(H)$, we have that $\mathcal{U}$ is normal in $N_G(H)$, and thus $N_G(H) \leq N_G(\mathcal{U})$. But $N_G(\mathcal{U}) = \mathcal{B}$. In particular, $Z' \leq \mathcal{B}$. Since Z' is either cyclic of order $q+1$, or is non-abelian of order co-prime to p, this is impossible.

Thus the assumption that $X \cap U \neq 1$ leads to a contradiction, and we have completed the proof of Proposition 14. □

In order to complete the proof of Theorem $1'$, we subdivide the remaining argument into two cases: $p = 2$ and p odd.

3.1. Case $\mathbf{p = 2}$.

Proposition 17. *If $p = 2$, the appropriate conclusions of Theorem $1'$ hold. That is, the only cocompact edge-transitive lattice in $G = SL_2(\mathbb{F}_q((t^{-1})))$ is the free product $\Gamma = C_{q+1} * C_{q+1}$, where C_{q+1} is the cyclic group of order $q+1$.*

Proof. Assume that $p = 2$. Then by Proposition 14 and Corollary 11 above, the vertex groups A_1 and A_2 of Γ are either both isomorphic to C_{q+1}, or both isomorphic to the dihedral group $D_{2(q+1)}$ of order $2(q+1)$. We first show that the amalgam with vertex groups $D_{2(q+1)}$ is not a cocompact edge-transitive lattice in G. By Lemma 12 above, it suffices to prove the following proposition.

Proposition 18. *Let $\mathbb{A}$ be the edge of groups*

$$\mathbb{A} = \; D_{2(q+1)} \quad - \quad C_2 \quad \longrightarrow \quad D_{2(q+1)}.$$

There is no covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$, where $\mathbb{G}$ is the edge of groups for $G = P_1 *_B P_2$.

Proof. Assume by contradiction that such a covering $\Phi : \mathbb{A} \to \mathbb{G}$ exists. Recall that the standard parahoric subgroups of G are $P_1 = SL_2(\mathbb{F}_q[[t^{-1}]])$ and

$$P_2 = \left\{ \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_1 \right\},$$

and that the Iwahori subgroup of G is

$$B = P_1 \cap P_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_1 \middle| c \equiv 0 \mod t^{-1} \right\}.$$

Let the edge group $A_0 \cong C_2$ be generated by an involution s. The vertex groups of $\mathbb{A}$ may then be given by the presentations

$$A_i = \langle s, t_i \mid s^2 = t_i^2 = (st_i)^{q+1} = 1 \rangle \cong D_{2(q+1)}$$

for $i = 1, 2$.

Let $\rho_0 : A_0 \hookrightarrow B$ and $\rho_i : A_i \hookrightarrow P_i$ $(i = 1, 2)$ be the monomorphisms as in Lemma 12. It follows that the elements $\rho_0(s) \in B$, $\rho_1(t_1) \in P_1$ and $\rho_2(t_2) \in P_2$ must all be involutions. By condition (2a) in Lemma 12, we have that for $i = 1, 2$

$$\rho_i(s) = \delta_i \rho_0(s) \delta_i^{-1}. \tag{1}$$

Now applying condition (2b) of Lemma 12, we observe that for $i = 1, 2$, the elements 1, t_i and st_i represent three distinct cosets of $A_0 = \langle s \rangle$ in $A_i = \langle s, t_i \rangle$. Hence the elements δ_i, $\rho_i(t_i)\delta_i$ and $\rho_i(s)\rho_i(t_i)\delta_i$ must represent three distinct cosets of B in P_i. Let

$$\gamma_i = \delta_i^{-1} \rho_i(t_i) \delta_i.$$

Then γ_i is an involution of P_i, but $\gamma_i \notin B$. Similarly, applying Equation (1) above, we have that

$$(\rho_i(t_i)\delta_i)^{-1} \rho_i(s) \rho_i(t_i) \delta_i = \delta_i^{-1} \rho_i(t_i) \delta_i \rho_0(s) \delta_i^{-1} \rho_i(t_i) \delta_i = \gamma_i \rho_0(s) \gamma_i \tag{2}$$

is an involution of $P_i - B$.

We next record the form of involutions of B, $P_1 - B$ and $P_2 - B$.

Lemma 19. *The involutions of the edge group B are as follows:*

$$\{g \in B \mid g^2 = 1\} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \ b \in \mathbb{F}_q[[t^{-1}]], \ \ b \neq 0 \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| \ c \in \mathbb{F}_q[[t^{-1}]], \ \ c \neq 0, \ \ t^{-1} | c \right\}$$
$$\cup \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| \ a, b, c \in \mathbb{F}_q[[t^{-1}]], \ \ b \neq 0, \ \ c \neq 0, \ \ t^{-1} | c, \ \ a^2 + bc = 1 \right\}.$$

The involutions of the vertex groups P_1 and P_2 which are not contained in the edge group B are as follows:

(1) $\{g \in P_1 - B \mid g^2 = 1\} =$
$$\left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| \ c \in \mathbb{F}_q[[t^{-1}]], \ \ c \neq 0, \ \ t^{-1} \nmid c \right\}$$
$$\cup \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| \ a, b, c \in \mathbb{F}_q[[t^{-1}]], \ \ b \neq 0, \ \ c \neq 0, \ \ t^{-1} \nmid c, \ \ a^2 + bc = 1 \right\}$$

(2) $\{g \in P_2 - B \mid g^2 = 1\} =$
$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \ b = b_{-1}t + b_0 + b_1 t^{-1} + \cdots, \ \ b_{-1} \neq 0 \right\}$$
$$\cup \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| \ a, c \in \mathbb{F}_q[[t^{-1}]], \, b = b_{-1}t + b_0 + b_1 t^{-1} + \cdots, \, b_{-1} \neq 0, c \neq 0, \, t^{-1} \nmid c, \, a^2 + bc = 1 \right\}.$$

Continuing with the proof of Proposition 18, suppose first that

$$\rho_0(s) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in B$$

with $b \neq 0$. Noting that involutions of P_2 have diagonal entries equal, we may let

$$\gamma_2 = \begin{pmatrix} e & f \\ g & e \end{pmatrix} \in P_2 - B.$$

Using $e^2 + fg = 1$, we compute

$$\gamma_2 \rho_0(s) \gamma_2 = \begin{pmatrix} 1 + beg & be^2 \\ bg^2 & 1 + beg \end{pmatrix}.$$

Now $b, e \in \mathbb{F}_q[[t^{-1}]]$ hence $be^2 \in \mathbb{F}_q[[t^{-1}]]$. But this contradicts $\gamma_2 \rho_0(s) \gamma_2 \in P_2 - B$. So $\rho_0(s)$ cannot be upper triangular. A similar computation using γ_1 shows that $\rho_0(s)$ cannot be lower triangular.

We are left with the possibility that

$$\rho_0(s) = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in B.$$

For $i = 1, 2$ let

$$\gamma_i = \begin{pmatrix} e_i & f_i \\ g_i & e_i \end{pmatrix} \in P_i - B.$$

We compute

$$\gamma_i \rho_0(s) \gamma_i = \begin{pmatrix} a + be_i g_i + ce_i f_i & be_i^2 + cf_i^2 \\ bg_i^2 + ce_i^2 & a + be_i g_i + ce_i f_i \end{pmatrix}.$$

Since $\gamma_1 \rho_0(s) \gamma_1 \in P_1 - B$, we have that $t^{-1} \nmid bg_1^2 + ce_1^2$. As $\rho_0(s) \in B$, we have $t^{-1} \mid c$, so t^{-1} cannot divide b. Since $b \in \mathbb{F}_q[[t^{-1}]]$, it follows that

$$b = b_0 + b_1 t^{-1} + \cdots$$

with $b_0 \neq 0$. Now $\gamma_2 \rho_0(s) \gamma_2 \in P_2 - B$, so

$$be_2^2 + cf_2^2 = k_{-1} t + \cdots$$

with the coefficient $k_{-1} \neq 0$. Since $b, e_2 \in \mathbb{F}_q[[t^{-1}]]$, the leading term $k_{-1}t$ must come from cf_2^2. Now $\gamma_2 \in P_2 - B$ so $f_2 = l_{-1} t + \cdots$ with $l_{-1} \neq 0$. Hence

$$c = m_1 t^{-1} + \cdots$$

where $m_1 l_{-1}^2 = k_{-1}$, in particular $m_1 \neq 0$. We now have

$$1 = a^2 + bc = (a_0 + a_1 t^{-1} + \cdots)^2 + (b_0 + b_1 t^{-1} + \cdots)(m_1 t^{-1} + \cdots) = a_0^2 + a_1^2 t^{-2} + \cdots + b_0 m_1 t^{-1} + \cdots$$

and the right-hand side can never equal 1, a contradiction.

We conclude that there is no covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$. □

Since the amalgam of dihedral groups $D_{2(q+1)}$ cannot embed as a cocompact edge-transitive lattice in G, the only possibility remaining is the amalgam of cyclic groups $\Gamma = C_{q+1} * C_{q+1}$. Lubotzky proves that this amalgam does embed as a cocompact lattice in G (Theorem 3.3 of [17]); alternatively one may easily construct a covering of graphs of groups as in Lemma 12 above. This completes the proof of Proposition 17. □

We remark that Proposition 17 above could have been proved using different results. For example, Lubotzky showed in [17] that the free product $\Gamma = C_{q+1} * C_{q+1}$ is the cocompact lattice of minimal covolume in G (when $p = 2$). By the discussion of covolumes in Section 1.4.4 above, it follows that an amalgam with vertex groups $D_{2(q+1)}$ cannot embed in G, since $|D_{2(q+1)}| > |C_{q+1}|$. Now, Lubotzky's result relied upon Theorem 7 above. It follows (with some work) from Theorem 7 that a cocompact lattice Γ in G cannot contain involutions, which also rules out the amalgam with vertex groups $D_{2(q+1)}$.

3.2. **Case p odd.**

Proposition 20. *If p is odd, the appropriate conclusions of Theorem $1'$ hold.*

Proof. Notice that in this case $Z(G) \cong C_2$ and $Z(G) \leq P_i$. In fact, $Z(G)$ is the unique subgroup of P_i of order 2. Now Proposition 14 together with Corollary 11 imply that $A_1 \cap A_2$ contains $Z(G)$, and that for $i = 1, 2$ both of the A_i are isomorphic to one of the subgroups listed in the conclusions to Corollary 11.

Suppose first that $q \equiv 3 \pmod 4$. If A_i is isomorphic to a normaliser of a non-split torus in $SL_2(q)$, then just as Lubotzky in the proof of Lemma 3.5 of [17], we may conclude that $\Gamma = A_1 *_{A_0} A_2$ with $A_0 = Z(G)$ is a cocompact edge-transitive lattice in G. In fact, if $p \notin \{7, 11, 19, 23, 59\}$, Lubotzky shows that Γ is the cocompact lattice of minimal covolume. For $p \in \{7, 11, 19, 23, 59\}$, again Lubotzky's argument shows that all the possibilities listed in Theorem $1'$ hold, and in fact unless $p = 11$ and $A_i \cong SL_2(3)$, they all are the cocompact lattices of minimal covolume in G.

Assume now that $q \equiv 1 \pmod 4$, and that A_i has order $2(q + 1)$ and is such that $A/Z(H) \cong D_{q+1}$. This time the argument of Lemma 3.5 [17] will not work, because as was shown in [18], A_i does not act transitively on the neighbours of x_i. In fact, now Lemma 13 above implies that G does not contain edge-transitive cocompact lattices unless possibly one of the following holds: $q = 5$ and $A_1 \cong A_2 \cong SL_2(3)$, or $q \in \{9, 29\}$ and $A_1 \cong A_2 \cong SL_2(5)$. If $q = 5$, then indeed $A_i \cong SL_2(3)$ acts transitively on the neighbours of x_i, as $|A_i \cap \mathrm{Stab}_G(x_{3-i})| = 4 = |A_1 \cap A_2|$, and so $|A_i : A_i \cap \mathrm{Stab}_G(x_{3-i})| = 6 = q + 1$, which means the conditions of Lemma 13 are satisfied. The obtained cocompact lattice is yet another from the series of examples of Lubotzky of lattices of minimal covolume. Similarly, if $q = 29$, $A_i \cong SL_2(5)$ acts transitively on the neighbours of x_i as $|A_i \cap \mathrm{Stab}_G(x_{3-i})| = 4 = |A_1 \cap A_2|$, and so $|A_i : A_i \cap St_G(x_{3-i})| = 30 = q + 1$, and again the resulting lattice is the one of minimal covolume.

Finally, suppose $q = 9$ and $A_i \cong SL_2(5)$. Assume $\Gamma = A_1 *_{A_1 \cap A_2} A_2$ is a cocompact edge-transitive lattice of G. Take $u \in A_1$ of order 3. By Lemma 16, u is conjugate to an element of $\mathcal{U}$, and so without loss of generality we may assume that $u \in \mathcal{U}$. And so $u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b \in \mathbb{F}_q((t^{-1}))$. Consider a sequence of elements $\{g_n\} \subseteq \mathcal{B}$ with $g_n := \begin{pmatrix} t^{-n} & 0 \\ 0 & t^n \end{pmatrix}$. Then $g_n u g_n^{-1} = \begin{pmatrix} 1 & t^{-2n}b \\ 0 & 1 \end{pmatrix}$. Clearly, as $n \to \infty$, $g_n u g_n^{-1} \to 1_G$. In particular, u^G is not closed, which contradicts Theorem 7 above. Hence Γ is not a cocompact lattice in G. We have now completed the proof of Proposition 20. □

A combination of Propositions 17 and 20 now completes the proof of Theorem 1′. □

4. Proof of the Main Result for Kac–Moody Groups

In this section we give a proof of our main result in its general setting. Let G be a topological Kac–Moody group of rank 2 with symmetric Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$, defined over a field $\mathbb{F} = \mathbb{F}_q$ with $q = p^a$, where p is a prime. Suppose Γ is a cocompact edge-transitive lattice in G with quotient a single edge. Since G naturally acts on its Bruhat–Tits building X, which in this case is a $(q+1)$–regular tree, we may apply Lemma 13 to conclude that $\Gamma = A_1 *_{A_0} A_2$ where A_1 and A_2 are finite groups of order divisible by $(q+1)$, since the edge group A_0 has index $(q+1)$ in both A_1 and A_2. Again the action of G on its Bruhat–Tits tree does not have to be faithful, and so we must take the kernel of the action $Z(G)$ into consideration. Thus what we are really looking for are finite subgroups A_i of G for $i = 0, 1, 2$, such that when we look at their images $\overline{A}_i$ in $G/Z(G)$, $\overline{A}_0$ has index $(q+1)$ in both $\overline{A}_1$ and $\overline{A}_2$, and both $\overline{A}_1$ and $\overline{A}_2$ are finite subgroups of $G/Z(G)$ of order divisible by $(q+1)$.

We first show in Section 4.1 below that Γ does not contain p–elements (Proposition 5 of the introduction). We will then use this to prove Proposition 40 below, which restricts the possible finite subgroups $\overline{A}_1$ and $\overline{A}_2$, in analogy with Proposition 14 for the case $G = SL_2(\mathbb{F}_q((t^{-1})))$ in Section 3 above.

4.1. **Cocompact lattices do not contain p–elements.** In the section we prove the following result, which was stated as Proposition 5 of the introduction.

Proposition 21. *If Γ is a cocompact lattice of G, Γ does not contain p–elements.*

Since $|Z(G)| \mid (q-1)^2$, while we are talking about p–elements, without loss of generality we may assume that $Z(G) = 1$, i.e., G is simple. To begin the proof, assume there exists $x \in \Gamma$ with $x^p = 1 \neq x$. Since x is an element of finite order, by the celebrated result of Serre (Proposition 6 above), x is contained in a parabolic/parahoric subgroup of G. Hence, without loss of generality we may suppose that $x \in \hat{B}_+$, a Borel (Iwahori) subgroup of G. In fact, as $\hat{B}_+ = H\hat{U}_+$ (see Proposition 9 above), we have $x \in \hat{U}_+$. We now, in Sections 4.1.1–4.1.3 below, prove the following important lemma:

Lemma 22. *Let x be a p–element of $\hat{U}_+$. Then x fixes an end of X.*

In Section 4.2 we will use Lemma 22 and its proof to establish a contradiction, and so complete the proof of Proposition 21 above.

4.1.1. *Real roots and the structure of U_+.* In this section we record several results concerning the real roots and associated root groups, and the structure of U_+.

Recall that the Weyl group W of G is the infinite dihedral group. The discussion in Section 1.4.2 above then implies that the set Φ^+ of positive real roots is the disjoint union of the sets

$$\Phi^1_+ := \{\alpha_1, w_1\alpha_2, w_1w_2\alpha_1, w_1w_2w_1\alpha_2, \ldots, (w_1w_2)^n\alpha_1, (w_1w_2)^n w_1\alpha_2, \ldots\}$$

and

$$\Phi^2_+ := \{\alpha_2, w_2\alpha_1, w_2w_1\alpha_2, w_2w_1w_2\alpha_1, \ldots, (w_2w_1)^n\alpha_2, (w_2w_1)^n w_2\alpha_1, \ldots\}.$$

Each real root may be identified with a half-apartment (half-line) of the standard apartment Σ. These identifications, for the positive real roots, are depicted in Figure 1 below.

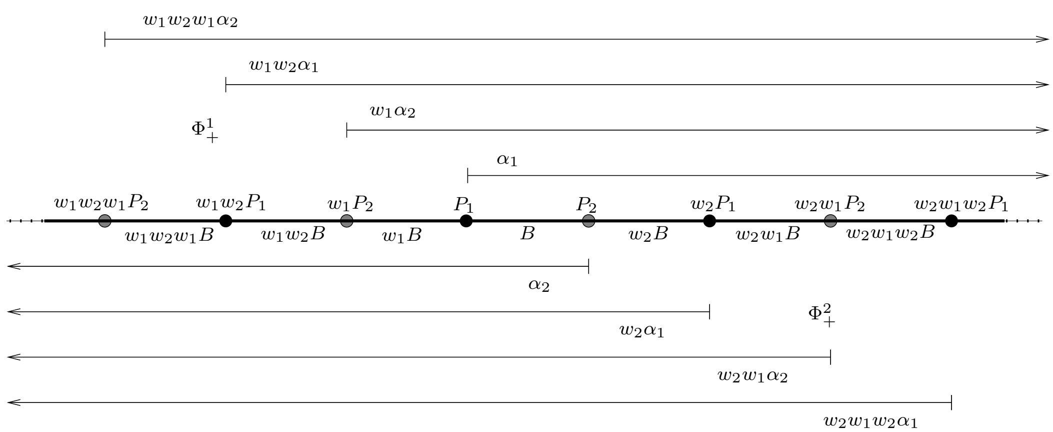

FIGURE 1. The sets of positive real roots Φ^1_+ and Φ^2_+, with each such root identified with a half-apartment of the standard apartment Σ.

We note that since the generalised Cartan matrix A is symmetric, for any two roots α and α' in Φ^1_+, the root groups U_α and $U_{\alpha'}$ commute. Similarly, for any two $\beta, \beta' \in \Phi^2_+$, U_β and U'_β commute. For $i = 1, 2$ let V_i be the abelian subgroup of U_+ defined by

$$V_i := \langle U_\alpha \mid \alpha \in \Phi^i_+ \rangle.$$

Lemma 23. *U_+ is the free product of V_1 and V_2.*

Proof. By Proposition 4 of Tits [24], the group U_+ is an amalgamated sum of V_1 and V_2. But $V_1 \cap V_2$ is trivial since there are no prenilpotent pairs of roots $\alpha \in \Phi^1_+$ and $\beta \in \Phi^2_+$. Hence $U_+ = V_1 * V_2$. □

For any positive integer n, denote by $(w_1, w_2; n)$ the element $w_1w_2w_1 \cdots$ of W which has n letters alternating between w_1 and w_2. Similarly, denote by $(w_2, w_1; n')$ the element $w_2w_1w_2 \cdots$ (n' letters). Put $(w_1, w_2; 0) = (w_2, w_1; 0) = 1$. Then every $w \in W$ is of the form $(w_1, w_2; n)$ or $(w_2, w_1; n')$ for some integer $n \geq 0$ or $n' \geq 0$. For $k \geq 0$ and $k' \geq 0$, define

$$i_k := \begin{cases} 1 & \text{if } k \text{ even} \\ 2 & \text{if } k \text{ odd} \end{cases} \quad \text{and} \quad i'_{k'} := \begin{cases} 2 & \text{if } k' \text{ even} \\ 1 & \text{if } k' \text{ odd.} \end{cases}$$

Now for $w = (w_1, w_2; n)$ and $w' = (w_2, w_1; n')$, define

$$U_w := \langle U_{(w_1,w_2;k)\alpha_{i_k}} \mid 0 \leq k \leq n \rangle \quad \text{and} \quad U_{w'} := \langle U_{(w_2,w_1;k')\alpha_{i'_{k'}}} \mid 0 \leq k' \leq n' \rangle.$$

The next result follows from Lemma 23 above.

Corollary 24. *If $w = (w_1, w_2; n)$ and $w' = (w_2, w_1; n')$, then $\langle U_w, U_{w'} \rangle = U_w * U_{w'}$.*

We also note that:

Lemma 25. *Let $w = (w_1, w_2; n)$. Then U_w is the direct product of the groups $U_{(w_1,w_2;k)\alpha_{i_k}}$, $0 \leq k \leq n$. Similarly for $w' = (w_2, w_1; n')$.*

4.1.2. *Action of root groups on X.* The group U_+ acts faithfully on X (Corollary on p. 34 of [7]). We now determine in some detail how the individual root groups act on the set of edges of the tree X.

We first introduce some convenient notation for the edges of X. We will say that an edge of X is a *left-hand edge* if it is closer to the vertex P_1 than to the vertex P_2, and a *right-hand edge* if it is closer to P_2 than to P_1. Then every edge of X except for B is either left-hand or right-hand. See Figure 2 below.

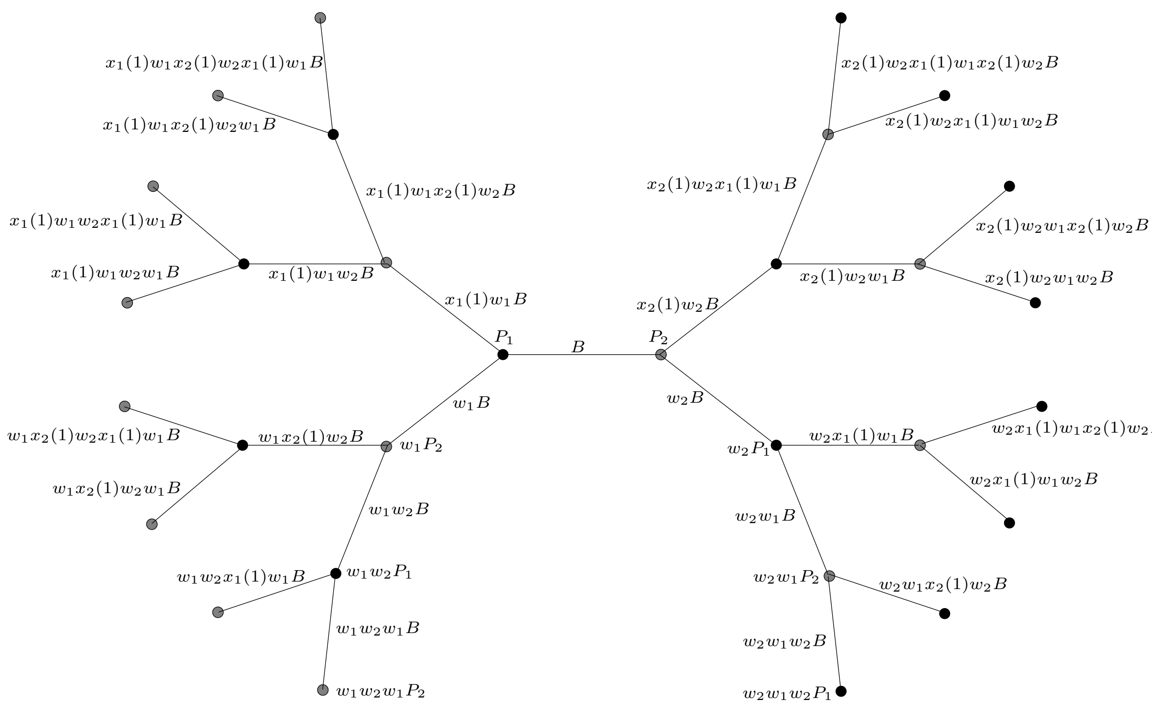

FIGURE 2. Ball$(B, 3)$ in the tree X, in the case $p = 2 = q$. The edges of X as well as the vertices of the standard apartment Σ are labelled.

By Lemma 9.1 of [9], the set of left-hand edges adjacent to the vertex P_1 is given by $\{x_1(l_1)w_1B \mid l_1 \in \mathbb{F}_q\}$. To simplify notation, we denote by (l_1) the left-hand edge $x_1(l_1)w_1B$. Similarly, the set of right-hand edges adjacent to P_2 is $\{x_2(r_1)w_2B \mid r_1 \in \mathbb{F}_q\}$, and we write (r_1) for $x_2(r_1)w_2B$.

By conjugating, for each $l_1 \in \mathbb{F}_q$ the set of left-hand edges in Ball$(B, 2)$ − Ball$(B, 1)$ which are adjacent to the edge $x_1(l_1)w_1B$ is given by $\{x_1(l_1)w_1x_2(l_2)w_2B \mid l_1, l_2 \in \mathbb{F}_q\}$. Denote by (l_1, l_2) the edge $x_1(l_1)w_1x_2(l_2)w_2B$. Similarly, denote by (r_1, r_2) the right-hand edge $x_2(r_1)w_2x_1(r_2)w_1B$, which is adjacent to $x_2(r_1)w_2B$.

Continuing in this way, for each integer $n \geq 1$ the set of left-hand edges in Ball(B, n) − Ball$(B, n-1)$ is the set $\{(l_1, \ldots, l_n) \mid l_j \in \mathbb{F}_q\}$, where $(l_1, \ldots, l_n)$ denotes the edge $x_1(l_1)w_1 \ldots x_i(l_n)w_iB$, with $i = 2$ if n is even and $i = 1$ if n is odd. Similarly for the set of right-hand edges in Ball(B, n) − Ball$(B, n-1)$. The standard apartment Σ then consists of the edge B together with all left-hand edges $(0, \ldots, 0)$ and all right-hand edges $(0, \ldots, 0)$.

Having established this notation, we will now show that root groups fix certain combinatorial balls in X. Lemma 26 below is a sharpening (for our case) of Lemmas 5 and 6 of Caprace–Rémy [8], which state that if α is a real root such that the distance from the base chamber B to the half-apartment determined by $-\alpha$ is at least $(4^{n+1} - 1)/3$, then the root group U_α fixes Ball$(B, n + 1)$.

Lemma 26. *Let wP_i be a vertex of the standard apartment Σ. If α is a real root such that α contains the vertex wP_i, and such that the distance from $-\alpha$ to the vertex wP_i is at least n, then U_α fixes* $\mathrm{Ball}(wP_i,n)$.

Proof. It suffices to show that if the distance from $-\alpha$ to wP_i is exactly n, then U_α fixes $\mathrm{Ball}(wP_i,n)$. For all real roots α, the root group U_α is a conjugate of either U_{α_1} or U_{α_2} by an element of W. Lemma 26 then follows from Lemma 27 below, which considers the case $\alpha=\alpha_1$. □

Lemma 27. *Let $n\geq 0$ be an integer, and put $i=2$ if n is even and $i=1$ if n is odd. Then U_{α_1} fixes* $\mathrm{Ball}((w_2,w_1;n)P_i,n+1)$.

Proof. The proof is by induction on n. We first show that U_{α_1} fixes $\mathrm{Ball}(P_2,1)$. Since U_{α_1} fixes the edges B and w_2B, it suffices to show that for all $t\in\mathbb{F}_q$ and for all $0\neq r_1\in\mathbb{F}_q$, $x_1(t)x_2(r_1)w_2B=x_2(r_1)w_2B$. For this, note that

$$w_2x_1(t)w_2=x_{w_2\alpha_1}(\epsilon t)\text{ and }w_2x_2(r_1)w_2=x_{-\alpha_2}(\epsilon' r_1)$$

for some $\epsilon,\epsilon'\in\{\pm1\}$. Hence $x_1(t)$ fixes $x_2(r_1)w_2B$ if and only if $x_{w_2\alpha_1}(\epsilon t)$ fixes $x_{-\alpha_2}(\epsilon' r_1)B$.

Since $r_1\neq 0$, the element $x_{-\alpha_2}(\epsilon' r_1)\in U_{-\alpha_2}$ does not fix any edge in $\mathrm{Ball}(P_2,1)$ except for w_2B. Thus $x_{-\alpha_2}(\epsilon' r_1)B$ is an edge in $\mathrm{Ball}(P_2,1)$ other than B and w_2B. Thus for some $0\neq r_1'\in\mathbb{F}_q$, we have $x_{-\alpha_2}(\epsilon' r_1)B=x_{\alpha_2}(r_1')w_2B$. We then compute

$$\begin{aligned}x_{w_2\alpha_1}(\epsilon t)x_{-\alpha_2}(\epsilon' r_1)B &= x_{w_2\alpha_1}(\epsilon t)x_{\alpha_2}(r_1')w_2B\\ &= x_{\alpha_2}(r_1')x_{w_2\alpha_1}(\epsilon t)w_2B\\ &= x_{\alpha_2}(r_1')w_2x_{\alpha_1}(t)B\\ &= x_{\alpha_2}(r_1')w_2B\\ &= x_{-\alpha_2}(\epsilon' r_1)B.\end{aligned}$$

Thus U_{α_1} fixes $\mathrm{Ball}(P_2,1)$.

Assume inductively that U_{α_1} fixes $\mathrm{Ball}((w_2,w_1;n)P_i,n+1)$, where $i=2$ if n is even and $i=1$ if n is odd. To prove that U_{α_1} fixes $\mathrm{Ball}((w_2,w_1;n+1)P_{3-i},n+2)$, suppose first that n is even, with $n=2k$ say. Then the inductive assumption is equivalent to the positive root group $(w_1w_2)^kU_{\alpha_1}(w_2w_1)^k=U_{(w_1w_2)^k\alpha_1}$ fixing $\mathrm{Ball}(P_2,n+1)$. We will show that $U_{(w_2w_1)^kw_2\alpha_1}$ fixes $\mathrm{Ball}(P_1,n+2)$, hence U_{α_1} fixes $\mathrm{Ball}((w_2,w_1;n+1)P_1,n+2)$ as required.

We first prove that $U_{(w_2w_1)^kw_2\alpha_1}$ fixes all left-hand edges $(l_1,\ldots,l_{n+2})$. Given $(l_1,\ldots,l_{n+2})$, there are constants $l_1',\ldots,l_{n+2}'\in\mathbb{F}_q$ such that

$$x_1(l_1)w_1x_2(l_2)w_2\cdots x_2(l_{n+2})w_2B=x_{-\alpha_1}(l_1')x_2(l_2')w_2x_1(l_3')w_1\cdots x_2(l_{n+2}')w_2B.$$

Now the edge $x_1(l_3')w_1\cdots x_2(l_{n+2}')w_2B$ is in $\mathrm{Ball}(P_2,n+1)$ hence by inductive assumption is fixed by $U_{(w_1w_2)^k\alpha_1}$. We then compute that for all $t\in\mathbb{F}_q$,

$$\begin{aligned}x_{(w_2w_1)^kw_2\alpha_1}(t)\cdot(l_1,l_2,\ldots,l_{n+2}) &= x_{(w_2w_1)^kw_2\alpha_1}(t)x_{-\alpha_1}(l_1')x_{\alpha_2}(l_2')w_2x_{\alpha_1}(l_3')w_1\cdots x_{\alpha_2}(l_{n+2}')w_2B\\ &= x_{-\alpha_1}(l_1')x_{\alpha_2}(l_2')x_{(w_2w_1)^kw_2\alpha_1}(t)w_2x_{\alpha_1}(l_3')w_1\cdots x_{\alpha_2}(l_{n+2}')w_2B\\ &= x_{-\alpha_1}(l_1')x_{\alpha_2}(l_2')w_2x_{(w_1w_2)^k\alpha_1}(\epsilon t)x_{\alpha_1}(l_3')w_1\cdots x_{\alpha_2}(l_{n+2}')w_2B\\ &= x_{-\alpha_1}(l_1')x_{\alpha_2}(l_2')w_2x_{\alpha_1}(l_3')w_1\cdots x_{\alpha_2}(l_{n+2}')w_2B\\ &= (l_1,l_2\ldots,l_{n+2}).\end{aligned}$$

Thus $U_{(w_2w_1)^kw_2\alpha_1}$ fixes all left-hand edges $(l_1,\ldots,l_{n+2})$.

We now show that $U_{(w_2w_1)^kw_2\alpha_1}$ fixes all right-hand edges $(r_1,\ldots,r_{n+1})$. For this, the inductive assumption implies that for all $t\in\mathbb{F}_q$,

$$\begin{aligned}x_{(w_2w_1)^kw_2\alpha_1}(t)\cdot(r_1,\ldots,r_{n+1}) &= x_{(w_2w_1)^kw_2\alpha_1}(t)x_{\alpha_2}(r_1)w_2\cdots x_{\alpha_2}(r_{n+1})w_2B\\ &= x_{\alpha_2}(r_1)x_{(w_2w_1)^kw_2\alpha_1}(t)w_2x_{\alpha_1}(r_2)w_1\cdots x_{\alpha_2}(r_{n+1})w_2B\\ &= x_{\alpha_2}(r_1)w_2x_{(w_1w_2)^k\alpha_1}(\epsilon t)x_{\alpha_1}(r_2)w_1\cdots x_{\alpha_2}(r_{n+1})w_2B\\ &= x_{\alpha_2}(r_1)w_2x_{\alpha_1}(r_2)w_1\cdots x_{\alpha_2}(r_{n+1})w_2B\\ &= (r_1,\ldots,r_{n+1}).\end{aligned}$$

Thus $U_{(w_2w_1)^k w_2\alpha_1}$ fixes all right-hand edges $(r_1,\dots,r_{n+1})$.

Since $U_{(w_2w_1)^k w_2\alpha_1}$ fixes all left-hand edges $(l_1,\dots,l_{n+2})$ and all right-hand edges $(r_1,\dots,r_{n+1})$, it fixes $\mathrm{Ball}(P_1,n+2)$. This completes the proof of the inductive step for n even. The proof of the inductive step for n odd is similar. This completes the proof that U_{α_1} fixes $\mathrm{Ball}((w_2,w_1;n)P_i,n+1)$ for all $n\geq 0$. □

Now that we have determined that certain combinatorial balls are fixed by root groups, we consider how these root groups act elsewhere on the tree X. For this, we first discuss how the root groups U_α for $\alpha\in\Phi^1_+$ act on left-hand edges, and how the root groups U_β for $\beta\in\Phi^2_+$ act on right-hand edges.

Lemma 28. *Let $(l_1,\dots,l_n)$ be a left-hand edge. Then for all $0\leq k\leq n-1$, there is an $\epsilon\in\{\pm1\}$ such that for all $t\in\mathbb{F}_q$,*

$$x_{(w_1,w_2;k)\alpha_{i_k}}(t)\cdot(l_1,\dots,l_n)=(l_1,\dots,l_k,l_{k+1}+\epsilon t,l_{k+2},\dots,l_n).$$

Similarly for the action of $U_{(w_2,w_1;k)\alpha_{i'_{k'}}}$ on right-hand edges.

Proof. It suffices to show by induction on $k\geq 0$ that for any $l_1,\dots,l_k\in\mathbb{F}_q$,

$$x_{(w_1,w_2;k)\alpha_{i_k}}(t)x_1(l_1)w_1x_2(l_2)w_2\cdots x_{i_k}(l_k)=x_1(l_1)w_1x_2(l_2)w_2\cdots x_{i_k}(l_{k+1}+\epsilon t). \tag{3}$$

In the case $k=0$, $x_1(t)x_1(l_1)=x_1(l_1+t)$ and we are done. For $k=1$, since U_{α_1} and $U_{w_1\alpha_2}$ commute, there is an $\epsilon\in\{\pm1\}$ such that

$$\begin{aligned}x_{w_1\alpha_2}(t)x_1(l_1)w_1x_2(l_2)&=x_1(l_1)x_{w_1\alpha_2}(t)w_1x_2(l_2)\\&=x_1(l_1)w_1x_2(\epsilon t)x_2(l_2)\\&=x_1(l_1)w_1x_2(l_2+\epsilon t).\end{aligned}$$

For $k\geq 2$ we compute that for some $\epsilon,\epsilon'\in\{\pm1\}$,

$$\begin{aligned}x_{(w_1,w_2;k)\alpha_{i_k}}(t)x_1(l_1)w_1x_2(l_2)w_2&=w_1x_{(w_2,w_1;k-1)\alpha_{i'_{(k-1)'}}}(\epsilon t)w_1x_1(l_1)w_1x_2(l_2)w_2\\&=w_1x_{(w_2,w_1;k-1)\alpha_{i'_{(k-1)'}}}(\epsilon t)x_{-1}(\epsilon' l_1)x_2(l_2)w_2\\&=w_1x_{-1}(\epsilon' l_1)x_2(l_2)x_{(w_2,w_1;k-1)\alpha_{i'_{(k-1)'}}}(\epsilon t)w_2\\&=x_1(l_1)w_1x_2(l_2)w_2x_{(w_1,w_2;k-2)\alpha_{i_{k-2}}}(t).\end{aligned}$$

The result then follows by induction. □

We now describe the action on certain left-hand edges of certain root groups U_β, where $\beta\in\Phi^2_+$. We first discuss the action of U_{α_2} on left-hand edges (l_1,l_2). For this, the following formula will be needed.

Lemma 29. *For $a,t\in\mathbb{F}_q$ with $a\neq 0$, the following statement holds:*

$$x_1(a)w_1x_2(t)w_2B=x_{-1}(a^{-1})x_2((-a)^{-m}t)w_2B.$$

Here $-m$ is the off-diagonal entry in the generalised Cartan matrix A for G.

Proof. To show that $x_1(a)w_1x_2(t)w_2B=x_{-1}(a^{-1})x_2((-a)^{-m}t)w_2B$ is equivalent to showing that

$$w_2x_2(-(-a)^{-m}t)x_{-1}(-a^{-1})x_1(a)w_1x_2(t)w_2\in B.\quad(*)$$

Now, denote by $x:=x_{-1}(-a^{-1})x_1(a)w_1$. Clearly $x\in L_1$, and in fact, $x\in M_1$. As M_1 is a homomorphic image of $SL_2(q)$, i.e., is isomorphic to either $SL_2(q)$ or to $PSL_2(q)$ under the natural identification $\begin{pmatrix}1&r\\0&1\end{pmatrix}\to x_1(r)$, $\begin{pmatrix}1&0\\r&1\end{pmatrix}\to x_{-1}(r)$ and $\begin{pmatrix}s&0\\0&s^{-1}\end{pmatrix}\to h_1(s)$, by explicit calculation we obtain that $x=x_1(-a)h_1(-a)$.

Thus $(*)$ is equivalent to proving that

$$w_2x_2(-(-a)^{-m}t)x_1(-a)h_1(-a)x_2(t)w_2\in B\quad(**)$$

But $h_1(-a)x_2(t)w_2 = h_1(-a)x_2(t)h_1(-a)^{-1}h_1(-a)w_2 = x_2((-a)^{-m}t)h_1(-a)w_2 = x_2((-a)^{-m}t)w_2h$ for some $h \in H$. Since $H \leq B$, $(**)$ is equivalent to

$$w_2x_2(-(-a)^{-m}t)x_1(-a)x_2((-a)^{-m}t)w_2 \in B$$

which is the same as

$$x_2(-(-a)^{-m}t)x_1(-a)x_2((-a)^{-m}t) = x_1(-a)^{x_2(-(-a)^{-m}t)} \in B^{w_2}.$$

Notice that $x_1(-a) \in U_2$, while $x_2(-(-a)^{-m}t) \in P_2$. But $U_2 \triangleleft P_2$, and so $x_1(-a)^{x_2(-(-a)^{-m}t)} \in U_2$. Therefore it remains to show that $U_2 \leq B^{w_2}$. Now, $B = HU_+$, and so $B^{w_2} = H^{w_2}U_+^{w_2}$. Hence if we can show that $U_2 \leq U_+^{w_2}$, we will be done. Recall that $U_+ = \langle U_\alpha \mid \alpha \in \Phi_+^1 \cup \Phi_+^2 \rangle$. Therefore

$$U_+^{w_2} = \langle U_\alpha \mid \alpha \in w_2(\Phi_+^1 \cup \Phi_+^2)\rangle = \langle U_\alpha \mid \alpha \in \Phi_+^1 \cup (\Phi_+^2 - \{\alpha_2\}) \cup \{-\alpha_2\}\rangle,$$

i.e., $U_+^{w_2} = U_2U_{-\alpha_2}$ which finishes the proof. □

We may now describe the action of U_{α_2} on the set of left-hand edges (l_1, l_2):

Corollary 30. *There is an automorphism ϕ of the additive group $(\mathbb{F}_q, +)$ such that for all $t \in F_q$ and all left-hand edges (l_1, l_2),*

$$x_2(t) \cdot (l_1, l_2) = \begin{cases} (l_1, l_2 + \phi(t)) & \text{if } l_1 \neq 0 \\ (l_1, l_2) & \text{if } l_1 = 0. \end{cases}$$

Proof. Consider $x_2(t) \cdot (l_1, l_2)$, where $l_1 \neq 0$. As $(l_1, l_2) = x_1(l_1)w_1x_2(l_2)w_2B$, using the previous lemma we have

$$x_2(t)x_1(l_1)w_1x_2(l_2)w_2B = x_2(t)x_{-\alpha_1}(l_1^{-1})x_2((-l_1)^{-m}l_2)w_2B = x_{-\alpha_1}(l_1^{-1})x_2(t + (-l_1)^{-m}l_2)w_2B.$$

Again using the previous lemma, we obtain that

$$x_{-\alpha_1}(l_1^{-1})x_2(t + (-l_1)^{-m}l_2)w_2B \quad = \quad x_1(l_1)w_1x_2(l_2 + (-l_1)^m t)w_2B.$$

Since for each $l_1 \in \mathbb{F}_q^*$, the map $\phi(t) := (-l_1)^m t$ is an automorphism of the additive group $(\mathbb{F}_q, +)$ as required, we obtain the desired result. The fact that $x_2(t)$ fixes $(0, l_2)$ is a consequence of Lemma 26 above. □

Corollary 31. *The action of $U_{w_1\alpha_2}$ on the set of left-hand edges $\{(l_1, l_2)\}$ commutes with that of U_{α_2}.*

Proof. By Lemma 26 above, the group U_{α_2} fixes each edge $(0, l_2)$. The corollary then follows by Lemmas 28 and Corollary 30 above. □

We will in fact need the following generalisation of Corollary 30 above. The proof of the following lemma is similar to that of Corollary 30.

Lemma 32. *For each $n \geq 0$ and $0 \leq k \leq n$, let $N = n + (n + 1 + k)$ and $\alpha' = (w_2, w_1; k)\alpha_{i'}$, where $i' = 2$ if k is even and $i' = 1$ if k is odd. Then there is an automorphism ϕ of the additive group $(\mathbb{F}_q, +)$ such that for all $t \in \mathbb{F}_q$ and all left-hand edges $(0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1})$,*

$$x_{\alpha'}(t) \cdot (0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1}) = \begin{cases} (0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1} + \phi(t)) & \text{if } l_{n+1} \neq 0 \\ (0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1}) & \text{if } l_{n+1} = 0. \end{cases}$$

Corollary 33. *In the notation of Lemma 32 above, the action of $U_{\alpha'}$ on the set of left-hand edges*

$$\{(0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1}) \mid l_j \in \mathbb{F}_q\}$$

commutes with that of each root group $U_{(w_1,w_2;j)\alpha_{i_j}}$, for $n < j \leq N$.

4.1.3. *Proof of Lemma 22.* We are now ready to prove Lemma 22 above, which says that a p–element of $\hat{U}_+$ must fix an end of the tree X. Let $x \in \hat{U}_+$ with $x^p = 1 \neq x$. Assume by contradiction that x does not fix any end of X. Then there is a smallest integer $n \geq 0$ such that no edge in $\mathrm{Ball}(B, n+1) - \mathrm{Ball}(B, n)$ is fixed by x. We first consider the case $n = 0$, and then generalise the argument for this case to the cases $n \geq 1$.

Suppose $n = 0$. Then every edge in $\mathrm{Ball}(B, 1)$ except for B is moved by x. For $i = 1, 2$, consider the restriction of x to $\mathrm{Ball}(P_i, 1)$. By Lemma 26 above, the only positive root group which acts nontrivially on $\mathrm{Ball}(P_i, 1)$ is U_{α_i}. Hence for $i = 1, 2$, there is a $0 \neq t_i \in \mathbb{F}_q$ such that

$$x|_{\mathrm{Ball}(P_i,1)} = x_i(t_i)|_{\mathrm{Ball}(P_i,1)}.$$

Now consider the restriction of x to $\mathrm{Ball}(B, 2)$. By Lemma 26 above,

$$x|_{\mathrm{Ball}(B,2)} = y|_{\mathrm{Ball}(B,2)} \text{ for some } y \in \langle U_{\alpha_1}, U_{\alpha_2}, U_{w_1\alpha_2}, U_{w_2\alpha_1}\rangle.$$

By Lemma 24 above,

$$\langle U_{\alpha_1}, U_{\alpha_2}, U_{w_1\alpha_2}, U_{w_2\alpha_1}\rangle = \langle U_{\alpha_1}, U_{w_1\alpha_2}\rangle * \langle U_{\alpha_2}, U_{w_2\alpha_1}\rangle = U_{w_1} * U_{w_2}.$$

Hence the element y can be written uniquely as a word in letters alternating between nontrivial elements of U_{w_1} and nontrivial elements of U_{w_2}. Moreover, for $i = 1, 2$ by Lemma 25 above each nontrivial element of U_{w_i} can be written uniquely as a product $x_i(s_i)x_{w_i\alpha_{3-i}}(s_i')$, with $s_i, s_i' \in \mathbb{F}_q$ and at least one of s_i and s_i' nonzero. Thus there is a canonical word for y with letters in U_{α_1}, U_{α_2}, $U_{w_1\alpha_2}$ and $U_{w_2\alpha_1}$.

Let $z \in U_{\alpha_1} * U_{\alpha_2}$ be the element obtained by deleting from this canonical word for y all nontrivial elements of $U_{w_1\alpha_2}$ and $U_{w_2\alpha_1}$. Note that z is well-defined, and that z can be written uniquely as a word

$$z = x_1(t_{1,1})x_2(t_{2,1})x_1(t_{1,2})x_2(t_{2,2}) \cdots x_1(t_{1,m})x_2(t_{2,m})$$

where for $i = 1, 2$ and $1 \leq j \leq m$ we have $t_{i,j} \in \mathbb{F}_q$, with possibly $t_{1,1} = 0$ and possibly $t_{2,m} = 0$, but all other $t_{i,j} \neq 0$. By definition of the elements y and z, and using Lemma 26 above again, for $i = 1, 2$

$$z|_{\mathrm{Ball}(P_i,1)} = y|_{\mathrm{Ball}(P_i,1)} = x|_{\mathrm{Ball}(P_i,1)} = x_i(t_i)|_{\mathrm{Ball}(P_i,1)}.$$

Hence for $i = 1, 2$

$$t_{i,1} + t_{i,2} + \cdots + t_{i,m} = t_i \neq 0.$$

Denote the set of left-hand edges in $\mathrm{Ball}(B, 2) - \mathrm{Ball}(B, 1)$ by

$$E := \{(l_1, l_2) \mid l_1, l_2 \in \mathbb{F}_q\}.$$

By Lemma 26, the root group $U_{w_2\alpha_1}$ fixes each edge in E. The root group $U_{w_1\alpha_2}$ commutes with U_{α_1}. By Corollary 31 above, the action of $U_{w_1\alpha_2}$ on E commutes with the action of U_{α_2} on E. Therefore there is a $t \in \mathbb{F}_q$ such that

$$x|_E = y|_E = x_{w_1\alpha_2}(t)z|_E.$$

Moreover,

$$x^p|_E = \mathrm{id}_E = (x_{w_1\alpha_2}(t))^p z^p|_E = z^p|_E.$$

That is, z^p fixes each edge in E. The following lemma shows that this is impossible, and we thus obtain a contradiction. Hence in the case $n = 0$, x must fix an end of X.

Lemma 34. *Let*

$$z = x_1(t_{1,1})x_2(t_{2,1})x_1(t_{1,2})x_2(t_{2,2}) \cdots x_1(t_{1,m})x_2(t_{2,m})$$

as above, with

$$t_{i,1} + t_{i,2} + \cdots + t_{i,m} = t_i$$

for $i = 1, 2$. Then z^p fixes each edge in the set of left-hand edges

$$E = \{(l_1, l_2) \mid l_1, l_2 \in \mathbb{F}_q\}$$

if and only if $t_2 = 0$.

Proof. Let $(l_1, l_2) \in E$. We first compute $z \cdot (l_1, l_2)$. To simplify notation, we assume that the automorphism ϕ in Corollary 30 above is the identity. For each $1 \leq j \leq m$, let $(L_{1,j}, L_{2,j}) \in E$ be the edge

$$(L_{1,j}, L_{2,j}) = x_1(t_{1,j})x_2(t_{2,j}) \cdots x_1(t_{1,m})x_2(t_{2,m}) \cdot (l_1, l_2).$$

Then for $1 \leq j \leq m$,

$$L_{1,j} = t_{1,j} + \cdots + t_{1,m} + l_1.$$

Putting $L_{1,m+1} = l_1$ and $L_{2,m+1} = l_2$, for $1 \leq j \leq m$

$$L_{2,j} = \begin{cases} L_{2,j+1} & \text{if } L_{1,j+1} = 0 \\ t_{2,j} + L_{2,j+1} & \text{if } L_{1,j+1} \neq 0. \end{cases}$$

Defining $t_{m+1} = 0$, it follows that for $1 \leq j \leq m$,

$$L_{2,j} = \begin{cases} L_{2,j+1} & \text{if } t_{1,j+1} + \cdots + t_{1,m} = -l_1 \\ t_{2,j} + L_{2,j+1} & \text{if } t_{1,j+1} + \cdots + t_{1,m} \neq -l_1. \end{cases}$$

Thus

$$z \cdot (l_1, l_2) = (L_{1,1}, L_{2,1}) = (t_{1,1} + \cdots + t_{1,m} + l_1, L_{2,1}) = (l_1 + t_1, L_{2,1})$$

where

$$L_{2,1} = l_2 + t_2 - \left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -l_1} t_{2,j} \right).$$

Using this computation and the fact that $pt_1 = pt_2 = 0$, we determine that for all $(l_1, l_2) \in E$,

$$z^p \cdot (l_1, l_2) = (l_1, l_2')$$

where

$$l_2' = l_2 - \left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -l_1} t_{2,j} + \sum_{t_{1,j+1} + \cdots + t_{1,m} = -(l_1 + t_1)} t_{2,j} + \cdots + \sum_{t_{1,j+1} + \cdots + t_{1,m} = -(l_1 + (p-1)t_1)} t_{2,j} \right).$$

Now z^p fixes each edge in E if and only if $l_2' = l_2$. In this case, for each $l_1 \in \mathbb{F}_q$,

$$\left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -l_1} t_{2,j} \right) + \left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -(l_1 + t_1)} t_{2,j} \right) + \cdots + \left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -(l_1 + (p-1)t_1)} t_{2,j} \right) = 0.$$

Recall that $q = p^a$, and choose a set of representatives $\{l_{1,1}, \ldots, l_{1,a}\} \subset \mathbb{F}_q$ of the orbits of the map $l \mapsto l + t_1$. By adding together the previous equation for each of $l_1 = l_{1,1}, \ldots, l_1 = l_{1,a}$, we obtain

$$\sum_{l \in \mathbb{F}_q} \left(\sum_{t_{1,j+1} + \cdots + t_{1,m} = -l} t_{2,j} \right) = 0.$$

Each sum $t_{1,j+1} + \cdots + t_{1,m}$ appears exactly once on the left-hand side, therefore z^p fixes each edge in E if and only if

$$t_{2,1} + \cdots + t_{2,m} = 0.$$

But $t_{2,1} + \cdots + t_{2,m} = t_2$, so we are done. □

Before considering the cases $n \geq 1$, we note the following corollary to Lemma 34 above.

Corollary 35. *Let z be as in Lemma 34 above. Then z^p fixes* $\mathrm{Ball}(B, 2)$ *if and only if* $t_1 = t_2 = 0$.

Proof. Since $pt_1 = pt_2 = 0$, by considering the action of U_{α_1} and U_{α_2} on $\mathrm{Ball}(B, 1)$ we see that z^p fixes $\mathrm{Ball}(B, 1)$ for all values of t_1 and t_2. A similar argument to that used for Lemma 34 above shows that z^p fixes the set of right-hand edges

$$\{(r_1, r_2) \mid r_1, r_2 \in \mathbb{F}_q\}$$

if and only if $t_1 = 0$. Hence z^p fixes the set of edges in $\mathrm{Ball}(B, 2) - \mathrm{Ball}(B, 1)$ if and only if $t_1 = t_2 = 0$, as required. □

We now generalise the argument for $n = 0$ to the cases $n \geq 1$. Recall that we chose n to be the smallest integer such that no edge in $\mathrm{Ball}(B, n+1) - \mathrm{Ball}(B, n)$ is fixed by x. By the minimality of n and the assumption $n \geq 1$, there is an edge in $\mathrm{Ball}(B, n) - \mathrm{Ball}(B, n-1)$ which is fixed by x. The next lemma follows from the proof of Lemma 14.1 in [9].

Lemma 36. *For any left-hand edge $(l_1, \ldots, l_n)$ in X, there is an element $u \in U_+$ such that $u \cdot (l_1, \ldots, l_n)$ is the left-hand edge $(0, \ldots, 0)$. For any right-hand edge $(r_1, \ldots, r_n)$ in X, there is a $v \in U_+$ such that $v \cdot (r_1, \ldots, r_n)$ is the right-hand edge $(0, \ldots, 0)$. Moreover, v can be chosen to fix all left-hand edges in the standard apartment Σ.*

Using Lemma 36, we may assume without loss of generality that x fixes the left-hand edge $(l_1, \ldots, l_n) = (0, \ldots, 0)$. Let $wP_i = (w_1, w_2; n)P_i$ be the vertex of Σ at distance n from P_1 on the left-hand side, and let $\alpha = (w_1, w_2; n)\alpha_i$ (here $i = 1$ if n is even and $i = 2$ if n is odd). Then by Lemma 26 above,

$$x|_{\mathrm{Ball}(wP_i,1)} = x_\alpha(t)|_{\mathrm{Ball}(wP_i,1)} \text{ for some } 0 \neq t \in \mathbb{F}_q.$$

Let $0 \leq k \leq n$ be the largest integer such that x fixes a right-hand edge in $\mathrm{Ball}(B, k) - \mathrm{Ball}(B, k-1)$. If $k \geq 1$ then by Lemma 36 above, we may assume that x fixes the right-hand edge $(0, \ldots, 0)$ in $\mathrm{Ball}(B, k) - \mathrm{Ball}(B, k-1)$. Let $w'P_{i'} = (w_2, w_1; k)P_{i'}$ be the vertex of Σ at distance k from P_2 on the right-hand side, and let $\alpha' = (w_2, w_1; k)\alpha_{i'}$ (here $i' = 2$ if k is even and $i' = 1$ if k is odd). Then by Lemma 26 above,

$$x|_{\mathrm{Ball}(w'P_{i'},1)} = x_{\alpha'}(t')|_{\mathrm{Ball}(w'P_{i'},1)} \text{ for some } 0 \neq t' \in \mathbb{F}_q.$$

Note that the distance from $wP_i = (w_1, w_2; n)P_i$ to $w'P_{i'} = (w_2, w_1; k)P_{i'}$ is exactly $n + 1 + k$. Let $N = n + (n + 1 + k)$, and consider the restriction of x to $\mathrm{Ball}(B, N + 1)$. (In the case $n = 0$ above, $k = 0$ and thus $N = 1$.) Then as in the case $n = 0$ above,

$$x|_{\mathrm{Ball}(B,N+1)} = y|_{\mathrm{Ball}(B,N+1)} \text{ for some } y \in U_{(w_1,w_2;N)} * U_{(w_2,w_1;N)}.$$

Note that $U_\alpha \leq U_{(w_1,w_2;N)}$ and $U_{\alpha'} \leq U_{(w_2,w_1;N)}$. Let $z \in U_\alpha * U_{\alpha'}$ be the element obtained by deleting from the word for y all letters except those in U_α or $U_{\alpha'}$. By definition of the elements y and z, and using Lemma 26 above again,

$$z|_{\mathrm{Ball}(wP_i,1)} = y|_{\mathrm{Ball}(wP_i,1)} = x|_{\mathrm{Ball}(P_i,1)} = x_\alpha(t)|_{\mathrm{Ball}(wP_i,1)}$$

and similarly for $\mathrm{Ball}(w'P_{i'}, 1)$ and $x_{\alpha'}(t')$.

Consider the following set of left-hand edges in $\mathrm{Ball}(B, N + 1) - \mathrm{Ball}(B, N)$:

$$E := \{(0, \ldots, 0, l_{n+1}, l_{n+2}, \ldots, l_N, l_{N+1}) \mid l_j \in \mathbb{F}_q\}.$$

Since x fixes the left-hand edge $(l_1, \ldots, l_n) = (0, \ldots, 0)$, the element x preserves E. Moreover, since x fixes the left-hand edge $(l_1, \ldots, l_n) = (0, \ldots, 0)$, which is *not* fixed by each of $U_{\alpha_1}, U_{w_1\alpha_2}, \ldots, U_{(w_1,w_2;n-1)\alpha_{i_{n-1}}}$,

$$x|_E = y'|_E \text{ for some } y' \in \langle U_{(w_1,w_2;n)\alpha_i}, \ldots, U_{(w_1,w_2;N)\alpha_{i_N}}\rangle * U_{(w_2,w_1;N)}.$$

Also, x fixes the right-hand edge $(0, \ldots, 0)$, which is *not* fixed by each of $U_{\alpha_2}, U_{w_2\alpha_1}, \ldots, U_{(w_2,w_1;k-1)\alpha_{i'_{k-1}}}$, so

$$x|_E = y''|_E \text{ for some } y'' \in \langle U_{(w_1,w_2;n)\alpha_i}, \ldots, U_{(w_1,w_2;N)\alpha_{i_N}}\rangle * \langle U_{(w_2,w_1;k)\alpha_{i'_k}}, \ldots, U_{(w_2,w_1;N)\alpha_{i'_N}}\rangle.$$

Now by Lemma 26 above, for all $j \geq k+1$ the root group $U_{(w_2,w_1;j)\alpha_{i'_j}}$ fixes $\mathrm{Ball}(wP_i, n+1+(k+1))$. Thus for all $j \geq k + 1$, the root group $U_{(w_2,w_1;j)\alpha_{i'_j}}$ fixes each edge in E. Hence, recalling that $U_{\alpha'} = U_{(w_2,w_1;k)\alpha_{i'}}$, we have that

$$x|_E = y'''|_E \text{ for some } y''' \in \langle U_{(w_1,w_2;n)\alpha_i}, \ldots, U_{(w_1,w_2;N)\alpha_{i_N}}\rangle * U_{\alpha'}.$$

Next, for each $n < j \leq N$, the root group $U_{(w_1,w_2;j)\alpha_{i_j}}$ commutes with $U_\alpha = U_{(w_1,w_2;n)\alpha_i}$. By Corollary 33 above, for each $n < j \leq N$ the action of the root group $U_{(w_1,w_2;j)\alpha_{i_j}}$ on E commutes with that of $U_{\alpha'}$ on E.

We conclude that there is a p–element $z' \in U_+$, which commutes with z on E, such that

$$x|_E = y|_E = y'|_E = y''|_E = y'''|_E = z'z|_E,$$

hence

$$x^p|_E = \mathrm{id}_E = (z')^p z^p|_E = z^p|_E.$$

That is, z^p fixes each edge in E. The proof that this is impossible with $t' \neq 0$ is similar to the case $n = 0$ above.

This completes the proof of Lemma 22. That is, we have shown that a p–element of $\hat{U}_+$ must fix an end of the tree X.

4.1.4. *Completing the proof of Proposition 21.* We now show how Lemma 22 may be used to complete the proof of Proposition 21. Since each real root α determines a half-line (half-apartment) in the standard apartment Σ of X, each real root α determines one of two possible ends of X. Denote by e_1 the end of X determined by α_1, and by e_2 the end of X determined by α_2. Then by definition of Φ^1_+ and Φ^2_+, each root in Φ^1_+ determines the end e_1, and each root in Φ^2_+ the end e_2. Define the group $-V_2$ by

$$-V_2 := \langle U_\alpha \mid -\alpha \in \Phi^2_+ \rangle.$$

Then the groups V_1 and $-V_2$ fix the end e_1.

Following the notation in Section 14 of [9], we put

$$\mathcal{U} := \widehat{V_1 \cup -V_2}$$

and

$$\mathcal{B}_{\mathcal{I}} := \bigcap_{w \in W} w\hat{B}w^{-1}.$$

Note that $\mathcal{B}_{\mathcal{I}}$ fixes the standard apartment Σ pointwise.

Now take $g \in G$ such that g induces the element $\tau := w_1 w_2 \in W$. The element τ acts as a translation along the standard apartment Σ, with translation length two edges (chambers), and with attracting fixed point e_1 and repelling fixed point e_2. Let R be the subgroup of G generated by g (in [9], the group generated by g was called T, but for us T is already a fixed maximal split torus). Finally define $\mathcal{B}$ be the stabiliser in G of the end e_1.

Proposition 37 (Theorem 14.1, [9])**.** *The group R normalises both $\mathcal{U}$ and $\mathcal{B}_{\mathcal{I}}$, and*

$$\mathcal{B} = \mathcal{B}_{\mathcal{I}}\mathcal{U}R = \mathcal{B}_{\mathcal{I}}R\mathcal{U} = \mathcal{U}R\mathcal{B}_{\mathcal{I}} = \mathcal{U}\mathcal{B}_{\mathcal{I}}R.$$

Proof. Although Carbone–Garland used a different completion of the Kac–Moody group Λ, their proof goes through in the building topology. □

We next consider the structure of $\mathcal{B}_{\mathcal{I}}$.

Lemma 38. *Let $b \in \mathcal{B}_{\mathcal{I}}$ be an element of finite order. Then the order of b divides $p - 1$.*

Proof. From the definition of $\mathcal{B}_{\mathcal{I}}$, we have $b \in \hat{B}^w$ for every $w \in W$. By Proposition 9 above, $\hat{B} = T \ltimes \hat{U}_+$. For all $w \in W$, $\hat{B}^w = T \ltimes \hat{U}_+^w$ as W normalises T. Thus $b \in T \ltimes \hat{U}_+^w$ for every $w \in W$. As $|T| \mid (q-1)^2$ and $\exp(T) = q - 1$, $y := b^{p-1} \in \hat{U}_+^w$ for every $w \in W$. In particular, y is an element of $\hat{U}_+$ which fixes the standard apartment Σ. Since for every $u \in \hat{U}_+$ either u is a p–element, or its order is infinite, it remains to show that y is not a p–element.

Assume by contradiction that $y^p = 1 \neq y$. Since y is in $\hat{U}_+$, there is a sequence of elements y_n in U_+ such that $\lim_{n\to\infty} y_n = y$. Without loss of generality, we may assume that y_n agrees with y on $\mathrm{Ball}(B, n)$. Then by Lemma 26 above, without loss of generality, y_n is an element of the free product of

$$U_{(w_1,w_2;n-1)} = \langle U_{\alpha_1}, U_{w_1\alpha_2}, \ldots, U_{(w_1,w_2;n-1)\alpha_{i_{n-1}}} \rangle$$

and

$$U_{(w_2,w_1;n-1)} = \langle U_{\alpha_2}, U_{w_2\alpha_1}, \ldots, U_{(w_2,w_1;n-1)\alpha_{i'_{n-1}}} \rangle.$$

Since $y^p = 1$, y_n^p fixes $\mathrm{Ball}(B, n)$. Put $z_n = y_n^p$. We claim that $z_n = 1$. We first consider the case $n = 1$. Then $y_1 \in U_{\alpha_1} * U_{\alpha_2}$, so y_1 has the same form as the element z in Lemma 34 above. Now y_1 fixes the intersection of the standard apartment with $\mathrm{Ball}(B, 1)$, so we have $t_1 = t_2 = 0$ in this case. Hence by

Corollary 35 above, the element $z_1 = y_1^p$ fixes $\mathrm{Ball}(B,2)$. But by Lemma 28 and Corollary 30 above, if the element z_1 of $U_{\alpha_1} * U_{\alpha_2}$ fixes every edge in $\mathrm{Ball}(B,2)$, then z_1 fixes every edge in X. Thus, as $\hat{U}_+$ acts faithfully, $z_1 = 1$. The proof that $z_n = 1$ for $n > 1$ is similar, using the analogous results for actions of root groups on larger balls in X.

We now have that $1 = z_n = y_n^p$, so each y_n is a p–element of U_+. By Lemma 23 above, U_+ is the free product of V_1 and V_2, hence any finite order element of U_+ is contained in a U_+–conjugate of either V_1 or V_2. Without loss of generality, pass to a subsequence of the y_n so that each y_n is an element of $u_n V_1 u_n^{-1}$ for some $u_n \in U_+$. Now y_n fixes the intersection of $\mathrm{Ball}(B,n)$ with the standard apartment Σ. Thus the root groups in $u_n V_1 u_n^{-1}$ include the first n root groups in V_1. Therefore y is in $\hat{V}_1$. But by the definition of the building topology, if $y \in \hat{V}_1$ fixes the standard apartment, then $y = 1$, a contradiction. □

Now consider our element $x \in \hat{U}_+ \cap \Gamma$ of order p. By Lemma 22 above, x fixes an end of X. By Lemma 36 above, the completed group $\hat{U}_+$ acts transitively on the set of left-hand ends of X (where a left-hand end is one corresponding to a ray emanating from P_1 and containing only left-hand edges). Thus we may assume without loss of generality that x fixes the end e_1, that is, $x \in \mathcal{B}$.

Using the decomposition given in Proposition 37 above, $x = ubr$ for some $u \in \mathcal{U}$, $b \in \mathcal{B}_{\mathcal{I}}$ and $r \in R$. The group R is infinite cyclic, normalises $\mathcal{B}_{\mathcal{I}}$ and $\mathcal{U}$, and has trivial intersection with $\mathcal{B}_{\mathcal{I}}$ and $\mathcal{U}$ (recall that $Z(G) = 1$). Since $x^p = 1$, we may thus assume that $r = 1$, so that $x = ub$. In fact, since $x \in \hat{U}_+$, $u \in \hat{V}_1$. Now for $y \in G$ define

$$f_n(y) := g^n y g^{-n}$$

where $\langle g \rangle = R$. Then for all $n \in \mathbb{N}$,

$$f_n(x) = f_n(u) f_n(b).$$

To complete our proof, we consider the limit of this expression as $n \to \infty$.

Lemma 39. *For each* $u \in \hat{V}_1$, $\lim_{n\to\infty} f_n(u) = 1_G$.

Proof. For $n \in \mathbb{N}$, let $\widehat{g^n V_1 g^{-n}}$ be the closure of the group $g^n V_1 g^{-n}$ in the building topology. Since conjugation by g is a homeomorphism, we have

$$\widehat{g^n V_1 g^{-n}} = g^n \hat{V}_1 g^{-n}. \tag{4}$$

By Lemma 26 above, if $-\alpha$ is a real root such that the distance from a vertex (either P_1 or P_2) of the base chamber B to the half-apartment determined by $-\alpha$ is at least $n+1$, then the root group U_α fixes $\mathrm{Ball}(B,n)$. For each $\alpha \in \Phi_+^1$, consider the group

$$f_n(U_\alpha) = g^n U_\alpha g^{-n} = U_{\tau^n \alpha}.$$

The root $-\tau^n\alpha$ is by definition the complement of the root $\tau^n\alpha = (w_1w_2)^n\alpha \in \Phi_+^1$. Since $\alpha \in \Phi_+^1$ and τ acts by translation by two edges with repelling fixed point e_2, the distance from $-\tau^n\alpha$ to the edge B is thus at least $2n \geq n+1$. Hence for each $\alpha \in \Phi_+^1$ and each $n \in \mathbb{N}$, the group $f_n(U_\alpha)$ fixes $\mathrm{Ball}(B,n)$ pointwise. Therefore for each $n \in \mathbb{N}$, the group $f_n(V_1)$ fixes $\mathrm{Ball}(B,n)$ pointwise.

By Equation (4) above and the definition of the building topology, it follows that for each $n \in \mathbb{N}$ the group $f_n(\hat{V}_1) = g^n \hat{V}_1 g^{-n}$ fixes $\mathrm{Ball}(B,n)$ pointwise. Hence for all $u \in \hat{V}_1$, $f_n(u) \to 1_G$ as required. □

Since $x = ub$ with $u \in \hat{V}_1$, Lemma 39 above implies that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(b).$$

Now, $b \in \mathcal{B}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ is a closed subgroup of $\mathcal{B}$ normalised by R. Therefore, $\lim_{n\to\infty} f_n(b) \in \mathcal{B}_{\mathcal{I}}$. Recall that by Lemma 38 above, if $y \in \mathcal{B}_{\mathcal{I}}$ is an element of finite order, then $o(y) \mid (q-1)$. Now, as $o(x) = p$, $o(f_n(x)) = p$ for all $n \in \mathbb{N}$. And so $o(\lim_{n\to\infty} f_n(x))$ divides p. That is, $o(\lim_{n\to\infty} f_n(x))$ is either 1 or p. But since $\lim_{n\to\infty} f_n(x)$ is in $\mathcal{B}_{\mathcal{I}}$, we may conclude that

$$\lim_{n\to\infty} f_n(x) = 1_G.$$

This contradicts Theorem 7 above. We conclude that a cocompact lattice Γ of G does not contain p–elements.

4.2. **Completing the proof of the Main Theorem.** Before we continue with the proof, let us recall that if P is a maximal parabolic/parahoric subgroup of G, then P has Levi decomposition $P = LU$, where U is an infinite pro-p group, while $L = TM$ where $T \leq P$ is a torus of G and $A_1(q) \cong M \triangleleft L$.

Proposition 40. *Let X be a finite subgroup of G which is contained in a cocompact lattice, and such that $|XZ(G)/Z(G)|$ is divisible by $(q+1)$. Then X is contained in a maximal parabolic/parahoric subgroup P of G. Moreover, X is isomorphic to a subgroup T_0H of a Levi complement L of P, where $T_0 \leq N_T(H)$ and $H \leq M$.*

For $p = 2$, $H \cong C_{q+1}$.

Let p be an odd prime. If $A_1(q)$ is universal, that is, if $M \cong SL_2(q)$, then H is a subgroup listed in the conclusions to Corollary 11. Otherwise $M \cong PSL_2(q)$ and $H \cong A/\langle -I \rangle$ where A is a conclusion to Corollary 11. But if $p = 3$ and $q = 9$, $H/Z(H) \ncong A_5$.

Proof. By a celebrated result of Serre (Proposition 6 above), each finite subgroup of G sits inside a standard parabolic/parahoric subgroup P of G. Since X is a subgroup of a cocompact lattice of G, by Proposition 21, X does not contain any p-elements. Hence, without loss of generality, $X \leq L$ and $(|X|, p) = 1$. The desired result now follows immediately from Corollary 11. □

We are now about to finish the proof of our main result. Clearly, if $p = 2$, it follows immediately by Proposition 40 and Lemma 13. Suppose now that p is odd.

First, let $q \equiv 3 \pmod 4$. By Lemma 13, we conclude that $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, A_i is a finite subgroup of G and conditions (1) and (2) of Lemma 13 hold. Invoking Proposition 40, we obtain a list of suitable candidates for the role of the A_i's, that is, $A_i = A_0H_i$ with $A_0 \leq N_T(H_i)$ and $H_i \cong H$ as above. Now, just as Lubotzky in the proof of Lemma 3.5 of [17], we conclude that $\Gamma = A_1 *_{A_0} A_2$ with $A_0 \leq N_T(H_i)$ is a cocompact edge-transitive lattice in G.

Assume though that $q \equiv 1 \pmod 4$ and H_i is again isomorphic H in the statement of Proposition 40. Suppose first that H_i is isomorphic to a normaliser of a non-split torus in $M_i \cong A_1(q)$. This time the argument of [19; Lemma 3.5] will not work, as was already shown in [20]. Let us briefly explain the reason. In its action on the tree, we assume that P_i fixes a vertex x_i. Now, L_i and thus A_i act on the set of neighbours Ω_i of x_i. Since $A_0 \leq T$, $A_i = A_0H_i$ intersects a one-point stabiliser B_i of L_i in a subgroup of index $4/d$, that is,

$$|A_i : A_i \cap B_i| = |A_0H_i : A_0H_i \cap T\,\mathrm{Stab}_{M_i}(x_j)| = |H_i : H_i \cap T| = 4/d$$

where $d = 2$ if $M_i \cong SL_2(q)$ and $d = 1$ if $M_i \cong PSL_2(q)$. Hence, the length of the orbit of A_i in its action on Ω_i is at most $\frac{2(q+1)/d}{4/d} = \frac{q+1}{2}$. That is A_i is not transitive on Ω_i. Now Lemma 12 implies that G does not contain edge-transitive cocompact lattices unless possibly one of the following holds: $q = 5$ and $B_1 \cong B_2 \cong A_1(3)$, or $q = 29$ and $B_1 \cong B_2 \cong A_1(5)$. If $q = 5$, then indeed A_i acts transitively on the neighbours of x_i as $|A_1 \cap \mathrm{Stab}_G(x_{3-i})| = 4 = |A_1 \cap A_2|$, and so $|A_i : A_1 \cap \mathrm{Stab}_G(x_{3-i})| = 6 = q + 1$, which means the conditions of Lemma 12 are satisfied. Similarly, if $q = 29$, A_i acts transitively on the neighbours of x_i as $|A_1 \cap St_G(x_{3-i})| = 4 = |A_1 \cap A_2|$, and so $|A_i : A_1 \cap St_G(x_{3-i})| = 30 = q + 1$, proving the result.

This completes the proof of Theorems 1 and 2 in the general case.

5. Refinements of Main Results and Volumes of Cocompact Lattices

In this section we prove Theorem 3 of the introduction, on the minimal covolume of cocompact lattices in G. The main results are Lemmas 43 and 45 below, which show that a cocompact lattice of minimal covolume in G is edge-transitive. While proving these, we will be able to refine the statements of our main results. We will restrict ourselves to the generic cases: either $p = 2$, or if p is odd, to $q \geq 60$ or even $q \geq 300$. Of course our discussion can be carried out in the same fashion for the case when p is odd and $q \leq 300$, but we decided to skip it in order to have "cleaner" statements.

As before, for $i = 1, 2$, let P_i be the stabiliser of a vertex in G, that is, a maximal parabolic/parahoric subgroup of G. Recall that $P_1 \cong P_2$, and if L_i is a Levi complement of P_i, then $L_i = TM_i$ where $T \leq B \leq P_1 \cap P_2$ is a torus of G, and $A_1(q) \cong M_i \triangleleft L_i$. Now M_i is normalised by T, and $T \cap M_i$ induces what are called inner-diagonal automorphisms on $M_i \cong A_1(q)$. However, there are clearly various possibilities for the action of elements of $T - T \cap M_i$ on M_i. In particular there are two obvious cases:

$$\textbf{Case 1}: \ \textit{For } i = 1,2, [T/T \cap M_i, M_i] = 1, \text{ and } \textbf{Case 2}: \ \textit{For } i = 1,2, \ [T/T \cap M_i, M_i] \neq 1.$$

In fact, we are going to organise our discussion based on this fairly trivial observation.

5.1. **Case 1.** In this case $L_i = M_i \circ T_i$, that is, L_i is a central (commuting) product of M_i and $T_i = C_T(M_i)$. It is possible but not necessary that $T_i \cap M_i = 1$.

Examples. Let G correspond to the generalised Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

(1) Let $p = 2$ and $G = G_u$, the universal version of the group. Then G is a central extension of $SL_2(\mathbb{F}_q((t^{-1})))$ by $\mathbb{F}_q^\times$, and so $L_i \cong C_{q-1} \times PSL_2(q)$ with $T_i \cap M_i = 1$ and $|T_i| = q - 1$.
(2) Let p be an odd prime, and $G \cong SL_2(\mathbb{F}_q((t^{-1})))$. Then $L_i \cong SL_2(q)$ with $T_i = T \cap M_i = \langle -I \rangle \cong C_2$.

Let Γ be an edge-transitive cocompact lattice of G. Then Γ is one of the conclusions of Theorem 1. Therefore up to isomorphism $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$, $A_i = A_0 \circ H_i$ with $A_i \leq P_i$, $H_i \leq M_i$, $A_0 \leq N_T(H_i)$ and H_i is as described by Theorem 1.

If $p = 2$, $A_0 \leq C_T(H_i) = C_T(M_i) = T_i$. If p is odd, then $q \equiv 3 \mod 4$, and H_i is a normaliser of a non-split torus in $M_i \cong A_1(q)$. Because of the structure of H_i, we have $A_0 \cap H_i = A_0 \cap M_i = Z(H_i) = Z(M_i)$. Furthermore, $N_{L_i}(H_i) \leq T_i H_i$ and $T_i \cap H_i = Z(H_i)$. Therefore, $A_0 = A_0 \cap T \leq T_i$.

Hence, in both cases discussed above, $A_0 \leq T_1 \cap T_2$, and so $[A_0, M_1] = [A_0, M_2] = 1$. As $\langle M_1, M_2 \rangle = \Lambda$, $A_0 \leq Z(\Lambda) = Z(G)$. In particular, our main results can be made more precise in the following way:

Theorem 41. *Let G be a topological Kac–Moody group of rank 2 defined over a field $\mathbb{F}_q$ of order $q = p^a$ where p is a prime, with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Suppose that either $p = 2$, or p is odd and $q \geq 60$. Suppose further that for a standard parabolic/parahoric P_i of G, its Levi complement $L_i = T_i \circ M_i$ where $M_i \cong A_1(q)$, $T_i = C_T(M_i)$ and $T \leq P_1 \cap P_2$ is a torus of G. Let Γ be an edge-transitive cocompact lattice in G. Then one of the following holds:*

If $p = 2$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$, $A_i = A_0 \times H_i$ with $H_i \cong C_{q+1}$ and A_0 a cyclic subgroup of $Z(G)$.

Suppose now that p is an odd prime.

If $q \equiv 1 \pmod 4$, then G does not contain any edge-transitive cocompact lattices.

If $q \equiv 3 \pmod 4$, and Γ is an edge-transitive cocompact lattice in G, then $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1,2$:

(1) *$A_i = A_0 \circ H_i$, with H_i isomorphic to the normaliser of a non-split torus in $A_1(q)$; and*
(2) *$A_0 \leq Z(G)$.*

An interesting and unusual consequence is the following observation.

Corollary 42. *Let G be a group as in Theorem 41 above. Suppose further that $q \equiv 3 \mod 4$. If $M_i \cong SL_2(q)$, then $Z(G) \neq 1$.*

Proof. If $M_i \cong SL_2(q)$, as discussed above $A_0 \geq Z(M_i) \cong C_2$. But $A_0 \leq Z(G)$, proving the result. □

Let us now discuss the question of covolumes. As described in Section 1.4, the covolume of an edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G may be calculated as follows:

$$\mu(\Gamma\backslash G) = \frac{1}{|A_1|} + \frac{1}{|A_2|} = \frac{1}{(q+1)|A_0|} + \frac{1}{(q+1)|A_0|} = \frac{2}{(q+1)|A_0|}.$$

In all conclusions to Theorem 41, the edge group A_0 satisfies $A_0 \leq Z(G)$. Now, among all the edge-transitive cocompact lattices in G, choose $\Gamma' = A_1' *_{A_0'} A_2'$ such that $|A_0'|$ is as large as possible. Then $A_0' = Z(G)$, and so for any other edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G, since $|A_0| \leq |Z(G)|$ we have

$$\mu(\Gamma\backslash G) \geq \mu(\Gamma'\backslash G) = \frac{2}{(q+1)|Z(G)|}.$$

And so among all the *edge-transitive* cocompact lattices in G, the lattice Γ' with edge group $A'_0 = Z(G)$ has the *smallest possible covolume.*

Now take Γ to be a cocompact, not necessarily edge-transitive, lattice in G. What happens then?

Lemma 43. *Let G be as in Theorem 41 with $q \not\equiv 1 \mod 4$. In fact, if p is odd, suppose that $q \geq 300$. If Γ is a cocompact lattice of G of minimal covolume, then Γ is edge-transitive.*

Combined with the discussion above, Lemma 43 proves Theorem 3 for this case.

Proof of Lemma 43. Since $Z(G)$ is finite, without loss of generality we may assume that $Z(G) = 1$. This together with Corollary 42 implies that $M_i \cong PSL_2(q)$ and $T_i \cap M_i = 1$. Moreover, $|T_i|$ is odd. If not, there exists $g \in T_i$ of order 2 (in particular, p is odd). Without loss of generality let $g \in T_1$. Then $[g, M_2] \neq 1$, for otherwise $g \in C_G(\langle M_1, M_2\rangle) = C_G(\Lambda) \leq Z(G) = 1$, a contradiction. Since $g \in T$, it normalises the root subgroups $U_{\pm\alpha_i}$, $i = 1, 2$. In particular, it normalises U_{α_2} and $U_{-\alpha_2}$, and so acts on M_2 as an element of the split torus, which is a contradiction as $M_2 \cong PSL_2(q)$ and its split torus is of odd order. It follows immediately that $|T|$ is odd too.

Let Γ be a cocompact lattice of G of minimal covolume. Since Γ is cocompact, the fundamental domain E for Γ contains at least two vertices x_1 and x_2 (connected by at least one edge) such that G_{x_i} is G-conjugate to P_i for $i = 1, 2$. By Proposition 8,

$$\mu(\Gamma\backslash G) = \sum_{s\in E} \frac{1}{|\Gamma_s|} \geq \frac{1}{|\Gamma_{x_1}|} + \frac{1}{|\Gamma_{x_2}|} \quad (*).$$

Since Γ is discrete, $|\Gamma_{x_i}|$ is finite, and so by Proposition 6, without loss of generality we may assume that $\Gamma_{x_i} \leq P_i$. But Γ is cocompact, and so Proposition 21 implies that in fact, we may suppose that Γ_{x_i} is a subgroup of $T_i \circ M_i$ of order coprime to p. Since Γ is a lattice of minimal covolume, by the discussion before this lemma, $|\Gamma_{x_i}| \geq (q + 1)$ for some $i \in \{1, 2\}$. In fact, if $|\Gamma_{x_i}| < q + 1$, then $|\Gamma_{x_j}| > q + 1$ where $\{i, j\} = \{1, 2\}$.

Let D_i denote a projection of Γ_{x_i} on T_i and H_i a projection of Γ_{x_i} on M_i. If $D_i = 1$ for $i \in \{1, 2\}$, then $\Gamma_{x_i} \leq M_i$. Choose i so that $|\Gamma_{x_i}| \geq q+1$. Hence, Γ_{x_i} is a subgroup of M_i whose order is at least $q+1$ and is co-prime to p. If $|\Gamma_{x_i}| > q+1$, Dickson's Theorem and its corollary give us that q is odd, Γ_{x_i} is a normaliser of a torus in M_i and $M_i \cong SL_2(q)$, a contradiction. Therefore $|\Gamma_{x_i}| = q + 1$ for $i = 1, 2$. By Theorem 41, Γ_{x_i} is isomorphic to C_{q+1} if $p = 2$, and a normaliser of a non-split torus in M_i if p is odd. And so it acts transitively on the set of neighbours of x_i and all the conditions of Lemma 4 hold. Thus Γ is edge-transitive, proving the result. Therefore, without loss of generality, we may assume that $D_1 \neq 1$. Now, if $H_i = 1$ for at least one of $i \in \{1, 2\}$, then the corresponding $\Gamma_{x_i} \leq T_i \leq T$. And so a subgroup generated by Γ_{x_1} and Γ_{x_2} is finite. This argument eventually contradicts the cocompactness of Γ. Hence, $H_1 \neq 1 \neq H_2$.

Assume first that $\Gamma_{x_1} \cap T_1 = 1$. As $D_1 \neq 1$, there exists a non-trivial element $g \in \Gamma_{x_1} - M_1$ of odd order which induces an inner-diagonal automorphism on M_1. Denote by $M_{x_1} := \Gamma_{x_1} \cap M_1$. Then $\langle g, M_{x_1}\rangle \leq \Gamma_{x_1}$. Since $H_1 \neq 1$ and $(|\Gamma_{x_1}|, p) = 1$, Dickson's Theorem and its corollary imply that either $M_{x_1} = 1$, or $M_{x_1} \neq 1$ and is either contained in the normaliser of a split torus of M_1, or p is odd and M_{x_1} is isomorphic to a non-abelian subgroup K of S_4 or A_5. In the former case $|\Gamma_{x_1}| \leq |T_1| \leq q - 1$. In the latter one, if p is odd and M_{x_1} is isomorphic to a non-abelian subgroup K of S_4 or A_5, then since g normalises but not centralises M_{x_1} $(C_{M_1}(K) = 1)$, $|\Gamma_{x_1}| \leq 60 < q + 1$. Finally, if M_{x_1} is contained in a normaliser of a split torus of M_1, since $\Gamma_{x_1} \cap T_1 = 1$, $m_r(\Gamma_{x_1}) = 1$ for all primes r. Hence $|\Gamma_{x_1}| \leq q - 1$. Therefore, $\Gamma_{x_1} \cap T_1 = 1$ implies $|\Gamma_{x_1}| < q + 1$. As an immediate consequence we obtain that $|\Gamma_{x_2}| > q + 1$ and consider $\Gamma_{x_2} \cap T_2$. Going through the same arguments for $i = 2$, we obtain that $\Gamma_{x_2} \cap T_2 \neq 1$. Therefore, we always have $\Gamma_{x_i} \cap T_i \neq 1$ for some $i \in \{1, 2\}$.

Without loss of generality we may assume that $\Gamma_{x_1} \cap T_1 \neq 1$ and choose a non-trivial element y_1 with $\langle y_1\rangle = \Gamma_{x_1} \cap T_1$. Since $y_1 \in T_1 \leq T$, $y_1 \in L_2$. Consider the action of y_1 on M_2. If $[y_1, M_2] = 1$, as $[y_1, M_1] = 1$, $y_1 \in Z(\Lambda) \leq Z(G) = 1$, a contradiction. Therefore y_1 acts non-trivially on M_2. Since T normalises U_{α_2} and $U_{-\alpha_2}$, y_1 acts on M_2 as an element of odd order of the split torus. Now consider $Y_1 := \langle \Gamma_{x_2}, y_1\rangle$. Since $Y_1 \leq L_2$ and $Y_1 \leq \Gamma$, Y_1 is a finite group of order prime to p. Therefore Γ_{x_2} acts of M_2 in one of the following ways: either as a subgroup of $N_{M_2}(T)$, or as a non-abelian subgroup K_2 of either S_4

or A_5 (in the latter case p is odd, $o(y_1)$ is either 3 or 5, and $\Gamma_{x_1} \cap T_1 = \langle y_1 \rangle$). Notice, that as $T \leq P_1 \cap P_2$, $\Gamma \cap T \leq \Gamma_{x_i}$ for $i = 1, 2$.

Let us first deal with the latter case. If $\Gamma_{x_2} \cap T_2 = 1$, $|\Gamma_{x_2}| \leq 60$. Now, $|\Gamma_{x_1}| \leq o(y_1)(q+1)$ with $o(y_1) \in \{3, 5\}$. Since Γ is a lattice of minimal covolume, equation $(*)$ implies that $q \leq 107$, a contradiction. Thus $\Gamma_{x_2} \cap T_2 \neq 1$. Hence there exists an element $y_2 \in \Gamma_{x_2} \cap T_2$ of odd order with $\langle y_2 \rangle \Gamma_{x_2} \cap T_2$. Using the same reasoning for y_2 as for y_1, we conclude that y_2 must act non-trivially on M_1 and Γ_{x_1} must act on M_1 either as a subgroup of $N_{M_1}(T)$, or as a non-abelian subgroup K_1 of S_4 or A_5 (in the latter case p is odd and $o(y_2)$ is either 3 or 5). In the latter case, the minimality of covolume together with formula $(*)$ give us that

$$\frac{2}{q+1} \geq \frac{2}{5 \cdot 60}$$

implying that $q < 300$, a contradiction. In the former one (i.e., Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(T)$) the fact that $\Gamma \cap T \leq \Gamma_{x_i}$ for $i = 1, 2$ implies that we can see all the torus involved in Γ_{x_i} in $\langle y_2 \rangle \times K_2$, and so $|\Gamma_{x_2}| \leq 5^2 \cdot 2$. This together with $(*)$ and the minimality implies that $q \leq 85$, a contradiction.

Therefore, $\Gamma_{x_2} \leq N_{L_2}(T)$. Again going through the same argument, we obtain similar conclusion for Γ_{x_1}: $\Gamma_{x_1} \leq N_{L_1}(T)$. As a result we have that if Γ is a cocompact lattice of minimal covolume which is not edge-transitive, $\Gamma \leq N$. But this is impossible, since N does not act cocompactly on X (its action preserves the standard apartment Σ, and so it has orbits which are at arbitrary distance from Σ). We conclude that a cocompact lattice of minimal covolume must be edge-transitive. □

5.2. **Case 2.** In this case T induces non-trivial outer-diagonal automorphisms on M_i for some i. Since $P_1 \cong P_2$, $L_1 \cong L_2$. As $M_i \cong A_1(q)$, p is **odd**, for if $p = 2$, $A_1(q) = SL_2(q) = PSL_2(q)$ does not admit outer-diagonal automorphisms. Moreover, L_i is isomorphic to a homomorphic image of $GL_2(q)$. In particular, $L_i = T_i M_i \langle t_i \rangle$ where $T_i = C_T(M_i)$, $T_i / T_i \cap M_i$ is a cyclic group of odd order and $t_i \in T$ is an involution with $L_i / T_i \cong PGL_2(q)$.

Since $q \geq 60$, there is not much more we can say about the edge-transitive lattices than we already did in the statement of Theorem 1.

However, as in the previous case, we will investigate the issue of the minimality of covolumes. Recall, that p is odd, $q \equiv 3 \mod 4$ and $q \geq 60$. As described in Section 1.4 and using the conclusion of Theorem 1, the covolume of an edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G may be calculated as follows:

$$\mu(\Gamma \backslash G) = \frac{1}{|A_1|} + \frac{1}{|A_2|} = \frac{1}{(q+1)|A_0|} + \frac{1}{(q+1)|A_0|} = \frac{2}{(q+1)|A_0|}$$

where $A_0 \leq N_T(H_i)$ for $i = 1, 2$ and H_i is the normaliser of a non-split torus of M_i. Assume there exists $i \in \{1, 2\}$ and $t \in A_0 \cap T_i$ of odd order. Then $[t, M_i] = 1$. Moreover, as $o(t) \mid (q-1)$, $[t, H_j] = 1$ where $\{i, j\} = \{1, 2\}$. But $t \in T$ and hence normalises M_j. Now the local structure of $A_1(q)$ implies that $[t, M_j] = 1$. Therefore, $t \in C_T(\langle M_1, M_2 \rangle) \leq Z(G)$.

Remark 44. *Notice that as $t_i \in T$ acts non-trivially on M_i, we have $2 \leq |T|_2 \leq 4$ and $|Z(G)|_2 \in \{1, 2\}$.*

Now, among all the edge-transitive cocompact lattices in G, choose $\Gamma' = A_1' *_{A_0'} A_2'$ such that $|A_0'|$ is as large as possible. Clearly, if $t \in T$ of order 2, then $t \in N_T(H_i)$, $i = 1, 2$, and so it is possible for $t \in A_0'$. Thus $|A_0'| = |Z(G)|2d$ if $|Z(G)|$ is odd (where $d = 1$ or 2 depending on the group), and $|A_0'| = |Z(G)|2$ if $|Z(G)|$ is even.

Therefore, for any other edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G, we have

$$\mu(\Gamma \backslash G) \geq \mu(\Gamma' \backslash G) = \frac{2}{2(q+1)|Z(G)|\delta}$$

where $\delta \in \{1, d\}$ as described above. And so among all the *edge-transitive* cocompact lattices in G, the lattice Γ' with edge group A_0' of order $2(q+1)|Z(G)|\delta$ has the *smallest possible covolume.*

Now take Γ to be a cocompact, not necessarily edge-transitive, lattice in G. What happens then? Again, to avoid small cases, we assume that q is large enough.

Lemma 45. *Let G be a topological Kac–Moody group of rank 2 defined over a field $\mathbb{F}_q$ of order $q = p^a$ where p is an odd prime, with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Suppose further that $q \not\equiv 1 \mod 4$ and $q \geq 300$. Finally, assume also that for a standard parabolic/parahoric P_i of G, its Levi complement L_i is a non-trivial non-abelian homomorphic image of $GL_2(q)$, i.e., $L_i/Z(L_i) \cong PGL_2(q)$.*

If Γ is a cocompact lattice of G of minimal covolume, then Γ is edge-transitive.

Again, the discussion above together with Lemma 45 proves Theorem 3 for this case.

Proof of Lemma 45. Since $Z(G)$ is finite, without loss of generality we may assume that $Z(G) = 1$. Let Γ be a cocompact lattice of G of minimal covolume. Since Γ is cocompact, the fundamental domain E for Γ contains at least two vertices x_1 and x_2 (connected by at least one edge) such that G_{x_i} is G-conjugate to P_i for $i = 1, 2$. By Proposition 8,

$$\mu(\Gamma\backslash G) = \sum_{s \in E} \frac{1}{|\Gamma_s|} \geq \frac{1}{|\Gamma_{x_1}|} + \frac{1}{|\Gamma_{x_2}|} \quad (**).$$

Since Γ is discrete, $|\Gamma_{x_i}|$ is finite, and so by Proposition 6, without loss of generality we may assume that $\Gamma_{x_i} \leq P_i$. But Γ is cocompact, and so Proposition 21 implies that in fact, we may suppose that Γ_{x_i} is a subgroup of L_i of order coprime to p. Since Γ is a lattice of minimal covolume, by the discussion before this lemma, $|\Gamma_{x_i}| \geq 2(q+1)\delta$ for some $i \in \{1, 2\}$. In fact, if $|\Gamma_{x_i}| < 2(q+1)\delta$, then $|\Gamma_{x_j}| > 2(q+1)\delta$ where $\{i, j\} = \{1, 2\}$.

Let D_i denote a projection of Γ_{x_i} on T_i and H_i a projection of Γ_{x_i} on $M_i\langle t_i \rangle$. If $D_i \leq Z(M_i)$ for $i \in \{1, 2\}$, then $\Gamma_{x_i} \leq M_i$. Choose i so that $|\Gamma_{x_i}| \geq 2(q+1)\delta$. Then Γ_{x_i} is a subgroup of $M_i\langle t_i \rangle$ whose order is at least $2(q+1)\delta$ and is co-prime to p. Using Dickson's Theorem and its corollary we obtain that equality holds and Γ_{x_i} is a normaliser of a torus in $M_i\langle t_i \rangle$. But now all the conditions of Lemma 4 holds, and so Γ is edge-transitive, proving the result. Therefore, without loss of generality, we may assume that $D_1 \neq 1$. Now, if $H_i = 1$ for at least on of $i \in \{1, 2\}$, then $\Gamma_{x_i} \leq T_i \leq T$. And so a subgroup generated by Γ_{x_1} and Γ_{x_2} is finite. This argument eventually contradicts the cocompactness of Γ. Hence, $H_1 \neq 1 \neq H_2$.

Assume first that $\Gamma_{x_1} \cap T_1 \leq Z(M_1)$. Then Γ_{x_1} is isomorphic to a subgroup of $M_1\langle t_1 \rangle$. As its order must be co-prime to p, we obtain that $|\Gamma_{x_1}| \leq 2(q+1)\delta$. If $\Gamma_{x_2} \cap T_2 \leq Z(M_2)$, similarly $|\Gamma_{x_2}| \leq 2(q+1)\delta$, and again we will obtain minimality when $|\Gamma_{x_i}| = 2(q+1)\delta$ for $i = 1, 2$, which in turn using Lemma 4 will give us that Γ has to be edge-transitive. Thus without loss of generality there exists $y_1 \in \Gamma_{x_1} \cap T_1$ of odd order. Repeating the discussion of our proof of Lemma 45 we obtain the desired result. □

References

[1] J. Alperin and R. Bell, *Groups and Representations*, (1991), Springer-Verlag, New York.
[2] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89** (1993), 3–47.
[3] H. Bass and R. Kulkarni, *Uniform tree lattices*, J. Amer. Math. Soc. **3** (1990), 843–902.
[4] A. Borel and J. Tits, *Elements unipotents et sous-groupes paraboliques de groupes reductifs*, Invent. Math. **12** (1971), 95–104.
[5] P.-E. Caprace, *On 2–spherical Kac–Moody groups and their central extensions*, Forum Math. **19** (2007), 763-781.
[6] P.-E. Caprace and N. Monod, *A lattice in more than two Kac–Moody groups is arithmetic*, preprint (2008), `arXiv:0812.1383v1`.
[7] P.-E. Caprace and B. Rémy, *Groups with a root group datum*, preprint (2008).
[8] P.-E. Caprace and B. Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009) 169–221.
[9] L. Carbone and H. Garland, *Existence of lattices in Kac–Moody groups over finite fields*, Commun. Contemp. Math. **5** (2003), 813–867.
[10] R. W. Carter, *Kac-Moody groups and their automorphisms,* in Groups, combinatorics and geometry (Durham, 1990), 218–228, London Math. Soc. Lecture Note Ser., **165**, Cambridge Univ. Press, Cambridge, 1992.
[11] D. Djoković and G. Miller, *Regular groups of automorphisms of cubic graphs*, J. Combin. Theory Ser. B **29** (1980), 195–230.
[12] J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. math. **150** (2002), 579–627.
[13] P. Fan, *Amalgams of prime index*, J. Algebra **98** (1986), 375–421.

[14] I. M. Gelfand, M. I. Graev and I. Piatetsky-Shapiro, *Representation theory and automorphic functions*, Saunders, San Francisco (1969).
[15] D. M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. **111** (1980), 377–406.
[16] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of Finite Simple Groups*, Number 3, Part I. Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI (1998).
[17] A. Lubotzky, *Lattices of minimal covolume in SL_2: a nonarchimedean analogue of Siegel's theorem* $\mu \geq \pi/21$, J. Amer. Math. Soc. **3** (1990), 961–975.
[18] A. Lubotzky and Th. Weigel, *Lattices of minimal covolume in SL_2 over local fields*, Proc. London Math. Soc. **78** (1999), 283–333.
[19] B. Rémy, *Groupes de Kac–Moody déployés et presques déployés*, Astérisque **277**, Soc. Math. France, 2002.
[20] B. Rémy and M. Ronan, *Topological groups of Kac–Moody type, right-angled twinnings and their lattices*, Comment. Math. Helv. **81** (2006), 191–219.
[21] J.-P. Serre, *Trees*, (2003), Springer-Verlag, Berlin.
[22] ———, *Cohomologie des groupes discrets*, Ann. of Math. Studies, no. 70, Princeton Univ. Press, Princeton, N.J., 1971, 77–169.
[23] C. L. Siegel, *Some remarks on discontinuous groups*, Ann. of Math. **46** (1945), 708–718.
[24] J. Tits, *Ensembles ordonnés, immeubles et sommes amalgamées*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 367–387.
[25] ———, *Uniqueness and presentation of Kac–Moody groups over fields*, J. Algebra **105** (1987), 542–573.

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