INDECOMPOSABLE PD_3-COMPLEXES

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Abstract. We show that if the fundamental group π of an indecomposable PD_3-complex is the fundamental group of a finite graph of finite groups then the vertex groups have periodic cohomology and the edge groups are metacyclic. If the vertex groups all have cohomological period dividing 4 then they are dihedral, the edge groups are $Z/2Z$, and the underlying graph is a tree. We also ask whether every PD_3-complex has a finite covering space which is homotopy equivalent to a closed orientable 3-manifold.

It is a well known consequence of the Sphere Theorem that every closed 3-manifold is a connected sum of indecomposable factors, which are either aspherical or have fundamental group Z or a finite group. There is a partial analogue for PD_3-complexes: Turaev showed that a PD_3-complex whose fundamental group is a free product is a connected sum [24], while Crisp showed that every indecomposable PD_3-complex is either aspherical or its fundamental group is the fundamental group of a finite graph of finite groups [3]. However the group may have infinitely many ends, in contrast to the situation for 3-manifolds. The only examples known thus far are orientable and have group $S_3 *_{Z/2Z} S_3$ [15].

Crisp showed also that if X is an orientable PD_3-complex then the centralizer of a nontrivial element of prime order must be finite. We shall use this observation repeatedly in §1 and §2. In Theorem 5 we show that if X is indecomposable and $\pi_1(X) = \pi\mathcal{G}$ for some finite graph $\mathcal{G}$ of finite groups then the vertex groups have periodic cohomology and the edge groups are metacyclic. If moreover the vertex groups are metacyclic then the edge groups are cyclic. In Theorem 9 we show that if the vertex groups all have cohomological period dividing 4 then they are dihedral, the edge groups have order 2 and the underlying graph is a tree. There are no known examples of nonorientable PD_3-complexes with virtually free fundamental group. In Theorem 11 we give a further constraint on any such example.

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In the final part of this paper we turn to the aspherical case. Here the main question is whether every aspherical PD_3-complex is homotopy equivalent to a closed 3-manifold. An equivalent question is whether every PD_3-complex has a finite covering space which is homotopy equivalent to a closed orientable 3-manifold. We suggest a reduction of this question to a question about Dehn surgery on links.

1. vertex groups have periodic cohomology

A *graph of groups* $(\mathcal{G},\Gamma)$ consists of a graph Γ with origin and target functions o and t from the set of edges $E(\Gamma)$ to the set of vertices $V(\Gamma)$, and a family $\mathcal{G}$ of groups G_v for each vertex v and subgroups $G_e \leq G_{o(e)}$ for each edge e, with monomorphisms $\phi_e : G_e \to G_{t(e)}$. (We shall usually suppress the maps ϕ_e from our notation.) The *fundamental group* of $(\mathcal{G},\Gamma)$ is the group $\pi\mathcal{G}$ with presentation

$$\langle G_v, t_e \mid t_e g t_e^{-1} = \phi_e(g)\ \forall\ g \in G_e, t_e = 1\ \forall\ e \in E(T)\rangle,$$

where T is some maximal tree for Γ. Different choices of maximal tree give isomorphic groups. The graph is minimal if each edge group is a proper subgroup of the adjacent vertex groups.

If G is a group G' and ζG shall denote the commutator subgroup and centre of G, while if $H \leq G$ is a subgroup $C_G(H)$ and $N_G(H)$ shall denote the centralizer and normalizer of H in G, respectively.

Lemma 1. *Let $\pi = \pi\mathcal{G}$, where $(\mathcal{G},\Gamma)$ is a finite graph of groups. If C is a subgroup of an edge group G_e with $N_{G_e}(C)$ properly contained in each of $N_{G_{o(e)}}(C)$ and $N_{G_{t(e)}}(C)$ then $N_\pi(C)$ is infinite.*

Proof. If $g_o \in G_{o(e)} - G_e$ and $g_t \in G_{t(e)} - G_e$ each normalize C then $g_o g_t$ normalizes C and has infinite order in π. □

A finitely generated group is the fundamental group of a finite graph of finite groups if and only if it is virtually free. Let F be a maximal free normal subgroup of π. Then $G = \pi/F$ is finite, and the canonical surjection $s : \pi \to G$ is injective on every finite subgroup of π. In particular, if H is a finite subgroup of π then the subgroup $FH = s^{-1}s(H)$ generated by F and H is a semidirect product $F \rtimes H$.

Theorem 2. *Let $\pi = \pi_1(X)$, where X is an orientable PD_3-complex, and suppose that $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G},\Gamma)$ is a minimal finite graph of finite nilpotent groups. Then all edge groups are trivial and the vertex groups are finite cyclic or direct products of cyclic groups of odd order with quaternionic 2-groups.*

Proof. If e is an edge the normalizer of G_e in each of $G_{o(e)}$ and $G_{t(e)}$ is strictly larger, since nilpotent groups satisfy the normalizer condition. (See Chapter 5.§2 of [20].) Therefore $N_\pi(G_e)$ is infinite. If $g \in G_e$ the centralizer $C_\pi(g)$ is also infinite, since $Aut(G_e)$ is finite. Hence $g = 1$, by Theorem 17 of [3], and so the edge groups are all trivial. Therefore π is the free product of finite groups. Since X has a connected sum decomposition inducing this splitting of π it follows that each vertex group is the fundamental group of a PD_3-complex. Hence it has periodic cohomology.

A finite nilpotent group has periodic cohomology if and only if it is cyclic or is the direct product of a cyclic group of odd order with a quaternionic 2-group $Q(2^k)$, for some $k \geq 3$. □

Finite cyclic groups have cohomological period 2, while direct products $Q(2^k) \times Z/dZ$ have cohomological period 4.

Corollary 3. *If X is finite and the vertex groups are cyclic or isomorphic to $Q(2^k)$ for some $k \geq 3$ then X is homotopy equivalent to a connected sum of $\mathbb{S}^3$-manifolds.*

Proof. All finite Swan complexes for cyclic groups or for such quaternionic 2-groups are homotopy equivalent to $\mathbb{S}^3$-manifolds [22]. □

Lemma 4. *Let G be a finite group with periodic cohomology. If the 2-Sylow subgroup of G is not cyclic then G has a central involution.*

Proof. This follows on examining the standard list of finite groups with periodic cohomology. □

If a group G has periodic cohomology all subgroups of order p^2 are cyclic. In particular, if $g \in G$ is an involution it is central if and only if it is the unique involution.

Theorem 5. *Let $\pi = \pi_1(X)$, where X is an orientable PD_3-complex, and suppose that $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is a finite graph of finite groups. Then the vertex groups have periodic cohomology and the edge groups are metacyclic. For each edge e the highest common factor of $[G_{o(e)} : G_e]$, $[G_{t(e)} : G_e]$ and $|G_e|$ is 1.*

Proof. Let F be a maximal free normal subgroup of π. If S is a Sylow p-subgroup of a vertex group G_v then FS is the fundamental group of a finite graph of finite p-groups. Hence FS is a free product of finite nilpotent groups with periodic cohomology, by Theorem 2. Therefore S has periodic cohomology. Since a finite group has periodic cohomology if and only if this holds for all its Sylow subgroups (see Proposition VI.9.3 of [2]) it follows that G_v has periodic cohomology.

If G_e is not metacyclic then the Sylow 2-subgroups of G_e, $G_{o(e)}$ and $G_{t(e)}$ are all non-cyclic. Hence G_e has a central involution which is also central in each of $G_{o(e)}$ and $G_{t(e)}$. As in Theorem 2 this contradicts Theorem 17 of [3].

Let p be a prime divisor of $|G_e|$, and let P be the p-Sylow subgroup of $G = \pi/F$. The projection s maps the p-Sylow subgroups of the edge and vertex groups into P. Thus the p-Sylow subgroup of G_e is an edge group in the induced graph of groups structure of $\pi_P = s^{-1}(P)$, and the adjacent vertex group groups are the p-Sylow subgroups of $G_{o(e)}$ and $G_{t(e)}$. On applying Theorem 2 we see that at least one of the two inclusions is an isomorphism. Thus p cannot divide both $[G_{o(e)} : G_e]$ and $[G_{t(e)} : G_e]$. □

If all the Sylow subgroups of a finite group are cyclic then it is metacyclic, with a presentation

$$\langle a, b \mid a^m = b^n = 1, bab^{-1} = a^r \rangle,$$

where $(n(r-1), m) = 1$. (See Proposition 10.1.10 of [20].) In such a group the commutator subgroup is the cyclic group of order m generated by the image of a.

Corollary 6. *If the vertex groups are all metacyclic then each edge group is cyclic.*

Proof. Every subgroup of the commutator subgroup of a metacyclic group is normal. Thus if an edge group is not cyclic its commutator subgroup is nontrivial, and is normal in each of the adjacent vertex groups. As in Theorem 5, this contradicts Theorem 17 of [3]. □

In particular, this corollary applies if π/F has odd order.

2. COHOMOLOGICAL PERIOD DIVIDING 4

The fact that the Sylow subgroups of G_v have cohomological period dividing 4 does not imply that G_v has cohomological period dividing 4. (The simplest example is the nonabelian group of order 21, which has cyclic Sylow subgroups and cohomological period 6. See Exercise VI.9.6 of [2].) However, if this holds for all the vertex groups we can say much more.

We recall that a group G has cohomological period dividing 4 if and only if it is a cyclic group or a direct product $B \times Z/dZ$, where B is a generalized quaternionic group $Q(8a, b, c)$, an extended binary polyhedral group T_k^*, O_k^* or I^*, or a metacyclic group $A(m, e)$, and $(d, |B|) = 1$ [4].

Lemma 7. *If $C < G < H$ are finite groups with cohomological period dividing 4 and $C = Z/pZ$ where p is an odd prime then $N_G(C) < N_H(C)$, unless $G = T_1^* \times Z/dZ$, $H = I^* \times Z/dZ$ and $p = 3$.*

Proof. It is easy to see from the list of such groups that if $p > 5$ then $N_G(C) = G$. If $p = 5$ this holds also unless $G = I^* \times Z/dZ$, in which case $N_G(C) = C \times Z/dZ$. If $p = 3$ then $N_G(C) = G$ unless $G = T_1^* \times Z/dZ$, $O_1^* \times Z/dZ$ or $I^* \times Z/dZ$, in which case $N_G(C) = C \times Z/dZ$, $S_3 \times Z/dZ$ or $C \times Z/dZ$, respectively. The only proper inclusion $G < H$ between such groups is induced by the inclusion of T_1^* as a subgroup of I^*. Thus $N_G(C) < N_H(C)$ except when $p = 3$, $G = T_1^* \times Z/dZ$ and $H = I^* \times Z/dZ$, in which case $N_G(C) = N_H(C) = C \times Z/dZ$. □

Lemma 8. *Let $\pi = A *_C B$, where A and B are indecomposable and $C \neq 1$. Then π is indecomposable.*

Proof. See pages 245-246 of [19]. □

Theorem 9. *Let $\pi = \pi_1(X)$, where X is an indecomposable orientable PD_3-complex, and suppose that $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is a finite graph of finite groups. Suppose that every vertex group has cohomological period dividing 4. Then the vertex groups are dihedral, the edge groups have order 2 and Γ is a tree.*

Proof. Let G_e be an edge group. If $G_e \cong T_1^* \times Z/dZ$ then G_e has a central involution g. But then g is also central in each of $G_{o(e)}$ and $G_{t(e)}$, and so $N_\pi(\langle g \rangle)$ is infinite. Thus it follows from Lemma 7 that G_e must be a 2-group. If g is the central involution of G_e it cannot be central in both of $G_{o(e)}$ and $G_{t(e)}$, for otherwise $C_\pi(\langle g \rangle)$ would be infinite, contradicting Theorem 17 of [3]. Hence one of them is a dihedral group $A(m, 1) \times Z/dZ$, by Lemma 4, and so G_e must have order 2, since $A(m, 1) \times Z/dZ$ has order $2md$ with md odd. Since the cyclic direct factor Z/dZ centralizes g we must in fact have $d = 1$.

Suppose that there are vertices $v \neq w$ such that G_v and G_w have involutions g_v and g_w with centralizers C_v and C_w of order > 2, and choose a (minimal) path connecting these vertices. Each vertex group has an unique conjugacy class of involutions, and so $g_w = a g_v a^{-1}$ for some a in the subgroup generated by the intermediate vertex groups along the path. Thus g_w is centralized by the subgroup generated by C_w and $a C_v a^{-1}$, which is infinite. This again contradicts Theorem 17 of [3].

If there is a nontrivial cycle γ in Γ incorporating an edge e we may choose a maximal tree T which does not contain e. (Here we do not require that the edges making up a cycle be compatibly oriented.) Therefore if g_e is a generator of G_e we have $t_e(g_e)t_e^{-1} = w g_e w^{-1}$, where w is a word in the union of the vertex groups along the rest of the cycle. The element $t_e w^{-1}$ has infinite order, and so the normalizer of g_e in π is infinite. This again contradicts Theorem 17 of [3].

The underlying graph is a tree T, by Theorem 6. Since each vertex group has an unique conjugacy class of involutions we may assume that T is linear, and that $A(m,1) = G_{o(e)}$ where $o(e)$ is an extremal vertex. Hence $\pi = H *_{Z/2Z} A(m,1)$, where H is the subgroup determined by the subgraph obtained by deleting the edge e and vertex $o(e)$. Suppose that $A(m,1)$ is generated by $\{a,z\}$ where a has order m and z is the involution in $H \cap A(m,1)$. Then π has a non-normal subgroup of index m, which has a graph of groups structure with vertex groups the conjugates of H by the powers of a and edge groups of order 2. On representing each conjugate of H as a graph-group we see that each of the original vertex groups other than $G_{o(e)} = A(m,1)$ appears m times as a vertex group, and all the edge groups are $Z/2Z$. Iterated applications of Lemma 8 show that this group is again indecomposable. Since it has finite index in π it is also the group of a PD_3-complex. Since each isomorphism class of vertex group occuring is represented more than once, there can be no non-dihedral vertex groups. □

Corollary 10. *The group π is a semidirect product $\pi \cong \pi' \rtimes Z/2Z$, π' is a free product of cyclic groups of odd order, and the associated covering space X' is homotopy equivalent to a connected sum of lens spaces.*

Proof. It is easy to see that $\pi/\pi' \cong Z/2Z$ and that π' is the normal subgroup generated by the images of the commutator subgroups of the vertex groups. The inclusion of any one of the edge groups is a splitting map. The final assertion follows from [24]. □

If the vertex groups are all isomorphic then $\pi \cong F \rtimes A(m,1)$ for some m. However the number of vertex groups may be arbitrarily large. Applying the final construction of Theorem 9 to $S_3 *_{Z/2Z} S_3$ gives an indecomposable subgroup of index 3 isomorphic to $S_3 *_{Z/2Z} S_3 *_{Z/2Z} S_3$. Iterating this process gives trees of unbounded length which are realized by indecomposable PD_3-complexes.

Must the vertex groups be all isomorphic? Is $S_3 *_{Z/2Z} A(5,1)$ the fundamental group of a PD_3-complex?

Are any PD_3-complexes of the type considered here homotopy equivalent to an infinite cyclic covering of a closed 4-manifold?

Although there are no known examples of nonorientable, indecomposable PD_3-complexes X with $\pi_1(X)$ virtually free, we have only the following explicit constraints.

Theorem 11. *Let X be a nonorientable PD_3-complex. If $\pi = \pi_1(X)$ is virtually free then it is infinite. If moreover* $\mathrm{Ker}(w_1(X)) \cong L \rtimes Z/2Z$ *where L/L' has odd order then $\pi/\pi' \cong Z/4Z$.*

Proof. The first assertion is clear, since PD_3-complexes with finite fundamental group are orientable [25]. The hypothesis in the second assertion implies that $\pi' = L$ and $\pi/\pi' \cong (Z/2Z)^2$ or $Z/4Z$. Suppose that $\pi/\pi' \cong (Z/2Z)^2$. The kernel of the map from the symmetric product $H^1(\pi;\mathbb{F}_2) \odot H^1(\pi;\mathbb{F}_2) \cong \mathbb{F}_2^3$ to $H^2(\pi;\mathbb{F}_2)$ induced by cup product is the dual of a quotient of $X^2(\pi)/[\pi, X^2(\pi)]$, where $X^2(\pi)$ is the verbal subgroup generated by squares [13]. It is easy to see that in this case $X^2(\pi) = \pi'$ and $[\pi, X^2(\pi)] = \pi'$. Hence $\beta_2(\pi;\mathbb{F}_2) \geq 3$. But this contradicts the bound $\beta_2(\pi;\mathbb{F}_2) \leq \beta_2(X;\mathbb{F}_2) = 2$ given by Poincaré duality. Thus we must have $\pi/\pi' \cong Z/4Z$. $\square$

In particular, the involutions of the free factors of π' must admit square roots, and so no such complex X has $\pi' \cong Z/3Z * Z/3Z$. Is there a PD_3-complex with fundamental group having the presentation

$$\langle a, b, c \mid a^4 = b^5 = c^5 = aba^{-1}b^2 = aca^{-1}c^2 = 1\rangle?$$

3. Is every aspherical PD_3-complex virtually a 3-manifold?

It is well known that every PD_2-complex is homotopy equivalent to a closed surface. The argument of Eckmann and Müller [7] for the cases with $\beta_1 \neq 0$ involves delicate combinatorial group theory. (The hypothesis $\beta_1 \neq 0$ is removed in [6].) More recently, Bowditch used geometric group theory to obtain the stronger result that an FP_2 group Γ with $H^2(\Gamma;\mathbb{Z}[\Gamma]) \cong Z$ acts properly discontinuously on $\mathbb{E}^2$ or $\mathbb{H}^2$ [1]. Higher dimensional considerations suggest another, more topological strategy, which can be justified *a posteriori.* The bordism Hurewicz homomorphism from $\Omega_n(X)$ to $H_n(X;\mathbb{Z})$ is an epimorphism in degrees $n \leq 4$. Therefore if X is an orientable PD_n-complex with $n \leq 4$ there is a degree-1 map $f : M \to X$ with domain a closed orientable n-manifold. (See [12] for the corresponding result for nonorientable PD_n-complexes, using w_1-twisted bordism and homology.) Choose compatible basepoints m_o and $x_o = f(m_o)$, and let $\pi = \pi_1(X, x_o)$ and $f_* = \pi_1(f)$. If X is a finite PD_2-complex then such a map f is a homotopy equivalence $\Leftrightarrow$ $\mathrm{Ker}(f_*) = 1 \Leftrightarrow \chi(M) = \chi(X)$. If $\mathrm{Ker}(f_*)$ contains the class of an essential simple closed curve γ we may reduce $\chi(M)$ by surgery on γ. Combining the results of [6, 7, 8] we see that there is always such a simple closed curve. Can this be shown directly, without appeal to [6, 7]?

We would like to study PD_3-complexes in a similar manner. Let X be a PD_3-complex and $f : M \to X$ a degree-1 map, where M is a closed 3-manifold. Then f is a homotopy equivalence $\Leftrightarrow \mathrm{Ker}(f_*) = 1$. Since $\pi_1(M)$ and $\pi_1(X)$ are finitely presentable, this kernel is normally generated by finitely many elements of $\pi_1(M)$, which may be represented by the components of a link $L \subset M$. We would like to modify M using such a link to render the kernel trivial. This *is* possible if X is homotopy equivalent to a closed orientable 3-manifold N, for M may then be obtained from N by Dehn surgery on a link whose components are null-homotopic in N [9]. Gadgil's argument appears to use the topology of the target space in an essential way.

Unfortunately there are PD_3-complexes which are not homotopy equivalent to 3-manifolds, so this strategy cannot be carried through in all cases. The known counter examples have virtually free fundamental groups [15, 21]. Since an orientable PD_3-complex with free fundamental group is homotopy equivalent to $\#^r(S^1 \times S^2)$ for some $r \geq 0$, it remains possible that every PD_3-complex is *virtually* a 3-manifold, i.e., has a finite covering space which is homotopy equivalent to a closed orientable 3-manifold. If this is true it must be possible to kill $\mathrm{Ker}(f_*)$ by surgery (and passing to finite covering spaces).

In general we might expect to encounter obstructions in $L_3(\pi, w)$ to obtaining a $\mathbb{Z}[\pi]$-homology equivalence by integral surgery. For instance, there are finite groups of cohomological period 4 which have finite Swan complexes but which do not act freely on homology 3-spheres [11]. However the validity of the Novikov conjecture for aspherical 3-manifolds suggests that such obstructions may never arise in the cases of most interest to us. (See [16, 18].) In any case, we allow the use of Dehn surgeries also.

4. Some reductions

Let X be a PD_3-complex and $f : M \to X$ be a degree-1 map with domain a closed 3-manifold M.

Lemma 12. *Let $X = X_1 \sharp X_2$ be a PD_3-complex which is the connected sum of PD_3-complexes which are virtually 3-manifolds. Then X is virtually a 3-manifold.*

Proof. Let $\widehat{X_i}$ be a finite regular covering space of X_i which is homotopy equivalent to a closed 3-manifold M_i, for each $i = 1$ or 2. Let $G_i = Aut(\widehat{X_i}/X_i)$ and let H be the kernel of the natural projection of $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$ onto $G_1 \times G_2$. Then the associated covering space X_H with fundamental group H is homotopy equivalent to a connected sum of copies of M_1 and M_2. $\square$

Thus we may assume that X is aspherical. There is then no need to pass to finite covers.

Lemma 13. *If an aspherical PD_3-complex X is virtually a 3-manifold then X is homotopy equivalent to a closed 3-manifold.*

Proof. Let $f : M \to \widehat{X}$ be a homotopy equivalence from a closed 3-manifold M to a finite regular covering space $\widehat{X}$. Then M is aspherical, and is either Seifert fibred, Haken or hyperbolic, by the Geometrization Theorem of Thurston and Perelman. If M is Seifert X is homotopy equivalent to a Seifert 3-manifold, by Theorem 15.1 of [1]. If M is hyperbolic the covering group $G = Aut(\widehat{X}/X)$ is isomorphic to a group of isometries of M, by Mostow rigidity. The group Γ of all lifts of such isometries to $\widetilde{M} = \mathbb{H}^3$ is isomorphic to $\pi_1(X)$ and acts properly discontinuously on $\mathbb{H}^3$. Since Γ is torsion-free the action is free, and so $\mathbb{H}^3/\Gamma$ is a closed 3-manifold homotopy equivalent to X. If M is an orientable Haken 3-manifold it has a canonical JSJ decomposition into Seifert fibred and hyperbolic pieces, and a similar conclusion holds [26]. (Zimmermann assumes M orientable, but his argument holds more generally.) $\square$

We shall however assume for simplicity that X and M are orientable. As every closed orientable 3-manifold is the target of a degree-1 map from a hyperbolic 3-manifold [17], we could also assume that M is aspherical. As we may lose asphericity of M under surgery, we shall settle for a simpler result.

Lemma 14. *Let X be an aspherical PD_3-complex and $f : M \to X$ a degree-1 map. Then we may assume that the irreducible factors of M are aspherical.*

Proof. Let $M = \sharp_{i=1}^{i=k} M_i$ be a factorization of M as a connected sum of irreducible 3-manifolds, with M_i aspherical if $i \leq r$ and $\pi_1(M_i)$ finite, Z or $Z \oplus (Z/2Z)$ if $i > r$. Since X is aspherical f extends to a map $F : \vee_{i=1}^{i=k} M_i \to X$. If $\pi_1(M_i)$ is finite then $F|_{M_i}$ is nullhomotopic, while if $\pi_1(M_i) \cong Z$ or $Z \oplus (Z/2Z)$ then $F|_{M_i}$ factors through S^1. In either case the restriction to such terms has degree 0. Hence F induces a degree-1 map from $\sharp_{i=1}^{i=r} M_i$ to X. $\square$

Let $L \subset M$ be a link whose components represent a subset of $\pi_1(M)$ whose normal closure is $\mathrm{Ker}(f_*)$. We may assume that the number of components of L is minimal among all such pairs (f, L).

We shall say that a link $L = \amalg_{i \leq m} L_i$ in a 3-manifold N with an open regular neighbourhood $n(L) = \amalg_{i \leq m} n(L_i)$ admits a *drastic* surgery if there is a family of slopes $\gamma_i \subset \partial\overline{n(L_i)}$ such that the normal closure of $\overline{\{[\gamma_1],\dots,[\gamma_m]\}}$ in $\pi_1(N-n(L))$ meets the image of each peripheral subgroup $\pi_1(\partial n(L_i))$ in a subgroup of finite index. If $f:M\to N$ is a degree-1 map of closed 3-manifolds $\mathrm{Ker}(f_*)$ is represented by a link which admits a drastic surgery [9]. (Gadgil's result is somewhat stronger.)

Lemma 15. *If X is an aspherical PD_3-complex and L admits a drastic surgery then X is homotopy equivalent to a closed 3-manifold.*

Proof. After a drastic surgery on L we may assume that $\mathrm{Ker}(f_*)$ is normally generated by finitely many elements of finite order. Let $M = N\#N'$ where $\pi_1(N)$ is torsion-free and the fundamental groups of the irreducible summands of N' are finite. As in the previous lemma f factors through the collapse of M onto N, and so induces a degree-1 map $g:N\to X$. This map is clearly π_1-injective, and so it is a homotopy equivalence. □

There are knots which admit no drastic surgery. The following example was suggested by Cameron Gordon. Let M be an orientable 3-manifold which is Seifert fibred over $S^2(p,q,r)$, where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 1$, and let K be a regular fibre. Let $\phi,\mu\subset\partial n(K)$ be a regular fibre and a meridian, respectively. Then surgery on the slope $s\mu+t\phi$ gives a 3-manifold N which is Seifert fibred over $S^2(p,q,r,s)$, if $s\neq 0$, or is a connected sum of lens spaces, if $s=0$. If $s\neq 0$ the image of ϕ has infinite order in $\pi_1(N)$; otherwise the image of μ has infinite order there. Thus no surgery on a regular fibre of M is drastic. (We may modify this example to obtain one with M not Seifert by replacing a tubular neighbourhood of another regular fibre by the exterior of a hyperbolic knot.)

However we have considerable latitude in our choice of link L representing $\mathrm{Ker}(f_*)$. In particular, we may modify L by a link homotopy, and so the key question may be:

is every knot $K\subset M$ *homotopic* to one admitting a drastic surgery?

The existence of PD_3-complexes which are not homotopy equivalent to 3-manifolds shows that we cannot expect a stronger result, in which "contains $\dots\overline{\pi_1(\partial n(L_i))}$" replaces "meets the image … finite index" in the definition of drastic surgery.

The argument for the existence of a degree-1 map $f:M\to X$ does not require us to assume *a priori* that X be finite, nor even that $\pi_1(X)$ be finitely presentable. The latter condition is needed to ensure that $\mathrm{Ker}(f_*)$ is represented by a link in M. In all dimensions $n\geq 4$ there are PD_n-groups of type FF which are not finitely presentable [5]. This leaves the question: are PD_3-groups finitely presentable? Our strategy does not address this issue.

References

[1] Bowditch, B.H. Planar groups and the Seifert conjecture, J. Reine u. Angew. Math. 576 (2004), 11–62.

[2] Brown, K.S. *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer-Verlag, Berlin - Heidelberg - New York (1982).

[3] Crisp, J.S. The decomposition of Poincaré duality complexes, Comment. Math. Helv. 75 (2000), 232–246.

[4] Davis, J.F. and Milgram, R.J. A survey of the spherical space form problem, Math. Reports 2(1985), 223–283.

[5] Davis, M.W. Poincaré duality groups, in *Surveys on Surgery Theory*, Vol. 1, Ann. Math. Study 145, Princeton Univ. Press, Princeton, NJ (2000), 167–193.

[6] Eckmann, B. and Linnell, P.A. Poincaré duality groups of dimension 2, II, Comment. Math. Helv. 58 (1981), 111–114.

[7] Eckmann, B. and Müller, H. Poincaré duality groups of dimension 2, Comment. Math. Helv. 55 (1980), 510–520.

[8] Gabai, D. The simple loop conjecture, J. Diff. Geom. 21 (1985), 143–149.

[9] Gadgil, S. Degree-one maps, surgery and four-manifolds, Bull. London Math. Soc. 39 (2007), 419–424.

[10] Gordon, C.McA. Dehn surgery on knots, *Proceedings of the International Congress of Mathematicians, (Kyoto, 1990)* vol. II, Math. Soc. Japan, Tokyo (1991), 631–642.

[11] Hambleton, I. and Madsen, I. Actions of finite groups on R^{n+k} with fixed point set R^k, Canad. J. Math. 38 (1986), 781–860.

[12] Hausmann, J.-C. and Vogel, P. *Geometry on Poincaré Spaces*, Mathematical Notes 41, Princeton University Press, Princeton, NJ (1993).

[13] Hillman, J.A. The kernel of integral cup product, J. Austral. Math. Soc. 43 (1987), 10–15.

[14] Hillman, J.A. On 3-dimensional Poincaré duality complexes and 2-knot groups, Math. Proc. Camb. Phil. Soc. 114 (1993), 215–218.

[15] Hillman, J.A. An indecomposable PD_3-complex: II, Alg. Geom. Top. 4 (2004), 1103–1109.

[16] Jahren, B. and Kwasik, S. Three-dimensional surgery theory, $UNil$-groups and the Borel conjecture, Topology 42 (2003), 1353–1369.

[17] Kawauchi, A. Almost identical imitations of $(3,1)$-dimensional manifold pairs, Osaka J. Math. 26 (1989), 743–758.

[18] Kirby, R.C. and Taylor, L.R. A survey of 4-manifolds through the eyes of surgery, in *Surveys on Surgery Theory, vol. 2* (edited by A.A.Ranicki), Ann. Math. Study 149, Princeton University Press, Princeton (2002), 387–421.

[19] Magnus, W., Karrass, A. and Solitar, D. *Combinatorial Group Theory*, Second revised edition, Dover Publications Inc, New York (1976).

[20] Robinson, D.J.S. *A Course in the Theory of Groups*, Graduate Texts in Mathematics 80,

Springer-Verlag, Berlin - Heidelberg - New York (1982).

[21] Swan, R.G. Periodic resolutions for finite groups, Ann. of Math. 72 (1960), 267–291.

[22] Thomas, C.B. Homotopy classification of free actions by finite groups on S^3, Proc. London Math. Soc. 40 (1980), 284–297.

[23] Thurston, W.P. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[24] Turaev, V.G. Three-dimensional Poincaré complexes: classification and splitting, Math. Sbornik 180 (1989), 809–830.

[25] Wall, C.T.C. Poincaré complexes. I, Ann. Math. 86 (1967), 213–245.

[26] Zimmermann, B. Das Nielsensche Realisierungsproblem für hinreichend grosse 3-Mannigfaltigkeiten, Math. Z. 180 (1982), 349–359.

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