GRADED CELLULAR BASES FOR THE CYCLOTOMIC KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF TYPE A

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Abstract. This paper constructs an explicit homogeneous cellular basis for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A over a field.

1. Introduction

In a groundbreaking series of papers Brundan and Kleshchev (and Wang) [**6**–**8**] have shown that the cyclotomic Hecke algebras of type $G(\ell,1,n)$, and their rational degenerations, are graded algebras. Moreover, they have extended Ariki's categorification theorem [**2**] to show over a field of characteristic zero the graded decomposition numbers of these algebras can be computed using the canonical bases of the higher level Fock spaces.

The starting point for Brundan and Kleshchev's work was the introduction of certain graded algebras $\mathscr{R}_n^\Lambda$ which arose from Khovanov and Lauda's [**23**, §3.4] categorification of the negative part of quantum group of an arbitrary Kac-Moody Lie algebra and, independently, in work of Rouquier [**31**]. In type A Brundan and Kleshchev [**6**] proved that the (degenerate and non-degenerate) cyclotomic Hecke algebras are $\mathbb{Z}$-graded by constructing explicit isomorphisms to $\mathscr{R}_n^\Lambda$.

The **cyclotomic Khovanov-Lauda–Rouquier algebra** $\mathscr{R}_n^\Lambda$ is generated by certain elements $\{\psi_1,\dots,\psi_{n-1}\}\cup\{y_1,\dots,y_n\}\cup\{\,e(\mathbf{i})\mid \mathbf{i}\in(\mathbb{Z}/e\mathbb{Z})^n\,\}$ which are subject to a long list of relations (see Definition 3.1). Each of these relations is homogeneous, so it follows directly from the presentation that $\mathscr{R}_n^\Lambda$ is $\mathbb{Z}$-graded. Unfortunately, it is not at all clear from the relations how to construct a homogeneous basis of $\mathscr{R}_n^\Lambda$, even using the isomorphism from $\mathscr{R}_n^\Lambda$ to the cyclotomic Hecke algebras.

The main result of this paper gives an explicit homogeneous basis of $\mathscr{R}_n^\Lambda$. In fact, this basis is cellular so our Main Theorem also proves a conjecture of Brundan, Kleshchev and Wang [**8**, Remark 4.12].

To describe this basis let $\mathscr{P}_n^\Lambda$ be the set of multipartitions of n, which is a poset under the dominance order. For each $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$ let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux (these terms are defined in §3.3). For each $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$ there is an idempotent $e_{\boldsymbol{\lambda}}$ and a homogeneous element $y_{\boldsymbol{\lambda}}\in K[y_1,\dots,y_n]$ (see Definition 4.15). Brundan, Kleshchev and Wang [**8**] have defined a combinatorial *degree* function $\deg:\coprod_{\boldsymbol{\lambda}}\operatorname{Std}(\boldsymbol{\lambda})\longrightarrow\mathbb{Z}$ and for each $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ there is a well-defined element $\psi_{d(\mathfrak{t})}\in\langle\psi_1,\dots,\psi_{n-1}\rangle$. Our Main Theorem is the following.

Main Theorem. *The cyclotomic Khovanov-Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ is a graded cellular algebra, with respect to the dominance order, with homogeneous cellular basis*

$$\{\,\psi^*_{d(\mathfrak{s})}e_{\boldsymbol{\lambda}}y_{\boldsymbol{\lambda}}\psi_{d(\mathfrak{t})}\mid\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda \text{ and } \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}.$$

Moreover, $\deg\left(\psi^*_{d(\mathfrak{s})}e_{\boldsymbol{\lambda}}y_{\boldsymbol{\lambda}}\psi_{d(\mathfrak{t})}\right)=\deg\mathfrak{s}+\deg\mathfrak{t}$.

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We prove our Main Theorem in the two really interesting cases where $\mathscr{R}_n^\Lambda$ is isomorphic to either a degenerate or a non-degenerate cyclotomic Hecke algebra. In particular, these results imply that this basis is a homogeneous basis for $\mathscr{R}_n^\Lambda$ if either $e = 0$ or e is a non-zero prime number. More generally, we show that over $\mathbb{Z}$ the algebra $\mathscr{R}_n^\Lambda(\mathbb{Z})$ can have p-torsion, for a prime p, only if p divides e. This implies that the ψ-basis is a homogeneous basis for the $\mathcal{O}$-algebra $\mathscr{R}_n^\Lambda(\mathcal{O}) = \mathscr{R}_n^\Lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$ whenever $\mathcal{O}$ is commutative integral domain $\mathcal{O}$ in which $e \cdot 1_{\mathcal{O}}$ is invertible.

The main difficulty in proving this theorem is that the graded presentation of the cyclotomic Khovanov-Lauda–Rouquier algebras hides many of the relations between the homogeneous generators. To overcome this we use the Brundan-Kleshchev isomorphism theorem to recast everything in terms of the representation theory of the cyclotomic Hecke algebras of type $G(\ell, 1, n)$. The key step is the realization that the idempotents $e(\mathbf{i})$ can be lifted to an integral form of the Hecke algebra defined over a discrete valuation ring $\mathcal{O}$, where they become natural sums of the seminormal basis elements [**28, 29**]. By lifting $e_{\boldsymbol\lambda} y_{\boldsymbol\lambda}$ to this integral form we are able to compare these elements with the standard basis of the cyclotomic Hecke algebras [**5, 12**], which allows us to prove our main theorem.

In fact, we give two graded cellular bases of the cyclotomic Khovanov-Lauda-Rouquier algebras $\mathscr{R}_n^\Lambda$. Intuitively, one of these bases is built from the *trivial* representation of the Hecke algebra and the other is built from its *sign* representation. We then show that these two bases are dual to each other, modulo more dominant terms. As a consequence, we deduce that the blocks of $\mathscr{R}_n^\Lambda$ are graded symmetric algebras (see Corollary 6.18), as conjectured by Brundan and Kleshchev [**7**, Remark 4.7].

This paper is organized as follows. In section 2 we define and develop the representation theory of *graded cellular algebras*, following and extending ideas of Graham and Lehrer [**18**]. Just as with the original definition of cellular algebras, graded cellular algebras are already implicit in the literature in the work of Brundan and Stroppel [**9, 10**]. In section 3, following Brundan and Kleshchev [**6**] we define the cyclotomic Khovanov-Lauda–Rouquier algebras of type $G(\ell, 1, n)$ and recall Brundan and Kleshchev's all important graded isomorphism theorem. In section 4 we shift gears and show how to lift the idempotents $e(\mathbf{i})$ to $\mathscr{H}_n^{\mathcal{O}}$, an integral form of the non-degenerate cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda$. We then use this observation to produce a family of non-trivial homogeneous elements of $\mathscr{R}_n^\Lambda \cong \mathscr{H}_n^\Lambda$, including $e_{\boldsymbol\lambda} y_{\boldsymbol\lambda}$, for $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$. In section 5 we lift the graded Specht modules of Brundan, Kleshchev and Wang to give a graded basis of $\mathscr{H}_n^\Lambda$ and then in section 6 we construct the dual graded basis and use this to show that the blocks of $\mathscr{H}_n^\Lambda$ are graded symmetric algebras. As an application we construct an isomorphism between the graded Specht modules and the dual graded Specht modules, which are defined using our second graded cellular basis of $\mathscr{H}_n^\Lambda$. In section 7 we consider the graded cellular basis of the degenerate cyclotomic Hecke algebras $H_n^\Lambda \cong \mathscr{R}_n^\Lambda$. We then apply all of our results to study the cyclotomic Khovanov-Lauda-Rouquier algebras $\mathscr{R}_n^\Lambda$ over more general rings. In an appendix, which was actually the starting point for this work, we use a different approach to explicitly describe the homogeneous elements which span the one dimensional two-sided ideals of $\mathscr{H}_n^\Lambda$.

2. Graded cellular algebras

This section defines graded cellular algebras and develops their representation theory, extending Graham and Lehrer's [**18**] theory of cellular algebras. Most of the arguments of Graham and Lehrer apply with minimal change in the graded setting. In particular, we obtain graded cell modules, graded simple and projective modules and a graded analogue of Brauer-Humphreys reciprocity.

§2.1. **Graded algebras.** Let R be a commutative integral domain with 1. In this paper a **graded R-module** is an R-module M which has a direct sum decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$. If $m \in M_d$, for $d \in \mathbb{Z}$, then m is **homogeneous** of **degree** d and we set $\deg m = d$. If M is a graded R-module let $\underline{M}$ be the ungraded R-module obtained by forgetting the grading on M. If M is a graded R-module and $s \in \mathbb{Z}$ let $M\langle s\rangle$ be the graded R-module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

A **graded R-algebra** is a unital associative R-algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ which is a graded R-module such that $A_dA_e \subseteq A_{d+e}$, for all $d, e \in \mathbb{Z}$. It follows that $1 \in A_0$ and that A_0 is a graded subalgebra of A. A graded (right) A-module is a graded R-module M such that $\underline{M}$ is an $\underline{A}$-module and $M_dA_e \subseteq M_{d+e}$, for all $d, e \in \mathbb{Z}$. Graded submodules, graded left A-modules and so on are all defined in the obvious way. Let A-**Mod** be the category of all finitely generated graded A-modules together with degree preserving homomorphisms; that is,

$$\operatorname{Hom}_A(M,N) = \{\, f \in \operatorname{Hom}_{\underline{A}}(\underline{M},\underline{N}) \mid f(M_d) \subseteq N_d \text{ for all } d \in \mathbb{Z}\,\},$$

for all $M, N \in A$-**Mod**. The elements of $\operatorname{Hom}_A(M,N)$ are homogeneous maps of degree 0. More generally, if $f \in \operatorname{Hom}_A(M\langle d\rangle, N) \cong \operatorname{Hom}_A(M, N\langle -d\rangle)$ then f is a homogeneous map from M to N of degree d and we write $\deg f = d$. Set

$$\operatorname{Hom}_A^{\mathbb{Z}}(M,N) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(M\langle d\rangle, N) \cong \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(M, N\langle -d\rangle)$$

for $M, N \in A$-**Mod**.

§2.2. **Graded cellular algebras.** Following Graham and Lehrer [**18**] we now define graded cellular algebras.

2.1. **Definition** (Graded cellular algebras)**.** Suppose that A is a $\mathbb{Z}$-graded R-algebra which is free of finite rank over R. A **graded cell datum** for A is an ordered quadruple $(\mathscr{P}, T, C, \deg)$, where $(\mathscr{P}, \rhd)$ is the **weight poset**, $T(\lambda)$ is a finite set for $\lambda \in \mathscr{P}$, and

$$C\colon \coprod_{\lambda\in\mathscr{P}} T(\lambda)\times T(\lambda) \longrightarrow A; (\mathfrak{s},\mathfrak{t}) \mapsto c^\lambda_{\mathfrak{s}\mathfrak{t}}, \quad \text{and} \quad \deg : \coprod_{\lambda\in\mathscr{P}} T(\lambda) \longrightarrow \mathbb{Z}$$

are two functions such that C is injective and

(GC1) $\{\, c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda), \lambda \in \mathscr{P}\,\}$ is an R-basis of A.

(GC2) Each basis element $c^\lambda_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg c^\lambda_{\mathfrak{s}\mathfrak{t}} = \deg\mathfrak{s} + \deg\mathfrak{t}$, for $\lambda \in \mathscr{P}$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda)$.

(GC3) If $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, for some $\lambda \in \mathscr{P}$, and $a \in A$ then there exist scalars $r_{\mathfrak{t}\mathfrak{v}}(a)$, which do not depend on $\mathfrak{s}$, such that

$$c^\lambda_{\mathfrak{s}\mathfrak{t}} a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) c^\lambda_{\mathfrak{s}\mathfrak{v}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda}$ is the R-submodule of A spanned by $\{\, c^\mu_{\mathfrak{a}\mathfrak{b}} \mid \mu \rhd \lambda \text{ and } \mathfrak{a},\mathfrak{b} \in T(\mu)\,\}$.

(GC4) The R-linear map $*\colon A \longrightarrow A$ determined by $(c^\lambda_{\mathfrak{s}\mathfrak{t}})^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \mathscr{P}$ and all $\mathfrak{s},\mathfrak{t} \in \mathscr{P}$, is an anti-isomorphism of A.

A **graded cellular algebra** is a graded algebra which has a graded cell datum. The basis $\{\, c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathscr{P} \text{ and } \mathfrak{s},\mathfrak{t} \in T(\lambda)\,\}$ is a **graded cellular basis** of A.

If we omit (GC2) then we recover Graham and Lehrer's definition of an (ungraded) cellular algebra. Therefore, by forgetting the grading, any graded cellular algebra is an (ungraded) cellular algebra in the original sense of Graham and Lehrer.

2.2. **Examples** a) Let $A = \mathfrak{gl}_2(R)$ be the algebra of 2×2 matrices over R. Let $\mathscr{P} = \{*\}$ and $T(*) = \{1,2\}$ and set

$$c_{11} = e_{12}, \quad c_{12} = e_{11}, \quad c_{21} = e_{22} \quad \text{and} \quad c_{22} = e_{21},$$

with $\deg(1) = 1$ and $\deg(2) = -1$. Then $(\mathscr{P}, T, C, \deg)$ is a graded cellular basis of A. In particular, taking R to be a field this shows that semisimple algebras can be given the structure of a graded cellular algebra with a non-trivial grading.
b) Brundan has pointed out that it follows from his results with Stroppel that the Khovanov diagram algebras [**9**, Cor. 3.3], their quasi-hereditary covers [**9**, Theorem 4.4], and the level two degenerate cyclotomic Hecke algebras [**10**, Theorem 6.6] are all graded cellular algebras in the sense of Definition 2.1. $\diamond$

2.3. **Definition** (Graded cell modules)**.** Suppose that A is a graded cellular algebra with graded cell datum $(\mathscr{P}, T, C, \deg)$, and fix $\lambda \in \mathscr{P}$. Then the **graded cell module** C^λ is the graded right A-module

$$C^\lambda = \bigoplus_{z\in\mathbb{Z}} C^\lambda_z,$$

where C^λ_z is the free R-module with basis $\{\, c^\lambda_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda) \text{ and } \deg \mathfrak{t} = z \,\}$ and where the action of A on C^λ is given by

$$c^\lambda_{\mathfrak{t}} a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) c^\lambda_{\mathfrak{v}},$$

where the scalars $r_{\mathfrak{t}\mathfrak{v}}(a)$ are the scalars appearing in (GC3).

Similarly, let $C^{*\lambda}$ be the left graded A-module which, as an R-module is equal to C^λ, but where the A-action is given by $a \cdot x := xa^*$, for $a \in A$ and $x \in C^{*\lambda}$.

It follows directly from Definition 2.1 that C^λ and $C^{*\lambda}$ are graded A-modules. Let $A^{\trianglerighteq\lambda}$ be the R-module spanned by the elements $\{\, c^\mu_{\mathfrak{u}\mathfrak{v}} \mid \mu \trianglerighteq \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu) \,\}$. It is straightforward to check that $A^{\trianglerighteq\lambda}$ is a graded two-sided ideal of A and that

$$A^{\trianglerighteq\lambda}/A^{\triangleright\lambda} \cong C^{*\lambda} \otimes_R C^\lambda \cong \bigoplus_{\mathfrak{s}\in T(\lambda)} C^\lambda\langle \deg \mathfrak{s}\rangle \tag{2.4}$$

as graded (A, A)-bimodules for the first isomorphism and as graded right A-modules for the second.

Let t be an indeterminate over $\mathbb{N}_0$. If $M = \oplus_{z\in\mathbb{Z}} M_z$ is a graded A-module such that each M_z is free of finite rank over R, then its **graded dimension** is the Laurent polynomial

$$\operatorname{Dim}_t M = \sum_{k\in\mathbb{Z}} (\dim_R M_k) t^k.$$

2.5. **Corollary.** *Suppose that A is a graded cellular algebra and $\lambda \in \mathscr{P}$. Then*

$$\operatorname{Dim}_t C^\lambda = \sum_{\mathfrak{s}\in T(\lambda)} t^{\deg \mathfrak{s}}.$$

Consequently, $\operatorname{Dim}_t A = \sum_{\lambda\in\mathscr{P}} \sum_{\mathfrak{s},\mathfrak{t}\in T(\lambda)} t^{\deg \mathfrak{s} + \deg \mathfrak{t}} = \sum_{\lambda\in\mathscr{P}} \big(\operatorname{Dim}_t C^\lambda\big)^2.$

Suppose that $\mu \in \mathscr{P}$. Then it follows from Definition 2.1, exactly as in [**18**, Prop. 2.4], that there is a bilinear form $\langle\ ,\ \rangle_\mu$ on C^μ which is determined by

$$c^\mu_{\mathfrak{a}\mathfrak{s}} c^\mu_{\mathfrak{t}\mathfrak{b}} \equiv \langle c^\mu_{\mathfrak{s}}, c^\mu_{\mathfrak{t}}\rangle_\mu c^\mu_{\mathfrak{a}\mathfrak{b}} \pmod{A^{\triangleright\mu}},$$

for any $\mathfrak{s}, \mathfrak{t}, \mathfrak{a}, \mathfrak{b} \in T(\mu)$. The next Lemma gives standard properties of this bilinear form $\langle\ ,\ \rangle_\mu$. Just as in the ungraded case (see, for example, [**27**, Prop. 2.9]) it follows directly from the definitions.

2.6. **Lemma.** *Suppose that $\mu \in \mathscr{P}$ and that $a \in A$, $x, y \in C^\mu$. Then*

$$\langle x, y\rangle_\mu = \langle y, x\rangle_\mu, \qquad \langle xa, y\rangle_\mu = \langle x, ya^*\rangle_\mu \qquad \text{and} \qquad xc^\mu_{\mathfrak{s}\mathfrak{t}} = \langle x, c^\mu_{\mathfrak{s}}\rangle_\mu c^\mu_{\mathfrak{t}},$$

for all $\mathfrak{s}, \mathfrak{t} \in T(\mu)$.

We consider the ring R as a graded R-module with trivial grading: $R = R_0$. Observe that $C^\mu \otimes C^\mu$ is a graded A-module with $\deg x \otimes y = \deg x + \deg y$.

2.7. **Lemma.** *Suppose that $\mu \in \mathscr{P}$. Then the induced map*

$$f : C^\mu \otimes_R C^\mu \longrightarrow R; x \otimes y \mapsto \langle x, y \rangle_\mu$$

is a homogeneous map of degree zero. In particular,

$$\operatorname{rad} C^\mu = \{\, x \in C^\mu \mid \langle x, y \rangle_\mu = 0 \text{ for all } y \in C^\mu \,\}.$$

is a graded submodule of C^μ.

Proof. By Lemma 2.6, $\operatorname{rad} C^\mu$ is a submodule of C^μ since $\langle\ ,\ \rangle_\mu$ is associative (with respect to the anti-automorphism $*$). It remains to show that the bilinear form defines a homogeneous map of degree zero. Suppose that $f(x \otimes y) \neq 0$, for some $x, y \in C^\mu$. Write $x = \sum_i x_i$ and $y = \sum_j y_j$, where x_i and y_i are both homogeneous of degree i. Then $\langle x_i, y_j \rangle_\mu \neq 0$ for some i and j. Now write $x_i = \sum_{\mathfrak{s}} a_{\mathfrak{s}} c^\mu_{\mathfrak{s}}$ and $y_j = \sum_{\mathfrak{t}} b_{\mathfrak{t}} c^\mu_{\mathfrak{t}}$, for $a_{\mathfrak{s}}, b_{\mathfrak{t}} \in R$ such that $a_{\mathfrak{s}} \neq 0$ only if $\deg \mathfrak{s} = i$ and $b_{\mathfrak{t}} \neq 0$ only if $\deg \mathfrak{t} = j$. Fix any $\mathfrak{v} \in T(\mu)$. Then by Lemma 2.6,

$$\langle x_i, y_j \rangle_\mu c^\mu_{\mathfrak{v}\mathfrak{v}} = \sum_{\mathfrak{s},\mathfrak{t}} a_{\mathfrak{s}} b_{\mathfrak{t}} \langle c^\mu_{\mathfrak{s}}, c^\mu_{\mathfrak{t}} \rangle_\mu c^\mu_{\mathfrak{v}\mathfrak{v}} \equiv \sum_{\mathfrak{s},\mathfrak{t}} a_{\mathfrak{s}} b_{\mathfrak{t}} c^\mu_{\mathfrak{v}\mathfrak{s}} c^\mu_{\mathfrak{t}\mathfrak{v}} \pmod{A^{\rhd\mu}}.$$

Taking degrees of both sides shows that $\langle x_i, y_j \rangle_\mu \neq 0$ only if $i + j = 0$. That is, $\langle x, y \rangle_\mu \neq 0$ only if $\deg(x \otimes y) = 0$ as we wanted to show. Finally, $\operatorname{rad} C^\mu$ is a graded submodule of C^μ because if $x = \sum_i x_i \in \operatorname{rad} C^\mu$ then $x_i \in \operatorname{rad} C^\mu$, for all i, since $\langle\ ,\ \rangle_\mu$ is homogeneous. □

The Lemma allows us to define a graded quotient of C^μ, for $\mu \in \mathscr{P}$.

2.8. **Definition.** Suppose that $\mu \in \mathscr{P}$. Let $D^\mu = C^\mu / \operatorname{rad} C^\mu$.

By definition, D^μ is a graded right A-module. Henceforth, let $R = K$ be a field and $A = \bigoplus_{z \in \mathbb{Z}} A_z$ a graded cellular K-algebra. Exactly as in the ungraded case (see [**18**, Prop. 2.6] or [**27**, Prop. 2.11-2.12]), we obtain the following.

2.9. **Lemma.** *Suppose that K is a field and that $D^\mu \neq 0$, for $\mu \in \mathscr{P}$. Then:*

a) *The right A-module D^μ is an absolutely irreducible graded A-module.*
b) *The (graded) Jacobson radical of C^μ is $\operatorname{rad} C^\mu$.*
c) *If $\lambda \in \mathscr{P}$ and M is a graded A-submodule of C^λ. Then*

$$\operatorname{Hom}^{\mathbb{Z}}_A(C^\mu, C^\lambda/M) \neq 0$$

only if $\lambda \unrhd \mu$. Moreover, if $\lambda = \mu$ then

$$\operatorname{Hom}^{\mathbb{Z}}_A(C^\mu, C^\mu/M) = \operatorname{Hom}_A(C^\mu, C^\mu/M) \cong K.$$

In particular, if M is a graded A-submodule of C^μ then every non-zero homomorphism from C^μ to C^μ/M is degree preserving.

Let $\mathscr{P}_0 = \{\, \lambda \in \mathscr{P} \mid D^\lambda \neq 0 \,\}$. Recall that if M is an A-module then $\underline{M}$ is the ungraded $\underline{A}$-module obtained by forgetting the grading.

2.10. **Theorem.** *Suppose that K is a field and that A is a graded cellular K-algebra.*

a) *If $\mu \in \mathscr{P}_0$ then D^μ is an absolutely irreducible graded A-module.*
b) *Suppose that $\lambda, \mu \in \mathscr{P}_0$. Then $D^\lambda \cong D^\mu\langle k \rangle$, for some $k \in \mathbb{Z}$, if and only if $\lambda = \mu$ and $k = 0$.*
c) *$\{\, D^\mu\langle k \rangle \mid \mu \in \mathscr{P}_0 \text{ and } k \in \mathbb{Z} \,\}$ is a complete set of pairwise non-isomorphic graded simple A-modules.*

Sketch of proof. Parts (a) and (b) follow directly from Lemma 2.9. For part (c), observe that, up to degree shift, every graded simple A-module is isomorphic to a quotient of A by a maximal graded right ideal. The graded cellular basis of A induces a graded filtration of A with all quotient modules isomorphic to direct sums of shifts of graded cell modules, so it is enough to show that every composition factor of C^λ is isomorphic to $D^\mu\langle k\rangle$, for some $\mu \in \mathscr{P}_0$ and some $k \in \mathbb{Z}$. Arguing exactly as in the ungraded case completes the proof; see [**18**, Theorem 3.4] or [**27**, Theorem 2.16]. □

In particular, just as Graham and Lehrer [**18**] proved in the ungraded case, every field is a splitting field for a graded cellular algebra.

2.11. **Corollary.** *Suppose that K is a field and A is a graded cellular algebra over K. Then $\{ \underline{D}^\mu \mid \mu \in \mathscr{P}_0 \}$ is a complete set of pairwise non-isomorphic ungraded simple A-modules.*

Proof. By Lemma 2.7, for each $\lambda \in \mathscr{P}$ the submodule $\operatorname{rad} C^\lambda$ is independent of the grading so the ungraded module $\underline{D}^\mu$ is precisely the module constructed by using the cellular basis of A obtained by forgetting the grading. Therefore, every (ungraded) simple module is isomorphic to $\underline{D}^\mu$ by forgetting the grading in Theorem 2.10 (or, equivalently, by [**18**, Theorem 3.4]). □

§2.3. **Graded decomposition numbers.** Recall that t is an indeterminate over $\mathbb{Z}$. If M is a graded A-module and D is a graded simple module let $[M : D\langle k\rangle]$ be the multiplicity of the simple module $D\langle k\rangle$ as a graded composition factor of M, for $k \in \mathbb{Z}$. Similarly, let $[\underline{M} : \underline{D}]$ the multiplicity of $\underline{D}$ as a composition factor of $\underline{M}$.

2.12. **Definition** (Graded decomposition matrices)**.** Suppose that A is a graded cellular algebra over a field. Then the **graded decomposition matrix** of A is the matrix $\mathbf{D}_A(t) = \big(d_{\lambda\mu}(t)\big)$, where

$$d_{\lambda\mu}(t) = \sum_{k\in\mathbb{Z}} [C^\lambda : D^\mu\langle k\rangle]\, t^k,$$

for $\lambda \in \mathscr{P}$ and $\mu \in \mathscr{P}_0$.

Using Lemma 2.9 we obtain the following.

2.13. **Lemma.** *Suppose that $\mu \in \mathscr{P}_0$ and $\lambda \in \mathscr{P}$. Then*

a) $d_{\lambda\mu}(t) \in \mathbb{N}_0[t, t^{-1}]$*;*
b) $d_{\lambda\mu}(1) = [\underline{C}^\lambda : \underline{D}^\mu]$*; and,*
c) $d_{\mu\mu}(t) = 1$ *and* $d_{\lambda\mu}(t) \neq 0$ *only if* $\lambda \trianglerighteq \mu$*.*

Next we study the graded projective A-modules with the aim of describing the composition factors of these modules using the graded decomposition matrix.

A graded A-module M has a **graded cell module filtration** if there exists a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$$

such that each M_i is a graded submodule of M and if $1 \le i \le k$ then $M_i/M_{i-1} \cong C^\lambda\langle k\rangle$, for some $\lambda \in \mathscr{P}$ and some $k \in \mathbb{Z}$. By [**17**, Theorem 3.2, Theorem 3.3], we know that every projective A-module is gradable.

2.14. **Proposition.** *Suppose that P is a projective A module. Then P has a graded cell module filtration.*

Proof. Fix a total ordering $\succ$ on $\mathscr{P} = \{\lambda_1 \succ \lambda_2 \succ \cdots \succ \lambda_N\}$ which is compatible with $\triangleright$ in the sense that if $\lambda \triangleright \mu$ then $\lambda \succ \mu$. Let $A(\lambda_i) = \bigcup_{j\le i} A^{\trianglerighteq\lambda_i}$. Then

$$0 \subset A(\lambda_1) \subset A(\lambda_2) \subset \cdots \subset A(\lambda_N) = A$$

is a filtration of A by graded two-sided ideals. Tensoring with P we have

$$0 \subseteq P \otimes_A A(\lambda_1) \subseteq P \otimes_A A(\lambda_2) \subseteq \cdots \subseteq P \otimes_A A(\lambda_N) = P,$$

a graded filtration of P. An easy exercise in the definitions (cf. [**27**, Lemma 2.14]), shows that there is a short exact sequence

$$0 \to A(\lambda_{i-1}) \to A(\lambda_i) \to A^{\trianglerighteq \lambda_i}/A^{\triangleright \lambda_i} \to 0.$$

Since P is projective, tensoring with P is exact so the subquotients in the filtration of P above are

$$P \otimes_A A(\lambda_i)/P \otimes_A A(\lambda_{i-1}) \cong P \otimes_A \big(A^{\trianglerighteq \lambda_i}/A^{\triangleright \lambda_i}\big) \cong P \otimes_A (C^{*\lambda_i} \otimes_R C^{\lambda_i}),$$

where the last isomorphism comes from (2.4). Hence, P has a graded cell module filtration as claimed. □

For each $\mu \in \mathscr{P}_0$ let P^μ be the projective cover of D^μ. Then for each $k \in \mathbb{Z}$, $P^\mu\langle k\rangle$ is the projective cover of $D^\mu\langle k\rangle$.

2.15. **Lemma.** *Suppose that $\lambda \in \mathscr{P}$ and $\mu \in \mathscr{P}_0$. Then:*

a) $d_{\lambda\mu}(t) = \operatorname{Dim}_t \operatorname{Hom}_A^{\mathbb{Z}}(P^\mu, C^\lambda)$.

b) $\operatorname{Hom}_A^{\mathbb{Z}}(P^\mu, C^\lambda) \cong P^\mu \otimes_A C^{*\lambda}$ *as $\mathbb{Z}$-graded K-modules.*

Proof. Part (a) follows directly from the definition of projective covers. Part (b) follows using essentially the same argument as in the ungraded case; see the proof of [**18**, Theorem 3.7(ii)]. □

2.16. **Definition** (Graded Cartan matrix)**.** Suppose that A is a graded cellular algebra over a field. Then the **graded Cartan matrix** of A is the matrix $\mathbf{C}_A(t) = \big(c_{\lambda\mu}(t)\big)$, where

$$c_{\lambda\mu}(t) = \sum_{k\in\mathbb{Z}} [P^\lambda : D^\mu\langle k\rangle]\, t^k,$$

for $\lambda, \mu \in \mathscr{P}_0$.

If $M = (m_{ij})$ is a matrix let $M^{\mathrm{tr}} = (m_{ji})$ be its transpose.

2.17. **Theorem** (Graded Brauer-Humphreys reciprocity)**.** *Suppose that K is a field and that A is a graded cellular K-algebra. Then $\mathbf{C}_A(t) = \mathbf{D}_A(t)^{tr}\mathbf{D}_A(t)$.*

Proof. Suppose that $\lambda, \mu \in \mathscr{P}_0$. Then by Proposition 2.14 and (2.4) we have

$$\begin{aligned}
c_{\lambda\mu}(t) &= \sum_{k\in\mathbb{Z}} [P^\lambda : D^\mu\langle k\rangle]\, t^k \\
&= \sum_{k\in\mathbb{Z}} \sum_{\nu\in\mathscr{P}} [(P^\lambda \otimes_A C^{*\nu}) \otimes_R C^\nu : D^\mu\langle k\rangle]\, t^k \\
&= \sum_{k\in\mathbb{Z}} \sum_{\nu\in\mathscr{P}} \operatorname{Dim}_t P^\lambda \otimes_A C^{*\nu} [C^\nu : D^\mu\langle k\rangle]\, t^k \\
&= \sum_{\nu\in\mathscr{P}} \operatorname{Dim}_t P^\lambda \otimes_A C^{*\nu} \sum_{k\in\mathbb{Z}} [C^\nu : D^\mu\langle k\rangle]\, t^k \\
&= \sum_{\nu\in\mathscr{P}} d_{\nu\lambda}(t) d_{\nu\mu}(t),
\end{aligned}$$

where we have used Lemma 2.15 in the last step. □

Let $K_0(A)$ be the (enriched) Grothendieck group of A. Thus, $K_0(A)$ is the free $\mathbb{Z}[t, t^{-1}]$-module generated by symbols $[M]$, where M runs over the finite dimensional graded A-modules, with relations $[M\langle k\rangle] = t^k[M]$, for $k \in \mathbb{Z}$, and $[M] = [N] + [P]$ whenever $0 \to N \to M \to P \to 0$ is a short exact sequence of graded A-modules. Then $K_0(A)$ is a free $\mathbb{Z}[t, t^{-1]}]$-module with distinguished bases

$\{\,[D^\mu] \mid \mu\in\mathscr{P}_0\,\}$ and $\{\,[C^\mu] \mid \mu\in\mathscr{P}_0\,\}$. Similarly, let $K_0^*(A)$ be the (enriched) Grothendieck group of finitely generated (graded) projective A-modules. Then $K_0^*(A)$ is free as a $\mathbb{Z}[t,t^{-1}]$-module with basis $\{\,[P^\mu] \mid \mu\in\mathscr{P}_0)\,\}$. Replacing $\mathscr{P}_0$ with $\mathscr{P}$ in the definition of $K_0(A)$, gives the free $\mathbb{Z}[t,t^{-1}]$-module $\mathscr{F}(A)$ which is generated by symbols $[\![C^\mu]\!]$ for $\mu\in\mathscr{P}$. Theorem 2.17 then says that the following diagram commutes:

$$\begin{tikzcd} K_0^*(A) \arrow[r, "\mathbf{D}_A(t)"] \arrow[dr, "\mathbf{C}_A(t)"'] & \mathscr{F}(A) \arrow[d, "\mathbf{D}_A(t)^{\mathrm{tr}}"] \\ & K_0(A) \end{tikzcd}$$

Recall from Definition 2.1 that A is equipped with a graded anti-automorphism $*$. Let M be a graded A-module. The **contragredient dual** of M is the graded A-module

$$M^{\circledast}=\operatorname{Hom}_A^{\mathbb{Z}}(M,K)=\bigoplus_{d\in\mathbb{Z}}\operatorname{Hom}_A(M\langle d\rangle,K)$$

where the action of A is given by $(fa)(m)=f(ma^*)$, for all $f\in M^{\circledast}$, $a\in A$ and $m\in M$. As a vector space, $M^{\circledast}_d=\operatorname{Hom}_A(M_{-d},K)$, so $\operatorname{Dim}_t M^{\circledast}=\operatorname{Dim}_{t^{-1}}M$.

2.18. **Proposition.** *Suppose that $\mu\in\mathscr{P}_0$. Then $D^\mu\cong(D^\mu)^{\circledast}$.*

Proof. By Lemma 2.7 $\langle\ ,\ \rangle_\mu$ restricts to give a non-degenerate homogeneous bilinear form of degree zero on D^μ. Therefore, if d is any non-zero element of D^μ then the map $D^\mu\longrightarrow(D^\mu)^{\circledast}$ given by $d\mapsto\langle d,-\rangle_\mu$, for $d\in D^\mu$, gives the desired isomorphism. $\square$

If M is a graded A-module then $(M\langle k\rangle)^{\circledast}\cong(M^{\circledast})\langle -k\rangle$ as K-vector spaces, for any $k\in\mathbb{Z}$. Consequently, contragredient duality induces a $\mathbb{Z}$-linear automorphism $^-\colon K_0(A)\longrightarrow K_0(A)$ which is determined by

$$\overline{t^k[M^{\circledast}]}=t^{-k}[M],$$

for all $M\in A$-**Mod** and all $k\in\mathbb{Z}$.

If $\mu\in\mathscr{P}_0$ then $\overline{[D^\mu]}=[D^\mu]$ by Proposition 2.18. Define polynomials $e_{\lambda\mu}(t)\in\mathbb{Z}[t,t^{-1}]$ by setting $(e_{\lambda\mu}(-t))=\mathbf{D}_A(t)^{-1}$. Then $e_{\mu\mu}=1$ and

$$[D^\mu]=[C^\mu]+\sum_{\substack{\nu\in\mathscr{P}_0\\ \mu\rhd\nu}}e_{\mu\nu}(-t)[C^\nu].$$

(Following the philosophy of the Kazhdan-Lusztig conjectures, we define the polynomials $e_{\lambda\mu}(-t)$ in the hope that $e_{\lambda\mu}(t)\in\mathbb{N}_0[t]$.) *A priori*, $d_{\lambda\mu}(t)\in\mathbb{N}_0[t,t^{-1}]$ and $e_{\lambda\mu}(t)\in\mathbb{Z}[t,t^{-1}]$. In contrast, we have a 'Kazhdan-Lusztig basis' for $K_0(A)$.

2.19. **Proposition.** *There exists a unique basis $\{\,[E^\mu] \mid \mu\in\mathscr{P}_0\,\}$ of $K_0(A)$ such that if $\mu\in\mathscr{P}_0$ then $\overline{[E^\mu]}=[E^\mu]$ and*

$$[E^\mu]=[C^\mu]+\sum_{\substack{\lambda\in\mathscr{P}_0\\ \mu\rhd\lambda}}f_{\mu\lambda}(-t)[C^\lambda],$$

for some polynomials $f_{\mu\lambda}(t)\in t\mathbb{Z}[t]$, for $\lambda\in\mathscr{P}_0$.

Proof. Using Proposition 2.18 it is easy to see that if $\lambda\in\mathscr{P}_0$ then there exist polynomials $r_{\lambda\mu}(t)\in\mathbb{Z}[t,t^{-1}]$, for $\mu\in\mathscr{P}_0$, such that

$$\overline{[C^\lambda]}=[C^\lambda]+\sum_{\substack{\mu\in\mathscr{P}_0\\ \lambda\rhd\mu}}r_{\lambda\mu}(t)[C^\mu].$$

The Corollary follows from this observation using a well-known inductive argument due to Kazhdan and Lusztig; see [**22**, Theorem 1.1] or [**13**, 1.2]. □

It seems unlikely to us that there is a mild condition on A which ensures that $[E^\mu] = [D^\mu]$, or equivalently, $d_{\lambda,\mu}(t) \in t\mathbb{N}_0[t]$ when $\lambda \rhd \mu$. We conclude this section by discussing a strong assumption on A which achieves this.

A graded A-module $M = \bigoplus_i M_i$ is **positively graded** if $M_i = 0$ whenever $i < 0$. It is easy to check that a graded cellular algebra A is positively graded if and only if $\deg \mathfrak{s} \geq 0$, for all $\mathfrak{s} \in T(\lambda)$, for $\lambda \in \mathscr{P}$. Consequently, if A is positively graded then so is each cell module of A.

A graded A-module $M = \bigoplus_i M_i$ is **pure of degree** d if $M = M_d$.

2.20. **Lemma.** *Suppose that A is a positively graded cellular algebra over a field K and suppose that $\lambda \in \mathscr{P}$ and $\mu \in \mathscr{P}_0$. Then:*

a) *D^μ is pure of degree 0; and,*
b) *$d_{\lambda\mu}(t) \in \mathbb{N}_0[t]$.*

Proof. The bilinear form $\langle\ ,\ \rangle$ on C^μ is homogeneous of degree 0 by Lemma 2.7. Therefore, if $x, y \in C^\mu$ and $\langle x, y\rangle_\mu \neq 0$ then $\deg x + \deg y = 0$, so that $x, y \in C^\mu_0$. This implies (a). In turn, this implies (b) because $D^\mu\langle k\rangle$ can only be a composition factor of C^λ if $k \geq 0$ (and $\lambda \unrhd \mu$) since A is positively graded. □

In the ungraded case, Graham and Lehrer [**18**, Remark 3.10] observed that a cellular algebra is quasi-hereditary if and only if $\mathscr{P} = \mathscr{P}_0$, which is naturally still true in the graded setting. Conversely, any graded split quasi-hereditary algebra that has a graded duality which fixes the simple modules is a graded cellular algebra by the arguments of Du and Rui [**14**, Cor. 6.2.2]. Similarly, it is easy to see that if A is a positively graded cellular algebra such that $\mathscr{P} = \mathscr{P}_0$ then A-**Mod** is a positively graded highest weight category with duality as defined in [**11**].

If $M = \bigoplus_{i\geq 0} M_i$ is a positive graded A-module let $M_+ = \bigoplus_{i>0} M_i$. If A is positively graded then M_+ is a graded A-submodule of M. Let $\operatorname{Rad} M$ be the Jacobson radical of M.

As the following Lemma indicates, there do exist positively graded quasi-hereditary cellular algebras such that, in the notation of Proposition 2.19, $[D^\mu] \neq [E^\mu]$ for all $\mu \in \mathscr{P} = \mathscr{P}_0$.

2.21. **Lemma.** *Suppose that A is a positive graded quasi-hereditary cellular algebra over a field. Then the following are equivalent:*

a) *$A_0 \cong A/A_+$ is a (split) semisimple algebra;*
b) *$\operatorname{Rad} A = A_+$;*
c) *$\operatorname{rad} C^\mu = C^\mu_+$, for all $\mu \in \mathscr{P}$;*
d) *$[D^\mu] = [E^\mu]$, for all $\mu \in \mathscr{P}$; and,*
e) *$d_{\lambda\mu}(t) \in t\mathbb{N}_0[t]$, for all $\lambda \neq \mu \in \mathscr{P}$.*

Proof. As A is quasi-hereditary, if $\mu \in \mathscr{P}$ then $D^\mu \neq 0$ and $\operatorname{rad} C^\mu = \operatorname{Rad} C^\mu$ by the general theory of cellular algebras (by Lemma 2.9). Therefore, since A is positively graded, all of the statements in the Lemma are easily seen to be equivalent to the condition that $D^\mu \cong C^\mu/C^\mu_+$, for all $\mu \in \mathscr{P}$. □

3. Khovanov-Lauda–Rouquier algebras and Hecke algebras

In this section, following [**6**], we set our notation and define the cyclotomic Khovanov-Lauda–Rouquier algebras of type A and recall Brundan and Kleshchev's graded isomorphism theorem.

§3.1. **Cyclotomic Khovanov-Lauda–Rouquier algebras.** As in section 2, let R be a commutative integral domain with 1.

Throughout this paper we fix an integer e such that either $e = 0$ or $e \geq 2$. Let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and with directed edges $i \longrightarrow i+1$, for all $i \in I$. Thus, Γ_e is the quiver of type A_∞ if $e = 0$, and it is the quiver of type $A_e^{(1)}$ if $e \geq 2$.

Let $(a_{i,j})_{i,j\in I}$ be the symmetric Cartan matrix associated with Γ_e, so that

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j \pm 1, \\ -1 & \text{if } e \neq 2 \text{ and } i = j \pm 1, \\ -2 & \text{if } e = 2 \text{ and } i = j + 1. \end{cases}$$

Following Kac [**21**, Chapt. 1], let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of the Cartan matrix, and $\{\, \alpha_i \mid i \in I \,\}$ the associated set of simple roots, $\{\, \Lambda_i \mid i \in I \,\}$ the fundamental dominant weights, and $(\cdot, \cdot)$ the bilinear form determined by

$$(\alpha_i, \alpha_j) = a_{i,j} \qquad \text{and} \qquad (\Lambda_i, \alpha_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

Finally, let $P_+ = \bigoplus_{i\in I} \mathbb{N}_0\Lambda_i$ be the dominant weight lattice of $(\mathfrak{h}, \Pi, \check{\Pi})$ and let $Q_+ = \bigoplus_{i\in I} \mathbb{N}_0\alpha_i$ be the positive root lattice.

For the remainder of this paper fix positive integers n and ℓ, a dominant weight $\Lambda \in P_+$, and a sequence of integers $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ such that

$$(\Lambda, \alpha_i) = \#\{\, 1 \leq s \leq \ell \mid \kappa_s \equiv i \pmod{e} \,\}, \qquad \text{for } i \in I,$$

and $\kappa_s - \kappa_{s+1} \geq n$, for $1 \leq s < \ell$. All of the bases considered in this paper will depend upon the choice of **multicharge** $\boldsymbol{\kappa}$. The assumption that $\kappa_s - \kappa_{s+1} \geq n$ is used only to streamline the choice of modular systems for the cyclotomic Hecke algebras in sections 4 and 7, respectively.

The following algebra has its origins in the work of Khovanov and Lauda [**23**], Rouquier [**31**] and Brundan and Kleshchev [**6**].

3.1. **Definition.** The **Khovanov-Lauda–Rouquier algebra** $\mathscr{R}_n^\Lambda$ of weight Λ and type Γ_e is the unital associative R-algebra with generators

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$$

and relations

$$y_1^{(\Lambda,\alpha_{i_1})} e(\mathbf{i}) = 0, \qquad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) = 1,$$

$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i}) \psi_r, \qquad y_r y_s = y_s y_r,$$

$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1,$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \text{if } |r - s| > 1,$$

$$\psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r \psi_r + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$y_{r+1} \psi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \pm 1, \\ (y_{r+1} - y_r) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1}) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r - 1, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\mathbf{i}), & \text{if } e = 2 \text{ and } i_{r+1} = i_r + 1 \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(\mathbf{i}), & \text{if } e\neq 2 \text{ and } i_{r+2}=i_r=i_{r+1}-1,\\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(\mathbf{i}), & \text{if } e\neq 2 \text{ and } i_{r+2}=i_r=i_{r+1}+1,\\ \big(\psi_{r+1}\psi_r\psi_{r+1}+y_r & \\ \qquad -2y_{r+1}+y_{r+2}\big)e(\mathbf{i}), & \text{if } e=2 \text{ and } i_{r+2}=i_r=i_{r+1}+1,\\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}), & \text{otherwise.} \end{cases}$$

for $\mathbf{i},\mathbf{j}\in I^n$ and all admissible r,s.

It is straightforward, albeit slightly tedious, to check that all of these relations are homogeneous with respect to the following degree function on the generators

$$\deg e(\mathbf{i})=0,\qquad \deg y_r=2\qquad\text{and}\qquad \deg\psi_s e(\mathbf{i})=-a_{i_s,i_{s+1}},$$

for $1\le r\le n$, $1\le s<n$ and $\mathbf{i}\in I^n$. Therefore, the Khovanov-Lauda–Rouquier algebra $\mathscr{R}_n^\Lambda$ is $\mathbb{Z}$-graded. From this presentation, however, it is not clear how to construct a basis for $\mathscr{R}_n^\Lambda$, or even what the dimension of $\mathscr{R}_n^\Lambda$ is.

§3.2. **Cyclotomic Hecke algebras.** Suppose now that K is a field and that $q\in K$ is a primitive e^{th} root of unity, if $e>0$, and not a root of unity if $e=0$. Note that if K is a field of characteristic $p>0$ then p does not divide e.

3.2. **Definition.** Suppose that $q\in K$ is a primitive e^{th} root of unity if $e>1$; or not a root of unity if $e=0$. The **cyclotomic Hecke algebra** $\mathscr{H}_n^\Lambda$ is the unital associative K-algebra with generators $T_0,T_1,\dots,T_{n-1}$ and relations

$$\begin{aligned} &\textstyle\prod_{i\in I}(T_0-q^i)^{(\Lambda,\alpha_i)}=0, && (T_r+1)(T_r-q)=0,\\ &T_0T_1T_0T_1=T_1T_0T_1T_0, && T_rT_s=T_sT_r,\ \text{if } |r-s|>1,\\ &T_sT_{s+1}T_s=T_{s+1}T_sT_{s+1}, && \end{aligned}$$

for $1\le r<n$ and $1\le s<n-1$.

For use throughout this paper we now set up the notation from [**6**] needed to describe the Brundan and Kleshchev's isomorphism $\mathscr{R}_n^\Lambda\cong\mathscr{H}_n^\Lambda$. Recall that $\ell=\sum_{i\in I}(\Lambda,\alpha_i)$. The algebra $\mathscr{H}_n^\Lambda$ is a cyclotomic Hecke algebra of type $G(\ell,1,n)$.

Let $\mathfrak{S}_n$ be the symmetric group of degree n and let $s_i=(i,i+1)\in\mathfrak{S}_n$, for $1\le i<n$. Then $\{s_1,\dots,s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. If $w\in\mathfrak{S}_n$ then the **length** of w is

$$\ell(w)=\min\{\,k\mid w=s_{i_1}\dots s_{i_k}\text{ for some }1\le i_1,\dots,i_k<n\,\}.$$

If $w=s_{i_1}\dots s_{i_k}$ with $k=\ell(w)$ then $s_{i_1}\dots s_{i_k}$ is a **reduced expression** for w. In this case, set $T_w:=T_{i_1}\dots T_{i_k}$. Then T_w is independent of the choice of reduced expression because the generators $T_1,\dots,T_{n-1}$ satisfy the braid relations of $\mathfrak{S}_n$; see, for example, [**27**, Theorem 1.8]. Set $L_1=T_0$ and $L_{i+1}=q^{-1}T_iL_iT_i$, for $i=1,\dots,n-1$. Then Ariki and Koike [**4**, Theorem 3.10] showed that

$$\{\,L_1^{a_1}\dots L_n^{a_n}T_w\mid 0\le a_1,\dots,a_n<\ell\text{ and }w\in\mathfrak{S}_n\,\}$$

is an K-basis of $\mathscr{H}_n^\Lambda$. This basis is not homogeneous in general.

Suppose that M is a finite dimensional $\mathscr{H}_n^\Lambda$-module. Then, by [**19**, Lemma 4.7], the eigenvalues of each L_m on M are of the form q^i for $i\in I$. So M decomposes as a direct sum $M=\bigoplus_{\mathbf{i}\in I^n}M_{\mathbf{i}}$ of its generalized eigenspaces, where

$$M_{\mathbf{i}}:=\{\,v\in M\mid v(L_r-q^{i_r})^k=0\text{ for }r=1,2,\cdots,n\text{ and }k\gg0\,\}.$$

(Clearly, we can take $k=\dim M$ here.) In particular, taking M to be the regular $\mathscr{H}_n^\Lambda$-module we get a system $\big\{e(\mathbf{i})\bigm|\mathbf{i}\in I^n\big\}$ of pairwise orthogonal idempotents in $\mathscr{H}_n^\Lambda$ such that $Me(\mathbf{i})=M_{\mathbf{i}}$ for each finite dimensional right $\mathscr{H}_n^\Lambda$-module M. Note that these idempotents are not, in general, primitive. Moreover, all but finitely many of the $e(\mathbf{i})$'s are zero and, by the relations, their sum is the identity element of $\mathscr{R}_n^\Lambda$.

Following Brundan and Kleshchev [**6**, §5] we now define elements of $\mathscr{H}_n^\Lambda$ which satisfy the relations of $\mathscr{R}_n^\Lambda$. For $r = 1, \dots, n$ define

$$y_r = \sum_{\mathbf{i}\in I^n} (1 - q^{-i_r} L_r) e(\mathbf{i}).$$

By (3.8), or less directly, [**6**, Lemma 2.1], $y_1, \dots, y_n$ are nilpotent elements of $\mathscr{H}_n^\Lambda$, so any power series in $y_1, \dots, y_n$ can be interpreted as elements of $\mathscr{H}_n^\Lambda$. Next, for every $\mathbf{i} \in I^n$, we set

$$y_r(\mathbf{i}) := q^{i_r}(1 - y_r) \in K[\![y_1, \dots, y_n]\!],$$

and define formal power series $P_r(\mathbf{i}) \in K[\![y_r, y_{r+1}]\!]$ by setting

$$P_r(\mathbf{i}) = \begin{cases} 1 & \text{if } i_r = i_{r+1}, \\ (1-q)\left(1 - y_r(\mathbf{i}) y_{r+1}(\mathbf{i})^{-1}\right)^{-1} & \text{if } i_r \neq i_{r+1}. \end{cases}$$

By a small generating function exercise, if $i_r \neq i_{r+1}$ then

$$P_r(\mathbf{i}) = \frac{1-q}{1-q^{i_r - i_{r+1}}} \left\{ 1 + \sum_{k \geq 1} \frac{q^{i_r - i_{r+1}} (y_{r+1} - y_r)(y_{r+1} - q^{i_r - i_{r+1}} y_r)^{k-1}}{(1 - q^{i_r - i_{r+1}})^k} \right\}. \tag{3.3}$$

Following [**6**] we also set

$$Q_r(\mathbf{i}) = \begin{cases} 1 - q - y_r + q y_{r+1} & \text{if } i_r = i_{r+1}, \\ (y_r(\mathbf{i}) - q y_{r+1}(\mathbf{i})))/(y_r(\mathbf{i}) - y_{r+1}(\mathbf{i})) & \text{if } i_{r+1} \neq i_r \pm 1, \\ (y_r(\mathbf{i}) - q y_{r+1}(\mathbf{i}))/(y_r(\mathbf{i}) - y_{r+1}(\mathbf{i}))^2 & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r + 1, \\ q^{i_r} & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r - 1, \\ q^{i_r}/(y_r(\mathbf{i}) - y_{r+1}(\mathbf{i})) & \text{if } e = 2 \text{ and } i_{r+1} = i_r + 1. \end{cases} \tag{3.4}$$

Brundan and Kleshchev note that in $K[\![y_r, y_{r+1}]\!]$ the numerators on the right hand side of equations are always divisible by the corresponding denominators so by canceling these common factors $Q_r(\mathbf{i})$ can be interpreted as an element of $\mathscr{H}_n^\Lambda$.

Finally, for $r = 1, \dots, n-1$ set

$$\psi_r = \sum_{\mathbf{i}\in I^n} (T_r + P_r(\mathbf{i})) Q_r(\mathbf{i})^{-1} e(\mathbf{i}).$$

We are abusing notation here because we are not distinguishing between the generators of the cyclotomic Khovanov-Lauda–Rouquier algebra and the elements that we have just defined in $\mathscr{H}_n^\Lambda$. This abuse is justified by the Brundan-Kleshchev graded isomorphism theorem.

3.5. **Theorem** (Brundan–Kleshchev [**6**, §4.5])**.** *The map $\mathscr{R}_n^\Lambda \longrightarrow \mathscr{H}_n^\Lambda$ which sends*

$$e(\mathbf{i}) \mapsto e(\mathbf{i}), \qquad y_r \mapsto y_r \qquad \text{and} \qquad \psi_s \mapsto \psi_s,$$

for $\mathbf{i} \in I^n$, $1 \leq r \leq n$ and $1 \leq s < n$, extends uniquely to an isomorphism of algebras. An inverse isomorphism is given by

$$L_r \mapsto \sum_{\mathbf{i}\in I^n} q^{i_r}(1 - y_r) e(\mathbf{i}), \qquad \text{and} \qquad T_s \mapsto \sum_{\mathbf{i}\in I^n} (\psi_s Q_s(\mathbf{i}) - P_s(\mathbf{i})) e(\mathbf{i}),$$

for $1 \leq r \leq n$ and $1 \leq s < n$.

Hereafter, we freely identify the algebras $\mathscr{R}_n^\Lambda$ and $\mathscr{H}_n^\Lambda$, and their generators, using this result. In particular, we consider $\mathscr{H}_n^\Lambda$ to be a $\mathbb{Z}$-graded algebra. All $\mathscr{H}_n^\Lambda$-modules will be $\mathbb{Z}$-graded unless otherwise noted.

§3.3. **Tableaux combinatorics and the standard basis.** We close this section by introducing some combinatorics and defining the standard basis of $\mathscr{H}_n^\Lambda$.

Recall that an **multipartition**, or ℓ-partition, of n is an ordered sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. The partitions $\lambda^{(1)}, \dots, \lambda^{(\ell)}$ are the **components** of $\boldsymbol{\lambda}$. Let $\mathscr{P}_n^\Lambda$ be the set of multipartitions of n. Then $\mathscr{P}_n^\Lambda$ is partially ordered by **dominance** where $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{i=1}^{j} \lambda_i^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{i=1}^{j} \mu_i^{(s)}$$

for all $1 \leq s \leq \ell$ and all $j \geq 1$. We write $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ or $\boldsymbol{\lambda} = \boldsymbol{\mu}$.

The **diagram** of an multipartition $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ is the set

$$[\boldsymbol{\lambda}] = \{ (r, c, l) \mid 1 \leq c \leq \lambda_r^{(l)}, r \geq 0 \text{ and } 1 \leq l \leq \ell \},$$

which we think of as an ordered ℓ-tuple of the diagrams of the partitions $\lambda^{(1)}, \dots, \lambda^{(\ell)}$. A **$\boldsymbol{\lambda}$-tableau** is a bijective map $\mathfrak{t} \colon [\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$. We think of $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(\ell)})$ as a labeling of the diagram of $\boldsymbol{\lambda}$. T his allows us to talk of the rows, columns and components of $\mathfrak{t}$. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau then set $\operatorname{Shape}(\mathfrak{t}) = \boldsymbol{\lambda}$.

A **standard $\boldsymbol{\lambda}$-tableau** is a $\boldsymbol{\lambda}$-tableau in which, in each component, the entries increase along each row and down each column. Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau let $\mathfrak{t}_k$ be the subtableau of $\mathfrak{t}$ labeled by $1, \dots, k$ in $\mathfrak{t}$. If $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$ then $\mathfrak{s}$ **dominates** $\mathfrak{t}$, and we write $\mathfrak{s} \unrhd \mathfrak{t}$, if $\operatorname{Shape}(\mathfrak{s}_k) \unrhd \operatorname{Shape}(\mathfrak{t}_k)$, for $k = 1, \dots, n$. Again, we write $\mathfrak{s} \rhd \mathfrak{t}$ if $\mathfrak{s} \unrhd \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. Extend the dominance partial ordering to pairs of partitions of the same shape by declaring that $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$, for $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}(\boldsymbol{\lambda})^2$ and $(\mathfrak{u}, \mathfrak{v}) \in \operatorname{Std}(\boldsymbol{\mu})^2$, if $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$ and either $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, or $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathfrak{u} \unrhd \mathfrak{s}$ and $\mathfrak{v} \unrhd \mathfrak{t}$.

Let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the unique standard $\boldsymbol{\lambda}$-tableau such that $\mathfrak{t}^{\boldsymbol{\lambda}} \unrhd \mathfrak{t}$ for all $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then $\mathfrak{t}^{\boldsymbol{\lambda}}$ has the numbers $1, \dots, n$ entered in order, from left to right and then top to bottom in each component, along the rows of $\boldsymbol{\lambda}$. The symmetric group acts on the set of $\boldsymbol{\lambda}$-tableaux. If $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ let $d(\mathfrak{t})$ be the permutation in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{t})$.

Recall from section 3.1 that we have fixed a multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell)$ which determines Λ.

3.6. **Definition** ([**12**, Definition 3.14])**.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Define $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} m_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}$, where

$$m_{\boldsymbol{\lambda}} = \prod_{s=2}^{\ell} \prod_{k=1}^{|\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|} (L_k - q^{\kappa_s}) \cdot \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w.$$

Here and below whenever an element of $\mathscr{H}_n^\Lambda$ is indexed by a pair of standard tableaux then these tableaux will always be assumed to have the same shape.

3.7. **Theorem** (The standard basis theorem [**12**, Theorem 3.26])**.** *The basis*

$$\{ m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \textit{ for } \boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda \}$$

is an ungraded cellular basis of $\mathscr{H}_n^\Lambda$*.*

Using the theory of (ungraded) cellular algebras from section 2 (or [**18**]), we could now construct Specht modules, or cell modules, for $\mathscr{H}_n^\Lambda$. We postpone doing this until section 5, however, where we are able to define *graded* Specht modules using Theorem 5.8 and the theory of graded cellular algebras developed in section 2.

Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\gamma = (r, c, l) \in [\boldsymbol{\lambda}]$. Then the **content** of γ is $\operatorname{cont}(\gamma) = \kappa_l + c - r \in \mathbb{Z}$ and the **residue** of γ is $\operatorname{res}(\gamma) = \operatorname{cont}(\gamma) + e\mathbb{Z}$. Thus, $\operatorname{cont}(\gamma) \in \mathbb{Z}$ and $\operatorname{res}(\gamma) \in I$. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau and $1 \leq k \leq n$ set $\operatorname{cont}_{\mathfrak{t}}(k) = \operatorname{cont}(\gamma)$

and $\mathrm{res}_{\mathfrak{t}}(k)=\mathrm{res}(\gamma)$, where γ is the unique node in $[\boldsymbol{\lambda}]$ such that $\mathfrak{t}(\gamma)=k$. Then, by [**20**, Prop. 3.7], there exist scalars $r_{\mathfrak{u}\mathfrak{v}}\in K$ such that

$$m_{\mathfrak{s}\mathfrak{t}}L_k = q^{\mathrm{res}_{\mathfrak{t}}(k)}m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}}. \tag{3.8}$$

If $\mathfrak{t}\in\mathrm{Std}(\boldsymbol{\lambda})$ is a standard $\boldsymbol{\lambda}$-tableau then its **residue sequence** $\mathrm{res}(\mathfrak{t})$ is the sequence

$$\mathrm{res}(\mathfrak{t}) = \big(\mathrm{res}_{\mathfrak{t}}(1),\dots,\mathrm{res}_{\mathfrak{t}}(n)\big).$$

We also write $\mathbf{i}^{\mathfrak{t}}=\mathrm{res}(\mathfrak{t})$. Set $\mathrm{Std}(\mathbf{i})=\coprod_{\boldsymbol{\lambda}\in\mathscr{P}_n^{\Lambda}}\{\,\mathfrak{t}\in\mathrm{Std}(\boldsymbol{\lambda})\mid\mathrm{res}(\mathfrak{t})=\mathbf{i}\,\}$.

4. The seminormal basis and homogeneous elements of $\mathscr{H}_n^{\Lambda}$

The aim of this section is to give an explicit description of the non-zero idempotents $e(\mathbf{i})$ in terms of certain primitive idempotents for the algebra $\mathscr{H}_n^{\Lambda}$ in the semisimple case. We then use this description to construct a family of homogeneous elements in $\mathscr{H}_n^{\Lambda}$ indexed by $\mathscr{P}_n^{\Lambda}$.

§4.1. **The Khovanov-Lauda–Rouquier idempotents.** Let $\mathscr{L}_n^{\Lambda}=\langle L_1,\dots,L_n\rangle$ be the subalgebra of $\mathscr{H}_n^{\Lambda}$ generated by the Jucys-Murphy elements of $\mathscr{H}_n^{\Lambda}$. Then $\mathscr{L}_n^{\Lambda}$ is a commutative subalgebra of $\mathscr{H}_n^{\Lambda}$.

The following simple Lemma indicates the difficulties of working with the homogeneous presentation of $\mathscr{H}_n^{\Lambda}$: we do not know how to prove this result without recourse to Brundan and Kleshchev's graded isomorphism $\mathscr{R}_n^{\Lambda}\cong\mathscr{H}_n^{\Lambda}$ (Theorem 3.5).

4.1. **Lemma.** *Suppose that $e(\mathbf{i})\neq 0$, for $\mathbf{i}\in I^n$. Then:*

a) *$e(\mathbf{i})$ is the unique idempotent in $\mathscr{H}_n^{\Lambda}$ such that $\mathscr{H}_{\mathbf{j}}e(\mathbf{i})=\delta_{\mathbf{i}\mathbf{j}}\mathscr{H}_{\mathbf{i}}$, for $\mathbf{j}\in I^n$;*
b) *$e(\mathbf{i})$ is a primitive idempotent in $\mathscr{L}_n^{\Lambda}$; and,*
c) *$\mathbf{i}=\mathrm{res}(\mathfrak{t})$ for some standard tableau $\mathfrak{t}$.*

Thus, the idempotents $\{\,e(\mathbf{i})\mid\mathbf{i}\in I^n\,\}\setminus\{0\}$ are the (central) primitive idempotents of $\mathscr{L}_n^{\Lambda}$.

Proof. By definition, $\mathscr{H}_{\mathbf{j}}e(\mathbf{i})=\delta_{\mathbf{i}\mathbf{j}}\mathscr{H}_{\mathbf{i}}$ so (a) follows since $e(\mathbf{i})\in\mathscr{H}_n^{\Lambda}e(\mathbf{i})$. Next, observe that every irreducible representation of $\mathscr{L}_n^{\Lambda}$ is one dimensional since $\mathscr{L}_n^{\Lambda}$ is a commutative algebra over a field. Further, modulo more dominant terms, L_k acts on the standard basis element $m_{\mathfrak{s}\mathfrak{t}}$ as multiplication by $q^{\mathrm{res}_{\mathfrak{t}}(k)}$ by (3.8). Therefore, the standard basis of $\mathscr{H}_n^{\Lambda}$ induces an $\mathscr{L}_n^{\Lambda}$-module filtration of $\mathscr{H}_n^{\Lambda}$ and the irreducible representations of $\mathscr{L}_n^{\Lambda}$ are indexed by the residue sequences $\mathrm{res}(\mathfrak{t})\in I^n$, for $\mathfrak{t}$ a standard $\boldsymbol{\lambda}$-tableau for some $\boldsymbol{\lambda}\in\mathscr{P}_n^{\Lambda}$. Consequently, the decomposition $\mathscr{H}_n^{\Lambda}=\bigoplus\mathscr{H}_{\mathbf{i}}$ is nothing more than the decomposition of $\mathscr{H}_n^{\Lambda}$ into a direct sum of block components when $\mathscr{H}_n^{\Lambda}$ is considered as an $\mathscr{L}_n^{\Lambda}$-module by restriction. Parts (b) and (c) now follow. □

4.2. **Corollary.** *As (graded) subalgebras of $\mathscr{H}_n^{\Lambda}$, $\mathscr{L}_n^{\Lambda}=\langle y_1,\dots,y_n,e(\mathbf{i})\mid\mathbf{i}\in I^n\rangle$.*

Proof. By Theorem 3.5, if $1\le r\le n$ then $y_r\in\mathscr{L}_n^{\Lambda}$ and $L_r\in\langle y_1,\dots,y_n,e(\mathbf{i})\mid\mathbf{i}\in I^n\rangle$. Further, by Lemma 4.1, $e(\mathbf{i})\in\mathscr{L}_n^{\Lambda}$, for $\mathbf{i}\in I^n$. Combining these two observations proves the Corollary. □

§4.2. **Idempotents and the seminormal form.** Lemma 4.2 of [**30**] explicitly constructs a family of idempotents in $\mathscr{H}_n^{\Lambda}$ which are indexed by the residue sequences of standard tableaux. As we now recall, these idempotents are defined by 'modular reduction' from the semisimple case.

Let x be an indeterminate over k and let $\mathcal{O}=K[x]_{(x)}$ be the localization of $K[x]$ at $x=0$. Then $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\pi=x\mathcal{O}$. Note that $x+q$ is invertible in $\mathcal{O}$ since $q\neq 0$. Let $\mathcal{K}=K(x)$ and consider $\mathcal{O}$ as a subring of $\mathcal{K}$.

Let $\mathscr{H}_n^{\mathcal{O}}$ be the Hecke algebra of type $G(\ell,1,n)$ with parameters $x+q$ and $Q_s=(x+q)^{\kappa_s}$, for $1\le s\le \ell$. That is, $\mathscr{H}_n^{\mathcal{O}}$ is the unital associative $\mathcal{O}$-algebra with generators $T_0,T_1,\dots,T_{n-1}$ and relations

$$\begin{gathered}(T_0-(x+q)^{\kappa_1})\dots(T_0-(x+q)^{\kappa_\ell})=0,\\ (T_r+1)(T_r-x-q)=0, \qquad T_0T_1T_0T_1=T_1T_0T_1T_0,\\ T_sT_{s+1}T_s=T_{s+1}T_sT_{s+1} \qquad \text{and } T_rT_s=T_sT_r,\end{gathered}$$

for $1\le r<n$ and $1\le s<n-1$ and where in the last equation $|r-s|>1$. Exactly as in section 3.2, we can define elements T_w and L_i of $\mathscr{H}_n^{\mathcal{O}}$, for $w\in\mathfrak{S}_n$ and $1\le i\le n$, and the main result of [**4**] says that $\mathscr{H}_n^{\mathcal{O}}$ is free as an $\mathcal{O}$-module with basis $\{\,L_1^{a_1}\dots L_n^{a_n}T_w \mid 0\le a_1,\dots,a_n<\ell \text{ and } w\in\mathfrak{S}_n\,\}$. Moreover, $\mathscr{H}_n^{\Lambda}\cong\mathscr{H}_n^{\mathcal{O}}\otimes_{\mathcal{O}}K$. Finally, let $\mathscr{H}_n^{\mathcal{K}}:=\mathscr{H}_n^{\mathcal{O}}\otimes_{\mathcal{O}}\mathcal{K}$. Then $\mathscr{H}_n^{\mathcal{K}}$ is a split semisimple algebra by Ariki's semisimplicity criterion [**1**] because in section 3.1 we fixed a multicharge $\boldsymbol{\kappa}$ with the property that $\kappa_i-\kappa_{i+1}\ge n$, for $1\le i<\ell$. We abuse notation and write T_w, L_k and $m_{\mathfrak{st}}$ for the elements of the three algebras $\mathscr{H}_n^{\mathcal{O}}\hookrightarrow\mathscr{H}_n^{\mathcal{K}}$ and $\mathscr{H}_n^{\Lambda}$ as the meaning should always be clear from context.

By [**20**, Prop. 3.7], the analogue of (3.8) for the algebras $\mathscr{H}_n^{\mathcal{O}}$ and $\mathscr{H}_n^{\mathcal{K}}$ is

$$m_{\mathfrak{st}}L_k=(x+q)^{\operatorname{cont}_{\mathfrak{t}}(k)}m_{\mathfrak{st}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})}r_{\mathfrak{uv}}m_{\mathfrak{uv}}. \tag{4.3}$$

This equation motivates the following definition.

4.4. **Definition** ([**29**, Defn 2.4])**.** Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_n^{\Lambda}$ and $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. Define

$$F_{\mathfrak{t}}=\prod_{k=1}^{n}\prod_{\substack{\kappa_\ell-n<c<\kappa_1+n\\ c\ne\operatorname{cont}_{\mathfrak{t}}(k)}}\frac{L_k-(x+q)^c}{(x+q)^{\operatorname{cont}_{\mathfrak{t}}(k)}-(x+q)^c}\in\mathscr{H}_n^{\mathcal{K}},$$

and set $f_{\mathfrak{st}}=F_{\mathfrak{s}}m_{\mathfrak{st}}F_{\mathfrak{t}}$.

By (4.3), $f_{\mathfrak{st}}=m_{\mathfrak{st}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})}r_{\mathfrak{uv}}m_{\mathfrak{uv}}$, for some $r_{\mathfrak{uv}}\in\mathcal{K}$. Therefore,

$$\{\,f_{\mathfrak{st}}\mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\text{ for }\boldsymbol{\lambda}\in\mathscr{P}_n^{\Lambda}\,\}$$

is a basis of $\mathscr{H}_n^{\mathcal{K}}$. This basis is the **seminormal basis** of $\mathscr{H}_n^{\mathcal{K}}$; see [**29**, Theorem 2.11]. The next definition, which is the key to what follows, allows us to write $F_{\mathfrak{t}}$ in terms of the seminormal basis and hence connect these elements with the graded representation theory.

Let $\boldsymbol{\lambda}$ be a multipartition. The node $\alpha=(r,c,l)\in[\boldsymbol{\lambda}]$ is an **addable node** of $\boldsymbol{\lambda}$ if $\alpha\notin[\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}]\cup\{\alpha\}$ is the diagram of a multipartition. Similarly, $\rho\in[\boldsymbol{\lambda}]$ is a **removable node** of $\boldsymbol{\lambda}$ if $[\boldsymbol{\lambda}]\setminus\{\rho\}$ is the diagram of a multipartition. Given two nodes $\alpha=(r,c,l)$ and $\beta=(s,d,m)$ then α is **below** β if either $l>m$, or $l=m$ and $r>s$.

4.5. **Definition** ([**20**, Defn. 3.15] and [**29**, (2.8)])**.** Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_n^{\Lambda}$ and $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. For $k=1,\dots,n$ let $\mathscr{A}_{\mathfrak{t}}(k)$ be the set of addable nodes of the multipartition $\operatorname{Shape}(\mathfrak{t}_k)$ which are *below* $\mathfrak{t}^{-1}(k)$. Similarly, let $\mathscr{R}_{\mathfrak{t}}(k)$ be the set of removable nodes of $\operatorname{Shape}(\mathfrak{t}_k)$ which are *below* $\mathfrak{t}^{-1}(k)$. Now define

$$\gamma_{\mathfrak{t}}=(x+q)^{\ell(d(\mathfrak{t}))+\delta(\boldsymbol{\lambda})}\prod_{k=1}^{n}\frac{\prod_{\alpha\in\mathscr{A}_{\mathfrak{t}}(k)}\big((x+q)^{\operatorname{cont}_{\mathfrak{t}}(k)}-(x+q)^{\operatorname{cont}(\alpha)}\big)}{\prod_{\rho\in\mathscr{R}_{\mathfrak{t}}(k)}\big((x+q)^{\operatorname{cont}_{\mathfrak{t}}(k)}-(x+q)^{\operatorname{cont}(\rho)}\big)}\quad\in\mathcal{K},$$

where $\delta(\boldsymbol{\lambda})=\frac12\sum_{s=1}^{\ell}\sum_{i\ge1}(\lambda_i^{(s)}-1)\lambda_i^{(s)}$.

It is an easy exercise in the definitions to check that the terms in the denominator of $\gamma_{\mathfrak{t}}$ are never zero so that $\gamma_{\mathfrak{t}}$ is a well-defined element of $\mathcal{K}$. As the algebra $\mathscr{H}_n^{\mathcal{K}}$ is semisimple we have the following.

4.6. **Lemma** ([**29**, Theorem 2.15]). *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ *and* $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$*. Then* $F_\mathfrak{t} = \frac{1}{\gamma_\mathfrak{t}} f_{\mathfrak{tt}}$ *is a primitive idempotent in* $\mathscr{H}_n^\mathcal{K}$*.*

For any standard tableau $\mathfrak{t}$ and an integer k, with $1 \le k \le n$, define sets $\mathscr{A}_\mathfrak{t}^\Lambda(k)$ and $\mathscr{R}_\mathfrak{t}^\Lambda(k)$ by

$$\begin{aligned} \mathscr{A}_\mathfrak{t}^\Lambda(k) &= \{\, \alpha \in \mathscr{A}_\mathfrak{t}(k) \mid \mathrm{res}(\alpha) = \mathrm{res}_\mathfrak{t}(k) \,\} \\ \text{and} \quad \mathscr{R}_\mathfrak{t}^\Lambda(k) &= \{\, \rho \in \mathscr{R}_\mathfrak{t}(k) \mid \mathrm{res}(\rho) = \mathrm{res}_\mathfrak{t}(k) \,\}. \end{aligned}$$

Using this notation we can give a non-recursive definition of the Brundan-Kleshchev-Wang degree function on standard tableaux.

4.7. **Definition** (Brundan, Kleshchev and Wang [**8**, Defn. 3.5])**.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and that $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau. Then

$$\deg \mathfrak{t} = \sum_{k=1}^n \Big(|\mathscr{A}_\mathfrak{t}^\Lambda(k)| - |\mathscr{R}_\mathfrak{t}^\Lambda(k)| \Big),$$

The next result connects the graded representation theory of $\mathscr{H}_n^\Lambda$ with the seminormal basis.

4.8. **Proposition.** *Suppose that* $e(\mathbf{i}) \neq 0$*, for some* $\mathbf{i} \in I^n$ *and let*

$$e(\mathbf{i})^\mathcal{O} := \sum_{\mathfrak{s} \in \mathrm{Std}(\mathbf{i})} \frac{1}{\gamma_\mathfrak{s}} f_{\mathfrak{ss}} \in \mathscr{H}_n^\mathcal{K}.$$

Then $e(\mathbf{i})^\mathcal{O} \in \mathscr{H}_n^\mathcal{O}$ *and* $e(\mathbf{i}) = e(\mathbf{i})^\mathcal{O} \otimes_\mathcal{O} 1_K$*.*

Proof. It is shown in [**30**, Lemma 4.2] that $e(\mathbf{i})^\mathcal{O}$ is an element of $\mathscr{H}_n^\mathcal{O}$. Therefore, we can reduce $e(\mathbf{i})^\mathcal{O}$ modulo π to obtain an element of $\mathscr{H}_n^\Lambda$: let $\hat{e}(\mathbf{i}) = e(\mathbf{i})^\mathcal{O} \otimes_\mathcal{O} 1_K$. Then $\{\, \hat{e}(\mathbf{j}) \mid \mathbf{j} \in I^n \,\}$ is a family of pairwise orthogonal idempotents in $\mathscr{H}_n^\Lambda$ such that $1_\mathscr{H} = \sum_\mathbf{j} \hat{e}(\mathbf{i})$ by [**30**, Cor. 4.7].

As in [**30**, Defn. 4.3], for every pair $(\mathfrak{s}, \mathfrak{t})$ of standard tableaux of the same shape define $g_{\mathfrak{st}} = \hat{e}(\mathbf{i}^\mathfrak{s}) m_{\mathfrak{st}} \hat{e}(\mathbf{i}^\mathfrak{t})$. Then $\{g_{\mathfrak{st}}\}$ is a (cellular) basis of $\mathscr{H}_n^\Lambda$ by [**30**, Theorem 4.5]. Moreover, by [**30**, Prop. 4.4], if $1 \le k \le n$ then

$$g_{\mathfrak{st}}(L_k - q^{\mathrm{res}_\mathfrak{t}(k)}) = \sum_{\substack{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t}) \\ \mathfrak{u} \in \mathrm{Std}(\mathbf{i}^\mathfrak{s}) \text{ and } \mathfrak{v} \in \mathrm{Std}(\mathbf{i}^\mathfrak{t})}} r_{\mathfrak{uv}} g_{\mathfrak{uv}},$$

for some $r_{\mathfrak{uv}} \in K$. It follows that $g_{\mathfrak{st}}(L_k - q^{\mathrm{res}_\mathfrak{t}(k)})^N = 0$ for $N \gg 0$. Therefore,

$$\mathscr{H}_\mathbf{i} = \sum_{\substack{\mathfrak{u} \text{ standard} \\ \mathfrak{v} \in \mathrm{Std}(\mathbf{i})}} K g_{\mathfrak{uv}} = \mathscr{H}_n^\Lambda \hat{e}(\mathbf{i}).$$

Hence, $e(\mathbf{i}) = \hat{e}(\mathbf{i})$ by Lemma 4.1(a) as required. □

§4.3. **Positive tableaux.** The idempotents $e(\mathbf{i})$ in the graded presentation of $\mathscr{H}_n^\Lambda \cong \mathscr{R}_n^\Lambda$ hide a lot of important information about the algebra. Proposition 4.8 gives us a way of accessing this information.

If $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ then set $\mathbf{i}_k = (i_1, \dots, i_k)$ so that $\mathbf{i}_k \in I^k$, for $1 \le k \le n$.

4.9. **Definition.** Suppose that $\mathbf{i} \in I^n$ and that $\mathfrak{s} \in \mathrm{Std}(\mathbf{i})$. Then $\mathfrak{s}$ is a **positive tableau** if, for $k = 1, \dots, n-1$:

a) $\mathscr{R}_\mathfrak{s}^\Lambda(k+1) = \emptyset$, and,
b) if $\alpha \in \mathscr{A}_\mathfrak{t}^\Lambda(k)$, where $\mathfrak{t} \in \mathrm{Std}(\mathbf{i}_k)$ and $\mathfrak{t} \rhd \mathfrak{s}_k$, then $\alpha \in \mathscr{A}_\mathfrak{s}^\Lambda(k)$ whenever α is below $\mathfrak{s}^{-1}(k+1)$.

Let $\operatorname{Std}^+(\mathbf{i}) = \{\,\mathfrak{s} \in \operatorname{Std}(\mathbf{i}) \mid \mathfrak{s} \text{ is positive}\,\}$. If $\mathfrak{s} \in \operatorname{Std}^+(\mathbf{i})$ then define

$$y_{\mathfrak{s}} = \prod_{k=1}^{n} y_k^{|\mathscr{A}_{\mathfrak{s}}^{\Lambda}(k)|} \in \mathscr{H}_n^{\Lambda}.$$

By definition, $\deg \mathfrak{s} \geq 0$ whenever $\mathfrak{s}$ is positive. The converse is false because there are many standard tableau $\mathfrak{t}$ which are not positive such that $\deg \mathfrak{t} \geq 0$.

4.10. **Examples** (a) Suppose that $e = 3$, $\ell = 1$ and $\mathbf{i} = (0, 1, 2, 2, 0, 1, 1, 2, 0)$. Then the positive tableaux in $\operatorname{Std}(\mathbf{i})$ are:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\,, \qquad \begin{array}{cccccc} 1 & 2 & 3 & 5 & 6 & 8 \\ 4 & 9 & & & & \\ 7 & & & & & \end{array}\,, \qquad \begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 8 & 9 \\ 4 & & & & & & \\ 7 & & & & & & \end{array}\,.$$

(b) Suppose that $e = 3$, $\ell = 1$ and let $\mathfrak{t} = \begin{array}{cccccc} 1 & 2 & 4 & 5 & 6 & 7 \\ 3 & & & & & \end{array}$. Then $\deg \mathfrak{t} = 0$, however, the tableau $\mathfrak{t}$ is not positive.

(c) Suppose that $e = 2$ and $\mathbf{i} = (0, 1, 1, 0, 1, 0)$ and let

$$\mathfrak{t} = \begin{array}{cccc} 1 & 2 & 4 & 5 \\ 3 & & & \\ 6 & & & \end{array} \qquad \text{and} \qquad \mathfrak{s} = \begin{array}{cc} 1 & 2 \\ 3 & 6 \\ 4 & \\ 5 & \end{array}.$$

Then $\mathfrak{s}$ and $\mathfrak{t}$ both belong to $\operatorname{Std}(\mathbf{i})$ and $\mathscr{R}_{\mathfrak{s}}^{\Lambda}(k) = \emptyset$, for $1 \leq k \leq 6$. However, $\mathfrak{s}$ is not a positive tableau because the node $(3, 1) = \mathfrak{t}^{-1}(6)$ is below $(2, 2) = \mathfrak{s}^{-1}(6)$ and $(3, 1)$ is not an addable node of $\mathfrak{s}_5$. $\Diamond$

Recall from section 3.2 that $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the unique standard $\boldsymbol{\lambda}$-tableau such that $\mathfrak{t}^{\boldsymbol{\lambda}} \trianglerighteq \mathfrak{t}$, for all $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The tableaux $\mathfrak{t}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$, are the most important examples of positive tableaux.

4.11. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. Then $\mathfrak{t}^{\boldsymbol{\lambda}}$ is positive.*

Proof. By definition, $\mathscr{R}_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{\Lambda}(k) = \emptyset$ for $1 \leq k \leq n$, so it remains to check condition (b) in Definition 4.9. Let $\beta = (r, c, l)$ be the lowest removable node of $\boldsymbol{\lambda}$, so that $\mathfrak{t}^{\boldsymbol{\lambda}}(\beta) = n$. By induction on n it suffices to show that $\alpha \in \mathscr{A}_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{\Lambda}(n-1)$ whenever $\alpha = (r', c', l')$ is below β and there exists a standard tableau $\mathfrak{t} \in \operatorname{Std}(\mathbf{i}_{n-1}^{\boldsymbol{\lambda}})$ such that $\mathfrak{t} \triangleright \mathfrak{t}_{n-1}^{\boldsymbol{\lambda}}$ and $\alpha \in \mathscr{A}_{\mathfrak{t}}^{\Lambda}(n-1)$.

Let $\boldsymbol{\mu} = \operatorname{Shape}(\mathfrak{t})$. Since $\mathfrak{t} \triangleright \mathfrak{t}_{n-1}^{\boldsymbol{\lambda}}$ we have that $\mu^{(k)} = (0)$ for $k > l$. Consequently, $\alpha \in \mathscr{A}_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{\Lambda}(n-1)$ if $l' > l$. As α is below β this leaves only the case when $l' = l$ in which case we have that $r' > r$. Since $\mathfrak{t} \triangleright \mathfrak{t}_{n-1}^{\boldsymbol{\lambda}}$ this forces $\alpha = (r + 1, 1, l)$ to be the addable node of $\boldsymbol{\lambda}$ in first column of the row directly below β, so $\alpha \in \mathscr{A}_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{\Lambda}(n-1)$ as required. $\square$

Suppose that $\mathfrak{s}$ is a positive tableau. To work with $e(\mathbf{i}^{\mathfrak{s}})y_{\mathfrak{s}}$ we have to choose the correct lift of it to $\mathscr{H}_n^{\mathcal{O}}$. Perhaps surprisingly, we choose a lift which depends on the tableau $\mathfrak{s}$ rather than choosing a single lift for each of the homogeneous elements $y_1, \dots, y_n$.

4.12. **Definition.** Suppose that $\mathbf{i} \in I^n$ and $\mathfrak{s} \in \operatorname{Std}^+(\mathbf{i})$. Define $y_{\mathfrak{s}}^{\mathcal{O}} = y_{\mathfrak{s},1}^{\mathcal{O}} \dots y_{\mathfrak{s},n}^{\mathcal{O}}$, where

$$y_{\mathfrak{s},k}^{\mathcal{O}} = \prod_{\alpha \in \mathscr{A}_{\mathfrak{s}}^{\Lambda}(k)} \big(1 - (x+q)^{-\operatorname{cont}(\alpha)} L_k\big) \quad \in \mathscr{H}_n^{\mathcal{O}},$$

for $k = 0, \dots, n$ (by convention, empty products are 1).

By definition, $y_{\mathfrak{s}}^{\mathcal{O}} \in \mathscr{H}_n^{\mathcal{O}}$. Moreover, $e(\mathbf{i}^{\mathfrak{s}})y_{\mathfrak{s}} = e(\mathbf{i}^{\mathfrak{s}})^{\mathcal{O}} y_{\mathfrak{s}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K \in \mathscr{H}_n^{\Lambda}$.

4.13. **Lemma.** *Suppose that $\mathbf{i} \in I^n$ and that $\mathfrak{s} \in \operatorname{Std}^+(\mathbf{i})$ and $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. Then:*

a) *If $\mathfrak{t} = \mathfrak{s}$ then $f_{\mathfrak{s}\mathfrak{s}} y_{\mathfrak{s}}^{\mathcal{O}} = u_{\mathfrak{s}}^{\mathcal{O}} \gamma_{\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}$, for some unit $u_{\mathfrak{s}}^{\mathcal{O}} \in \mathcal{O}$.*

b) *If $\mathfrak{t}\neq\mathfrak{s}$ then there exists an integer $d_\mathfrak{t}\ge\deg\mathfrak{s}$ and an invertible element $u^\mathcal{O}_\mathfrak{t}\in\mathcal{O}$ such that*

$$f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_\mathfrak{s}=\begin{cases}u^\mathcal{O}_\mathfrak{t}x^{d_\mathfrak{t}}f_{\mathfrak{t}\mathfrak{t}}, & \text{if } \mathfrak{t}\rhd\mathfrak{s},\\ 0, & \text{otherwise,}\end{cases}$$

and where $d_\mathfrak{t}=\deg\mathfrak{s}$ whenever $\mathcal{K}$ is a field of characteristic zero.

Proof. By [**29**, Prop. 2.6], if $1\le k\le n$ then $f_{\mathfrak{t}\mathfrak{t}}L_k=(x+q)^{\mathrm{cont}_\mathfrak{t}(k)}f_{\mathfrak{t}\mathfrak{t}}$ in $\mathscr{H}^\mathcal{K}_n$, so $f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_\mathfrak{s}$ is a scalar multiple of $f_{\mathfrak{t}\mathfrak{t}}$ and it remains to determine the multiple.

(a) Observe that $\mathscr{R}_\mathfrak{s}(k)=\emptyset$ for $1\le k\le n$. Further, if $\alpha\in\mathscr{A}_\mathfrak{s}(k)$ and $\alpha\notin\mathscr{A}^\Lambda_\mathfrak{s}(k)$ then the factor that α contributes to $\gamma_\mathfrak{s}$ is a unit in $\mathcal{O}$. Therefore, applying Definition 4.5 and Definition 4.12 shows that

$$f_{\mathfrak{s}\mathfrak{s}}y^\mathcal{O}_\mathfrak{s}=\prod_{k=1}^n\prod_{\alpha\in\mathscr{A}^\Lambda_\mathfrak{s}(k)}\big(1-(x+q)^{\mathrm{cont}_\mathfrak{s}(k)-\mathrm{cont}(\alpha)}\big)\cdot f_{\mathfrak{s}\mathfrak{s}}=u^\mathcal{O}_\mathfrak{s}\gamma_\mathfrak{s}f_{\mathfrak{s}\mathfrak{s}},$$

for some invertible element $u^\mathcal{O}_\mathfrak{s}\in\mathcal{O}$, proving (a).

(b) Suppose that $1\le k\le n$. Then we claim that

$$f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_{\mathfrak{s},1}\dots y^\mathcal{O}_{\mathfrak{s},k}=\begin{cases}u^\mathcal{O}_{\mathfrak{t},k}x^{d_k}f_{\mathfrak{t}\mathfrak{t}}, & \text{if } \mathfrak{t}_k\trianglerighteq\mathfrak{s}_k,\\ 0, & \text{otherwise,}\end{cases}$$

where $d_k\ge\deg\mathfrak{s}_k$ and $u^\mathcal{O}_{\mathfrak{t},k}\in\mathcal{O}$ is invertible. If $k=0$ then there is nothing to prove so we may assume by induction that the claim is true for $f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_{\mathfrak{s},1}\dots y^\mathcal{O}_{\mathfrak{s},k}$ and consider $f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_{\mathfrak{s},1}\dots y^\mathcal{O}_{\mathfrak{s},k+1}$.

If $\mathfrak{t}_k\ntrianglerighteq\mathfrak{s}_k$ then, by induction, both sides of the claim are zero, so we may assume that $\mathfrak{t}_k\trianglerighteq\mathfrak{s}_k$ Let $\rho=\mathfrak{t}^{-1}(k+1)$ be the node labeled by $k+1$ in $\mathfrak{t}$ and β be the node labeled by $k+1$ in $\mathfrak{s}$.

Suppose first that $\mathfrak{t}_{k+1}\ntrianglerighteq\mathfrak{s}_{k+1}$. As $\mathfrak{t}_k\trianglerighteq\mathfrak{s}_k$ this can happen if and only if ρ is below β. However, since $\mathfrak{s}$ is positive, every addable node of $\mathfrak{t}_k$ below β is an addable node of $\mathfrak{s}_k$. Hence, $\rho\in\mathscr{A}^\Lambda_\mathfrak{s}(k+1)$ and, consequently (since $\mathrm{res}(\mathfrak{s})=\mathrm{res}(\mathfrak{t})$), $\mathrm{cont}_\mathfrak{t}(k+1)=\mathrm{cont}(\alpha)$, for some $\alpha\in\mathscr{A}^\Lambda_\mathfrak{s}(k+1)$. Therefore, the coefficient of $f_{\mathfrak{t}\mathfrak{t}}$ in $f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_{\mathfrak{s},1}\dots y^\mathcal{O}_{\mathfrak{s},k+1}$ is zero, as we needed to show.

Next, suppose that $\mathfrak{t}_{k+1}\trianglerighteq\mathfrak{s}_{k+1}$. Then ρ is not below β. Consequently, if $\alpha\in\mathscr{A}^\Lambda_\mathfrak{s}(k+1)$ then $(1-(x+q)^{\mathrm{cont}_\mathfrak{t}(k+1)-\mathrm{cont}(\alpha)})=ux^d$, for some $d\ge1$ and some unit $u\in\mathcal{O}$, since $\mathrm{res}(\alpha)=\mathrm{res}_\mathfrak{s}(k+1)$. This shows that $f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_{\mathfrak{s},1}\dots y^\mathcal{O}_{\mathfrak{s},k+1}$ can be written in the required form and so proves the claim and completes the proof of the Lemma. □

4.14. **Theorem.** *Suppose that $\mathbf{i}\in I^n$ and that $\mathfrak{s}\in\mathrm{Std}^+(\mathbf{i})$. Then there exists a non-zero scalar $c\in K$ such that*

$$e(\mathbf{i})y_\mathfrak{s}=cm_{\mathfrak{s}\mathfrak{s}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{s})}r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}},$$

some $r_{\mathfrak{u}\mathfrak{v}}\in K$. In particular, $y_\mathfrak{s}$ is a non-zero homogeneous element of degree $2\deg\mathfrak{s}$.

Proof. To prove the theorem we work in $\mathscr{H}^\mathcal{O}_n$ and in $\mathscr{H}^\mathcal{K}_n$. By Lemma 4.13, inside $\mathscr{H}^\mathcal{K}_n$ we have

$$e(\mathbf{i})^\mathcal{O}y^\mathcal{O}_\mathfrak{s}=\sum_{\mathfrak{t}\in\mathrm{Std}(\mathbf{i})}\frac{1}{\gamma_\mathfrak{t}}f_{\mathfrak{t}\mathfrak{t}}y^\mathcal{O}_\mathfrak{s}=u^\mathcal{O}_\mathfrak{s}f_{\mathfrak{s}\mathfrak{s}}+\sum_{\substack{\mathfrak{t}\in\mathrm{Std}(\mathbf{i})\\ \mathfrak{t}\rhd\mathfrak{s}}}\frac{u^\mathcal{O}_{\mathfrak{t},n}x^{d_\mathfrak{t}}}{\gamma_\mathfrak{t}}f_{\mathfrak{t}\mathfrak{t}}.$$

for some invertible elements $u_{\mathfrak{t}}^{\mathcal{O}} \in \mathcal{O}$ and where $d_{\mathfrak{t}} \geq \deg \mathfrak{s}$ for each $\mathfrak{t}$. Rewriting this equation in terms of the standard basis we see that

$$e(\mathbf{i})^{\mathcal{O}} y_{\mathfrak{s}}^{\mathcal{O}} = u_{\mathfrak{s}}^{\mathcal{O}} m_{\mathfrak{s}\mathfrak{s}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s}\mathfrak{s})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathcal{K}$. However, $e(\mathbf{i})^{\mathcal{O}} y_{\mathfrak{s}}^{\mathcal{O}} \in \mathscr{H}_n^{\mathcal{O}}$, by Proposition 4.8, and $m_{\mathfrak{u}\mathfrak{v}} \in \mathscr{H}_n^{\mathcal{O}}$ for all $(\mathfrak{u}, \mathfrak{v})$. So, in fact, $r_{\mathfrak{u}\mathfrak{v}} \in \mathcal{O}$ for all $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{s})$ and reducing this equation modulo $\pi = x\mathcal{O}$ gives the first statement in the Theorem.

Finally, since $y_{\mathfrak{s}} \neq 0$ we have that $\deg y_{\mathfrak{s}} = 2 \deg \mathfrak{s}$ by Definition 4.7 — recall that $\mathfrak{s}$ is positive only if $\mathscr{R}_{\mathfrak{s}}^{\Lambda}(k) = \emptyset$, for $1 \leq k \leq n$. □

By Lemma 4.11, the tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ is positive for any $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. Therefore, we have the following important special case of Definition 4.9.

4.15. **Definition.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. Set $e_{\boldsymbol{\lambda}} = e(\mathbf{i}^{\boldsymbol{\lambda}})$ and $y_{\boldsymbol{\lambda}} = y_{\mathfrak{t}^{\boldsymbol{\lambda}}}$.

As in section 2, if $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$ let $\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}$ be the two-sided ideal spanned by the $m_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in \mathscr{P}_n^{\Lambda}$ with $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$.

Then using Theorem 4.14 we obtain:

4.16. **Corollary.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. Then $y_{\boldsymbol{\lambda}}$ is a non-zero homogeneous element of degree $2 \deg \mathfrak{t}^{\boldsymbol{\lambda}}$. Moreover, there exists a non-zero scalar $c_{\boldsymbol{\lambda}} \in K$ such that $e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \equiv c_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} \pmod{\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}}$.*

Equivalently, $e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \equiv c_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}} \pmod{\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}}$. From small examples it is plausible that $e_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}} \in \mathscr{L}_n^{\Lambda}$, for all $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$. This would give a partial explanation for the last result.

5. A graded cellular basis of $\mathscr{H}_n^{\Lambda}$

In this section we build on Theorem 4.14 to prove our Main Theorem which shows that $\mathscr{H}_n^{\Lambda}$ is a graded cellular algebra. Brundan, Kleshchev and Wang [**8**] have already constructed a graded Specht module for $\mathscr{H}_n^{\Lambda}$. The main result of this section essentially 'lifts' the Brundan, Kleshchev and Wang's construction of the graded Specht modules to a graded cellular basis of $\mathscr{H}_n^{\Lambda}$.

§5.1. **Lifting the graded Specht modules to $\mathscr{H}_n^{\Lambda}$.** As Brundan and Kleshchev note [**7**, §4.5], it follows directly from Definition 3.1 that $\mathscr{H}_n^{\Lambda}$ has a unique K-linear anti-automorphism $*$ which fixes each of the graded generators. We warn the reader that, in general, $*$ is different from the anti-automorphism of $\mathscr{H}_n^{\Lambda}$ determined by the (ungraded) cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$.

Inspired partly by Brundan, Kleshchev and Wang's [**8**, §4.2] construction of the graded Specht modules we make the following definition.

5.1. **Definition.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and fix reduced expressions $d(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ and $d(\mathfrak{t}) = s_{j_1} \dots s_{j_m}$ for $d(\mathfrak{s})$ and $d(\mathfrak{t})$, respectively. Define

$$\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{d(\mathfrak{s})}^{*} e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{d(\mathfrak{t})},$$

where $\psi_{d(\mathfrak{s})} = \psi_{i_1} \dots \psi_{i_k}$ and $\psi_{d(\mathfrak{t})} = \psi_{j_1} \dots \psi_{j_m}$.

An immediate and very useful consequence of this definition and the homogeneous relations of $\mathscr{H}_n^{\Lambda}$ is the following.

5.2. **Lemma.** *Suppose that $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$, and that $\mathbf{i}, \mathbf{j} \in I^n$. Then*

$$e(\mathbf{i}) \psi_{\mathfrak{s}\mathfrak{t}} e(\mathbf{j}) = \begin{cases} \psi_{\mathfrak{s}\mathfrak{t}}, & \textit{if } \operatorname{res}(\mathfrak{s}) = \mathbf{i} \textit{ and } \operatorname{res}(\mathfrak{t}) = \mathbf{j}, \\ 0, & \textit{otherwise.} \end{cases}$$

The next two results combine Corollary 4.16 with Brundan, Kleshchev and Wang's results for the graded Specht modules to describe the homogeneous elements $\psi_{\mathfrak{s}\mathfrak{t}}$.

5.3. **Lemma** (cf. [**8**, Cor. 3.14]). *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ *and* $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. *Then*

$$\deg \psi_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}.$$

Proof. By [**8**, Cor. 3.14], if $d(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ is a reduced expression for $d(\mathfrak{s})$ then $\deg \mathfrak{s} - \deg \mathfrak{t}^{\boldsymbol{\lambda}} = \deg(e_{\boldsymbol{\lambda}} \psi_{\mathfrak{s}})$. Therefore,

$$\deg \psi_{\mathfrak{s}\mathfrak{t}} = \deg(\psi_{\mathfrak{s}}^* e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{\mathfrak{t}}) = \deg(e_{\boldsymbol{\lambda}} \psi_{\mathfrak{s}}) + \deg y_{\boldsymbol{\lambda}} + \deg(e_{\boldsymbol{\lambda}} \psi_{\mathfrak{t}}) = \deg \mathfrak{s} + \deg \mathfrak{t},$$

where the last equality follows because $\deg y_{\boldsymbol{\lambda}} = 2 \deg \mathfrak{t}^{\boldsymbol{\lambda}}$ by Corollary 4.16. □

We note that it is possible to prove Lemma 5.3 directly by induction on the dominance ordering on standard tableaux. We now show that $\psi_{\mathfrak{s}\mathfrak{t}}$ is non-zero.

5.4. **Lemma** (cf. [**8**, Prop. 4.5]). *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ *and that* $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. *Then there exists a non-zero scalar* $c \in K$*, which does not depend upon the choice of reduced expressions for* $d(\mathfrak{s})$ *and* $d(\mathfrak{t})$*, such that*

$$\psi_{\mathfrak{s}\mathfrak{t}} = c m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in K$.

Proof. This is an immediate consequence of Corollary 4.16 and [**8**, Theorem 4.10a], however, we sketch the proof as this result is central to this paper.

Let $d(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ and $d(\mathfrak{t}) = s_{j_1} \dots s_{j_m}$ be the reduced expressions for $d(\mathfrak{s})$ and $d(\mathfrak{t})$, respectively, that we fixed in Definition 5.1.

By Corollary 4.16, $e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}}$ is a homogeneous element of $\mathscr{H}_n^\Lambda$ and

$$e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{d(\mathfrak{t})} \equiv c_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} \pmod{\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}}.$$

Using Theorem 3.5 and the homogeneous relations of $\mathscr{H}_n^\Lambda$ it is easy to prove that $e_{\boldsymbol{\lambda}} \psi_{d(\mathfrak{t})}$ is equal to a linear combination of terms of the form $e_{\boldsymbol{\lambda}} f_w(y) T_w$, where $f_w(y) \in K[y_1, \dots, y_n]$ for some $w \in \mathfrak{S}_n$ with $w \le d(\mathfrak{t})$, and where $f_{d(\mathfrak{t})}(y)$ is invertible. By (3.8), $m_{\boldsymbol{\lambda}} y_r \equiv m_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}} y_r \equiv 0 \pmod{\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}}$, for $1 \le r \le n$. Now if $w \in \mathfrak{S}_n$ then, modulo $\mathscr{H}_n^{\rhd \lambda}$, $m_\lambda T_w$ can be written as a linear combination of elements of the form $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}}$, where $\mathfrak{v} \in \operatorname{Std}(\lambda)$ and $d(\mathfrak{v}) \le w$, by Theorem 3.7. Therefore, just as in [**8**, Prop. 4.5], we obtain

$$e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{d(\mathfrak{t})} \equiv c' m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \\ \mathfrak{v} \rhd \mathfrak{t}}} r_{\mathfrak{v}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}}$$

for some $c', r_{\mathfrak{v}} \in K$ with $c' \ne 0$. The scalar c' depends only on $\mathfrak{t}$ and $\boldsymbol{\lambda}$, and not on the choice of reduced expression for $d(\mathfrak{t})$, by [**8**, Prop. 2.5(i)]. Similarly, multiplying the last equation on the left with $\psi_{d(\mathfrak{s})}^* e_{\boldsymbol{\lambda}}$, and again using (3.8) and the fact that $\{m_{\mathfrak{u}\mathfrak{v}}\}$ is a cellular basis, we obtain

$$\psi_{\mathfrak{s}\mathfrak{t}} \equiv c m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u},\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \\ (\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})}} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \pmod{\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}}$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in K$ and some non-zero scalar $c \in K$ which depends only on $d(\mathfrak{s})$, $d(\mathfrak{t})$ and $\boldsymbol{\lambda}$. This completes the proof. □

Recall from section 4.3 that $\mathscr{H}_n^{\rhd \boldsymbol{\lambda}}$ is the two-sided ideal of $\mathscr{H}_n^\Lambda$ with basis the of standard basis elements $\{m_{\mathfrak{u}\mathfrak{v}}\}$, where $\mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\mu})$ and $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$.

5.5. **Corollary.** *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$. *Then* $\mathscr{H}_n^{\rhd\boldsymbol{\lambda}}$ *is a homogeneous two-sided ideal of* $\mathscr{H}_n^\Lambda$ *with basis* $\{\,\psi_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u},\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\mu}),$ *for* $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$ *with* $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}\}$.

As the next example shows, the elements $\psi_{\mathfrak{s}\mathfrak{t}}$ do, in general, depend upon the choice of the reduced expressions for $d(\mathfrak{s})$ and $d(\mathfrak{t})$.

5.6. **Example** Suppose that $e = 3$, $\Lambda = \Lambda_0$ and $n = 9$ so that we are considering the Iwahori-Hecke algebra of $\mathfrak{S}_9$ at a third root of unity (for any suitable field). Take $\lambda = (4, 3, 1^2)$ and set

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1&2&3&9\\\hline 4&6&8\\\cline{1-3} 5\\\cline{1-1} 7\\\cline{1-1}\end{array} \qquad \text{and} \qquad \mathfrak{u} = \begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 4&6&8\\\cline{1-3} 5\\\cline{1-1} 9\\\cline{1-1}\end{array}\,.$$

Then $d(\mathfrak{t}) = s_4s_5s_7s_6s_5s_7s_8s_7 = s_4s_5s_7s_6s_5s_8s_7s_8$. Now, $\mathrm{res}_{\mathfrak{t}}(7) = \mathrm{res}_{\mathfrak{t}}(9)$ so applying the last relation in Definition 3.1 (the graded analogue of the braid relation),

$$e_\lambda y_\lambda \psi_4\psi_5\psi_7\psi_6\psi_5\psi_7\psi_8\psi_7 = e_\lambda y_\lambda\big(\psi_4\psi_5\psi_7\psi_6\psi_5\psi_8\psi_7\psi_8 + \psi_4\psi_5\psi_7\psi_6\psi_5\big).$$

Consequently, if $\mathfrak{s} \in \mathrm{Std}(\lambda)$ and we define $\psi_{\mathfrak{s}\mathfrak{t}}$ using the first reduced expression for $d(\mathfrak{t})$ above and $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ using the second reduced expression then $\psi_{\mathfrak{s}\mathfrak{t}} = \hat\psi_{\mathfrak{s}\mathfrak{t}} + \psi_{\mathfrak{s}\mathfrak{u}}$. Therefore, different choices of reduced expression for $d(\mathfrak{t})$ can give different elements $\psi_{\mathfrak{s}\mathfrak{t}}$, for any $\mathfrak{s} \in \mathrm{Std}(\lambda)$. ◇

We do not actually need the next result, but given Example 5.6 it is reassuring. Brundan, Kleshchev and Wang prove an analogue of this result as part of their construction of the graded Specht modules [**8**, Theorem 4.10]. They have to work much harder, however, as they have to simultaneously prove that the grading on their modules is well-defined.

5.7. **Lemma** (cf. [**8**, Theorem. 4.10a]). *Suppose that* $\psi_{\mathfrak{s}\mathfrak{t}}$ *and* $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ *are defined using different reduced expressions for* $d(\mathfrak{s})$ *and* $d(\mathfrak{t})$, *where* $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda} \in \mathscr{P}$. *Then*

$$\psi_{\mathfrak{s}\mathfrak{t}} - \hat\psi_{\mathfrak{s}\mathfrak{t}} = \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})} s_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}},$$

where $s_{\mathfrak{u}\mathfrak{v}} \neq 0$ *only if* $\mathrm{res}(\mathfrak{u}) = \mathrm{res}(\mathfrak{s})$, $\mathrm{res}(\mathfrak{v}) = \mathrm{res}(\mathfrak{t})$ *and* $\deg\mathfrak{u} + \deg\mathfrak{v} = \deg\mathfrak{s} + \deg\mathfrak{t}$.

Proof. Using two applications of (5.4), we can write

$$\psi_{\mathfrak{s}\mathfrak{t}} - \hat\psi_{\mathfrak{s}\mathfrak{t}} = \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}} = \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})} s_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}}, s_{\mathfrak{u}\mathfrak{v}} \in K$. Multiplying on the left and right by $e(\mathbf{i}^{\mathfrak{s}})$ and $e(\mathbf{i}^{\mathfrak{t}})$, respectively, and using Lemma 5.2, shows that $s_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $\mathrm{res}(\mathfrak{u}) = \mathrm{res}(\mathfrak{s})$ and $\mathrm{res}(\mathfrak{v}) = \mathrm{res}(\mathfrak{t})$. Finally, by Lemma 5.4, the $\psi_{\mathfrak{u}\mathfrak{v}}$ appearing on the right hand are all linearly independent and $\psi_{\mathfrak{s}\mathfrak{t}}$ and $\hat\psi_{\mathfrak{s}\mathfrak{t}}$ are non-zero homogeneous elements of the same degree by Lemma 5.3. Therefore, so if $s_{\mathfrak{u}\mathfrak{v}} \neq 0$ then $\deg\mathfrak{u}+\deg\mathfrak{v} = \deg\psi_{\mathfrak{u}\mathfrak{v}} = \deg\psi_{\mathfrak{s}\mathfrak{t}} = \deg\mathfrak{s} + \deg\mathfrak{t}$, as required. □

We can now prove the main result of this paper. The existence of a graded cellular basis for $\mathscr{H}_n^\Lambda$ was conjectured by Brundan, Kleshchev and Wang [**8**, Remark 4.12]

5.8. **Theorem** (Graded cellular basis). *The algebra* $\mathscr{H}_n^\Lambda$ *is a graded cellular algebra with weight poset* $(\mathscr{P}_n^\Lambda, \trianglerighteq)$ *and graded cellular basis* $\{\,\psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ *for* $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda\,\}$. *In particular,* $\deg\psi_{\mathfrak{s}\mathfrak{t}} = \deg\mathfrak{s} + \deg\mathfrak{t}$, *for all* $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$.

Proof. By (5.4), the transition matrix between the set $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ and the standard basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ is an invertible triangular matrix (when suitably ordered!). Therefore, $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $\mathscr{H}_n^\Lambda$ giving (GC1) from Definition 2.1. By definition $\psi_{\mathfrak{s}\mathfrak{t}}$ is homogeneous and $\deg\psi_{\mathfrak{s}\mathfrak{t}} = \deg\mathfrak{s} + \deg\mathfrak{t}$, by Lemma 5.3, establishing (GC2).

To prove (GC4), recall that $*$ is the unique anti-isomorphism of $\mathscr{H}_n^\Lambda$ which fixes each of the graded generators. By definition, $(e_{\boldsymbol\lambda} y_{\boldsymbol\lambda})^* = e_{\boldsymbol\lambda} y_{\boldsymbol\lambda}$ since $e_{\boldsymbol\lambda}$ and $y_{\boldsymbol\lambda}$ commute. Therefore, $\psi_{\mathfrak{s}\mathfrak{t}}^* = \psi_{\mathfrak{t}\mathfrak{s}}$, for all $\mathfrak{s}$ and $\mathfrak{t}$. Consequently, the anti-automorphism of $\mathscr{H}_n^\Lambda$ induced by the basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$, as in (GC4), coincides with the anti-isomorphism $*$. In particular, (GC4) holds.

It remains then to check that the basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ satisfies (GC3), for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol\lambda)$ and $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$. By definition, $\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{d(\mathfrak{s})}^* \psi_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{t}}$. Suppose that $h \in \mathscr{H}_n^\Lambda$. Using Lemma 5.4 twice, together with Corollary 5.5 and the fact that $\{m_{\mathfrak{u}\mathfrak{v}}\}$ is a cellular basis of $\mathscr{H}_n^\Lambda$, we find

$$\begin{aligned}
\psi_{\mathfrak{s}\mathfrak{t}} h = \psi_{d(\mathfrak{s})}^* \psi_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{t}} h &\equiv \psi_{d(\mathfrak{s})}^* \sum_{\mathfrak{v}\trianglerighteq\mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{v}} h \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}}\\
&\equiv \psi_{d(\mathfrak{s})}^* \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol\lambda)} s_{\mathfrak{v}} m_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{v}} \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}}\\
&\equiv \psi_{d(\mathfrak{s})}^* \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol\lambda)} t_{\mathfrak{v}} \psi_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{v}} \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}}\\
&\equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol\lambda)} t_{\mathfrak{v}} \psi_{\mathfrak{s}\mathfrak{v}} \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}}
\end{aligned}$$

for some scalars $r_{\mathfrak{v}}, s_{\mathfrak{v}}, t_{\mathfrak{v}} \in K$. Hence, $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is a graded cellular basis and $\mathscr{H}_n^\Lambda$ is a graded cellular algebra, as required. □

Applying Corollary 2.5, we obtain the graded dimension of $\mathscr{H}_n^\Lambda$

$$\operatorname{Dim}_t \mathscr{H}_n^\Lambda = \sum_{\boldsymbol\lambda\in\mathscr{P}_n^\Lambda} \sum_{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol\lambda)} t^{\deg\mathfrak{s}+\deg\mathfrak{t}}.$$

This result is due to Brundan and Kleshchev [**7**, Theorem 4.20]. See also [**8**, Remark 4.12]. This can be further refined to compute $\operatorname{Dim}_t e(\mathbf{i})\mathscr{H}_n^\Lambda e(\mathbf{j})$, for $\mathbf{i}, \mathbf{j} \in I^n$, using Lemma 5.2.

§5.2. **The graded Specht modules.** Now that $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is known to be a graded cellular basis we can define the graded cell modules $S^{\boldsymbol\lambda}$ of $\mathscr{H}_n^\Lambda$, for $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$.

5.9. **Definition** (Graded Specht modules)**.** Suppose that $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$. The **graded Specht module** $S^{\boldsymbol\lambda}$ is the graded cell module associated with $\boldsymbol\lambda$ as in Definition 2.3.

Thus, $S^{\boldsymbol\lambda}$ has basis $\{\, \psi_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol\lambda) \,\}$ and the action of $\mathscr{H}_n^\Lambda$ on $S^{\boldsymbol\lambda}$ comes from its action on $\mathscr{H}_n^{\trianglerighteq\boldsymbol\lambda}/\mathscr{H}_n^{\rhd\boldsymbol\lambda}$.

In the absence of a graded cellular basis Brundan, Kleshchev and Wang [**8**] have already defined a graded Specht module $S_{BKW}^{\boldsymbol\lambda}$, for $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$. The two notions of graded Specht modules coincide.

5.10. **Corollary.** *Suppose that $\boldsymbol\lambda \in \mathscr{P}_n^\Lambda$. Then $S^{\boldsymbol\lambda} \cong S_{BKW}^{\boldsymbol\lambda}$ as $\mathbb{Z}$-graded $\mathscr{H}_n^\Lambda$-modules.*

Proof. Brundan, Kleshchev and Wang [**8**] actually define the graded left module $S_{BKW}^{*\boldsymbol\lambda}$, however, it is an easy exercise to switch their notation to the right. Mirroring the notation of [**8**, §4.2], set $\dot v_{\boldsymbol\lambda} = e_{\boldsymbol\lambda} y_{\boldsymbol\lambda} + \mathscr{H}_n^{\rhd\boldsymbol\lambda} = \psi_{\mathfrak{t}^{\boldsymbol\lambda}\mathfrak{t}^{\boldsymbol\lambda}} + \mathscr{H}_n^{\rhd\boldsymbol\lambda}$. By Theorem 5.8 the graded right module $\dot v_{\boldsymbol\lambda}\mathscr{H}_n^\Lambda$ has basis $\{\, \dot v_{\boldsymbol\lambda}\psi_{d(\mathfrak{t})} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol\lambda) \,\}$. Comparing this construction with [**8**, §4.2] and Definition 2.3 it is immediate that

$$S_{BKW}^{\boldsymbol\lambda} \cong \dot v_{\boldsymbol\lambda}\mathscr{H}_n^\Lambda\langle -\deg\mathfrak{t}^{\boldsymbol\lambda}\rangle \cong S^{\boldsymbol\lambda}.$$

In the notation of [**8**], the first isomorphism is given by $v_{\mathfrak{t}} \mapsto \dot v_{\boldsymbol\lambda}\psi_{d(\mathfrak{t})}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol\lambda)$. There is a degree shift for the middle term because $\deg \dot v_{\boldsymbol\lambda} = 2\deg\mathfrak{t}^{\boldsymbol\lambda}$ by Corollary 4.16. □

By Lemma 5.4 and Corollary 5.5, the ungraded module $\underline{S}^{\boldsymbol{\lambda}}$ coincides with the ungraded Specht module determined by the standard basis (Theorem 3.7), because the transition matrix between the graded cellular basis and the standard basis is unitriangular.

Let $\dot{D}^{\boldsymbol{\mu}}$ be the ungraded simple $\mathscr{H}_n^\Lambda$-module which is defined using the standard basis of $\mathscr{H}_n^\Lambda$, for $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Define a multipartition $\boldsymbol{\mu}$ to be **Λ-Kleshchev** if $\dot{D}^{\boldsymbol{\mu}} \neq 0$. Although we will not need it, there is an explicit combinatorial characterization of the Λ-Kleshchev multipartitions; see [**3**] or [**7**, (3.27)] (where they are called restricted multipartitions).

By Theorem 2.10, and the remarks of the last paragraph, the graded irreducible $\mathscr{H}_n^\Lambda$-modules are naturally labeled by the Λ-Kleshchev multipartitions of n. Notice, however, that this does not immediately imply that $D^{\boldsymbol{\mu}}$ is non-zero if and only if $\boldsymbol{\mu}$ is a Λ-Kleshchev multipartition: the problem is that the homogeneous bilinear form on the graded Specht module, which is induced by the graded basis (see Lemma 2.6), could be different to the bilinear form on the ungraded Specht module, which is induced by the standard basis. Our next result shows, however, that these two forms are essentially equivalent because their radicals coincide.

The following result is almost equivalent to [**7**, Theorem 5.10].

5.11. **Corollary.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Then $\dot{D}^{\boldsymbol{\mu}} = \underline{D}^{\boldsymbol{\mu}}$, for all $\boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Consequently, $D^{\boldsymbol{\mu}} \neq 0$ if and only if $\boldsymbol{\mu}$ is a Λ-Kleshchev multipartition.*

Proof. We argue by induction on dominance. If $\boldsymbol{\mu}$ is minimal in the dominance order then $D^{\mu} = S^{\boldsymbol{\mu}}$ and $\dot{D}^{\boldsymbol{\mu}} = \underline{S}^{\boldsymbol{\mu}}$ by Lemma 2.13(c). Hence, $\dot{D}^{\boldsymbol{\mu}} = \underline{D}^{\boldsymbol{\mu}}$ in this case. Now suppose that $\boldsymbol{\mu}$ is not minimal with respect to dominance. Using Lemma 2.13(c) again, $D^{\boldsymbol{\mu}} = 0$ if and only if every composition factor of $S^{\boldsymbol{\mu}}$ is isomorphic to $D^{\boldsymbol{\nu}}$ for some multipartition $\boldsymbol{\nu}$ with $\boldsymbol{\mu} \rhd \boldsymbol{\nu}$. Similarly, $\dot{D}^{\boldsymbol{\mu}} = 0$ if and only if every composition factor of $\underline{S}^{\boldsymbol{\mu}}$ is isomorphic to $\dot{D}^{\boldsymbol{\nu}}$, where $\boldsymbol{\mu} \rhd \boldsymbol{\nu}$. By induction, $\dot{D}^{\boldsymbol{\nu}} = \underline{D}^{\boldsymbol{\nu}}$ so the result follows. □

§5.3. **The blocks of $\mathscr{H}_n^\Lambda$.** We now show how Theorem 5.8 restricts to give a basis for the blocks, or the indecomposable two-sided ideals, of $\mathscr{H}_n^\Lambda$. Recall that $Q_+ = \bigoplus_{i \in I} \mathbb{N}_0 \alpha_i$ is the positive root lattice. Fix $\beta \in Q_+$ with $\sum_{i \in I}(\Lambda_i, \beta) = n$ and let

$$I^\beta = \{\, \mathbf{i} \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \beta \,\}.$$

Then I^β is an $\mathfrak{S}_n$-orbit of I^n and it is not hard to check that every $\mathfrak{S}_n$-orbit can be written uniquely in this way for some $\beta \in Q_+$. Define

$$\mathscr{H}_\beta^\Lambda = e_\beta \mathscr{H}_n^\Lambda, \qquad \text{where } e_\beta = \sum_{\mathbf{i} \in I^\beta} e(\mathbf{i}).$$

Then by the main result of [**25**], $\mathscr{H}_\beta^\Lambda$ is a block of $\mathscr{H}_n^\Lambda$. That is,

$$\mathscr{H}_n^\Lambda = \bigoplus_{\beta \in I^n,\, I^\beta \neq \emptyset} \mathscr{H}_\beta^\Lambda.$$

is the decomposition of $\mathscr{H}_n^\Lambda$ into a direct sum of indecomposable two-sided ideals. Let $\mathscr{P}_\beta^\Lambda = \{\, \boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda \mid \mathbf{i}^{\boldsymbol{\lambda}} \in I^\beta \,\}$. It follows from the combinatorial classification of the blocks of $\mathscr{H}_n^\Lambda$ that $\coprod_{\mathbf{i} \in I^\beta} \operatorname{Std}(\mathbf{i}) = \coprod_{\boldsymbol{\lambda} \in \mathscr{P}_\beta^\Lambda} \operatorname{Std}(\boldsymbol{\lambda})$. Hence, by Lemma 5.2 and Theorem 5.8 we obtain the following.

5.12. **Corollary.** *Suppose that $\beta \in Q_+$. Then*

$$\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \textit{ for } \boldsymbol{\lambda} \in \mathscr{P}_\beta^\Lambda \,\}$$

is a graded cellular basis of $\mathscr{H}_\beta^\Lambda$. In particular, $\mathscr{H}_\beta^\Lambda$ is a graded cellular algebra.

6. A dual graded cellular basis and a homogeneous trace form

In this section we construct a second graded cellular basis $\{\psi'_{\mathfrak{s}\mathfrak{t}}\}$ for the algebras $\mathscr{H}_n^\Lambda$ and $\mathscr{H}_\beta^\Lambda$. Using both the ψ-basis and the ψ'-basis we then show that $\mathscr{H}_\beta^\Lambda$ is a graded symmetric algebra, proving another conjecture of Brundan and Kleshchev [**7**, Remark 4.7].

§6.1. **The dual Murphy basis.** The main idea is that the ψ-basis is, via the standard basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$, built from the trivial representation of $\mathscr{H}_n^\Lambda$. The new basis that we will construct is, via the $\{n_{\mathfrak{s}\mathfrak{t}}\}$ basis defined below, modeled on the sign representation of $\mathscr{H}_n^\Lambda$.

6.1. **Definition** (Du and Rui [**15**, (2.7)]). Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$ and $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. Define $n_{\mathfrak{s}\mathfrak{t}}=(-q)^{-\ell(d(\mathfrak{s}))-\ell(d(\mathfrak{t}))}T_{d(\mathfrak{s})^{-1}}n_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$, where

$$n_{\boldsymbol{\lambda}}=\prod_{s=1}^{\ell-1}\prod_{k=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(\ell-s)}|}(L_k-q^{\kappa_s})\cdot\sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}}(-q)^{-\ell(w)}T_w.$$

(The normalization of $n_{\mathfrak{s}\mathfrak{t}}$ by a power of $-q^{-1}$ is for compatibility with the results from [**29**] that we use below. The asymmetry in the definitions of the basis elements $m_{\mathfrak{s}\mathfrak{t}}$ and $n_{\mathfrak{s}\mathfrak{t}}$ arises because the relations $(T_r-q)(T_r+1)=0$, for $1\le r<n$ are asymmetric. Renormalizing these relations to $(\hat{T}_r-v)(\hat{T}_r+v^{-1})=0$, where $q=v^2$, makes the definition of these elements symmetric; see, for example, [**28**, §3].)

It follows from Theorem 3.7 that $\{n_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis of $\mathscr{H}_n^\Lambda$; see [**29**, (3.1)]. We now recall how $L_1,\dots,L_n$ acts on this basis. To describe this requires some more notation.

If $\lambda=(\lambda_1,\lambda_2,\dots)$ is a partition then its **conjugate** is the partition $\lambda'=(\lambda'_1,\lambda'_2,\dots)$, where $\lambda'_i=\#\{\,j\ge1\mid\lambda_j\ge i\,\}$. If $\mathfrak{t}$ is a standard λ-tableau let $\mathfrak{t}'$ be the standard λ'-tableau given by $\mathfrak{t}'(r,c)=\mathfrak{t}(c,r)$. Pictorially, λ' and $\mathfrak{t}'$ are obtained by interchanging the rows and the columns of λ and $\mathfrak{t}$, respectively.

Similarly, if $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(\ell)})$ is a multipartition then the **conjugate multipartition** is the multipartition $\boldsymbol{\lambda}'=(\lambda^{(\ell)\prime},\dots,\lambda^{(1)\prime})$. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau then the **conjugate tableau** $\mathfrak{t}'$ is the standard $\boldsymbol{\lambda}'$-tableau given by $\mathfrak{t}'(r,c,l)=\mathfrak{t}(c,r,\ell-l+1)$.

By [**29**, Prop. 3.3], if $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ and $1\le k\le n$ then there exist scalars $r_{\mathfrak{u}\mathfrak{v}}\in K$ such that

$$n_{\mathfrak{s}\mathfrak{t}}L_k=q^{\operatorname{res}_{\mathfrak{t}'}(k)}n_{\mathfrak{s}\mathfrak{t}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})}r_{\mathfrak{u}\mathfrak{v}}n_{\mathfrak{u}\mathfrak{v}}.\tag{6.2}$$

As in section 4.2, fix a modular system $(\mathcal{K},\mathcal{O},K)$ for $\mathscr{H}_n^\Lambda$. Until noted otherwise we will work in $\mathscr{H}_n^{\mathcal{K}}$. Following Definition 4.4, define $f'_{\mathfrak{s}\mathfrak{t}}=F_{\mathfrak{s}'}n_{\mathfrak{s}\mathfrak{t}}F_{\mathfrak{t}'}$, for $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$. Moreover, by (6.2), if $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$, then

$$f'_{\mathfrak{s}\mathfrak{t}}=n_{\mathfrak{s}\mathfrak{t}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})}r_{\mathfrak{u}\mathfrak{v}}n_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}}\in K$. Therefore, $\{f'_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $\mathscr{H}_n^{\mathcal{K}}$, as was noted in [**29**, §3].

We now retrace our steps from section 4.2 replacing the $f_{\mathfrak{s}\mathfrak{t}}$ basis with the $f'_{\mathfrak{s}\mathfrak{t}}$ basis.

Recall from section 4.2 that if $\alpha=(r,c,l)$ and $\beta=(s,d,m)$ are two nodes then α is below β if either $l>m$, or $l=m$ and $r>s$. Dually, we say that β is **above** α. With this notation we can define a 'dual' version of the scalars $\gamma_{\mathfrak{t}}\in\mathcal{K}$.

6.3. **Definition** (cf. Definition 4.5). Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda$ and $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. For $k=1,\dots,n$ let $\mathscr{A}_{\mathfrak{t}}(k)'$ be the set of addable nodes of the multipartition $\operatorname{Shape}(\mathfrak{t}_k)$

which are *above* $\mathfrak{t}^{-1}(k)$. Similarly, let $\mathscr{R}_\mathfrak{t}(k)'$ be the set of removable nodes of $\operatorname{Shape}(\mathfrak{t}_k)$ which are *above* $\mathfrak{t}^{-1}(k)$. Now define

$$\gamma'_\mathfrak{t} = (x+q)^{-\ell(d(\mathfrak{t}))-\delta(\boldsymbol{\lambda})} \prod_{k=1}^{n} \frac{\prod_{\alpha\in\mathscr{A}_{\mathfrak{t}'}(k)'}\big((x+q)^{\operatorname{cont}_{\mathfrak{t}'}(k)}-(x+q)^{\operatorname{cont}(\alpha)}\big)}{\prod_{\rho\in\mathscr{R}_{\mathfrak{t}'}(k)'}\big((x+q)^{\operatorname{cont}_{\mathfrak{t}'}(k)}-(x+q)^{\operatorname{cont}(\rho)}\big)} \quad \in \mathcal{K}.$$

Suppose that $\mathbf{i}\in I^n$ and that $\operatorname{Std}(\mathbf{i})\neq\emptyset$. Define $\mathbf{i}'=\operatorname{res}(\mathfrak{s}')$, where $\mathfrak{s}$ is any element of $\operatorname{Std}(\mathbf{i})$. Then $\mathbf{i}'\in I^n$ and $\mathbf{i}'$ is independent of the choice of $\mathfrak{s}$.

Recall that Proposition 4.8 defines the idempotent $e(\mathbf{i})^{\mathcal{O}}\in\mathscr{H}_n^{\mathcal{O}}$, for $\mathbf{i}\in I^n$.

6.4. **Lemma.** *Suppose that $\mathbf{i}\in I^n$ with $e(\mathbf{i})\neq 0$. Then, in $\mathscr{H}_n^{\mathcal{O}}$,*

$$e(\mathbf{i}')^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\gamma'_\mathfrak{s}} f'_{\mathfrak{s}\mathfrak{s}}.$$

Proof. By [**29**, Remark 3.6], if $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ then $\frac{1}{\gamma'_\mathfrak{s}}f'_{\mathfrak{s}\mathfrak{s}} = \frac{1}{\gamma_{\mathfrak{s}'}}f_{\mathfrak{s}'\mathfrak{s}'}$ in $\mathscr{H}_n^{\mathcal{K}}$. So, the result is just a rephrasing of Proposition 4.8. (Note that $\gamma'_\mathfrak{t}$, as defined in Definition 6.3, is the specialization at the parameters of $\mathscr{H}_n^{\mathcal{K}}$ of the element $\gamma'_\mathfrak{t}$ defined in [**29**, §3]; see the remarks before [**29**, Prop. 3.4].) □

Definition 4.9 defines a homogeneous element $y_\mathfrak{s}\in\mathscr{H}_n^{\Lambda}$ for each positive tableau $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$, $\mathbf{i}\in I^n$. To construct the dual basis we lift $e(\mathbf{i}')y_\mathfrak{s}$ to $\mathscr{H}_n^{\mathcal{O}}$.

6.5. **Definition.** Suppose that $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ is a positive tableau. Let $\mathscr{A}^{\Lambda}_{\mathfrak{s}'}(k)' = \{\,\alpha\in\mathscr{A}_{\mathfrak{s}'}(k)' \mid \operatorname{res}(\alpha)=\operatorname{res}_{\mathfrak{s}'}(k)\,\}$ and define $(y'_\mathfrak{s})^{\mathcal{O}} = (y'_{\mathfrak{s},1})^{\mathcal{O}}\dots(y'_{\mathfrak{s},n})^{\mathcal{O}}$ where

$$(y'_{\mathfrak{s},k})^{\mathcal{O}} = \prod_{\alpha\in\mathscr{A}^{\Lambda}_{\mathfrak{s}'}(k)'} \big(1-(x+q)^{-\operatorname{cont}(\alpha)}L_k\big) \in \mathscr{H}_n^{\mathcal{O}},$$

for $k=1,\dots,n$.

Observe that if $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ is a positive tableau then $e(\mathbf{i}')y_\mathfrak{s} = e(\mathbf{i}')^{\mathcal{O}}(y'_\mathfrak{s})^{\mathcal{O}}\otimes_{\mathcal{O}}1_K$ because $|\mathscr{A}^{\Lambda}_{\mathfrak{s}}(k)| = |\mathscr{A}^{\Lambda}_{\mathfrak{s}'}(k)'|$, for $1\le k\le n$. Note, however, that $(y'_\mathfrak{s})^{\mathcal{O}}\neq y^{\mathcal{O}}_\mathfrak{s}$ in general.

The following two results are analogues of Lemma 4.13 and Theorem 4.14, respectively. We leave the details to the reader because they can be proved by repeating the arguments from section 4, the only real difference being that Lemma 6.4 is used instead of Proposition 4.8.

6.6. **Lemma.** *Suppose that $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\mathbf{i})$, where $\mathbf{i}\in I^n$, and that $\mathfrak{s}$ is a positive tableau. Then:*

a) *If $\mathfrak{t}=\mathfrak{s}$ then $f'_{\mathfrak{t}\mathfrak{t}}(y'_\mathfrak{s})^{\mathcal{O}} = u^{\mathcal{O}}_\mathfrak{s}\gamma'_\mathfrak{s}f'_{\mathfrak{s}\mathfrak{s}}$, for some unit $u^{\mathcal{O}}_\mathfrak{s}\in\mathcal{O}$.*

b) *If $\mathfrak{t}\neq\mathfrak{s}$ then there exists an integer $d_\mathfrak{t}\ge\deg\mathfrak{s}$ and an invertible element $u^{\mathcal{O}}_\mathfrak{t}\in\mathcal{O}$ such that*

$$f'_{\mathfrak{t}\mathfrak{t}}(y'_\mathfrak{s})^{\mathcal{O}} = \begin{cases} u^{\mathcal{O}}_\mathfrak{t}x^{d'_\mathfrak{t}}f'_{\mathfrak{t}\mathfrak{t}}, & \textit{if } \mathfrak{t}\rhd\mathfrak{s},\\ 0, & \textit{otherwise,}\end{cases}$$

and where $d'_\mathfrak{t}=\deg\mathfrak{s}$ whenever $\mathcal{K}$ is a field of characteristic zero.

As a consequence, we can repeat the proof of Theorem 4.14 to deduce the following.

6.7. **Proposition.** *Suppose that $\mathfrak{s}\in\operatorname{Std}^+(\mathbf{i})$ is a positive tableau, for $\mathbf{i}\in I^n$. Then there exists a non-zero $c\in K$ such that*

$$e(\mathbf{i}')y_\mathfrak{s} = cn_{\mathfrak{s}\mathfrak{s}} + \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{s})} r_{\mathfrak{u}\mathfrak{v}}n_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}}\in K$.

§6.2. **The dual graded basis.** If $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ then $\mathfrak{t}^{\boldsymbol{\lambda}}$ is a positive tableau by Lemma 4.11. Recall that $e_{\boldsymbol{\lambda}} = e(\mathbf{i}^{\boldsymbol{\lambda}})$. Define $e'_{\boldsymbol{\lambda}} = e(\mathbf{i}')$, where $\mathbf{i} = \mathbf{i}^{\boldsymbol{\lambda}}$. Then as a special case of Proposition 6.7, there is a non-zero $c \in K$ such that

$$e'_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} = c n_{\boldsymbol{\lambda}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t}^{\boldsymbol{\lambda}})} r_{\mathfrak{u}\mathfrak{v}} n_{\mathfrak{u}\mathfrak{v}}, \tag{6.8}$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in K$. This is what we need to define the dual graded basis of $\mathscr{H}_n^\Lambda$.

6.9. **Definition.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and recall that we have fixed reduced expressions $d(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ and $d(\mathfrak{t}) = s_{j_1} \dots s_{j_m}$ for $d(\mathfrak{s})$ and $d(\mathfrak{t})$, respectively. Define $\psi'_{\mathfrak{s}\mathfrak{t}} = \psi_{i_k} \dots \psi_{i_1} e'_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{j_1} \dots \psi_{j_m}$.

By definition, $\psi'_{\mathfrak{s}\mathfrak{t}}$ is a homogeneous element of $\mathscr{H}_n^\Lambda$. Just as with $\psi_{\mathfrak{s}\mathfrak{t}}$, the element $\psi'_{\mathfrak{s}\mathfrak{t}}$ will, in general, depend upon the choice of reduced expressions for $d(\mathfrak{s})$ and $d(\mathfrak{t})$. Arguing just as in section 5.1 we obtain the following facts. We leave the details to the reader.

6.10. **Proposition.** *Suppose that $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for some $\lambda \in \mathscr{P}_n^\Lambda$. Then*

a) *If $\mathbf{i}, \mathbf{j} \in I^n$ then*
$$e(\mathbf{i}')\psi'_{\mathfrak{s}\mathfrak{t}} e(\mathbf{j}') = \begin{cases} \psi'_{\mathfrak{s}\mathfrak{t}}, & \text{if } \operatorname{res}(\mathfrak{s}) = \mathbf{i} \text{ and } \operatorname{res}(\mathfrak{t}) = \mathbf{j}, \\ 0, & \text{otherwise.} \end{cases}$$

b) $\deg \psi'_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}$.

c) $\psi'_{\mathfrak{s}\mathfrak{t}} = c n_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} n_{\mathfrak{u}\mathfrak{v}}$, *for some $r_{\mathfrak{u}\mathfrak{v}} \in K$ and $0 \neq c \in K$.*

d) *If $\hat{\psi}'_{\mathfrak{s}\mathfrak{t}}$ is defined using a different choice of reduced expressions for $d(\mathfrak{s})$ and $d(\mathfrak{t})$ then*
$$\psi'_{\mathfrak{s}\mathfrak{t}} - \hat{\psi}'_{\mathfrak{s}\mathfrak{t}} = \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} \psi'_{\mathfrak{u}\mathfrak{v}},$$
where $r_{\mathfrak{u}\mathfrak{v}} \in K$ is non-zero only if $\operatorname{res}(\mathfrak{u}) = \operatorname{res}(\mathfrak{s})$, $\operatorname{res}(\mathfrak{v}) = \operatorname{res}(\mathfrak{t})$ and $\deg \mathfrak{u} + \deg \mathfrak{v} = \deg \mathfrak{s} + \deg \mathfrak{t}$.

Using Proposition 6.10, and arguing exactly as in the proof of Theorem 5.8 we obtain the graded dual basis of $\mathscr{H}_n^\Lambda$.

6.11. **Theorem.** *The basis $\{\, \psi'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda \,\}$ is a graded cellular basis of $\mathscr{H}_n^\Lambda$.*

The basis $\{\psi'_{\mathfrak{s}\mathfrak{t}}\}$ is the **dual graded basis** of $\mathscr{H}_n^\Lambda$. We note that the unique anti-isomorphism of $\mathscr{H}_n^\Lambda$ which fixes the homogeneous generators of $\mathscr{H}_n^\Lambda$ coincides with the graded anti-isomorphisms coming from both the graded cellular basis and the dual graded cellular basis, via (GC4) of Definition 2.1.

As with the graded basis, the dual graded basis restricts to give a graded cellular basis for the blocks of $\mathscr{H}_n^\Lambda$.

6.12. **Corollary.** *Suppose that $\beta \in Q_+$. Then*
$$\{\, \psi'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda}' \in \mathscr{P}_\beta^\Lambda \,\}$$
is a graded cellular basis of $\mathscr{H}_\beta^\Lambda$.

§6.3. **Graded symmetric algebras.** Recall that a **trace form** on a K-algebra A is a K-linear map $\tau : A \longrightarrow K$ such that $\tau(ab) = \tau(ba)$, for all $a, b \in A$. The algebra A is **symmetric** if A is equipped with a non-degenerate symmetric bilinear form $\theta : A \times A \to K$ which is associative in the following sense:

$$\theta(xy, z) = \theta(x, yz), \quad \text{for all } x, y, z \in A.$$

Define a trace form $\tau : A \to K$ on A by setting $\tau(a) = \theta(a, 1)$ for any $a \in A$. Note that $\ker\tau$ cannot contain any non-zero left or right ideals because θ is non-degenerate. We leave the next result for the reader.

6.13. **Lemma.** *Suppose that A is a finite dimensional K-algebra which is equipped with an anti-automorphism σ of order 2. Then A is symmetric if and only if there is a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : A \times A \longrightarrow K$ which is associative in the sense $\langle ab, c\rangle = \langle a, cb^\sigma\rangle$ for any $a, b, c \in A$.*

A graded algebra A is a **graded symmetric** algebra if there exists a homogeneous non-degenerate trace form $\tau : A \longrightarrow K$. Apart from providing a second graded cellular basis of $\mathscr{H}_n^\Lambda$, the dual graded basis of $\mathscr{H}_n^\Lambda$ is useful because we can use it to show that the algebras $\mathscr{H}_\beta^\Lambda$, for $\beta \in Q_+$, are **graded symmetric algebras**.

Following Brundan and Kleshchev [**8**, (3.4)], if $\beta \in Q_+$ then the **defect** of β is

$$\operatorname{def}\beta = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta),$$

where $(\ ,\ )$ is the non-degenerate pairing on the root lattice introduced in section 3.1. If $\ell = 1$ then $\operatorname{def}\beta$ is the *e-weight* of the block $\mathscr{H}_\beta^\Lambda$. If $\ell > 1$ then $\operatorname{def}\beta$ coincides with Fayers [**16**] definition of weight for the algebras $\mathscr{H}_\beta^\Lambda$.

In what follows, the following result of Brundan, Kleshchev and Wang's will be very important. (In [**8**, §3], $\deg\mathfrak{s}'$ is called the *codegree* of $\mathfrak{s}$.)

6.14. **Lemma** (Brundan, Kleshchev and Wang [**8**, Lemma 3.12])**.** *Suppose that $\boldsymbol{\mu} \in \mathscr{P}_\beta^\Lambda$ and that $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\mu})$. Then $\deg\mathfrak{s} + \deg\mathfrak{s}' = \operatorname{def}\beta$.*

To define the homogeneous trace form τ_β on $\mathscr{H}_\beta^\Lambda$ recall that, by the main result of [**26**], $\mathscr{H}_n^\Lambda$ is a symmetric algebra with induced trace form $\tau : \mathscr{H}_n^\Lambda \longrightarrow K$, where τ is the K-linear map determined by

$$\tau(L_1^{a_1} \dots L_n^{a_n} T_w) = \begin{cases} 1, & \text{if } a_1 = \dots = a_n = 0 \text{ and } w = 1,\\ 0, & \text{otherwise,} \end{cases}$$

where $0 \le a_1, \dots, a_n < \ell$ and $w \in \mathfrak{S}_n$. In general, the map τ is not homogeneous, however, we can use τ to define a homogeneous trace form on $\mathscr{H}_\beta^\Lambda$ since $\mathscr{H}_\beta^\Lambda$ is a subalgebra of $\mathscr{H}_n^\Lambda$.

6.15. **Definition** (Homogeneous trace)**.** Suppose that $\beta \in Q_+$. Then $\tau_\beta : \mathscr{H}_\beta^\Lambda \longrightarrow K$ is the map which on a homogeneous element $a \in \mathscr{H}_\beta^\Lambda$ is given by

$$\tau_\beta(a) = \begin{cases} \tau(a), & \text{if } \deg(a) = 2\operatorname{def}\beta,\\ 0, & \text{otherwise.} \end{cases}$$

It is an easy exercise to verify that τ_β is a trace form on $\mathscr{H}_\beta^\Lambda$. By definition, τ is homogeneous of degree $-2\operatorname{def}\beta$. To show that τ_β is induced from a non-degenerate symmetric bilinear form on $\mathscr{H}_\beta^\Lambda$ we need the following fact.

6.16. **Lemma** ([**28**, Lemma 5.4 and Theorem 5.5])**.** *Suppose that $\mathfrak{a}, \mathfrak{b} \in \operatorname{Std}(\boldsymbol{\mu})$ and $\mathfrak{c}, \mathfrak{d} \in \operatorname{Std}(\boldsymbol{\nu})$, for $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_\beta^\Lambda$. Then $m_{\mathfrak{a}\mathfrak{b}} n_{\mathfrak{c}\mathfrak{d}} \ne 0$ only if $\mathfrak{c}' \trianglerighteq \mathfrak{b}$. Further, there exists a non-zero scalar $u_{\boldsymbol{\lambda}} \in K$, which depends only on $\boldsymbol{\lambda}$, such that*

$$\tau(m_{\mathfrak{a}\mathfrak{b}} n_{dc}) = \begin{cases} u_{\boldsymbol{\lambda}}, & \textit{if } (\mathfrak{c}', \mathfrak{d}') = (\mathfrak{a}, \mathfrak{b}),\\ 0, & \textit{if } (\mathfrak{c}', \mathfrak{d}') \ntrianglerighteq (\mathfrak{a}, \mathfrak{b}). \end{cases}$$

Define a homogeneous bilinear form $\langle\ ,\ \rangle_\beta$ on $\mathscr{H}_\beta^\Lambda$ of degree $-2\operatorname{def}\beta$ by

$$\langle a, b\rangle_\beta = \tau_\beta(ab^*).$$

By definition, $\langle\ ,\ \rangle_\beta$ is symmetric and associative in the sense that $\langle a, bc\rangle_\beta = \langle ac^*, b\rangle_\beta$ for any $a, b, c \in \mathscr{H}_\beta^\Lambda$.

6.17. **Theorem.** *Suppose that $\beta \in Q_+$ and that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\Lambda_\beta$. If $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\boldsymbol{\mu})$ then*

$$\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi'_{\mathfrak{u}\mathfrak{v}} \rangle_\beta = \begin{cases} u, & \text{if } (\mathfrak{u}', \mathfrak{v}') = (\mathfrak{s}, \mathfrak{t}), \\ 0, & \text{if } (\mathfrak{u}', \mathfrak{v}') \ntrianglerighteq (\mathfrak{s}, \mathfrak{t}), \end{cases}$$

for some non-zero scalar $u \in K$.

Proof. By Lemma 5.4 and Proposition 6.10(c), there exist non-zero scalars $c, c' \in K$ and $r_{\mathfrak{a}\mathfrak{b}}, r'_{\mathfrak{d}\mathfrak{c}} \in K$ such that

$$(\dagger) \qquad \psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{v}\mathfrak{u}} = \Big(c m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{a},\mathfrak{b}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{a}\mathfrak{b}} m_{\mathfrak{a}\mathfrak{b}} \Big) \Big(c' n_{\mathfrak{v}\mathfrak{u}} + \sum_{(\mathfrak{d},\mathfrak{c}) \rhd (\mathfrak{v},\mathfrak{u})} r'_{\mathfrak{d}\mathfrak{c}} n_{\mathfrak{d}\mathfrak{c}} \Big).$$

Therefore, $\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi'_{\mathfrak{u}\mathfrak{v}} \rangle_\beta = 0$ unless $\mathfrak{v}' \trianglerighteq \mathfrak{t}$ by Lemma 6.16. Now,

$$\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi'_{\mathfrak{u}\mathfrak{v}} \rangle_\beta = \tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{v}\mathfrak{u}}) = \tau_\beta(\psi'_{\mathfrak{v}\mathfrak{u}} \psi_{\mathfrak{s}\mathfrak{t}}) = \tau_\beta(\psi_{\mathfrak{t}\mathfrak{s}} \psi'_{\mathfrak{u}\mathfrak{v}}) = \langle \psi_{\mathfrak{t}\mathfrak{s}}, \psi'_{\mathfrak{v}\mathfrak{u}} \rangle_\beta,$$

where we have used the easily checked fact that $\tau_\beta(h) = \tau_\beta(h^*)$ for the third equality. Combined with (†), this shows that $\langle \psi_{\mathfrak{s}\mathfrak{t}}, \psi'_{\mathfrak{u}\mathfrak{v}} \rangle_\beta = 0$ unless $(\mathfrak{u}', \mathfrak{v}') \trianglerighteq (\mathfrak{s}, \mathfrak{t})$.

To complete the proof it remains to consider the case when $(\mathfrak{u}', \mathfrak{v}') = (\mathfrak{s}, \mathfrak{t})$. By Lemma 6.16, (†) now reduces to the equation $\psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{t}'\mathfrak{s}'} = cc' m_{\mathfrak{s}\mathfrak{t}} n'_{\mathfrak{t}'\mathfrak{s}'}$. By Lemma 5.3, Proposition 6.10(b) and Lemma 6.14, we have

$$\deg(\psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{t}'\mathfrak{s}'}) = \deg \mathfrak{s} + \deg \mathfrak{t} + \deg \mathfrak{s}' + \deg \mathfrak{t}' = 2 \operatorname{def} \beta,$$

Therefore, we can replace τ_β with τ and use Lemma 6.16 to obtain

$$\tau_\beta(\psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{t}'\mathfrak{s}'}) = \tau(\psi_{\mathfrak{s}\mathfrak{t}} \psi'_{\mathfrak{t}'\mathfrak{s}'}) = cc' \tau(m_{\mathfrak{s}\mathfrak{t}} n_{\mathfrak{t}'\mathfrak{s}'}) = cc' u_{\boldsymbol{\lambda}}.$$

As $cc'u_{\boldsymbol{\lambda}} \neq 0$ this completes the proof. □

Applying Lemma 6.13, we deduce that $\mathscr{H}^\Lambda_\beta$ is a graded symmetric algebra. This was conjectured by Brundan and Kleshchev [**7**, Remark 4.7],

6.18. **Corollary.** *Suppose that $\beta \in Q_+$. Then $\mathscr{H}^\Lambda_\beta$ is a graded symmetric algebra with homogeneous trace form τ_β of degree $-2 \operatorname{def} \beta$.*

We remark that the two graded bases $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ and $\{\psi'_{\mathfrak{u}\mathfrak{v}}\}$ are almost certainly not dual with respect to $\langle\ ,\ \rangle_\beta$. We call $\{\psi'_{\mathfrak{u}\mathfrak{v}}\}$ the *dual* graded basis because Theorem 6.17 shows that these two bases are dual modulo more dominant terms. As far as we are aware, if $\ell > 2$ then there are no known pairs of dual bases for $\mathscr{H}^\Lambda_n$, even in the ungraded case.

§6.4. **Dual graded Specht modules.** Using the graded cellular basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ we defined the graded Specht module $S^{\boldsymbol{\lambda}}$. Similarly, if $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$ then the **dual graded Specht module** $S_{\boldsymbol{\lambda}}$ is the graded cell module associated with $\boldsymbol{\lambda}$, via Definition 2.3, using the dual graded basis $\{\psi'_{\mathfrak{s}\mathfrak{t}}\}$. Thus, $S_{\boldsymbol{\lambda}}$ has a homogeneous basis $\{\, \psi'_{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$, with the action of $\mathscr{H}^\Lambda_n$ being induced by its action on the dual graded basis.

By [**28**, Cor. 5.7], it was shown that $\underline{S}^{\boldsymbol{\lambda}}$ and $\underline{S}_{\boldsymbol{\lambda}'}$ are dual to each other with respect to the contragredient duality induced on $\mathscr{H}^\Lambda_n$**-Mod** by the cellular algebra anti-isomorphism defined by the standard cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$. We generalize this result to the graded setting.

Let $\mathscr{H}^{\prime \rhd \boldsymbol{\lambda}}_n = \langle \psi_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\boldsymbol{\mu}) \text{ where } \boldsymbol{\mu} \rhd \boldsymbol{\lambda} \rangle_K$ be the graded two-sided ideal of $\mathscr{H}^\Lambda_n$ spanned by the elements of the cellular basis $\{\psi'_{\mathfrak{u}\mathfrak{v}}\}$ of more dominant shape. Then $\mathscr{H}^{\prime \rhd \boldsymbol{\lambda}}_n$ is also spanned by the elements $\{n_{\mathfrak{u}\mathfrak{v}}\}$, where $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\boldsymbol{\mu})$ and $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$ by Proposition 6.10(c).

6.19. **Proposition.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_\beta$. Then $S^{\boldsymbol{\lambda}} \cong S^{\circledast}_{\boldsymbol{\lambda}'} \langle \operatorname{def} \beta \rangle$ as graded $\mathscr{H}^\Lambda_\beta$-modules.*

Proof. By Theorem 6.17 the graded two-sided ideals $\mathscr{H}_\beta^{\rhd\boldsymbol{\lambda}}$ and $\mathscr{H}_\beta'^{\rhd\boldsymbol{\lambda}'}$ of $\mathscr{H}_\beta^\Lambda$ are orthogonal with respect to the trace form $\langle\ ,\ \rangle_\beta$. By construction $S^{\boldsymbol{\lambda}}\langle\deg\mathfrak{t}^{\boldsymbol{\lambda}}\rangle\cong(\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}+\mathscr{H}_n^{\rhd\boldsymbol{\lambda}})\mathscr{H}_n^\Lambda$ and $S_{\boldsymbol{\lambda}'}\langle\deg\mathfrak{t}_{\boldsymbol{\lambda}'}\rangle\cong(\psi'_{\mathfrak{t}_{\boldsymbol{\lambda}'}\mathfrak{t}^{\boldsymbol{\lambda}'}}+\mathscr{H}_n'^{\rhd\boldsymbol{\lambda}})\mathscr{H}_n^\Lambda$, where $\mathfrak{t}_{\boldsymbol{\lambda}'}=(\mathfrak{t}^{\boldsymbol{\lambda}})'$. Therefore, $\langle\ ,\ \rangle_\beta$ induces an homogeneous associative bilinear form

$$\langle\ ,\ \rangle_{\beta,\boldsymbol{\lambda}}\colon S^{\boldsymbol{\lambda}}\langle\deg\mathfrak{t}^{\boldsymbol{\lambda}}\rangle\times S_{\boldsymbol{\lambda}'}\langle\deg\mathfrak{t}_{\boldsymbol{\lambda}'}\rangle\longrightarrow K;\langle a+\mathscr{H}_n^{\rhd\boldsymbol{\lambda}},b+\mathscr{H}_n'^{\rhd\boldsymbol{\lambda}'}\rangle_{\beta,\boldsymbol{\lambda}}=\langle a,b\rangle_\beta.$$

In particular, if $\mathfrak{s},\mathfrak{t}'\in\operatorname{Std}(\boldsymbol{\lambda})$ then, by Theorem 6.17,

$$\langle\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}}+\mathscr{H}_n^{\rhd\boldsymbol{\lambda}},\psi'_{\mathfrak{t}_{\boldsymbol{\lambda}'}\mathfrak{t}}+\mathscr{H}_n'^{\rhd\boldsymbol{\lambda}'}\rangle_{\beta,\boldsymbol{\lambda}}=\begin{cases}u, & \text{if } \mathfrak{s}=\mathfrak{t}',\\ 0, & \text{unless } \mathfrak{t}'\trianglerighteq\mathfrak{s},\end{cases}$$

for some $0\neq u\in K$. Therefore, $\langle\ ,\ \rangle_{\beta,\boldsymbol{\lambda}}$ is a homogeneous non–degenerate pairing of degree $-2\operatorname{def}\beta$ and, since taking duals reverses the grading,

$$S^{\boldsymbol{\lambda}}\cong S_{\boldsymbol{\lambda}'}^{\circledast}\langle 2\operatorname{def}\beta-\deg\mathfrak{t}_{\boldsymbol{\lambda}'}-\deg\mathfrak{t}^{\boldsymbol{\lambda}}\rangle=S_{\boldsymbol{\lambda}'}^{\circledast}\langle\operatorname{def}\beta\rangle,$$

since $\operatorname{def}\beta=\deg\mathfrak{t}^{\boldsymbol{\lambda}}+\deg\mathfrak{t}_{\boldsymbol{\lambda}'}$ by Lemma 6.14. □

During the proof of Theorem 6.17 we showed that $m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{t}'\mathfrak{s}'}=c\psi_{\mathfrak{s}\mathfrak{t}}\psi'_{\mathfrak{t}'\mathfrak{s}'}$, for some non-zero constant $c\in K$. Hence, we have the following interesting fact.

6.20. **Corollary** (of Theorem 6.17). *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda$ and that $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. Then $m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{t}'\mathfrak{s}'}$ is a homogeneous element of $\mathscr{H}_n^\Lambda$ of degree $2\operatorname{def}\beta$.*

Let $\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda$. Recall that by definition, $e_{\boldsymbol{\lambda}}=e(\mathbf{i}^{\mathfrak{t}^{\boldsymbol{\lambda}}})$ and $e'_{\boldsymbol{\lambda}'}=e(\mathbf{i}^{\mathfrak{t}_{\boldsymbol{\lambda}}})$, where $\mathfrak{t}_{\boldsymbol{\lambda}}=(\mathfrak{t}^{\boldsymbol{\lambda}'})'$. Let $w_{\boldsymbol{\lambda}}=d(\mathfrak{t}_{\boldsymbol{\lambda}})$ and define $z_{\boldsymbol{\lambda}}=m_{\boldsymbol{\lambda}}T_{w_{\boldsymbol{\lambda}}}n_{\boldsymbol{\lambda}'}$.

6.21. **Corollary.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_\beta^\Lambda$. Then*

$$z_{\boldsymbol{\lambda}}=e_{\boldsymbol{\lambda}}z_{\boldsymbol{\lambda}}e'_{\boldsymbol{\lambda}'}=ce_{\boldsymbol{\lambda}}y_{\boldsymbol{\lambda}}\psi_{w_{\boldsymbol{\lambda}}}y_{\boldsymbol{\lambda}'}=cy_{\boldsymbol{\lambda}}\psi_{w_{\boldsymbol{\lambda}}}y_{\boldsymbol{\lambda}'}e'_{\boldsymbol{\lambda}'},$$

for some $0\neq c\in K$. In particular, $z_{\boldsymbol{\lambda}}$ is a homogeneous element of $\mathscr{H}_n^\Lambda$ of degree $\operatorname{def}\beta+\deg(\mathfrak{t}^{\boldsymbol{\lambda}})+\deg(\mathfrak{t}^{\boldsymbol{\lambda}'})$.

Proof. By Corollary 4.16 and (6.8) there exist $0\neq c\in K$ such that

$$e_{\boldsymbol{\lambda}}y_{\boldsymbol{\lambda}}\psi_{w_{\boldsymbol{\lambda}}}\equiv ce_{\boldsymbol{\lambda}}m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}_{\boldsymbol{\lambda}}}+\sum_{\substack{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\\ \ell(d(\mathfrak{t}))<\ell(w_{\boldsymbol{\lambda}})}}a_{\mathfrak{t}}e_{\boldsymbol{\lambda}}m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}}\pmod{\mathscr{H}_n^{\rhd\boldsymbol{\lambda}}},$$

for some $a_{\mathfrak{t}}\in K$. Further, $e'_{\boldsymbol{\lambda}'}y_{\boldsymbol{\lambda}'}\equiv c'e'_{\boldsymbol{\lambda}'}n_{\boldsymbol{\lambda}'}\pmod{\mathscr{H}_n'^{\rhd\boldsymbol{\lambda}'}}$, for some non-zero $c'\in K$, by Proposition 6.7. By definition $\mathfrak{t}\trianglerighteq\mathfrak{t}_{\boldsymbol{\lambda}}$ for all $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$, so if $\mathfrak{t}\neq\mathfrak{t}_{\boldsymbol{\lambda}}$ then $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}}n_{\boldsymbol{\lambda}'}=0$ by Lemma 6.16 since $(\mathfrak{t}^{\boldsymbol{\lambda}'})'=\mathfrak{t}_{\boldsymbol{\lambda}}\not\trianglerighteq\mathfrak{t}$. Hence, multiplying these two equations together gives the Corollary. □

There may well be a more direct proof of the last two results because these elements are already well-known in the representation theory of $\mathscr{H}_n^\Lambda$. Note that

$$m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{t}'\mathfrak{s}'}=T_{d(\mathfrak{s})^{-1}}m_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}T_{d(\mathfrak{t}')^{-1}}n_{\boldsymbol{\lambda}'}T_{d(\mathfrak{s}')}=T_{d(\mathfrak{s})^{-1}}z_{\boldsymbol{\lambda}}T_{d(\mathfrak{s}')},$$

because $d(\mathfrak{t})d(\mathfrak{t}')^{-1}=w_{\boldsymbol{\lambda}}$, with the lengths adding; see, for example, [**28**, Lemma 5.1]. It follows from [**29**, Prop. 4.4] that $(T_{d(\mathfrak{s})^{-1}}z_{\boldsymbol{\lambda}}T_{d(\mathfrak{s})})^2=rT_{d(\mathfrak{s})^{-1}}z_{\boldsymbol{\lambda}}T_{d(\mathfrak{s})}$, for some $r\in K$, such that $r\neq 0$ if and only if the Specht module $S^{\boldsymbol{\lambda}}$ is projective. If $r=0$ then these elements are nilpotent and they belong the radical of $\mathscr{H}_n^\Lambda$. We invite the reader to check that the map

$$S_{\boldsymbol{\lambda}'}\langle\operatorname{def}\beta+\deg\mathfrak{t}^{\boldsymbol{\lambda}}\rangle\xrightarrow{\ \sim\ }z_{\boldsymbol{\lambda}}\mathscr{H}_n^\Lambda;\psi'_{\mathfrak{t}}\mapsto z_{\boldsymbol{\lambda}}\psi'_{d(\mathfrak{t})},$$

for $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda}')$, is a isomorphism of graded $\mathscr{H}_n^\Lambda$-modules. Similarly, there is a graded isomorphism $S^{\boldsymbol{\lambda}}\langle\operatorname{def}\beta+\deg\mathfrak{t}^{\boldsymbol{\lambda}'}\rangle\xrightarrow{\ \sim\ }n_{\boldsymbol{\lambda}'}T_{w_{\boldsymbol{\lambda}'}}m_{\boldsymbol{\lambda}}\mathscr{H}_n^\Lambda$. By Corollary 6.21, $z_{\boldsymbol{\lambda}}^*=ce_{\boldsymbol{\lambda}'}\psi_{w_{\boldsymbol{\lambda}'}}e_{\boldsymbol{\lambda}}$ is homogeneous of degree $\operatorname{def}\beta+\deg(\mathfrak{t}^{\boldsymbol{\lambda}})+\deg(\mathfrak{t}^{\boldsymbol{\lambda}'})$, for some non-zero $c \in K$. Arguing as in Corollary 6.21 shows that $z^*_{\boldsymbol{\lambda}} = n_{\boldsymbol{\lambda}'} T_{w_{\boldsymbol{\lambda}'}} m_{\boldsymbol{\lambda}}$. Consequently, on the elements $z_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_n$, the graded cellular anti-automorphism $*$ of $\mathscr{H}^\Lambda_n$ coincides with the ungraded cellular algebra anti-isomorphism which is induced by the standard basis $\{m_{\mathfrak{uv}}\}$ of $\mathscr{H}^\Lambda_n$.

7. The degenerate and integral Khovanov-Lauda–Rouquier algebras

The Khovanov-Lauda–Rouquier algebras $\mathscr{R}^\Lambda_n$ are defined over an arbitrary commutative integral domain R, however, so far we have produced a cellular basis for $\mathscr{R}^\Lambda_n$ only when $R = K$ is a field of characteristic coprime to e. In this section we extend Theorem 5.8 to a more general class of rings. Before we can do this, however, we need to reprove Theorem 5.8 for the degenerate cyclotomic Hecke algebras of type A.

§7.1. **The degenerate cyclotomic Hecke algebra.** For this section, suppose that p is prime, or zero, and let K be a field of characteristic p. In this section we set $e = p$ so that $I = \mathbb{Z}/p\mathbb{Z}$.

The **degenerate cyclotomic Hecke algebra** H^Λ_n is the K-algebra generated by $x_1, \dots, x_n$ and $t_1, \dots, t_{n-1}$ subject to the relations

$$\begin{aligned}
\textstyle\prod_{i\in I}(x_1 - i)^{(\Lambda,\alpha_i)} &= 0, & t_i^2 &= 1,\\
t_s t_{s+1} t_s &= t_{s+1} t_s t_{s+1}, & t_i t_k &= t_k t_i, \text{ if } |i-k| > 1,\\
t_i x_{i+1} &= x_i t_i + 1, & t_i x_k &= x_k t_i \text{ if } |i-k| > 1,\\
x_i x_k &= x_i x_k,
\end{aligned}$$

for $1 \le k \le n$, $1 \le s < n-1$ and $1 \le i < n$.

The generators $t_1, \dots, t_{n-1}$ satisfy the braid relations of $\mathfrak{S}_n$. Therefore, if $w \in \mathfrak{S}_n$ and $w = s_{i_1} \dots s_{i_k}$ is a reduced expression then element $t_w = t_{i_1} \dots t_{i_k}$ depends only on w and not on the choice of reduced expression. By [**5**, Theorem 6.1] or [**24**, Theorem 7.5.6], $\{\, x_1^{a_1} \dots x_n^{a_n} t_w \mid 0 \le a_i < \ell \text{ and } w \in \mathfrak{S}_n \,\}$ is a basis of H^Λ_n.

Next, following Brundan and Kleshchev [**6**, §3.2], for $1 \le r < n$ and $1 \le s \le n$ define elements of H^Λ_n

$$\psi_r = \sum_{\mathbf{i}\in I^n} (t_r + p_r(\mathbf{i})) q_r(\mathbf{i})^{-1} e(\mathbf{i}) \quad \text{and} \quad y_s = \sum_{\mathbf{i}\in I^n} (x_s - i_s) e(\mathbf{i}),$$

where $p_r(\mathbf{i})$ and $q_r(\mathbf{i})$ are power series in $K[\![y_r, y_{r+1}]\!]$ with similar definitions to the elements $P_r(\mathbf{i})$ and $Q_r(\mathbf{i})$ given by (3.3) and (3.4); for the precise details see [**6**, 3.22 and 3.30]

As in the non-degenerate case, by [**24**, Lemma 7.1.2] there exists a family of idempotents $\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ in H^Λ_n such that if M is any H^Λ_n-module then

$$M_{\mathbf{i}} = M e(\mathbf{i}) = \{\, m \in M \mid m(x_r - i_r)^k = 0 \text{ for } k \gg 0 \,\},$$

and $M = \bigoplus_{\mathbf{i}\in I^n} M_{\mathbf{i}}$ is the decomposition of M into a direct sum of generalized eigenspaces for the commutative subalgebra $\langle x_1, \dots, x_n \rangle$ of H^Λ_n. Just as in Lemma 4.1 it follows that the non-zero $e(\mathbf{i})$ are the primitive idempotents in $\langle x_1, \dots, x_n \rangle$.

Brundan and Kleshchev's graded isomorphism in the degenerate case is the following.

7.1. **Theorem** (Brundan–Kleshchev [**6**, §3.5]). *Suppose that $e = p$ is prime. Then the map $\mathscr{R}^\Lambda_n \longrightarrow H^\Lambda_n$ which sends*

$$e(\mathbf{i}) \mapsto e(\mathbf{i}), \qquad y_r \mapsto y_r \qquad \text{and} \qquad \psi_s \mapsto \psi_s,$$

for $\mathbf{i} \in I^n$, $1 \le r \le n$ and $1 \le s < n$, extends uniquely to an isomorphism of algebras.

Following [**5**, §6], and paralleling the notation introduced for the non-degenerate case in Definition 3.6, for $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ define $m_{\mathfrak{s}\mathfrak{t}} = t_{d(\mathfrak{s})^{-1}} m_{\boldsymbol{\lambda}} t_{d(\mathfrak{t})}$, where

$$m_{\boldsymbol{\lambda}} = \prod_{s=2}^{\ell} \prod_{k=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (x_k - \kappa_s) \cdot \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} t_w.$$

Then $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda \,\}$ is a cellular basis of H_n^Λ by [**5**, Theorem 6.3]. (The multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell)$ is defined in section 3.1.)

These elements behave very similarly to the standard basis elements of $\mathscr{H}_n^\Lambda$ (see Theorem 3.7), which is why we use the same notation for them. In particular, if $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$, then by [**5**, Lemma 6.6]

$$m_{\mathfrak{s}\mathfrak{t}} x_k = \operatorname{res}_{\mathfrak{t}}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some scalars $r_{\mathfrak{u}\mathfrak{v}} \in K$.

Next we want to introduce a seminormal basis for H_n^Λ. To do this we need an integral form of H_n^Λ. To define this we have to assume that $e = p$ is prime and that K is a field of characteristic $p > 0$ because we do not know how to define an integral form for H_n^Λ directly when $e = 0$. (We can do this indirectly, however, because if $e = 0$ then H_n^Λ is isomorphic to a non-degenerate cyclotomic Hecke algebra by [**6**, Cor. 2], so we are back in the situation considered in section 5.)

Suppose then that $e = p$ is prime and that K is a field of characteristic p. In fact, we can assume that K is the field with p elements since every field is a splitting field for H_n^Λ by cellularity (see the remarks after Theorem 2.10). Let $\mathcal{O} = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime p and let $H_n^{\mathcal{O}}$ be the associative unital $\mathbb{Z}$-algebra with generators $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ which satisfy the same relations as above except that

$$(x_1 - \kappa_1)(x_1 - \kappa_2) \dots (x_1 - \kappa_\ell) = 0.$$

Then $H_n^{\mathcal{O}}$ is free as an $\mathcal{O}$-module, $H_n^\Lambda \cong H_n^{\mathcal{O}} \otimes_{\mathcal{O}} K$ and $H_n^{\mathbb{Q}} = H_n^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{Q}$ is semisimple.

The point of the relation for x_1 in $H_n^{\mathcal{O}}$ is that the content functions $\operatorname{cont}_{\mathfrak{t}}(k)$ (see above 3.8), separate the standard tableaux: that is, $\operatorname{cont}_{\mathfrak{s}}(k) = \operatorname{cont}_{\mathfrak{t}}(k)$, for $1 \le k \le n$, if and only if $\mathfrak{s} = \mathfrak{t}$. Once again, the analogous elements $\{m_{\mathfrak{s}\mathfrak{t}}\}$ give a basis of $H_n^{\mathcal{O}}$ and we have

$$m_{\mathfrak{s}\mathfrak{t}} x_k = \operatorname{cont}_{\mathfrak{t}}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} s_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for $s_{\mathfrak{u}\mathfrak{v}} \in \mathbb{Z}$. Hence, we can define a seminormal basis for $H_n^{\mathbb{Q}}$: define

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{\kappa_\ell - n < c < \kappa_1 + n \\ c \neq \operatorname{cont}_{\mathfrak{t}}(k)}} \frac{x_k - c}{\operatorname{cont}_{\mathfrak{t}}(k) - c} \in H_n^{\mathbb{Q}},$$

and set $f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$. Then $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $H_n^{\mathbb{Q}}$ by [**5**, Prop. 6.8].

Recall that in Definition 4.5 we defined certain sets $\mathscr{A}_{\mathfrak{t}}(k)$ and $\mathscr{R}_{\mathfrak{t}}(k)$ for each standard tableau $\mathfrak{t}$ and each integer k, with $1 \le k \le n$.

7.2. **Definition** (cf. Definition 4.5)**.** Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_n^\Lambda$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Define

$$\gamma_{\mathfrak{t}} = \prod_{k=1}^{n} \frac{\prod_{\alpha \in \mathscr{A}_{\mathfrak{t}}(k)} \big(\operatorname{cont}_{\mathfrak{t}}(k) - \operatorname{cont}(\alpha)\big)}{\prod_{\rho \in \mathscr{R}_{\mathfrak{t}}(k)} \big(\operatorname{cont}_{\mathfrak{t}}(k) - \operatorname{cont}(\rho)\big)} \quad \in \mathbb{Q}.$$

We have that $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}}$ in $H_n^{\mathbb{Q}}$ by [**5**, Prop. 6.8] and an easy inductive argument using [**5**, Lemma 610]. This brings us to the analogue of Proposition 4.8.

7.3. **Proposition.** *Suppose that $e(\mathbf{i}) \neq 0$, for some $\mathbf{i} \in I^n$ and let*

$$e(\mathbf{i})^{\mathcal{O}} := \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}} \in H_n^{\mathbb{Q}}.$$

Then $e(\mathbf{i})^{\mathcal{O}} \in H_n^{\mathcal{O}}$ and $e(\mathbf{i}) = e(\mathbf{i})^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K$.

Proof. The degenerate cyclotomic Hecke algebras fit into the general framework considered in [**30**], so $e(\mathbf{i})^{\mathcal{O}} \in H_n^{\mathcal{O}}$ by [**30**, Lemma 4.2]. The proof can be completed by repeating the argument of Proposition 4.8. ☐

For each positive tableau $\mathfrak{s}$ define $y_{\mathfrak{s}} = \prod_{k=1}^n y_k^{|\mathscr{A}_{\mathfrak{t}}^{\Lambda}(k)|} \in H_n^{\Lambda}$, where we are implicitly using the Brundan-Kleshchev graded isomorphism theorem in the degenerate case. If $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ is positive then lift $e(\mathbf{i})y_{\mathfrak{s}}$ to $H_n^{\mathcal{O}}$ by defining

$$e(\mathbf{i})^{\mathcal{O}} y_{\mathfrak{s}}^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}} \prod_{k=1}^{n} \prod_{\alpha\in\mathscr{A}_{\mathfrak{s}}^{\Lambda}(k)} \big(x_k - \operatorname{cont}_{\mathfrak{s}}(\alpha)\big).$$

Then, by repeating the argument of Theorem 4.14 we find that, in H_n^{Λ},

$$e(\mathbf{i})y_{\mathfrak{s}} = c m_{\mathfrak{s}\mathfrak{s}} + \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{s})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some $0 \neq c, r_{\mathfrak{u}\mathfrak{v}} \in K$. In particular, $e_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}}$ is equal to a multiple of $m_{\boldsymbol{\lambda}}$ plus a linear combination of more dominant terms.

Defining the elements $\psi_{\mathfrak{s}\mathfrak{t}} \in H_n^{\Lambda}$, for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda}$ exactly as before (see Definition 5.1), we can now repeat the proof of Theorem 5.8 to obtain the following.

7.4. **Theorem.** *Suppose that $e = p$ is prime and that K is a field of characteristic $p > 0$. Then the degenerate Hecke algebra H_n^{Λ} is a graded cellular algebra with graded cellular basis $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathscr{P}_n^{\Lambda} \,\}$.*

Using the results in this section, and those in section 5.3, it is an easy exercise to show that if $e = p$ is prime then $\{\, \psi'_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda}' \in \mathscr{P}_{\beta}^{\Lambda} \,\}$ is a graded cellular basis of $\mathscr{H}_{\beta}^{\Lambda}$, for $\beta \in Q_+$. The argument of Corollary 6.18 can now be repeated to show that $H_n^{\Lambda} \cong \mathscr{R}_n^{\Lambda}$ is a graded symmetric algebra. We leave the details to the reader.

§7.2. **Integral forms.** The Khovanov-Lauda–Rouquier algebras $\mathscr{R}_n^{\Lambda}$ can be defined over any commutative integral domain. So far we have produced graded cellular bases only when $\mathscr{R}_n^{\Lambda}$, via the Brundan-Kleshchev isomorphism theorems, is isomorphic to a degenerate or non-degenerate cyclotomic Hecke algebra over certain fields. We now consider more general rings.

Throughout this section, let $\mathscr{R}_n^{\Lambda}(\mathbb{Z})$ be the Khovanov-Lauda–Rouquier algebra of type $\Gamma = \Gamma_e$ defined over $\mathbb{Z}$, and let $\hat{\mathscr{R}}_n^{\Lambda}(\mathbb{Z})$ be the torsion free part of $\mathscr{R}_n^{\Lambda}(\mathbb{Z})$. Unlike in the last section, $e \in \{0, 2, 3, 4, \dots\}$ is not necessarily prime. If $\mathcal{O}$ is any commutative integral domain let $\mathscr{R}_n^{\Lambda}(\mathcal{O}) \cong \mathscr{R}_n^{\Lambda}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$ be the Khovanov-Lauda–Rouquier algebra over $\mathcal{O}$.

The following result is implicit in [**6**, Theorem 6.1]. It arose out of discussions with Alexander Kleshchev.

7.5. **Lemma.** a) *Suppose that $e = 0$ or that e is prime. Then $\mathscr{R}_n^{\Lambda}(\mathbb{Z}) = \hat{\mathscr{R}}_n^{\Lambda}(\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $\ell^n n!$.*

b) *Suppose that $e > 0$ is not prime. Then $\mathscr{R}_n^{\Lambda}(\mathbb{Z})$ has p-torsion, for a prime p, only if p divides e.*

Proof. First, observe that by Theorem 3.5

$$\operatorname{rank}\hat{\mathscr{R}}_n^\Lambda(\mathbb{Z}) = \dim_{\mathbb{Q}}(\mathscr{R}_n^\Lambda(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}) = \dim_{\mathbb{Q}}\mathscr{R}_n^\Lambda(\mathbb{Q}) = \ell^n n!,$$

where we take q to be a primitive e^{th} root of unity in $\mathbb{C}$ if $e\neq 0$ and not a root of unity if $e=0$.

Next suppose that $e=0$ and p is any prime. Let K be an infinite field of characteristic p and let $q\in K$ be a transcendental element of K. Then $\mathscr{H}_n^\Lambda \cong \mathscr{R}_n^\Lambda(K)\cong\mathscr{R}_n^\Lambda(\mathbb{Z})\otimes_{\mathbb{Z}} K$ by Theorem 3.5, so that $\mathscr{R}_n^\Lambda(\mathbb{Z})$ has no p-torsion.

Now suppose that $e>0$ and that p is prime not dividing e. Let K be a field of characteristic p which contains a primitive e^{th} root of unity q and let $\mathscr{H}_n^\Lambda$ be the non-degenerate cyclotomic Hecke algebra with parameter q. Then $\mathscr{H}_n^\Lambda\cong\mathscr{R}_n^\Lambda(K)\cong \mathscr{R}_n^\Lambda(\mathbb{Z})\otimes_{\mathbb{Z}} K$ by Brundan and Kleshchev's isomorphism Theorem (3.5). Hence, $\mathscr{R}_n^\Lambda(\mathbb{Z})$ has no p-torsion.

Finally, consider the case when $e=p$ is prime and let K be a field of characteristic p. Then $H_n^\Lambda\cong\mathscr{R}_n^\Lambda(K)\cong\mathscr{R}_n^\Lambda(\mathbb{Z})\otimes_{\mathbb{Z}}K$, so once again $\mathscr{R}_n^\Lambda(\mathbb{Z})$ has no p-torsion. Hence, $\mathscr{R}_n^\Lambda(\mathbb{Z})$ can have p-torsion only if $e>0$ is not prime and p divides e. □

The graded cellular basis $\{\psi_{\mathfrak{st}}\}$ is defined in terms of the generators of $\mathscr{R}_n^\Lambda(\mathbb{Z})$. Moreover, if $e=0$ and K is any field, or if $e>0$ and K is a field containing a primitive e^{th} root of 1, then $\{\psi_{\mathfrak{st}}\otimes 1_K\}$ is a graded cellular basis of the algebra $\mathscr{R}_n^\Lambda(K)\cong\mathscr{H}_n^\Lambda$. Further, if $e=p$ is prime then $\{\psi_{\mathfrak{st}}\otimes_{\mathbb{Z}}1_K\}$ is a graded cellular basis of $\mathscr{R}_n^\Lambda(K)\cong H_n^\Lambda$ whenever K is a field of characteristic p. Hence, applying Lemma 7.5, Theorem 5.8 and Theorem 7.4, we obtain our final result.

7.6. **Theorem.** *Let $\mathcal{O}$ be a commutative integral domain and suppose that either $e=0$, e is non-zero prime, or that $e\cdot 1_{\mathcal{O}}$ is invertible in $\mathcal{O}$. Then $\mathscr{R}_n^\Lambda(\mathcal{O})\cong\mathscr{R}_n^\Lambda(\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{O}$ is a graded cellular algebra with graded cellular basis*

$$\{\,\psi_{\mathfrak{st}}\otimes 1_{\mathcal{O}}\mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\text{ and }\boldsymbol{\lambda}\in\mathscr{P}_n^\Lambda\,\}.$$

It seems likely to us that the ψ-basis is a graded cellular basis of $\mathscr{R}_n^\Lambda(\mathbb{Z})$.

Appendix A. One dimensional homogeneous representations

Using Theorem 5.8 it is straightforward to give an explicit homogeneous basis for the one dimensional two-sided ideals of $\mathscr{H}_n^\Lambda$. In this appendix, which may be of independent interest, we give a proof of this result without appealing to Theorem 5.8. We consider the non-degenerate case only and leave easy modifications required in the degenerate case for the reader.

We remark that it is possible to prove an analogue of Theorem 5.8 using the ideas in this appendix. However, using these techniques we were only able to show that the basis $\{\psi_{\mathfrak{st}}\}$ was a graded cellular basis with respect to the *lexicographic* order on $\mathscr{P}_n^\Lambda$.

A1. **Definition.** Suppose that $1\le s\le e$ and $(\Lambda,\alpha_s)>0$ and set

$$u_{n,s}=\prod_{i\in I}\big((L_1-q^i)\dots(L_n-q^i)\big)^{(\Lambda,\alpha_i)-\delta_{is}},$$

$$x_{(n)}=\sum_{w\in\mathfrak{S}_n}T_w\quad\text{and}\quad x'_{(n)}=\sum_{w\in\mathfrak{S}_n}(-q)^{-\ell(w)}T_w.$$

Finally, define $z_n^{+,s}=u_{n,s}x_{(n)}$ and $z_n^{-,s}=u_{n,s}x'_{(n)}$, for $1\le s\le e$.

The following result is well-known and easily verified.

A2. **Lemma.** *Suppose that $1\le s\le e$ and that $\varepsilon\in\{+,-\}$. Then*

$$T_w z_n^{\varepsilon,s}=z_n^{\varepsilon,s}T_w=(-1)^{\frac12(1-\varepsilon1)\ell(w)}q^{\frac12(1+\varepsilon1)\ell(w)}z_n^{\varepsilon,s},$$
$$L_k z_n^{\varepsilon,s}=z_n^{\varepsilon,s}L_k=q^{s+\varepsilon(k-1)}z_n^{\varepsilon,s},$$

for all $w \in \mathfrak{S}_n$ and $1 \le k \le n$. In particular, $Kz_n^{\pm,s}$ is a one dimensional two-sided ideal of $\mathscr{H}_n^\Lambda$. Moreover, every one dimensional two-sided ideal is isomorphic to $Kz_n^{\varepsilon,s}$, for some s, and

$$Kz_n^{\varepsilon,s} = \left\{ h \in \mathscr{H}_n^\Lambda \;\middle|\; \begin{array}{c} T_0 h = q^s h = hT_0 \text{ and} \\ T_i h = hT_i = (-1)^{\frac12(1-\varepsilon 1)} q^{\frac12(1+\varepsilon 1)} h \text{ for } 1 \le i < n \end{array} \right\}.$$

A3. **Proposition.** *Suppose that Kz is a two sided ideal $\mathscr{R}_n^\Lambda$, for some non-zero element $z \in \mathscr{H}_n^\Lambda$. Then z is homogeneous.*

Proof. Write $z = \sum_{i\in\mathbb{Z}} z_i$, where z_i is a homogeneous element of degree i, for each $i \in \mathbb{Z}$, with only finitely many z_i being non-zero. Let $h \in \mathscr{H}_n^\Lambda$ be any homogeneous element. Then $hz = fz$, for some $f \in K$, so that

$$\sum_{i\in\mathbb{Z}} f z_i = hz = \sum_{i\in\mathbb{Z}} hz_i.$$

By assumption, either $hz_i = 0$ or $\deg(hz_i) = \deg h + \deg z_i$, for each i. Therefore, if $\deg h > 0$ and $hz \ne 0$ then $hz_i = fz_j$ for some $j > i$, which is a contradiction since this forces $hz = fz$ to have fewer homogeneous summands than z. Therefore, $hz = 0$ if $\deg h > 0$. Similarly, $hz = 0$ if $\deg h < 0$. Therefore, for any $h \in \mathscr{H}_n^\Lambda$ we have that $hz_i = fz_i$, for all $i \in \mathbb{Z}$, so that $z_i = z_n^{\pm,s}$, for some s by Lemma A2. Since the non-zero z_i have different degrees they must be linearly independent, so it follows from Lemma A2 that $z = z_i$ for a unique i. In particular, z is homogeneous as claimed. □

The following definition will be used to give the degree of the elements $z_{n,s}^\varepsilon$ and to explicitly describe them as a product of the homogeneous generators of $\mathscr{H}_n^\Lambda$.

We extend our use of the Kronecker delta by writing, for any statement S, $\delta_S = 1$ if S is true and $\delta_S = 0$ otherwise.

A4. **Definition** (cf. Definition 4.9)**.** Suppose that $1 \le s \le e$ and let $\varepsilon \in \{+,-\}$. Let $\mathbf{i}_n^{\varepsilon,s} = (i_1^{\varepsilon,s}, \dots, i_n^{\varepsilon,s}) \in I^n$, where $i_k^{\varepsilon,s} = s + \varepsilon(k-1) \pmod e$. For $1 \le k \le n$ set

$$d_k^{\varepsilon,s} = \#\{\, 1 \le t \le \ell \mid i_k^{\varepsilon,s} = t \text{ and } (\Lambda, \alpha_t) > \delta_{st} \,\} + \delta_{e|k}.$$

Finally, define $y_n^{\varepsilon,s} = \prod_{k=1}^n y_k^{d_k^{\varepsilon,s}}$.

Brundan, Kleshchev and Wang [**8**, (4.5)] note that the natural embedding $\mathscr{H}_n^\Lambda \hookrightarrow \mathscr{H}_{n+1}^\Lambda$ is an embedding of graded algebras. Explicitly, the graded embedding is determined by

$$\text{(A5)} \qquad \psi_s \mapsto \psi_s, \qquad y_r \mapsto y_r, \qquad \text{and} \qquad e(\mathbf{i}) \mapsto \sum_{j\in I} e(\mathbf{i} \vee j),$$

where $1 \le r \le n$, $1 \le s < n$, $\mathbf{i} \in I^n$ and $\mathbf{i} \vee i = (i_1, \dots, i_n, i)$.

We can now explicitly describe $z_n^{\varepsilon,s}$ as a product of homogeneous elements and hence determine its degree.

A6. **Theorem.** *Suppose that $1 \le s \le e$, $(\Lambda, \alpha_s) > 0$ and that $\varepsilon \in \{+,-\}$. Then*

$$z_n^{\varepsilon,s} = Ce(\mathbf{i}_n^{\varepsilon,s}) y_n^{\varepsilon,s},$$

for some non-zero constant $C \in K$. In particular, $\deg z_n^{\varepsilon,s} = 2(d_1^{\varepsilon,s} + \cdots + d_n^{\varepsilon,s})$.

Proof. As $Kz_n^{\varepsilon,s}$ is a two-sided ideal we have that $e(\mathbf{i}_n^{\varepsilon,s}) z_n^{\varepsilon,s} e(\mathbf{i}_n^{\varepsilon,s}) \in Kz_n^{\varepsilon,s}$. Further, it is well-known and easy to check (*cf.* [**28**, §4]), that $Kz_n^{\varepsilon,s} \cong S(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and

$$\lambda^{(t)} = \begin{cases} (n), & \text{if } t = s \text{ and } \varepsilon = +, \\ (1^n), & \text{if } t = s \text{ and } \varepsilon = -, \\ (0), & \text{otherwise.} \end{cases}$$

Therefore, as $\mathbf{i}_n^{\varepsilon,s} = \mathbf{i}^{\boldsymbol{\lambda}}$ it follows from the construction of the graded Specht modules in section 5.2 (or [**8**, Theorem 4.10]), that $z_n^{\varepsilon,s} e(\mathbf{i}_n^{\varepsilon,s}) \neq 0$, so we see that $z_n^{\varepsilon,s} = e(\mathbf{i}_n^{\varepsilon,s}) z_n^{\varepsilon,s} = z_n^{\varepsilon,s} e(\mathbf{i}_n^{\varepsilon,s}) = e(\mathbf{i}_n^{\varepsilon,s}) z_n^{\varepsilon,s} e(\mathbf{i}_n^{\varepsilon,s})$ as claimed.

It remains to write $z_n^{\varepsilon,s}$ as a product of homogeneous elements. To ease the notation we treat only the case when $\varepsilon = +$ and we write $z_n = z_n^{\varepsilon,s}$, $\mathbf{i}_n = \mathbf{i}_n^{\varepsilon,s}$ and $d_n = d_n^{\varepsilon,s}$. The case when $\varepsilon = -$ follows by exactly the same argument (and, in fact, the same constants appear below), the only difference is that the products $T_{n-1} \dots T_j$ must be replaced by $(-q)^{j-n} T_{n-1} \dots T_j$ below.

Suppose, first, that $n = 1$. By definition, $d_1 = (\Lambda, \alpha_s) - 1$. Recall that $L_1 = \sum_{\mathbf{i}} q^{i_1}(1 - y_1) e(\mathbf{i})$ by Theorem 3.5. Therefore, we have

$$\begin{aligned} z_1 e(\mathbf{i}_n) &= \prod_{t \in I} (L_1 - q^t)^{(\Lambda,\alpha_t) - \delta_{st}} e(\mathbf{i}_n) = \prod_{t \in I} (q^s - q^t - q^s y_1)^{(\Lambda,\alpha_t) - \delta_{st}} e(\mathbf{i}_n) \\ &= \prod_{t \neq s} (q^s - q^t - q^s y_1)^{(\Lambda,\alpha_t)} e(\mathbf{i}_n) \cdot (-q^s y_1)^{(\Lambda,\alpha_s) - 1} e(\mathbf{i}_n) \\ &= \prod_{t \neq s} (q^s - q^t)^{(\Lambda,\alpha_t)} \cdot (-q^s y_1)^{(\Lambda,\alpha_s) - 1} e(\mathbf{i}_n) \end{aligned}$$

where the last equality follows because the 'cyclotomic relation' $y_1^{(\Lambda,\alpha_s)} e(\mathbf{i}_n) = 0$, holds in $\mathscr{R}_n^\Lambda$. Thus, the Theorem holds when $n = 1$.

Now suppose that $n > 1$ and that the Theorem holds for smaller n. Then, using the definitions,

$$\begin{aligned} z_n &= e(\mathbf{i}_n) \prod_{t \in I} (L_n - q^t)^{(\Lambda,\alpha_t) - \delta_{st}} \cdot z_{n-1} \cdot \Big(1 + \sum_{j=1}^{n-1} T_{n-1} \dots T_j\Big) e(\mathbf{i}_n) \\ &= \prod_{t \in I} (L_n - q^t)^{(\Lambda,\alpha_t) - \delta_{st}} \cdot e(\mathbf{i}_n) z_{n-1} \cdot \Big(1 + \sum_{j=1}^{n-1} T_{n-1} \dots T_j\Big) e(\mathbf{i}_n). \end{aligned}$$

By induction and (A5), there exists a scalar non-zero $C \in K$ such that

$$\begin{aligned} e(\mathbf{i}_n) z_{n-1} = z_{n-1} e(\mathbf{i}_n) &= C y_{n-1}^{\varepsilon,s} \prod_{i \in I} e(\mathbf{i}_{n-1} \vee i) \cdot e(\mathbf{i}_n) \\ &= C y_{n-1}^{\varepsilon,s} e(\mathbf{i}_n) \end{aligned}$$

Let $d'_n = d_n - \delta_{e|n}$. Then there exist constants $C'_a \in K$, for $a \geq d'_n$, such that

$$\begin{aligned} &\prod_{t \in I} (L_n - q^t)^{(\Lambda,\alpha_t) - \delta_{st}} \cdot e(\mathbf{i}_n) z_{n-1} \\ &\qquad = C \prod_{t \in I} \big(q^{s+(n-1)}(1 - y_n) - q^t\big)^{(\Lambda,\alpha_t) - \delta_{st}} \cdot y_{n-1}^{\varepsilon,s} e(\mathbf{i}_n) \\ &\qquad = e(\mathbf{i}_n) y_{n-1}^{\varepsilon,s} \sum_{a \geq d'_n} C_a y_n^a, \end{aligned}$$

with $C_{d'_n} = C(-q)^{(s+(n-1))d'_n} \prod_t (q^{s+(n-1)} - q^t)^{(\Lambda,\alpha_t) - \delta_{st}}$, where the product is over those $t \in I$ with $t \not\equiv s + (n-1) \pmod{e\mathbb{Z}}$. In particular, $C_{d'_n} \neq 0$. Next, recall from Theorem 3.5 that

$$T_k e(\mathbf{i}_n) = \big(\psi_k Q_k(\mathbf{i}_n) - P_k(\mathbf{i}_n)\big) e(\mathbf{i}_n),$$

for $1 \leq k \leq n$. Applying the relations in (3.1), if $1 \leq k_1 < \dots < k_p < n$ then

$$e(\mathbf{i}_n) \psi_{k_p} \dots \psi_{k_1} e(\mathbf{i}_n) = \psi_{k_p} \dots \psi_{k_1} e(s_{k_1} \dots s_{k_p} \cdot \mathbf{i}_n) e(\mathbf{i}_n) = 0.$$

Moreover, by the proof of Proposition A3 we know that $z_{n-1} y_i = 0$, for $1 \leq i < n$. Therefore, when we expand $P_j(\mathbf{i}_n)$ as a power series in $K[\![y_1, \dots, y_n]\!]$ only those terms in $K[\![y_n]\!]$ contribute to z_n. Putting all of this together we find that

$$z_n = e(\mathbf{i}_n) y_{n-1}^{\varepsilon,s} \sum_{a \geq d'_n} C'_a y_n^a$$

for some $C'_a \in K$. Notice that only one of these terms can survive since z_n is homogeneous by Proposition A3. By (3.3) the constant term of $P_j(\mathbf{i}_n)$ is $-(1-q)/(1-q^{-1}) = q$, so

$$\frac{C'_{d'_n}}{C_{d'_n}} = 1 + \sum_{j=1}^{n-1} q^t = 1 + q + \cdots + q^{n-1}.$$

Therefore, $C'_{d'_n} \neq 0$ if and only if $e \nmid n$, which is exactly the case when $d'_n = d_n$ so the Theorem holds when $e \nmid n$.

Finally, suppose that $e|n$. Then $C'_{d'_n} = 0$, by what we have just shown, and $d_n = d'_n + 1$, so we need to show that $C'_{d'_n+1} \neq 0$. This time the degree one term of $P_n(\mathbf{i}_n)$ and the degree zero terms of $P_j(\mathbf{i}_n)$, for $1 \leq j < n$, contribute to $C'_{d'_n+1}$. Using (3.3) again, we find that

$$\frac{C'_{d'_n+1}}{C_{d'_n}} = \frac{q}{q-1}\big(q + q^2 + \cdots + q^{n-1}\big) = \frac{q}{1-q} \neq 0.$$

This completes the proof of the Theorem. □

We remark that we do not know how to prove Theorem A6 using the relations directly. One problem, for example, is that it is not clear from the proof of Theorem A6 that $C_{d'_n+1} = 0$ when $e \nmid n$ – note that if $C_{d'_n+1} \neq 0$ then z_n would not be homogeneous since $C_{d'_n} \neq 0$ when $e \nmid n$. We are able to prove Theorem A6 only because we already know that z_n is homogeneous by Proposition A3.

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