MINIMAL FAITHFUL PERMUTATION DEGREES FOR IRREDUCIBLE COXETER GROUPS

NEIL SAUNDERS

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia

E-mail address: `neils@maths.usyd.edu.au`

ABSTRACT. The minimal faithful degree of a finite group G, denoted by $\mu(G)$, is the least non-negative integer n such that G embeds inside $Sym(n)$. In this article we calculate the minimal faithful permutation degree for all of the irreducible Coxeter groups.

1. INTRODUCTION

The minimal faithful permutation degree $\mu(G)$ of a finite group G is the least non-negative integer n such that G embeds in the symmetric group $Sym(n)$. It is well known that $\mu(G)$ is the smallest value of $\sum_{i=1}^{n} |G : G_i|$ for a collection of subgroups $\{G_1, \ldots, G_n\}$ satisfying $\bigcap_{i=1}^{n} \text{core}(G_i) = \{1\}$, where $\text{core}(G_i) = \bigcap_{g \in G} G_i^g$.

We will often denote such a collection of subgroups by $\mathscr{R}$ and refer it as the representation of G. The elements of $\mathscr{R}$ are called *transitive constituents* and if $\mathscr{R}$ consists of just one subgroup G_0 say, then we say that $\mathscr{R}$ is transitive in which case faithfulness requires that G_0 is core-free.

The study of this area dates back to Johnson [4] where he proved that one can construct a minimal faithful representation $\{G_1, \ldots, G_n\}$ consisting entirely of so called *primitive* subgroups. These cannot be expressed as the intersection of subgroups that properly contain them.

Here we give a theorem due to Karpilovsky [5], which also serves as an introductory example. We will make use of this theorem later and the proof of it can be found in [4] or [4].

Theorem 1.1. *Let A be a finite abelian group and let $A \cong A_1 \times \ldots \times A_n$ be its direct product decomposition into non-trivial cyclic groups of*

AMS subject classification (2000): 20B35, 20F55 .
Keywords: Faithful Permutation Representations, Coxeter Groups.

prime power order. Then

$$\mu(A) = a_1 + \ldots + a_n,$$

where $|A_i| = a_i$ *for each* i.

A theme of Johnson [4] and Wright [10] was to investigate the inequality

$$\mu(G \times H) \leq \mu(G) + \mu(H) \tag{1}$$

that clearly holds for finite groups G and H. Johnson and Wright first investigated under what conditions equality holds in (1). In [10] Wright proved the following.

Theorem 1.2. *Let* G *and* H *be non-trivial nilpotent groups. Then* $\mu(G \times H) = \mu(G) + \mu(H)$.

Further in [10], Wright defined a class of finite groups $\mathscr{C}$ in the following way; for all $G \in \mathscr{C}$, there exists a nilpotent subgroup G_1 of G such that $\mu(G_1) = \mu(G)$. It is a consequence of Theorem (1.2) that $\mathscr{C}$ is closed under direct products and so equality in (1) holds for any two groups $H, K \in \mathscr{C}$. Wright proved that $\mathscr{C}$ contains all nilpotent, symmetric, alternating and dihedral groups, though the extent of it is still an open problem. In [2], Easdown and Praeger showed that equality in (1) holds for all finite simple groups.

In the closing remarks of [10], Wright flagged the question whether equality in (1) holds for all finite groups. The referee to that paper then provided an example where strict inequality holds.

This example involved the monomial reflection group $G(5,5,3)$ which has minimal degree 15, and the centralizer of its embedded image in $Sym(15)$, which turns out to be cyclic of order 5. It was observed by the referee that

$$\mu(G(5,5,3)) = \mu(G(5,5,3) \times C_{Sym(15)}(G(5,5,3))) = 15 < 20,$$

thus providing strict inequality in (1). This motivated the author of this article in [8] to consider minimal degrees of the complex reflection groups $G(p,p,q)$ thereby providing an infinite class of examples where strict inequality holds.

In [9], the author proved that a similar scenario occurs with the groups $G(4,4,3)$ and $G(2,2,5)$. That is for G either one of these groups, we have

$$\mu(G) = \mu(G \times C(G))$$

where $C(G)$ is the centralizer of G in its minimal embedding, which is non-trivial, $\mu(G(4,4,3)) = 12$ and $\mu(G(2,2,5)) = 10$. So we obtain two more examples of strict inequality in (1) of degrees 12 and 10 respectively. However the author does not know whether 10 is the smallest degree for which strict inequality occurs.

It is well known that $G(2, 2, n)$ is isomorphic to the Coxeter group $W(D_n)$ and so in pursuit of more examples of strict inequality in (1), the author turned to the irreducible Coxeter groups. The absence of a complete account of the minimal faithful permutation degrees of these groups in the literature is addressed in the remainder of this article. However, finding more examples of strict inequality in (1) remains a work in progress.

2. The Classical Groups $W(A_n), W(B_n)$ and $W(D_n)$

The Coxeter group $W(A_n)$ is the symmetric group $Sym(n+1)$ and so it has minimal degree $n+1$. The Coxeter group $W(B_n)$ is the full wreath product $C_2 \wr Sym(n)$ which acts faithfully as signed permutations on $\{\pm 1, \pm 2, \ldots, \pm n\}$, which shows that $\mu(W(B_n)) \leq 2n$. On the other hand, the base group is an elementary abelian 2-group of rank n of minimal degree $2n$ by Theorem 1.1, so $\mu(W(B_n))$ is at least $2n$, proving $\mu(W(B_n)) = 2n$.

The calculation for the Coxeter group $W(D_n)$ is a little harder and so we appeal to an argument given in [6] to do most of this calculation. We first need to establish some definitions and preliminary results relating to permutation actions.

Definition 2.1. Let p be a prime and n an integer. The *permutation module* for the symmetric group $Sym(n)$ is the direct sum of n copies of the cyclic group of order p denoted by C_p^n where $Sym(n)$ under the usual action that permutes coordinates. Define two submodules of C_p^n,

$$\begin{aligned} U &= \{(a_1, a_2, \ldots, a_n) \in C_p^n \mid \prod_{i=1}^{n} a_i = 1\} \\ V &= \{(a, a, \ldots, a) \mid a \in C_p\}. \end{aligned}$$

In the above, U is a submodule of dimension $(n-1)$ and is called the *deleted permutation module.*

The next result is a direct calculation.

Proposition 2.2. *U and V are the only proper invariant modules under the action of the alternating group $Alt(n)$.*

The Coxeter group $W(D_n)$ is the split extension of the deleted permutation module U, when $p = 2$, with the symmetric group $Sym(n)$. It can be also realized as the group of even signed permutations of the set $\{\pm 1, \ldots, \pm n\}$ and so is a subgroup of index 2 in the group $W(B_n)$.

Theorem 2.3.

$$\mu(W(D_n)) = \begin{cases} 4 & \textit{if} \quad n = 3 \\ 2n & \textit{if} \quad n \neq 3. \end{cases}$$

Proof. When $n = 2$, this group is the Klein four group and so $\mu(W(D_2)) = 4$. For $n = 3$, we have $W(D_3) \cong (C_2 \times C_2) \rtimes Sym(3)$ which is isomorphic to the symmetric group $Sym(4)$ and so $\mu(W(D_3)) = 4$.

For $n = 4$, we observe the left action of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ on $\mathbb{H}$ (see Section 3 for more details) considered as a 4-dimensional real vector space with basis $\{1, i, j, k\}$. The matrices for this action are the 4 by 4 identity matrix, and the matrices:
for i,

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for j,

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and for k,

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that these elements are contained in the subgroup of even signed permutation matrices and so are elements of $W(D_4)$ and moreover, the group they generate is isomorphic to Q_8. Now Q_8 has minimal degree 8, since all non-trivial subgroups intersect at the centre, so its minimal degree is given by the Cayley representation. Therefore

$$8 = \mu(Q_8) \leq \mu(W(D_4)) \leq \mu(W(B_4)) = 8,$$

proving $\mu(W(D_4)) = 8$.

For n greater or equal to 5, the proof that $\mu(W(D_n)) = 2n$ follows from a special case of [6, Proposition 5.2.8], where they calculate minimal permutation degree of $U \rtimes Alt(n)$ where $n \geq 5$ relying on the simplicity of $Alt(n)$. Since $U \rtimes Alt(n)$ is a proper subgroup of $U \rtimes Sym(n)$, and has minimal degree $2n$, it follows immediately that $U \rtimes Sym(n)$ does too. □

3. Real Reflection Subgroups

The Coxeter groups $W(H_3), W(H_4)$ and $W(F_4)$ can be realized as reflection groups on real 4-dimensional space, that is, subgroups of $O_4(\mathbb{R})$. The reflection subgroups of real three and four dimensional space have been studies extensively and the reader is referred to [7].

We will first deal with the Coxeter group $W(H_3)$. This calculation is easy by the following result.

Theorem 3.1. *[2, Theorem 3.1] Let $S_1 \times \ldots \times S_r$ be a direct product of simple groups. Then*

$$\mu(S_1 \times \ldots \times S_r) = \mu(S_1) + \ldots + \mu(S_r).$$

The Coxeter group $W(H_3)$ is isomorphic to the direct product $C_2 \times Alt(5)$ (see [7]) which are clearly simple groups. Moreover, it is easy to see that the minimal degree of $Alt(5)$ is 5 and so by Theorem 3.1, we have $\mu(W(H_3)) = 2 + 5 = 7$.

For the groups $W(H_4)$ and $W(F_4)$, we first recall the quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1\}.$$

Earlier we treated $\mathbb{H}$ as a 4-dimensional real vector space, however, it is well-known that it may also be treated as an algebra equipped with a norm function. That is, for every quaternion $h = a+bi+cj+dk$, the norm of h is $a^2 + b^2 + c^2 + d^2$. The set of quaternions of norm 1 form a subgroup of the multiplicative group $\mathbb{H}^*$ called the *unit quaternions*, denoted by S^3. It is well-known that all finite subgroups of $\mathbb{H}^*$ are contained in S^3.

There is a surjection of the semidirect product $(S^3 \times S^3) \rtimes C_2$ onto the orthogonal group $O_4(\mathbb{R})$ (where C_2 acts by interchanging components). The kernel of this homomorphism is the diagonal subgroup of the centre of $S^3 \times S^3$ and so we have an isomorphism $(S^3 \circ S^3) \rtimes C_2 \cong O_4(\mathbb{R})$ where $\circ$ denotes central product. Thus the finite reflection subgroups of $O_4(\mathbb{R})$ are tightly controlled by the finite subgroups of S^3. This is helpful as there are only five classes of finite subgroups of S^3.

Proposition 3.2. *[7, Theorem 5.14]*
Every finite subgroup of S^3 is conjugate in S^3 to one of the following groups:

1. *the cyclic group $\mathcal{C}_m$ of order m,*
2. *the binary dihedral group $\mathcal{D}_m$ of order $4m$,*
3. *the binary tetrahedral group $\mathcal{T}$ of order 24,*

(4) *the binary octahedral group* $\mathcal{O}$ *of order* 48,

(5) *the binary icosahedral group* $\mathcal{I}$ *of order* 120.

We recall a well known fact that every Coxeter group acts faithfully on its associated root system. Moreover when the Coxeter group is irreducible, roots of any given length are contained in a single orbit, on which the group also acts faithfully (see [3]). Thus the size of any orbit in the root system is always an upper bound for the minimal degree of a Coxeter group. It is the case that for $W(H_4)$ and $W(F_4)$, we cannot do any better than the size of their root systems for their minimal degrees. Below we state convenient structures of $W(H_4)$ and $W(F_4)$ that allow us to calculate their minimal degree. A proof of these structures can be found in [7].

Proposition 3.3. *We have these abstract isomorphisms;*

$W(H_4) \cong (\mathcal{I} \circ \mathcal{I}) \rtimes C_2 \qquad W(F_4) \cong (Q_8 \circ Q_8) \rtimes (Sym(3) \times Sym(3)),$

where $\circ$ *denotes central product.*

Lemma 3.4. *Let* G *be a finite group and* p *is prime. Suppose that the centre* $Z(G)$ *is cyclic of order* p *and that it is the unique minimal normal subgroup of* G*. Then the central product* $G \circ G$ *has a unique minimal normal subgroup isomorphic to* C_p*, namely* $Z(G) \circ Z(G)$*.*

Proof. Let $\bar{N}$ be a non-trivial normal subgroup of $G \circ G$. Let N be the preimage of $\bar{N}$ in $G \times G$, a normal subgroup of $G \times G$ strictly containing the diagonal copy of $Z(G)$.

Let (x, g) be any element of N not contained in the diagonal copy of $Z(G)$. If $x, g \in Z(G)$, then (x, g) and this diagonal copy generate $Z(G) \times Z(G)$, which is therefore contained in N. Otherwise, one of x and g, say g, is not contained in $Z(G)$, so there is some element $h \in G$ such that $g^{-1}h^{-1}gh \neq 1$. But N contains $= (x, g)^{-1}(x, g)^{(1,h)} = (1, g^{-1}h^{-1}gh)$ as well as its normal closure. So N contains $\{1\} \times Z(G)$ and therefore contains $Z(G) \times Z(G)$ in this case also. Passing to the quotient, we find $Z(G) \circ Z(G)$ is contained in $\bar{N}$. □

We now deal with the Coxeter group $W(F_4)$. Firstly, we note that the binary tetrahedral group $\mathcal{T}$ is abstractly isomorphic to $Q_8 \rtimes C_3$ and a presentation for it can be given thus

$$\langle x, y, b \mid b^3 = 1,\ x^2 = y^2,\ x^y = x^{-1},\ x^b = y,\ y^b = xy \rangle.$$

Also note that $Sym(3)$ is abstractly isomorphic to $C_3 \rtimes C_2$. Thus we may present another decomposition for $W(F_4)$ which is more convenient to work with.

Lemma 3.5. $\mathcal{T} \times \mathcal{T} \cong (Q_8 \times Q_8) \rtimes (C_3 \times C_3)$ *and so* $\mathcal{T} \circ \mathcal{T} \cong (Q_8 \circ Q_8) \rtimes (C_3 \times C_3)$.

Proof. The first isomorphism is clear since, under the appropriate actions,

$$\begin{aligned}\mathcal{T}\times\mathcal{T} &\cong (Q_8 \rtimes C_3)\times(Q_8\rtimes C_3)\\ &\cong (Q_8\times Q_8)\rtimes(C_3\times C_3).\end{aligned}$$

For the second isomorphism, recall that the central product of a group by itself is formed by taking the quotient of the direct product by the diagonal subgroup of the center. Since $Z(\mathcal{T}) = Z(Q_8) \cong C_2$, we have that $\mathcal{T}\times\mathcal{T}$ surjects onto $(Q_8\circ Q_8)\rtimes(C_3\times C_3)$ with kernel $Z_D(\mathcal{T}\times\mathcal{T})$, the diagonal subgroup of the centre of $\mathcal{T}\times\mathcal{T}$. Thus the second isomorphism follows. □

This lemma implies that $\mathcal{T}\circ\mathcal{T}$ has a unique Sylow 2-subgroup, namely $Q_8\circ Q_8$. Writing $Sym(3)$ as $C_3\rtimes C_2$, we may give a presentation for $Q_8\rtimes Sym(3)$ as the group generated by $\{x,y,b,a\}$ with x, y, and b obeying all relations given above with the following:

$$a^2=1,\ b^a=b^{-1},\ x^a=y^{-1}.$$

With this we may rearrange the above isomorphisms to get:

$$\begin{aligned}W(F_4) &\cong (Q_8\circ Q_8)\rtimes(Sym(3)\times Sym(3))\\ &\cong (Q_8\circ Q_8)\rtimes((C_3\rtimes C_2)\times(C_3\rtimes C_2))\\ &\cong (Q_8\circ Q_8)\rtimes((C_3\times C_3)\rtimes(C_2\times C_2))\\ &\cong ((Q_8\circ Q_8)\rtimes(C_3\times C_3))\rtimes(C_2\times C_2)\\ &\cong (\mathcal{T}\circ\mathcal{T})\rtimes(C_2\times C_2)\end{aligned}$$

Now the root system of the Coxeter group $W(F_4)$ consists of 48 roots of two lengths; 24 long and 24 short roots (see [3]). Thus 24 is an upper bound for the minimal degree of $W(F_4)$ and we will prove that in fact it is 24 by showing $\mu(\mathcal{T}\circ\mathcal{T}) = 24$.

Observe that Lemma 3.4 applies to $\mathcal{T}$ and so $\mathcal{T}\circ\mathcal{T}$ has a unique minimal normal subgroup isomorphic to C_2. Thus every minimal faithful representation is given by a core-free subgroup. Below we prove that any such subgroup must have index at least 24.

Since $\mathcal{T}\circ\mathcal{T}$ is a proper subgroup of $W(F_4)$, 24 is an upper bound for its minimal degree. On the other hand, the central product $Q_8\circ Q_8$ is a proper subgroup of $\mathcal{T}\circ\mathcal{T}$ which has minimal degree 16 by [2, Proposition 2.4] and so

$$16 \le \mu(\mathcal{T}\circ\mathcal{T}) \le 24.$$

Proposition 3.6. *If L is a core-free subgroup of $\mathcal{T}\circ\mathcal{T}$, then $|\mathcal{T}\circ\mathcal{T} : L| \ge 24$.*

Proof. Suppose for a contradiction that $\mathrm{core}(L) = \{1\}$ and $|\mathcal{T} \circ \mathcal{T} : L| < 24$. Then, since $|\mathcal{T} \circ \mathcal{T}| = 2^5 3^2$, $12 < |L| \leq 18$. This leaves two cases:

Case (i): $|L| = 16$.

Then L is a 2-group and is thus contained in the unique Sylow 2-subgroup of $\mathcal{T} \circ \mathcal{T}$, namely $Q_8 \circ Q_8$. Now $Q_8 \circ Q_8$ is a nilpotent group and L is a subgroup of index 2 and thus is normal in $Q_8 \circ Q_8$. Therefore L contains the center of $Q_8 \circ Q_8$ which is also the center of $\mathcal{T} \circ \mathcal{T}$, which contradicts that $\mathrm{core}(L)$ is trivial.

Case (ii): $|L| = 18$.

Then L has a set of Sylow 3-subgroups of order 9. By Sylow's theorem, the number of Sylow 3-subgroups divides 2 and is congruent to 1 mod 3. Therefore, there is a unique Sylow 3-subgroup $Syl(3)$, and thus L is a semidirect product of $Syl(3)$ by cyclic group subgroup of order 2.

Write $L = Syl(3) \rtimes \langle w \rangle$. Now by Lemma 3.5, any element of order 3 in $\mathcal{T} \circ \mathcal{T}$ normalizes $Q_8 \circ Q_8$. But on the other hand, w normalizes $Syl(3)$ in L. Thus

$$[w, Syl(3)] = w^{-1} Syl(3)^{-1} w Syl(3) \subset (Q_8 \circ Q_8) \cap Syl(3) = \{1\}.$$

So w in fact must commute with $Syl(3)$ and since the only element which does this is the central involution, we have $\langle w \rangle = Z(\mathcal{T} \circ \mathcal{T})$. Again this contradicts that $\mathrm{core}(L) = \{1\}$.

Therefore any core-free subgroup of $\mathcal{T} \circ \mathcal{T}$ has index at least 24 as required. □

Theorem 3.7. *The minimal faithful permutation degree of $\mathcal{T} \circ \mathcal{T}$ and $W(F_4)$ is 24.*

We now turn our attention to $W(H_4)$. Note that the size of the root system is 120 (see [3]). Our method for calculating the minimal degree of this Coxeter group is to show that the minimal degree of its proper subgroup $\mathcal{I} \circ \mathcal{I}$ is 120. Now the binary icosahedral group has a unique minimal normal subgroup, namely its centre which is isomorphic to C_2. Lemma 3.4 implies that the central product $\mathcal{I} \circ \mathcal{I}$ also has a unique minimal normal subgroup isomorphic to C_2. Therefore, every minimal faithful permutation representation of $\mathcal{I} \circ \mathcal{I}$ is necessarily transitive. Thus as before finding a minimal permutation representation reduces to simply searching through the subgroup lattice for core-free subgroups and selecting the one of minimal index. The *LowIndexSubgroups* command in MAGMA (see [1]) returns that the smallest such index is 120, thus proving $\mu(\mathcal{I} \circ \mathcal{I}) = 120$.

Remark 3.8. In the process of calculating the minimal degree of $W(F_4)$ and $W(H_4)$, we have exhibited examples of groups where the minimal degree of a proper quotient is larger than the minimal degree of the group. Specifically, it can be shown that $\mu(\mathcal{T} \times \mathcal{T}) = 16$, yet $\mu(\mathcal{T} \circ \mathcal{T}) = 24$. Similarly, we have $\mu(\mathcal{I} \circ \mathcal{I}) > \mu(\mathcal{I} \times \mathcal{I})$.

4. The Groups $W(E_6), W(E_7)$ and $W(E_8)$

In this section we will use the fact that for every Coxeter group, there is a well-defined length function which induces a *sign* homomorphism $W \longrightarrow C_2; w \mapsto (-1)^{l(w)}$. The kernel of this homomorphism is an index 2 subgroup denoted by W^+ called the *rotation* subgroup. It is the case that for the groups $W(E_6)$ and $W(E_7)$ their rotation subgroups are simple groups. Calculating the minimal permutation degree for a simple group reduces to finding the maximal subgroups, which have been well studied.

Let us first deal with the group $W(E_6)$. By Humphreys [3], its rotation subgroup $W(E_6)^+$ is a simple group isomorphic to $SU_4(\mathbb{F}_2)$ and by [6, Table 5.2.A], it has minimal degree 27. On the other hand, $W(E_6)$ acts faithfully on the set of positive/negative roots of E_7 that are not contained in E_6. By inspection of the size of the root systems, this set has size 27 as well. Therefore the minimal degrees of $W(E_6)$ and its rotation subgroup co-inside, both being 27.

We now turn our attention to $W(E_7)$. Again by [3] this group is a split extension of its rotation subgroup by a cyclic group of order 2. Now its rotation subgroup $W(E_7)^+$ is a simple group isomorphic to $O_7(\mathbb{F}_2)$ and again by [6, Table 5.2.A], its minimal degree is 28. So we have the following decomposition

$$W(E_7) \cong O_7(\mathbb{F}_2) \times C_2,$$

where each direct factor is a simple group. Thus by Theorem 3.1 we have $\mu(W(E_7)) = 28 + 2 = 30$.

Before we deal with the group $W(E_8)$, we require a lemma about covering groups.

Lemma 4.1. *Let G be a p to 1 non-split central extension of the simple group S. Then*

$$\mu(G) \geq p\mu(S).$$

Proof. We have that G maps surjectively onto S with kernel C_p. Let this mapping be φ. It quickly follows that this kernel is the unique minimal normal subgroup of G since if N is a non-trivial proper normal subgroup of G distinct from $\ker(\varphi)$, then $\varphi(N)$ is a non-trivial proper normal subgroup of S contradicting simplicity. So the minimal degree $\mu(G)$ is the index of the largest subgroup which does not contain the kernel. Let this subgroup be L and observe that L is isomorphic to its image in S under φ. Therefore we have

$$\mu(G) = |G : L| = p|S : \varphi(L)| \geq p\mu(S),$$

where the last inequality is necessary since $\varphi(L)$ need not be maximal in S. □

Now the rotation subgroup of $W(E_8)$ is a $2:1$ non-split central extension of the simple group $O_8(\mathbb{F}_2)$. By [6, Table 5.2.A] this group has minimal degree 120 and so by Lemma 4.1,

$$\mu(W(E_8))^+ \geq 2\mu(O_8(\mathbb{F}_2)) = 240.$$

On the other hand, the root system of $W(E_8)$ has size 240 as well and so we deduce that $\mu(W(E_8)^+) = \mu(W(E_8)) = 240$.

5. The Groups $W(I_2(m))$

The groups $W(I_2(m))$ are isomorphic to the dihedral groups of order $2m$. The minimal degrees of these groups were calculated by Easdown and Praeger in [2].

Proposition 5.1. *For any integer $k = \prod_{i=1}^{m} p_i^{\alpha_i} > 1$, with the p_i distinct primes, define $\psi(k) = \sum_{i=1}^{m} p_i^{\alpha_i}$, with $\psi(1) = 0$. Then for the dihedral group $D_{2^r n}$ of order $2^r n$, with n odd, we have*

$$\mu(D_{2^r n}) = \begin{cases} 2^r & \text{if } n = 1, 1 \leq r \leq 2 \\ 2^{r-1} & \text{if } n = 1, r > 2 \\ \psi(n) & \text{if } n > 1, r = 1 \\ 2^{r-1} + \psi(n) & \text{if } n > 1, r > 1. \end{cases}$$

6. Direct Products

Recall that a finite Coxeter group is the direct product of irreducible Coxeter groups. While in many areas of study it is enough just to concentrate on the irreducible components, this is not the case when dealing with minimal degrees. Thus knowledge of the minimal degrees for all the irreducible Coxeter groups does not imply the minimal degree for an arbitrary Coxeter group.

We can make the following observations with what is known about minimal degrees of direct products. The groups $W(A_n)$ and $W(B_n)$

are contained in the Wright class $\mathscr{C}$ and so

$$\mu(W(A_n) \times W(B_n)) = \mu(W(A_n)) + \mu(W(B_n)).$$

Also since $W(H_3)$ and $W(E_7)$ are direct products of simple groups, we have by Theorem 3.1

$$\mu(W(H_3) \times W(E_7)) = \mu(W(H_3)) + \mu(W(E_7)).$$

On the other hand, we have the following isomorphism for odd n greater than or equal to 5; $W(D_n) \times W(A_1) \cong W(B_n)$. So we have

$$\mu(W(D_n) \times W(A_1)) = \mu(W(B_n))$$

but $\mu(W(D_n)) + \mu(W(A_1)) = 2n + 2 > 2n = \mu(W(B_n))$.

7. Acknowledgements

The author thanks his supervisors David Easdown and Anthony Henderson for their expert guidance and many helpful discussions.

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