Scalar and Vector Spherical Harmonic Spectral Equations of Rotating Non-Linear and Linearised Magnetohydrodynamics

D. J. Ivers, School of Mathematics and Statistics and
C. G. Phillips, Mathematics Learning Centre,
University of Sydney, NSW 2006, Australia

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Abstract

Vector spherical harmonic analyses have been used successfully to solve laminar and mean-field magnetohydrodynamic dynamo problems with interactions, such as the laminar induction term, anisotropic alpha-effects and anisotropic diffusion, which are difficult to analyse spectrally in spherical geometries. Spectral forms of the non-linear rotating Boussinesq and anelastic momentum, magnetic field and heat equations are derived for spherical geometries from vector spherical harmonic expansions of the velocity, magnetic induction, vorticity, electrical current and gravitational acceleration, and from scalar spherical harmonic expansions of the pressure and temperature. Combining the vector spherical harmonic forms of the momentum equation and the magnetic induction equation with poloidal-toroidal representations of the velocity and the magnetic field, non-linear spherical harmonic spectral equations are also derived for the poloidal-toroidal potentials of the velocity, or the momentum density in the anelastic approximation, and the magnetic field. Both compact and spectral interaction expansion forms are given.

Vector spherical harmonic spectral forms of the linearised rotating magnetic induction, momentum and heat equations for a general basic state can be obtained by linearising the corresponding non-linear spectral equations. Similarly, the spherical harmonic spectral equations for the poloidal-toroidal potentials of the velocity and the magnetic field may be linearised. However, for computational applications, new alternative hybrid linearised spectral equations are derived herein. The algorithmically simpler hybrid equations depend on vector spherical harmonic expansions of the velocity, magnetic field, vorticity, electrical current and gravitational acceleration of the basic state, and scalar spherical harmonic expansions of the poloidal-toroidal potentials of the perturbation velocity, magnetic field and temperature. The spectral equations derived herein may be combined with the corresponding spectral forms of anisotropic diffusion terms derived in Phillips and Ivers (2000).

KEYWORDS: magnetohydrodynamics, vector spherical harmonic, spectral equation, toroidal, poloidal, anelastic approximation.

1 Introduction

It is generally accepted that the Earth and the planets, Mercury, Jupiter, Saturn, Neptune and Uranus, possess planetary magnetic fields, which are generated by the magnetohydrodynamic dynamo action of the motions in their electrically conducting fluid cores. The core physics underlying the dynamo process may be modelled by the equations of magnetohydrodynamics, with the simplifications of the Boussinesq or anelastic approximations. The purpose of the present work is to derive several different angular spectral forms of the nonlinear and linearised magnetohydrodynamic equations in both approximations useful for their solution in spherical geometries. The spectral equations are based on scalar and vector spherical harmonics together with toroidal-poloidal vector field representations. Although complicated and difficult to derive, they are extremely important in applications.

The momentum, magnetic induction and temperature equations governing the velocity $\mathbf{v}$, magnetic induction field $\mathbf{B}$ and temperature Θ of an electrically-conducting Boussinesq fluid in a frame rotating with angular velocity $\mathbf{\Omega}$ are

$$\rho\left(\frac{\partial \mathbf{v}}{\partial \tau} + \boldsymbol{\omega}\times\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} + \frac{d\mathbf{\Omega}}{d\tau}\times\mathbf{r}\right) = -\nabla P + \mathbf{J}\times\mathbf{B} - \rho\alpha_\Theta\Theta\mathbf{g}_e + \mathbf{F}_\nu \qquad (1.1)$$

$$\frac{\partial \mathbf{B}}{\partial \tau} - \eta\nabla^2\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) \qquad (1.2)$$

$$\rho c_p\left(\frac{\partial\Theta}{\partial\tau}+\mathbf{v}\cdot\nabla\Theta-\kappa\nabla^2\Theta\right)=Q+Q_\nu+\mathbf{J}^2/\sigma\,, \tag{1.3}$$

where τ is the time and $\mathbf{r}$ is the position vector, together with the mass conservation equation and Gauss' Law,

$$\nabla\cdot\mathbf{v}=0\,,\qquad \nabla\cdot\mathbf{B}=0\,. \tag{1.4}$$

Dimensional equations are used throughout, since the non-dimensionalisation of the linearised equations depends strongly on the basic state. In equation (1.1) $\mathbf{F}_\nu := \rho\nu\nabla^2\mathbf{v}$ is the viscous volume force. The modified pressure P is related to the non-hydrostatic pressure p by $P = p+\frac{1}{2}\rho\mathbf{v}^2$, $\boldsymbol{\omega}=\nabla\times\mathbf{v}$ is the vorticity and $\mathbf{J}=\nabla\times\mathbf{B}/\mu_0$ is the electrical current density. In V, $\mathbf{J}$ and $\mathbf{B}$ are also related to the electric field $\mathbf{E}$ by Ohm's Law for a moving conductor,

$$\mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,. \tag{1.5}$$

The advection term has been written as a cross-product to simplify implementation in computer codes. The effective gravitational acceleration $\mathbf{g}_e$ includes the centripetal acceleration and may be non spherically-symmetric. In general, $\mathbf{g}_e$ is derived from an effective potential U_e, $\mathbf{g}_e=-\nabla U_e$, where $U_e := U-\frac{1}{2}(\boldsymbol{\Omega}\times\mathbf{r})^2$ and U is the gravitational potential. The potential U is sourced from the density by $\nabla^2U=4\pi G\rho$, where G is the gravitational constant. In the Boussinesq equations the density is uniform, except in the buoyancy force. However, the potential U may still be non spherically-symmetric due to lateral density variations occurring in a mantle surrounding V. In (1.3) Q is the rate of radiogenic heat production per unit volume and $Q_\nu := 2\rho\nu(\nabla\mathbf{v})_S : (\nabla\mathbf{v})_S$ is the viscous volume heating (Landau & Lifshitz 1959). The subscript S indicates the symmetric part of the gradient and the colon denotes the double scalar product of rank-2 tensors $\mathbf{F}$ and $\mathbf{G}$ defined in terms of cartesian components by $\mathbf{F}:\mathbf{G}=\sum_{i,j}F_{ij}G_{ij}$. Both viscous and Ohmic heating have been retained in the temperature equation, although their neglect is consistent with the Boussinesq approximation (Malkus 1964). The kinematic viscosity ν, magnetic diffusivity η, thermal diffusivity κ, thermal expansivity α_Θ, specific heat capacity c_p, electrical conductivity σ and magnetic permeability μ_0 are uniform. In the anelastic equations the density is spherically-symmetric, except in the buoyancy term. The anelastic approximation and its modifications to $\mathbf{F}_\nu$ and Q_ν are described in Section 7.

The vector spherical harmonic analysis of equations (1.1)–(1.4) forms the basis of several useful methods for solving a range of problems in spherical geometries. Let (r,θ,ϕ) be spherical polar coordinates with co-latitude θ, east-longitude ϕ, unit vectors $(\mathbf{1}_r,\mathbf{1}_\theta,\mathbf{1}_\phi)$ and $\mathbf{r}=r\mathbf{1}_r$. Three types of (surface) vector spherical harmonic have been commonly used in the literature. The first type, which are closely related to the scaloidal-poloidal-toroidal representation of vector fields, are defined by

$$\mathbf{P}_n^m=Y_n^m\mathbf{1}_r\,,\quad \mathbf{B}_n^m=\frac{r}{\sqrt{n(n+1)}}\nabla Y_n^m\,,\quad \mathbf{C}_n^m=-\frac{i}{\sqrt{n(n+1)}}\mathbf{r}\times\nabla Y_n^m\,,$$

where the spherical harmonic Y_n^m is defined by equation (2.1) (see Brink & Satchler 1968; also Morse & Feshbach 1953). The second type, $\mathbf{Y}_{n,n_1}^m$, defined by equation (2.5), are simply related to **PBC** and will be referred to as **Y**'s or simply vector spherical harmonics. These are the herein preferred set of vector spherical harmonics (see Brink & Satchler 1968; also James 1974), primarily because of their useful differentiation properties given in Section 2. The third type of vector spherical harmonic is based on the cartesian unit vectors at $(r,0,0)$ on the z-axis rotated to the basis vectors $(\mathbf{1}_\theta-i\mathbf{1}_\phi)/\sqrt{2}$, $\mathbf{1}_r$, $-(\mathbf{1}_\theta+i\mathbf{1}_\phi)/\sqrt{2}$ (Gelfand & Shapiro 1956; see also Burridge 1969). Surface vector spherical harmonics should not be confused with various solid vector spherical harmonics, which have also been defined (see Morse & Feshbach 1953; Backus 1958) and not used herein.

The **Y**-forms, (3.12) and (3.49), of the non-linear equations, (1.1) and (1.3), are derived in Section 3. The **Y**-form (3.34) of the magnetic vector potential equation (3.32) is also derived in Section 3. The **Y**-form of the magnetic induction equation (1.2) is then an easy consequence using (2.17)–(2.19). The **Y**-vector spherical harmonic equations have been derived for various problems: the non-diffusive mass, momentum and temperature equations of meteorology and aeronomy (Moses 1974), including the Coriolis, centrifugal and non-linear advective term; the magnetic induction equation (James 1974); and the Laplace tidal equations (Swarztrauber & Kasahara 1985). The **PBC** spectral equations have been derived for linear rotating fluids (Rieutord 1987, 1991) and for atmospheric oscillations, incorporating the Pedersen conductivity region and the Hall region, and containing the Coriolis force and special cases of the Lorentz force (Jones 1970, 1971a,b). The Gelfand-Shapiro vector spherical harmonic form of the magnetic induction equation (Oprea, Chossat & Armbruster 1997) and related equations for core surface motions (Jackson & Bloxham 1991) have been derived. The derivation of the spectral form of thermal anisotropic diffusion uses vector spherical harmonics

(Phillips & Ivers 2000). The vector spherical harmonic analysis of magnetic anisotropic diffusion (Phillips 1995) and viscous anisotropic diffusion (Phillips & Ivers 2000, 2001, 2003) requires the additional use of tensor spherical harmonics at several intermediate steps.

The vector spherical harmonic spectral equations can be further developed. The solenoidal conditions (1.4) are not automatically satisfied by vector spherical harmonic expansions of the magnetic field and the velocity, but must be imposed as additional conditions on the coefficients. Conditions (1.4) may be met intrinsically in spherical geometries by the toroidal-poloidal representations,

$$\mathbf{B} = \mathbf{T}\{T\} + \mathbf{S}\{S\}\,, \quad \mathbf{v} = \mathbf{T}\{t\} + \mathbf{S}\{s\}\,, \tag{1.6}$$

where toroidal and poloidal fields with potentials T and S are defined, respectively, by

$$\mathbf{T}\{T\} := \nabla\times\{T\mathbf{r}\}\,, \quad \mathbf{S}\{S\} := \nabla\times\mathbf{T}\{S\}\,. \tag{1.7}$$

From the identity $\nabla\times\mathbf{S}\{S\} = \mathbf{T}\{-\nabla^2 S\}$, it follows that the toroidal-poloidal representations of the electric current and the vorticity are given in terms of the magnetic and velocity potentials S, T, s and t by

$$\mu_0\mathbf{J} = \mathbf{T}\{-\nabla^2 S\} + \mathbf{S}\{T\}\,, \qquad \boldsymbol{\omega} = \mathbf{T}\{-\nabla^2 s\} + \mathbf{S}\{t\}\,.$$

Boundary conditions are also often simpler in terms of toroidal and poloidal potentials, e.g. matching a magnetic field to an insulating exterior. There are two important and useful developments of the vector spherical harmonic equations using toroidal-poloidal representations. The first development is a compact form of the toroidal-poloidal spectral equations, in which products are expressed as convolution sums of vector spherical harmonic coefficients and coupling integrals, but the remaining linear terms are expressed explicitly in terms of scalar spherical harmonic coefficients of the toroidal and poloidal potentials. James (1974) derived compact spectral toroidal-poloidal magnetic induction equations. Analogous compact spectral toroidal-poloidal momentum equations are derived in Section 4. The second development is the interaction expansion form of the toroidal-poloidal spectral equations, in which products are expanded as sums of interactions between the scalar spherical harmonic coefficients of the magnetic potentials S and T and the velocity potentials s and t. Interaction expansion forms of the magnetic induction equation (1.2) for the evolution of the magnetic toroidal and poloidal potentials S and T, were derived by Bullard & Gellman (1954); see equations (5.10) and (5.11). In Section 5 analogous toroidal-poloidal spectral-interaction forms of the momentum equations (5.6) and (5.8) are derived for the velocity toroidal and poloidal potentials. Merilees (1968) derived the spectral inviscid radial vorticity equation and horizontal divergence equation together with the spectral potential temperature and mass equations. Frazer (1974) and Pekeris & Accad (1975) derived these equations, without non-linear advection and with $\mathbf{v}\cdot\nabla\mathbf{v}$, respectively. In Section 5 a spectral PT-interaction form of (5.12) is derived for the heat equation in terms of the toroidal and poloidal potentials of the magnetic field and the velocity.

The convolution sums produced by product terms make the spectral equations derived in Sections 3–5 unsuitable for time-dependent non-linear applications using Faedo-Galerkin time-stepping methods. Fast-Fourier transform based spectral techniques currently have the advantage in time-stepping problems, but massively parallel computers may shift the advantage to convolution sums in the future. However, the spectral forms of the non-linear magnetohydrodynamic (MHD) equations are practically useful in steady conditions. Further, the MHD equations linearised about a steady basic state $(\mathbf{v}_0, \mathbf{B}_0, \Theta_0)$ are important in many applications and the linearised forms of the spectral equations may be efficiently applied to the (generalised) eigenproblems, non-steady and inhomogeneous steady problems arising from such applications. The linearised equations, which govern the perturbation fields $\mathbf{v}'$, $\mathbf{B}'$ and Θ', are

$$\rho\left(\frac{\partial\mathbf{v}'}{\partial\tau} + \boldsymbol{\omega}_0\times\mathbf{v}' + \boldsymbol{\omega}'\times\mathbf{v}_0 + 2\boldsymbol{\Omega}\times\mathbf{v}'\right) = -\nabla P' + \mathbf{J}_0\times\mathbf{B}' + \mathbf{J}'\times\mathbf{B}_0 - \rho\alpha_\Theta\Theta'\mathbf{g}_e + \rho\nu\nabla^2\mathbf{v}' \tag{1.8}$$

$$\frac{\partial\mathbf{B}'}{\partial\tau} = \eta\nabla^2\mathbf{B}' + \nabla\times(\mathbf{v}_0\times\mathbf{B}') + \nabla\times(\mathbf{v}'\times\mathbf{B}_0) \tag{1.9}$$

$$\rho c_p\left(\frac{\partial\Theta'}{\partial\tau} + \mathbf{v}_0\cdot\nabla\Theta' + \mathbf{v}'\cdot\nabla\Theta_0\right) = \rho c_p\kappa\nabla^2\Theta' + Q' + 2\rho\nu(\nabla\mathbf{v}_0)_S : (\nabla\mathbf{v}')_S + 2\mathbf{J}_0\cdot\mathbf{J}'/\sigma\,, \tag{1.10}$$

where $P' = p' + \rho\mathbf{v}_0\cdot\mathbf{v}'$, together with the solenoidal conditions,

$$\nabla\cdot\mathbf{v}' = 0\,, \qquad \nabla\cdot\mathbf{B}' = 0\,. \tag{1.11}$$

The vector spherical harmonic form and both compact and toroidal-poloidal spectral interaction forms of equations (1.8)–(1.10) can be obtained by linearising the non-linear vector spherical harmonic spectral equations or the toroidal-poloidal spectral equations, respectively. However, it is preferable for computational purposes, to use algorithmically simpler hybrid vector harmonic/spherical harmonic toroidal-poloidal forms of the equations, than either the linearised vector spherical harmonic spectral equations or the linearised toroidal-poloidal spectral equations. The number of distinct terms in the spectral toroidal-poloidal-interaction forms of the momentum, magnetic induction and heat equations, (5.6), (5.8), (5.10), (5.11) and (5.12), complicate their use in computer programs. The corresponding vector spectral momentum equation (3.12), magnetic vector potential equation (3.34) and heat equation (3.49) contain substantially fewer terms and can be used directly in applications, but require more field variables since, as noted above, vector spherical harmonic expansions do not automatically impose the solenoidal condition on the magnetic or velocity fields. In Section 6 we derive new hybrid spectral forms for the linearised momentum equation, (6.1) and (6.2), the magnetic induction equation, (6.11) and (6.12), and the heat equation (6.15), in which the basic state is described mathematically by the vector fields $\mathbf{v}_0$, $\boldsymbol{\omega}_0$, $\mathbf{B}_0$, $\mathbf{J}_0$, $\nabla\Theta_0$ and $\mathbf{g}$, but the perturbation state is given by the scalar fields s', t', S', T' and Θ'. In the hybrid equations the vector fields of the basic state are expanded in vector spherical harmonics and the perturbation fields in scalar spherical harmonics, defined in (2.1), but interaction terms are kept in vector spherical form. Truncation of the fields and equations gives a Galerkin approximation in angle.

In Section 7 modifications to the spectral equations are outlined for the anelastic approximation. A final aim of the paper is to explain and demonstrate the use and application of vector spherical harmonic techniques so they can be effectively applied to more elaborate models.

2 Vector Spherical Harmonics

The properties of vector spherical harmonics, which are needed subsequently, are given in this section. The scalar spherical harmonic in colatitude θ and east-longitude ϕ is defined by

$$Y_n^m(\theta,\phi) := (-)^m \sqrt{\frac{(2n+1)(n-m)!}{(n+m)!}}\, P_{n,m}(\cos\theta) e^{im\phi}\,, \tag{2.1}$$

where $P_{n,m}$ is the the Neumann associated Legendre function defined by

$$P_{n,m}(z) := (-)^n \frac{(1-z^2)^{m/2}}{2^n n!} \frac{d^{m+n}(1-z^2)^n}{dz^{m+n}}\,.$$

Under complex conjugation, indicated by the asterisk,

$$(Y_n^m)^* = (-)^m Y_n^{-m}\,. \tag{2.2}$$

The spherical harmonics (2.1) are orthonormal with respect to the inner-product on scalar functions of θ and ϕ,

$$(f,g) := \frac{1}{4\pi}\oint f g^*\, d\Omega\,,$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is the element of solid angle. A lowercase Greek subscript on a scalar quantity will denote the 2-index of a spherical harmonic. For example, Y_α will denote $Y_{n_\alpha}^{m_\alpha}$ and the orthonormalisation condition is

$$(Y_\alpha, Y_\beta) = \delta_{\alpha\beta}\,, \tag{2.3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Scalar fields can be expanded in series of spherical harmonics, which form a complete orthonormal set. Thus the poloidal and toroidal potentials of the magnetic field and the velocity, the pressure, the temperature and the effective gravitational potential have the spherical harmonic expansions

$$f = \sum_\alpha f_\alpha Y_\alpha\,, \quad f = S, T, s, t, P, \Theta, U_e\,. \tag{2.4}$$

The summations are over $n_\alpha = 0, 1, 2, \ldots$ and $m_\alpha = -n_\alpha : n_\alpha$ in general, but for poloidal and toroidal potentials the $n_\alpha = 0$ term does not contribute to the vector field and is omitted. The Y-coefficient f_α in the spherical harmonic expansion of a scalar field f is obtained by taking the inner-product of the field with Y_α, i.e. $f_\alpha = (f, Y_\alpha)$.

The vector spherical harmonics used herein are defined by (see James 1974),

$$\mathbf{Y}^m_{n,n_1} := (-)^{n-m}\sqrt{2n+1}\sum_{m_1,\mu}\begin{pmatrix} n & n_1 & 1 \\ m & -m_1 & -\mu \end{pmatrix} Y^{m_1}_{n_1}\mathbf{e}_\mu\,, \tag{2.5}$$

where the complex basis vectors $\mathbf{e}_\mu$ are defined in terms of the unit vectors $\mathbf{1}_x$, $\mathbf{1}_y$, $\mathbf{1}_z$ of the cartesian coordinate system $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ by $\mathbf{e}_0 := \mathbf{1}_z$ and $\mathbf{e}_{\pm1} := \mp(\mathbf{1}_x \pm i\mathbf{1}_y)/\sqrt{2}$. In particular,

$$\mathbf{Y}^\mu_{n,0} = \delta_{n1}\mathbf{e}_\mu\,, \qquad \mathbf{Y}^0_{0,1} = -\mathbf{1}_r\,. \tag{2.6}$$

In spherical polar coordinates,

$$\sqrt{n(2n+1)}\;\mathbf{Y}^m_{n,n-1} = \mathbf{1}_r nY^m_n + \mathbf{1}_\theta\partial_\theta Y^m_n + \mathbf{1}_\phi \csc\theta\;\partial_\phi Y^m_n \tag{2.7}$$

$$\sqrt{n(n+1)}\;\mathbf{Y}^m_{n,n} = i\mathbf{1}_\theta\csc\theta\;\partial_\phi Y^m_n - i\mathbf{1}_\phi\partial_\theta Y^m_n \tag{2.8}$$

$$\sqrt{(n+1)(2n+1)}\;\mathbf{Y}^m_{n,n+1} = -\mathbf{1}_r(n+1)Y^m_n + \mathbf{1}_\theta\partial_\theta Y^m_n + \mathbf{1}_\phi\csc\theta\;\partial_\phi Y^m_n\,. \tag{2.9}$$

Under complex conjugation,

$$(\mathbf{Y}^m_{n,n_1})^* = (-)^{n+n_1+m+1}\mathbf{Y}^{-m}_{n,n_1}\,. \tag{2.10}$$

A lowercase Greek subscript on a vector quantity will denote the 3-index of a vector spherical harmonic. Thus $\mathbf{Y}_\alpha$ will denote $\mathbf{Y}^{m_\alpha}_{n_\alpha,n_{\alpha1}}$. The vector spherical harmonics (2.5) are orthonormal with respect to the inner-product

$$(\mathbf{F},\mathbf{G}) := \frac{1}{4\pi}\oint \mathbf{F}\cdot\mathbf{G}^*\,d\Omega$$

of complex vector functions $\mathbf{F}$ and $\mathbf{G}$ in θ and ϕ. Hence

$$(\mathbf{Y}_\alpha,\mathbf{Y}_\beta) = \delta_{\alpha\beta}\,, \tag{2.11}$$

where $\delta_{\alpha\beta} := \delta_{n_\alpha n_\beta}\delta_{n_{1\alpha}n_{1\beta}}\delta_{m_\alpha m_\beta}$.

The magnetic field, electric current, velocity, vorticity, temperature gradient $\mathbf{q} := \nabla\Theta$ and effective gravitational acceleration have the vector spherical harmonic expansions,

$$\mathbf{F} = \sum_\alpha F_\alpha\mathbf{Y}_\alpha\,, \qquad \mathbf{F} = \mathbf{B},\mathbf{J},\mathbf{v},\boldsymbol{\omega},\mathbf{q},\mathbf{g}_e\,. \tag{2.12}$$

The $\mathbf{Y}$-coefficient F_α in the vector spherical harmonic expansion of a vector field $\mathbf{F}$ is independent of θ and ϕ, and is obtained by taking the inner-product of the field with $\mathbf{Y}_\alpha$, i.e. $F_\alpha = (\mathbf{F},\mathbf{Y}_\alpha)$. The summations in equation (2.12) are over $n_\alpha = 0, 1, 2, \ldots$, $n_{1\alpha} = n_\alpha, n_\alpha\pm1$ and $m_\alpha = -n_\alpha\!: n_\alpha$. For example, $\mathbf{r} = -r\mathbf{Y}^0_{0,1}$ so $r^0_{0,1} = -r$ with all other $r_\alpha = 0$ and $\boldsymbol{\Omega} = (\Omega_x + i\Omega_y)/\sqrt{2}\mathbf{Y}^{-1}_{1,0} + \Omega_z\mathbf{Y}^0_{1,0} - (\Omega_x - i\Omega_y)/\sqrt{2}\mathbf{Y}^1_{1,0}$, i.e. $\Omega^{-1}_{1,0} = (\Omega_x + i\Omega_y)/\sqrt{2}$, $\Omega^0_{1,0} = \Omega_z$, $\Omega^1_{1,0} = -(\Omega_x - i\Omega_y)/\sqrt{2}$ with all other $\Omega_\alpha = 0$.

The scalar and vector spherical harmonics, (2.1) and (2.5), have the following useful differentiation properties. The gradient of a scalar function is given by

$$\nabla(fY^m_n) = \sqrt{\frac{n}{2n+1}}\;\mathbf{Y}^m_{n,n-1}\partial^{n-1}_n f - \sqrt{\frac{n+1}{2n+1}}\;\mathbf{Y}^m_{n,n+1}\partial^{n+1}_n f\,, \tag{2.13}$$

where f is a function of r and

$$\partial^{n_1}_n := \begin{cases} \dfrac{\partial}{\partial r} + \dfrac{n+1}{r}, & \text{if } n_1 = n-1; \\[2ex] \dfrac{\partial}{\partial r} - \dfrac{n}{r}, & \text{if } n_1 = n+1. \end{cases} \tag{2.14}$$

In 3-index notation, let $\partial_\gamma := \partial^{n_{1\gamma}}_{n_\gamma}$ and $\partial^\gamma := \partial^{n_\gamma}_{n_{1\gamma}}$. Formulae for the divergence are

$$\nabla\cdot(f\mathbf{Y}^m_{n,n-1}) = \sqrt{\frac{n}{2n+1}}Y^m_n\partial^n_{n-1}f\,, \quad \nabla\cdot(f\mathbf{Y}^m_{n,n}) = 0\,, \quad \nabla\cdot(f\mathbf{Y}^m_{n,n+1}) = -\sqrt{\frac{n+1}{2n+1}}Y^m_n\partial^n_{n+1}f\,. \tag{2.15}$$

In particular, if a vector field $\mathbf{F}$ is solenoidal, the coefficient of Y^m_n in $\nabla\cdot\mathbf{F}$ must vanish,

$$\sqrt{n}\;\partial^n_{n-1}F^m_{n,n-1} - \sqrt{n+1}\;\partial^n_{n+1}F^m_{n,n+1} = 0\,, \tag{2.16}$$

and the $n_1 = n \pm 1$ **Y**-coefficients of **F** are not independent. The curl formulae are

$$\nabla\times(f\mathbf{Y}^m_{n,n-1}) = i\sqrt{\frac{n+1}{2n+1}}\mathbf{Y}^m_{n,n}\partial^n_{n-1}f \tag{2.17}$$

$$\nabla\times(f\mathbf{Y}^m_{n,n}) = \frac{i}{\sqrt{2n+1}}\left\{\sqrt{n}\mathbf{Y}^m_{n,n+1}\partial^{n+1}_n f + \sqrt{n+1}\mathbf{Y}^m_{n,n-1}\partial^{n-1}_n f\right\} \tag{2.18}$$

$$\nabla\times(f\mathbf{Y}^m_{n,n+1}) = i\sqrt{\frac{n}{2n+1}}\mathbf{Y}^m_{n,n}\partial^n_{n+1}f\,. \tag{2.19}$$

The scalar Laplacian satisfies

$$\nabla^2 fY^m_n = Y^m_n D_n f\,, \tag{2.20}$$

where

$$D_n := \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{n(n+1)}{r^2}\,.$$

Note

$$D_n = \partial^n_{n\pm1}\partial^{n\pm1}_n\,,\quad \partial^{n\pm1}_n D_n = D_{n\pm1}\partial^{n\pm1}_n\,,\quad \partial^n_{n\pm1}D_{n\pm1} = D_n\partial^n_{n\pm1}\,. \tag{2.21}$$

The vector Laplacian is

$$\nabla^2 f\mathbf{Y}^m_{n,n_1} = \mathbf{Y}^m_{n,n_1}D_{n_1}f\,. \tag{2.22}$$

If α is a 2-index, let $D_\alpha = D_{n_\alpha}$, and if α is a 3-index, let $D_{1\alpha} = D_{n_{1\alpha}}$. Hence $\nabla^2 fY_\alpha = Y_\alpha D_\alpha f$ and $\nabla^2 f\mathbf{Y}_\alpha = \mathbf{Y}_\alpha D_{1\alpha}f$.

3 Vector Spherical Harmonic Forms of the Non-Linear Equations

The **Y**-vector spherical harmonic forms (3.12), (3.34) and (3.49), respectively, of the momentum equation (1.1), the magnetic vector potential equation (3.32) and the temperature equation (1.3) are derived. The equations incorporate a non spherically-symmetric gravitation volume force, viscous and ohmic heating. The vector spherical harmonic form of a vector (scalar) equation is obtained by expanding all vector fields in vector spherical harmonics, all scalar fields in scalar spherical harmonics and taking the inner-product of the equation with a free vector (scalar) spherical harmonic. The **Y**-vector spherical harmonic form (3.42)–(3.44) of the magnetic induction equation (1.2), first derived by James (1974), is simply deduced from **Y**-magnetic vector potential equation (3.34). Together equations (3.12), (3.42)–(3.44) and (3.49) determine the three velocity coefficients $v^m_{n,n-1}$, $v^m_{n,n}$, $v^m_{n,n+1}$, the pressure coefficient P^m_n, the three magnetic coefficients $B^m_{n,n-1}$, $B^m_{n,n}$, $B^m_{n,n+1}$ and the temperature coefficient Θ^m_n for each degree n and order m. These equations must be supplemented by the incompressible flow and solenoidal magnetic field conditions (1.4), and any boundary conditions. By the divergence formulae (2.15), the coefficient of Y^m_n in the **Y**-form of (1.4)(a) must vanish, which imposes the following restriction on $v^m_{n,n-1}$ and $v^m_{n,n+1}$,

$$\sqrt{n}\partial^n_{n-1}v^m_{n,n-1} - \sqrt{n+1}\partial^n_{n+1}v^m_{n,n+1} = 0\,. \tag{3.1}$$

Similarly, the coefficients $B^m_{n,n-1}$ and $B^m_{n,n+1}$ of **B** are also dependent,

$$\sqrt{n}\;\partial^n_{n-1}B^m_{n,n-1} - \sqrt{n+1}\;\partial^n_{n+1}B^m_{n,n+1} = 0\,. \tag{3.2}$$

The analogous relations connecting coefficients of $\boldsymbol{\omega}$ and **J** are clearly identically satisfied when expressed in terms of **v** and **B**.

Non-linear terms and products produce convolution sums over coupling integrals of three scalar or vector spherical harmonics. The momentum, magnetic induction and temperature equations lead to only three types of coupling integral, (3.13), (3.14) and the integral in (3.49). Integrals (3.14) and that in (3.49) are simply related by (3.50) but they are usually implemented more efficiently individually.

The coupling integrals simplify in special cases and can be analytically evaluated. This is done for the Coriolis force, the Poincaré force and the buoyancy force with spherically-symmetric gravitation.

3.1 The Momentum Equation

The **Y**-momentum equation (3.12) is derived by taking the inner-product of (1.1) with $\mathbf{Y}_\gamma$, where the 3-index γ is a free index, and considering each term separately. Simplified vector spherical harmonic forms are also given for the Coriolis volume force (3.22)–(3.24), the Poincaré volume force (3.26) and a spherically-symmetric gravitational volume force (3.28).

The **Y**-form of the time-derivative and the viscous force per unit mass is

$$(\partial \mathbf{v}/\partial\tau - \nu\nabla^2\mathbf{v}, \mathbf{Y}_\gamma) = (\partial/\partial\tau - \nu D_{1\gamma})v_\gamma\,, \tag{3.3}$$

since the vector Laplacian satisfies property (2.22).

From the gradient formula (2.13) the **Y**-coefficients of the pressure gradient $(\nabla P)^m_{n,n_1}$ are related to the spherical harmonic coefficients of the pressure P^m_n by

$$(\nabla P)^m_{n,n-1} = f_P(n,n-1)\partial^{n-1}_n P^m_n\,,\quad (\nabla P)^m_{n,n} = 0\,,\quad (\nabla P)^m_{n,n+1} = f_P(n,n+1)\partial^{n+1}_n P^m_n\,, \tag{3.4}$$

where the factor f_P for the pressure gradient field is

$$f_P(n,n_1) := \begin{cases} \sqrt{n/(2n+1)}, & \text{if } n_1 = n-1; \\ 0, & \text{if } n_1 = n; \\ -\sqrt{(n+1)/(2n+1)}, & \text{if } n_1 = n+1. \end{cases} \tag{3.5}$$

In terms of 3-indices the pressure gradient coefficients are $(\nabla P)_\gamma := (\nabla P, \mathbf{Y}_\gamma) = f_P(\gamma)\partial_\gamma P_\gamma$.

The **Y**-coefficients of the vector-product of two **Y**-expansions can easily be found by taking the inner-product of $\mathbf{F}\times\mathbf{G} = \sum_\alpha F_\alpha \mathbf{Y}_\alpha \times \sum_\beta G_\beta \mathbf{Y}_\beta$ with $\mathbf{Y}_\gamma$. This yields

$$(\mathbf{F}\times\mathbf{G}, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} F_\alpha G_\beta (\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\,. \tag{3.6}$$

Thus the **Y**-coefficients of the four terms, $\boldsymbol{\omega}\times\mathbf{v}$, $2\boldsymbol{\Omega}\times\mathbf{v}$, $d\boldsymbol{\Omega}/d\tau\times\mathbf{r}$ and $\mathbf{J}\times\mathbf{B}$ can be expressed in terms of the same coupling integral $(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma)$,

$$(\boldsymbol{\omega}\times\mathbf{v}, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} \omega_\alpha v_\beta (\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) \tag{3.7}$$

$$(2\boldsymbol{\Omega}\times\mathbf{v}, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} 2\Omega_\alpha v_\beta (\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) \tag{3.8}$$

$$(d\boldsymbol{\Omega}/d\tau\times\mathbf{r}, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} (d\Omega_\alpha/d\tau) r_\beta (\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) \tag{3.9}$$

$$(\mathbf{J}\times\mathbf{B}, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} J_\alpha B_\beta (\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma)\,. \tag{3.10}$$

Similarly the **Y**-coefficients of the buoyancy volume force can be expressed in terms of the different coupling integral $(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)$,

$$(\rho\alpha_\Theta\Theta\mathbf{g}_e, \mathbf{Y}_\gamma) = \sum_{\alpha,\beta} \rho\alpha_\Theta g^e_\alpha \Theta_\beta (\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\,. \tag{3.11}$$

Combining the **Y**-coefficients for each term, (3.3), (3.4), (3.7)–(3.10) and (3.11), yields the **Y**-momentum equation,

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_{1\gamma}\right)v_\gamma = -f_P(\gamma)\partial_\gamma P_\gamma + \sum_{\alpha,\beta}\{[-\rho\omega_\alpha v_\beta - \rho 2\Omega_\alpha v_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta](\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g^e_\alpha\Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,. \tag{3.12}$$

The two coupling integrals of three harmonics, $(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma)$ and $(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)$, which occur in (3.12), have been evaluated in closed form (Adams 1900; James 1973, 1976) in terms of $3j$-, $6j$- and $9j$-symbols,

$$(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) = (-)^{n_\alpha+n_\beta+n_{1\gamma}+m_\gamma}\sqrt{6}i\Lambda(\alpha,\beta,\gamma)\begin{Bmatrix} n_\alpha & n_\beta & n_\gamma \\ n_{1\alpha} & n_{1\beta} & n_{1\gamma} \\ 1 & 1 & 1 \end{Bmatrix}\begin{pmatrix} n_{1\alpha} & n_{1\beta} & n_{1\gamma} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & -m_\gamma \end{pmatrix}, \tag{3.13}$$

and

$$(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma) = (-)^{m_\gamma+1}\Lambda(\alpha,\beta,\gamma)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\gamma} & n_\gamma & n_\beta \end{Bmatrix}\begin{pmatrix} n_{1\alpha} & n_\beta & n_{1\gamma} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & -m_\gamma \end{pmatrix}. \tag{3.14}$$

The Λ factor is given by

$$\Lambda(\alpha,\beta,\gamma) := \sqrt{(2n_\alpha+1)(2n_{1\alpha}+1)(2n_\beta+1)(2n_{1\beta}+1)(2n_\gamma+1)(2n_{1\gamma}+1)}$$

for 3-indices α, β and γ, as in (3.13). In the case of a 2-index, such as the β index in (3.14), the corresponding factor $(2n_{1\beta}+1)$ is omitted from Λ.

To complete the development of the $\mathbf{Y}$-momentum equation the coefficients ω_α and J_α in (3.12) must be expressed in terms of the Y-coefficients of $\mathbf{v}$ and $\mathbf{B}$. The curl formulae (2.17)–(2.19) relate the $\mathbf{Y}$-coefficients of the velocity and the vorticity by

$$\omega^m_{n,n-1} = i\sqrt{\frac{n+1}{2n+1}}\partial^{n-1}_n v^m_{n,n} \tag{3.15}$$

$$\omega^m_{n,n} = \frac{i}{\sqrt{2n+1}}\left\{\sqrt{n}\partial^n_{n+1}v^m_{n,n+1} + \sqrt{n+1}\partial^n_{n-1}v^m_{n,n-1}\right\} \tag{3.16}$$

$$\omega^m_{n,n+1} = i\sqrt{\frac{n}{2n+1}}\partial^{n+1}_n v^m_{n,n}. \tag{3.17}$$

Similarly, the $\mathbf{Y}$-coefficients of the magnetic field and the electric current are related by

$$\mu_0 J^m_{n,n-1} = i\sqrt{\frac{n+1}{2n+1}}\partial^{n-1}_n B^m_{n,n} \tag{3.18}$$

$$\mu_0 J^m_{n,n} = \frac{i}{\sqrt{2n+1}}\left\{\sqrt{n}\partial^n_{n+1}B^m_{n,n+1} + \sqrt{n+1}\partial^n_{n-1}B^m_{n,n-1}\right\} \tag{3.19}$$

$$\mu_0 J^m_{n,n+1} = i\sqrt{\frac{n}{2n+1}}\partial^{n+1}_n B^m_{n,n}. \tag{3.20}$$

The gravitational coefficients g^e_α may be prescribed apriori or derived from an effective gravitational potential U_e. By the gradient formula (2.13),

$$\mathbf{g}_e = -\sum_{n,m}\left\{\sqrt{\frac{n}{2n+1}}\mathbf{Y}^m_{n,n-1}\partial^{n-1}_n U^m_{e,n} - \sqrt{\frac{n+1}{2n+1}}\mathbf{Y}^m_{n,n+1}\partial^{n+1}_n U^m_{e,n}\right\}. \tag{3.21}$$

The general buoyancy term is most easily evaluated in the form (3.12) using (3.21).

The coupling integrals (3.13) and (3.14) simplify in special cases. The angular velocity of the reference frame can be approximated by diurnal and precessional parts, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_d + \boldsymbol{\Omega}_p$. Since the precessional period is about 27,500 years, $\Omega_p/\Omega_d \sim 10^{-7}$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}_d$ is a good approximation, except in the time derivative, $d\boldsymbol{\Omega}/d\tau$. If the z-axis is fixed along the diurnal rotation axis, then $\boldsymbol{\Omega} = \Omega_d \mathbf{Y}^0_{1,0}$ from (2.6)(a) and hence $\Omega^0_{1,0} = \Omega_d$ with all other $\Omega_\alpha = 0$. The time derivative, which is the same in fixed and rotating frames, is $d\boldsymbol{\Omega}/d\tau = \dot{\Omega}_d \mathbf{1}_z + \boldsymbol{\Omega}_p \times \boldsymbol{\Omega}_d$. Further, for fixed $n_\gamma = n$ and $m_\gamma = m$, there are then at most seven non-zero coupling integrals in the Coriolis term. The associated coefficients, apart from a factor $\rho 2\Omega$, are

$$(\mathbf{1}_z \times \mathbf{v}, \mathbf{Y}^m_{n,n-1}) = -iK^m_{-n}v^m_{n-1,n-1} - \frac{im}{n}v^m_{n,n-1} \tag{3.22}$$

$$(\mathbf{1}_z \times \mathbf{v}, \mathbf{Y}^m_{n,n}) = iK^m_n v^m_{n-1,n} - \frac{im}{n(n+1)}v^m_{n,n} - iK^m_{-n-1}v^m_{n+1,n} \tag{3.23}$$

$$(\mathbf{1}_z \times \mathbf{v}, \mathbf{Y}^m_{n,n+1}) = \frac{im}{n+1}v^m_{n,n+1} + iK^m_{n+1}v^m_{n+1,n+1}, \tag{3.24}$$

where

$$K^m_n := \frac{1}{n}\sqrt{\frac{(n+1)(n^2-m^2)}{2n+1}}. \tag{3.25}$$

Since the angular velocity $\boldsymbol{\Omega}$ is spatially uniform, $d\boldsymbol{\Omega}/d\tau = \sum_m (d\Omega^m_{1,0}/d\tau)\mathbf{Y}^m_{1,0}$. Note that it is not simply directed along the z-axis. Hence the $\mathbf{Y}$-spectral form of the Poincaré force reduces to

$$(\rho\, d\boldsymbol{\Omega}/d\tau \times \mathbf{r}, \mathbf{Y}_\gamma) = -\rho\sum_m r(d\Omega^m_{1,0}/d\tau)(\mathbf{Y}^m_{1,0} \times \mathbf{Y}^0_{0,1}, \mathbf{Y}_\gamma),$$

noting (2.6)(b). There are only three non-zero coupling integrals, namely $(\mathbf{Y}_{1,0}^m \times \mathbf{Y}_{0,1}^0, \mathbf{Y}_{1,1}^m) = i\sqrt{2/3}$ for $m = -1, 0, 1$, which are equivalent to the identity $\mathbf{1}_r \times \mathbf{Y}_{1,0}^m = i\sqrt{2/3}\mathbf{Y}_{1,1}^m$. The coefficients of the Poincaré volume force are thus

$$(\rho\, d\mathbf{\Omega}/d\tau \times \mathbf{r}, \mathbf{Y}_{1,1}^m) = -i\sqrt{2/3}\rho r d\Omega_{1,0}^m/d\tau\,. \tag{3.26}$$

If the gravitational acceleration $\mathbf{g}_e$ is spherically symmetric, then $\mathbf{g}_e = -g(r)\mathbf{1}_r$, $\mathbf{g}_e = g\mathbf{Y}_{0,1}^0$ from (2.6)(b) and the buoyancy volume force (3.11) greatly simplifies. The $\mathbf{Y}$-coefficient is $(\rho\alpha_\Theta\Theta\mathbf{g}_e, \mathbf{Y}_\gamma) = \rho g\alpha_\Theta \sum_\beta \Theta_\beta(\mathbf{Y}_{0,1}^0 Y_\beta, \mathbf{Y}_\gamma)$. For fixed $n_\gamma = n$ and $m_\gamma = m$ only two coupling integrals are non-zero, $(\mathbf{Y}_{0,1}^0 Y_n^m, \mathbf{Y}_{n,n-1}^m) = -\sqrt{n/(2n+1)}$ and $(\mathbf{Y}_{0,1}^0 Y_n^m, \mathbf{Y}_{n,n+1}^m) = \sqrt{(n+1)/(2n+1)}$, which are equivalent to the identity (James 1974),

$$Y_n^m \mathbf{1}_r = \sqrt{\frac{n}{2n+1}}\mathbf{Y}_{n,n-1}^m - \sqrt{\frac{n+1}{2n+1}}\mathbf{Y}_{n,n+1}^m\,, \tag{3.27}$$

The associated coefficients are $(\Theta\mathbf{g}_e, \mathbf{Y}_{n,n-1}^m) = -g\Theta_n^m\sqrt{n/(2n+1)}$ and $(\Theta\mathbf{g}_e, \mathbf{Y}_{n,n+1}^m) = g\Theta_n^m\sqrt{(n+1)/(2n+1)}$. Thus the $\mathbf{Y}$-expansion of the buoyancy volume force for a spherically-symmetric gravitational acceleration simplifies to

$$\rho\alpha_\Theta\Theta\mathbf{g}_e = -\rho g\alpha_\Theta \sum_{n,m}\Theta_n^m \left\{\sqrt{\frac{n}{2n+1}}\mathbf{Y}_{n,n-1}^m - \sqrt{\frac{n+1}{2n+1}}\mathbf{Y}_{n,n+1}^m\right\}\,. \tag{3.28}$$

Using (3.22)–(3.24), (3.26) and (3.28) in (3.12), the $n_{1\gamma} = n_\gamma, n_\gamma \pm 1$ component momentum equations with spherically-symmetric gravitational acceleration and $\mathbf{\Omega}$ parallel to $\mathbf{1}_z = \mathbf{Y}_{1,0}^0$ are, omitting γ subscripts,

$$\begin{aligned}\rho\left(\frac{\partial}{\partial\tau} - \nu D_{n-1}\right)v_{n,n-1}^m - iK_{-n}^m\rho 2\Omega v_{n-1,n-1}^m - \frac{im}{n}\rho 2\Omega v_{n,n-1}^m = -\sqrt{\frac{n}{2n+1}}(\partial_n^{n-1}P_n^m - \rho\alpha_\Theta g\Theta_n^m)\\ + \sum_{\alpha,\beta}(-\rho\omega_\alpha v_\beta + J_\alpha B_\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_{n,n-1}^m)\end{aligned} \tag{3.29}$$

$$\begin{aligned}\rho\left(\frac{\partial}{\partial\tau} - \nu D_n\right)v_{n,n}^m + iK_n^m\rho 2\Omega v_{n-1,n}^m - \frac{im}{n(n+1)}\rho 2\Omega v_{n,n}^m - iK_{-n-1}^m\rho 2\Omega v_{n+1,n}^m + i\sqrt{\tfrac{2}{3}}\,\rho r(d\Omega_{1,0}^m/d\tau)\delta_n^1 =\\ \sum_{\alpha,\beta}(-\rho\omega_\alpha v_\beta + J_\alpha B_\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_{n,n}^m)\end{aligned} \tag{3.30}$$

$$\begin{aligned}\rho\left(\frac{\partial}{\partial\tau} - \nu D_{n+1}\right)v_{n,n+1}^m + \frac{im}{n+1}\rho 2\Omega v_{n,n+1}^m + iK_{n+1}^m\rho 2\Omega v_{n+1,n+1}^m = \sqrt{\frac{n+1}{2n+1}}(\partial_n^{n+1}P_n^m - \rho\alpha_\Theta g\Theta_n^m)\\ + \sum_{\alpha,\beta}(-\rho\omega_\alpha v_\beta + J_\alpha B_\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_{n,n+1}^m)\,.\end{aligned} \tag{3.31}$$

The Coriolis volume force couples the coefficients v_{n,n_1}^m with the same n_1 (and m) but different n. If the incompressible condition (3.1) is included the coefficients v_{n,n_1}^m couple into two infinite chains with either $|n_1 - n| = \mathrm{mod}(n, 2)$ or $|n_1 - n| = \mathrm{mod}(n+1, 2)$. If $\mathbf{\Omega}$ is not parallel to $\mathbf{1}_z$, the Coriolis volume force induces stronger coupling over m between the coefficients v_{n,n_1}^m. There are then several more non-zero coupling integrals of the form $(\mathbf{Y}_{1,0}^\mu \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)$, $\mu = \pm 1$, which can also be evaluated analytically. The Poincaré volume force only drives $v_{1,1}^m$.

3.2 The Magnetic Vector Potential and Induction Equations

The $\mathbf{Y}$-vector spherical harmonic form (3.33), or in component form (3.35)–(3.37), of the magnetic vector potential equation (3.32) is derived. The $\mathbf{Y}$-vector spherical harmonic form of the magnetic induction equation (1.2) (James 1974) is also deduced in component form (3.42)–(3.44). The magnetic vector potential equation closely resembles the momentum equation, the uncurled form of the well-known analogy between the magnetic induction equation and the vorticity equation (Elsasser 1950). The similar $\mathbf{Y}$-spectral analysis of the momentum and magnetic vector potential equations provides checks on the spectral equations and simplifies computer implementation. It is also easier to spectrally analyze terms of the form $\mathbf{F} \times \mathbf{G}$ and then apply the curl formulae (2.17)–(2.19), rather than analyze $\nabla \times (\mathbf{F} \times \mathbf{G})$ directly. Hence the derivation of the $\mathbf{Y}$-spectral equations for $\mathbf{B}$ starts from the magnetic vector potential equation rather than the magnetic induction equation.

The electric field $\mathbf{E}$ is given in terms of the magnetic vector potential $\mathbf{A}$, where $\nabla\times\mathbf{A} = \mathbf{B}$, and the electric scalar potential Φ, by $\mathbf{E} = -\partial\mathbf{A}/\partial\tau - \nabla\Phi$. Eliminating $\mathbf{E}$ transforms Ohm's Law (1.5) into the magnetic vector potential equation,

$$\partial\mathbf{A}/\partial\tau = -\eta\nabla\times\mathbf{B} + \mathbf{v}\times\mathbf{B} - \nabla\Phi\,. \tag{3.32}$$

The magnetic induction equation (1.2) is the curl of (3.32). Expanding $\mathbf{v}$ and $\mathbf{B}$ in terms of vector spherical harmonics and using (3.6),

$$(\mathbf{v}\times\mathbf{B},\mathbf{Y}_\gamma) = \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}_\gamma)\,,$$

where the coupling integral $(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}_\gamma)$ is given by (3.13). The inner-product of (3.32) with $\mathbf{Y}_\gamma$, where γ is a free index, yields

$$(\partial\mathbf{A}/\partial\tau,\mathbf{Y}_\gamma) = -\eta(\nabla\times\mathbf{B},\mathbf{Y}_\gamma) + \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}_\gamma) - (\nabla\Phi)_\gamma\,. \tag{3.33}$$

where $(\nabla\Phi)_\gamma := (\nabla\Phi,\mathbf{Y}_\gamma)$ are the $\mathbf{Y}$-coefficients of $\nabla\Phi$. The first term on the right in (3.33) can be simplified using the vector identity, $\nabla\times\nabla\times\mathbf{A} = -\nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A})$, the relation $\nabla\times\mathbf{A} = \mathbf{B}$ and properties (2.21). Thus the $\mathbf{Y}$-spectral form of the magnetic vector potential equation is obtained,

$$\left(\frac{\partial}{\partial\tau} - \eta D_\gamma\right)A_\gamma = \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}_\gamma) - (\nabla\{\Phi + \eta\nabla\cdot\mathbf{A}\})_\gamma\,. \tag{3.34}$$

The degree n and order m component equations of (3.34) are

$$\left(\frac{\partial}{\partial\tau} - \eta D_{n-1}\right)A^m_{n,n-1} = \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n-1}) - \sqrt{\frac{n}{2n+1}}\partial^{n-1}_n(\Phi^m_n + \eta\{\nabla\cdot\mathbf{A}\}^m_n) \tag{3.35}$$

$$\left(\frac{\partial}{\partial\tau} - \eta D_n\right)A^m_{n,n} = \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n}) \tag{3.36}$$

$$\left(\frac{\partial}{\partial\tau} - \eta D_{n+1}\right)A^m_{n,n+1} = \sum_{\alpha,\beta} v_\alpha B_\beta(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n+1}) + \sqrt{\frac{n+1}{2n+1}}\partial^{n+1}_n(\Phi^m_n + \eta\{\nabla\cdot\mathbf{A}\}^m_n)\,. \tag{3.37}$$

The gradient in (3.34), which has been evaluated using the gradient formula (2.13), makes no contribution to the $A^m_{n,n}$-equation. From the curl formulae (2.17)–(2.19) the $\mathbf{Y}$-coefficients of $\mathbf{B}$ are given in terms of the $\mathbf{Y}$-coefficients of $\mathbf{A}$ by

$$B^m_{n,n-1} = i\sqrt{\frac{n+1}{2n+1}}\partial^{n-1}_n A^m_{n,n} \tag{3.38}$$

$$B^m_{n,n} = \frac{i}{\sqrt{2n+1}}\left\{\sqrt{n}\partial^n_{n+1}A^m_{n,n+1} + \sqrt{n+1}\partial^n_{n-1}A^m_{n,n-1}\right\} \tag{3.39}$$

$$B^m_{n,n+1} = i\sqrt{\frac{n}{2n+1}}\partial^{n+1}_n A^m_{n,n}\,. \tag{3.40}$$

The divergence of $\mathbf{A}$ has the spectral form,

$$(\nabla\cdot\mathbf{A})^m_n = \sqrt{\frac{n}{2n+1}}\,\partial^n_{n-1}A^m_{n,n-1} - \sqrt{\frac{n+1}{2n+1}}\,\partial^n_{n+1}A^m_{n,n+1}\,. \tag{3.41}$$

Equations (3.35)–(3.37), together with (3.38)–(3.41), determine the vector potential $\mathbf{Y}$-coefficients A_γ and the scalar potential Y-coefficients Φ_γ. They provide the simplest starting point for the derivations of the spectral toroidal-poloidal induction equations in §§4–6.

The $\mathbf{Y}$-vector spherical harmonic form of the magnetic induction equation for the magnetic field $\mathbf{Y}$-coefficients B_γ follows from (3.35)–(3.37) and (3.38)–(3.40). By (3.38) applying $i\sqrt{(n+1)/(2n+1)}\partial^{n-1}_n$ to equation (3.36) and using (2.21) yields the $B^m_{n,n-1}$-equation,

$$\left(\frac{\partial}{\partial\tau} - \eta D_{n-1}\right)B^m_{n,n-1} = i\sqrt{\frac{n+1}{2n+1}}\sum_{\alpha,\beta}\partial^{n-1}_n(v_\alpha B_\beta)(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n})\,. \tag{3.42}$$

Similarly, by (3.39) applying $i\sqrt{(n+1)/(2n+1)}\,\partial^n_{n-1}$ to equation (3.35) and $i\sqrt{n/(2n+1)}\,\partial^n_{n+1}$ to equation (3.37), and adding yields the $B^m_{n,n}$-equation,

$$\left(\frac{\partial}{\partial\tau}-\eta D_n\right)B^m_{n,n}=i\sqrt{\frac{n+1}{2n+1}}\sum_{\alpha,\beta}\partial^n_{n-1}(v_\alpha B_\beta)(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n-1})$$
$$+\,i\sqrt{\frac{n}{2n+1}}\sum_{\alpha,\beta}\partial^n_{n+1}(v_\alpha B_\beta)(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n+1})\,. \quad (3.43)$$

Lastly, by (3.40) applying $i\sqrt{n/(2n+1)}\,\partial^{n+1}_n$ to (3.36) yields the $B^m_{n,n+1}$-equation,

$$\left(\frac{\partial}{\partial\tau}-\eta D_{n+1}\right)B^m_{n,n+1}=i\sqrt{\frac{n}{2n+1}}\sum_{\alpha,\beta}\partial^{n+1}_n(v_\alpha B_\beta)(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta,\mathbf{Y}^m_{n,n})\,. \quad (3.44)$$

The gradient terms in (3.35) and (3.37) have been eliminated and equations (3.42)–(3.44) are expressed purely in terms of the $\mathbf{Y}$-coefficients of $\mathbf{B}$ and $\mathbf{v}$. However, equations (3.42) and (3.44) are not independent, since $B^m_{n,n-1}$ and $B^m_{n,n+1}$ must also satisfy condition (3.2). Thus (3.43) and only one of (3.42) and (3.44) determine the magnetic field. Alternatively, (3.42) and (3.44) can be reduced to a single equation using the toroidal-poloidal representation (1.6)(a) as in §4.

3.3 The Heat Equation

The scalar spherical harmonic spectral equation for the temperature (3.49) is derived by taking the inner-product of (1.3) with Y_γ, where γ is a free index. Combining the time-derivative and the thermal diffusion term, which simplifies using the Laplacian property (2.20), gives

$$(\partial\Theta/\partial\tau-\kappa\nabla^2\Theta,Y_\gamma)=(\partial/\partial\tau-\kappa D_\gamma)\Theta_\gamma\,. \quad (3.45)$$

To derive the spectral form of the advective term $\mathbf{v}\cdot\nabla\Theta$, substitute the vector spherical harmonic expansions for the velocity, (2.12), and for the temperature gradient $\mathbf{q}:=\nabla\Theta$,

$$\mathbf{v}\cdot\nabla\Theta=\sum_{\alpha,\beta}v_\alpha q_\beta\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta\,.$$

Hence

$$(\mathbf{v}\cdot\nabla\Theta,Y_\gamma)=\sum_{\alpha,\beta}v_\alpha q_\beta(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)\,. \quad (3.46)$$

The Ohmic heating term can be expanded in a similar way, $\mathbf{J}\cdot\mathbf{J}=\sum_{\alpha,\beta}J_\alpha J_\beta\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta$, and hence

$$(\mathbf{J}^2,Y_\gamma)=\sum_{\alpha,\beta}J_\alpha J_\beta(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)\,. \quad (3.47)$$

The spectral form of the viscous heating term,

$$Q_\nu=\tfrac{1}{2}\rho\nu\sum_{i,j}(\partial_i v_j+\partial_j v_i)^2=\rho\nu[\nabla\mathbf{v}:\nabla\mathbf{v}+\nabla\mathbf{v}:(\nabla\mathbf{v})^T]\,,$$

can be found using the identities, $\nabla\mathbf{v}:\nabla\mathbf{v}=\frac{1}{2}\nabla^2\mathbf{v}^2-\mathbf{v}\cdot\nabla^2\mathbf{v}$ and $\nabla\mathbf{v}:(\nabla\mathbf{v})^T=\nabla\mathbf{v}:\nabla\mathbf{v}-\boldsymbol{\omega}^2$. By the vector Laplacian property (2.22), $\nabla^2\mathbf{v}^2=\nabla^2\sum_{\alpha,\beta,\gamma}v_\alpha v_\beta(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)Y_\gamma=\sum_{\alpha,\beta,\gamma}D_\gamma(v_\alpha v_\beta)(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)Y_\gamma$. Also $\mathbf{v}\cdot\nabla^2\mathbf{v}=\sum_{\alpha,\beta}v_\alpha\mathbf{Y}_\alpha\cdot\nabla^2(v_\beta\mathbf{Y}_\beta)=\sum_{\alpha,\beta,\gamma}v_\alpha(D_{1\beta}v_\beta)(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)Y_\gamma$. Thus, treating $\boldsymbol{\omega}^2$ similarly to $\mathbf{J}^2$ in (3.47),

$$(Q_\nu,Y_\gamma)=\rho\nu(\nabla^2\mathbf{v}^2-2\mathbf{v}\cdot\nabla^2\mathbf{v}-\boldsymbol{\omega}^2,Y_\gamma)=\rho\nu\sum_{\alpha,\beta}\{D_\gamma(v_\alpha v_\beta)-2v_\alpha D_{1\beta}v_\beta-\omega_\alpha\omega_\beta\}(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta,Y_\gamma)\,. \quad (3.48)$$

The spherical harmonic spectral heat equation is obtained by combining equations (3.45)–(3.48) and simplifying the viscous volume heating using the symmetry of the coupling integral in α and β,

$$\rho c_p\Big(\frac{\partial}{\partial\tau}-\kappa D_\gamma\Big)\Theta_\gamma = Q_\gamma + \sum_{\alpha,\beta}\Big\{-\rho c_p v_\alpha q_\beta + \rho\nu\Big[2\frac{\partial v_\alpha}{\partial r}\frac{\partial v_\beta}{\partial r} + (p_{1\alpha}+p_{1\beta}-p_\gamma)\frac{v_\alpha v_\beta}{r^2} - \omega_\alpha\omega_\beta\Big] + J_\alpha J_\beta/\sigma\Big\}(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta, Y_\gamma)\,, \quad (3.49)$$

where $p_{1\alpha} := n_{1\alpha}(n_{1\alpha}+1)$, etc. The coupling integral $(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta, Y_\gamma)$ in (3.49) is related to the integral (3.14) using the complex conjugate properties (2.2) and (2.10),

$$(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta, Y_\gamma) = (-)^{n_\alpha+n_{1\alpha}+m_\alpha+1+m_\gamma}(\mathbf{Y}_\beta Y_{n_\gamma}^{-m_\gamma}, \mathbf{Y}_{n_\alpha,n_{1\alpha}}^{-m_\alpha})\,. \quad (3.50)$$

The components q_β of the temperature gradient $\mathbf{q}$ are given in terms of the radial derivatives of the spherical harmonic coefficients of the temperature by

$$q_{n,n-1}^m = f_q(n,n-1)\partial_n^{n-1}\Theta_n^m\,,\quad q_{n,n}^m = 0\,,\quad q_{n,n+1}^m = f_q(n,n+1)\partial_n^{n+1}\Theta_n^m\,, \quad (3.51)$$

where $f_q(n,n_1) = f_P(n,n_1)$, with f_P given by (3.5), is a factor for the temperature gradient field.

4 Compact Non-Linear Toroidal-Poloidal Spectral Equations

The incompressible flow and solenoidal magnetic field conditions (1.4), or the equivalent $\mathbf{Y}$-spectral equations (3.1) and (3.2), can be satisfied identically by using the toroidal-poloidal representations (1.6) and (1.7). The Y-coefficients of the magnetic potentials S and T in (2.4) are related to the $\mathbf{Y}$-coefficients of the magnetic field in (2.12) by

$$B_{n,n_1}^m = f_B(n,n_1)\begin{cases}\partial_n^{n_1}S_n^m\,, & \text{if } n_1 = n\pm 1;\\ T_n^m\,, & \text{if } n_1 = n;\end{cases} \quad (4.1)$$

where the factor f_B for the magnetic field is given by

$$f_B(n,n_1) := \begin{cases}(n+1)\sqrt{n/(2n+1)}\,, & \text{if } n_1 = n-1;\\ -i\sqrt{n(n+1)}\,, & \text{if } n_1 = n;\\ n\sqrt{(n+1)/(2n+1)}\,, & \text{if } n_1 = n+1.\end{cases} \quad (4.2)$$

The two dependent variables $B_{n,n\pm1}^m$ are substituted by a single variable S_n^m and $B_{n,n}^m$ is substituted by T_n^m. Thus evolution of the magnetic field is governed by only two independent equations of degree n and order m, namely the spectral toroidal-poloidal induction equations. Compact forms (4.21) and (4.22) of these equations, in which the product and non-linear terms are expanded in vector spherical harmonic coefficients, are derived in this section from the magnetic vector potential spectral equations (3.35)–(3.37).

The Y-coefficients of the velocity potentials s and t in (2.4) are related to the $\mathbf{Y}$-coefficients of the velocity (2.12) by equations analogous to (4.1),

$$v_{n,n_1}^m = f_v(n,n_1)\begin{cases}\partial_n^{n_1}s_n^m\,, & \text{if } n_1 = n\pm 1;\\ t_n^m\,, & \text{if } n_1 = n;\end{cases} \quad (4.3)$$

where $f_v(n,n_1) = f_B(n,n_1)$. The two dependent variables $v_{n,n\pm1}^m$ are replaced by a single variable s_n^m, $v_{n,n}^m$ is replaced by t_n^m. However, unlike the magnetic case, the $n_1 = n\pm1$ momentum equations in (3.12) are not dependent and there is still the pressure to consider. If the pressure is eliminated, then evolution of the velocity is also governed by only two independent equations of degree n and order m, the spectral toroidal-poloidal momentum equations. Compact forms (4.7) and (4.13) of these equations are also derived from the spectral momentum equations (3.12).

Of course, only vector equations can have toroidal-poloidal component equations so the heat equation is not considered.

4.1 The Momentum Equation

The compact forms (4.7) and (4.13) of the spectral toroidal-poloidal momentum equations with general rotation rate $\mathbf{\Omega}$ and gravitational acceleration $\mathbf{g}_e$ are derived for the toroidal-poloidal velocity potentials t and s. Solving (4.3) for t_n^m,

$$t_n^m = e_v(n,n)v_{n,n}^m\,, \quad (4.4)$$

where the factor e_v for the velocity equation is defined by

$$e_v(n, n_1) := \begin{cases} 1/\sqrt{n(2n+1)}, & \text{if } n_1 = n-1; \\ i/\sqrt{n(n+1)}, & \text{if } n_1 = n; \\ 1/\sqrt{(n+1)(2n+1)}, & \text{if } n_1 = n+1. \end{cases} \tag{4.5}$$

It is clear from the time derivative in (3.12), that the $v_{n,n}^m$-equation governs the evolution of the toroidal velocity potential coefficient t_n^m. Thus the spectral toroidal momentum equation, or t_n^m-equation, is produced by

$$t_n^m\text{-equation} = e_v(n, n) \times v_{n,n}^m\text{-equation}\,. \tag{4.6}$$

Substituting (4.3) into the left side of (3.12) and using (4.6) gives the compact spectral toroidal momentum equation,

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_\gamma\right)t_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}=n_\gamma}} e_v(\gamma)\{[-\rho\omega_\alpha v_\beta - \rho 2\Omega_\alpha v_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta](\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g_\alpha^e \Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,. \tag{4.7}$$

The time derivative and viscous terms are on the left, and the non-linear and other product interaction terms, including the remaining linear terms, are on the right side of (4.7).

In terms of S_n^m and T_n^m, the $\mathbf{Y}$-coefficients of the electric current are given by

$$\mu_0 J_{n,n_1}^m = f_J(n, n_1) \begin{cases} \partial_n^{n_1} T_n^m\,, & \text{if } n_1 = n \pm 1; \\ D_n S_n^m\,, & \text{if } n_1 = n; \end{cases} \tag{4.8}$$

where the factor f_J for the current field is

$$f_J(n, n_1) := \begin{cases} (n+1)\sqrt{n/(2n+1)}\,, & \text{if } n_1 = n-1; \\ i\sqrt{n(n+1)}\,, & \text{if } n_1 = n; \\ n\sqrt{(n+1)/(2n+1)}\,, & \text{if } n_1 = n+1. \end{cases} \tag{4.9}$$

Similarly, the $\mathbf{Y}$-coefficients of the vorticity, in terms of s_n^m and t_n^m, are

$$\omega_{n,n_1}^m = f_\omega(n, n_1) \begin{cases} \partial_n^{n_1} t_n^m\,, & \text{if } n_1 = n \pm 1; \\ D_n s_n^m\,, & \text{if } n_1 = n; \end{cases} \tag{4.10}$$

where $f_\omega(n, n_1) = f_J(n, n_1)$.

The derivation of the compact spectral poloidal momentum equation (4.13) is more involved. Solving (4.3) for s_n^m,

$$D_n s_n^m = e_v(n, n-1)\partial_{n-1}^n v_{n,n-1}^m + e_v(n, n+1)\partial_{n+1}^n v_{n,n+1}^m\,, \tag{4.11}$$

where the factor e_v is defined by (4.5). From the time derivative it is seen that applying $e_v(n, n-1)\partial_{n-1}^n$ to the $v_{n,n-1}^m$-equation in (3.12) and $e_v(n, n-1)\partial_{n+1}^n$ to the $v_{n,n+1}^m$-equation in (3.12), and adding, yields an equation which governs the evolution of $D_n s_n^m$. This equation, called the compact spectral poloidal momentum equation or the s_n^m-equation, is produced by

$$s_n^m\text{-equation} = e_v(n, n-1)\partial_{n-1}^n v_{n,n-1}^m\text{-equation} + e_v(n, n+1)\partial_{n+1}^n v_{n,n+1}^m\text{-equation}\,. \tag{4.12}$$

The combination of operators here eliminates the modified pressure P_n^m. A purely algebraic combination for s_n^m is possible from (4.3), namely

$$s_n^m = \frac{r}{\sqrt{n(n+1)}}\left(\frac{v_{n,n-1}^m}{\sqrt{(n+1)(2n+1)}} - \frac{v_{n,n+1}^m}{\sqrt{n(2n+1)}}\right),$$

but the corresponding combination of the $v_{n,n\pm1}^m$-equations does not eliminate the pressure. The diffusion term in the s_n^m-equation takes the simple form,

$$e_v(n, n-1)\partial_{n-1}^n D_{n-1} v_{n,n-1}^m + e_v(n, n+1)\partial_{n+1}^n D_{n+1} v_{n,n+1}^m = D_n D_n s_n^m\,.$$

From (3.12) the compact spectral poloidal momentum equation is thus

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_\gamma\right) D_\gamma s_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}=n_\gamma\pm 1}} e_v(\gamma)\partial^\gamma\{[-\rho\omega_\alpha v_\beta - \rho 2\Omega_\alpha v_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta](\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g^e_\alpha \Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,, \quad (4.13)$$

where ∂^γ is defined following (2.14). The **Y**-coefficients on the right are given in terms of the Y-coefficients of the toroidal-poloidal potentials of $\mathbf{v}$ and $\mathbf{B}$ by (4.1), (4.3), (4.8) and (4.10).

4.2 The Magnetic Vector Potential and Induction Equations

Compact spectral toroidal-poloidal magnetic induction equations (4.21) and (4.22) are derived for the toroidal-poloidal magnetic potentials T and S.

Uncurling the toroidal-poloidal representation (1.6) and (1.7) yields $\mathbf{A} = T\mathbf{r} + \nabla\times S\mathbf{r} + \nabla R$, where the second term on the right side is the toroidal part of $\mathbf{A}$, and the first and third terms comprise the poloidal and scaloidal parts of $\mathbf{A}$. In component form,

$$A^m_{n,n-1} = \sqrt{\frac{n}{2n+1}}(rT^m_n + \partial^{n-1}_n R^m_n)\,, \quad (4.14)$$

$$A^m_{n,n} = -i\sqrt{n(n+1)}S^m_n\,, \quad (4.15)$$

$$A^m_{n,n+1} = -\sqrt{\frac{n+1}{2n+1}}(rT^m_n + \partial^{n+1}_n R^m_n)\,. \quad (4.16)$$

It follows from (4.15) and the time derivative in (3.36) that the $A^m_{n,n}$-equation (3.36) is $i/\sqrt{n(n+1)}$ times the spectral poloidal magnetic equation or S^m_n-equation, i.e.

$$S^m_n\text{-equation} = e_B(n,n) \times A^m_{n,n}\text{-equation}\,, \quad (4.17)$$

where e_B is the magnetic equation factor,

$$e_B(n,n_1) := \begin{cases} -1/\sqrt{n(2n+1)}, & \text{if } n_1 = n-1; \\ i/\sqrt{n(n+1)}, & \text{if } n_1 = n; \\ -1/\sqrt{(n+1)(2n+1)}, & \text{if } n_1 = n+1. \end{cases} \quad (4.18)$$

To obtain the T^m_n-equation, R^m_n must be eliminated from equations (4.14) and (4.16). By the definition (2.14) of $\partial^{n_1}_n$ and (2.21),

$$T^m_n = e_B(n,n-1)\partial^n_{n-1}A^m_{n,n-1} + e_B(n,n+1)\partial^n_{n+1}A^m_{n,n+1}\,. \quad (4.19)$$

Thus the spectral toroidal magnetic induction equation or T^m_n-equation is extracted from (3.35) and (3.37) by operating on the $A^m_{n,n-1}$-equation (3.35) with $e_B(n,n-1)\partial^n_{n-1}$ and on the $A^m_{n,n+1}$-equation (3.37) with $e_B(n,n+1)\partial^n_{n+1}$, i.e.

$$T^m_n\text{-equation} = e_B(n,n-1)\partial^n_{n-1}A^m_{n,n-1}\text{-equation} + e_B(n,n+1)\partial^n_{n+1}A^m_{n,n+1}\text{-equation}\,. \quad (4.20)$$

The gradient term in (3.32) makes no contribution to the T^m_n-equation. Moreover, the diffusion term simplifies to

$$e_B(n,n-1)\partial^n_{n-1}D_{n-1}A^m_{n,n-1} + e_B(n,n+1)\partial^n_{n+1}D_{n+1}A^m_{n,n+1} = D_n T^m_n\,,$$

using (2.21). Thus

$$\left(\frac{\partial}{\partial\tau} - \eta D_\gamma\right) S_\gamma = \sum_{\substack{\alpha,\beta \\ (n_{1\gamma}=n_\gamma)}} e_B(\gamma)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)v_\alpha B_\beta \quad (4.21)$$

$$\left(\frac{\partial}{\partial\tau} - \eta D_\gamma\right) T_\gamma = \sum_{\substack{\alpha,\beta,n_{1\gamma} \\ (n_{1\gamma}=n_\gamma\pm 1)}} e_B(\gamma)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\partial^\gamma(v_\alpha B_\beta)\,. \quad (4.22)$$

Of course the **Y**-coefficients of **v** and **B** on the right sides of (4.21) and (4.22) are given in terms of S_γ and T_γ by (4.1) and (4.2).

5 Non-Linear Toroidal-Poloidal Spectral-Interaction Equations

In this section the toroidal-poloidal spectral-interaction forms, (5.6) and (5.8), of the momentum equation are derived. Toroidal and poloidal potentials are substituted for the velocity and magnetic field in the non-linear and product terms on the right sides of (4.7) and (4.13) and the expressions are simplified. In this section only the case, where $\mathbf{\Omega}$ is parallel to $\mathbf{1}_z$, is considered. The toroidal-poloidal spectral-interaction equations are analogous to the Bullard & Gellman (1954) form of the magnetic induction equation, which are also given in (5.10) and (5.11).

In equations (4.7) and (4.13) the sums over the 3-indices α and β, i.e. over the six quantities n_α, $n_{1\alpha}$, m_α, n_β, $n_{1\beta}$ and m_β, can be reduced to sums over four indices by evaluating the sums over $n_{1\alpha}$ and $n_{1\beta}$ and expressing the coupling integrals in terms of the *Adams-Gaunt integral*,

$$A_{\alpha\beta\gamma} := \oint Y_\alpha Y_\beta Y_\gamma \, d\Omega \, , \tag{5.1}$$

and the *Elsasser dynamo integral*,

$$E_{\alpha\beta\gamma} := \oint \frac{1}{\sin\theta}(\partial_\theta Y_\alpha \ \partial_\phi Y_\beta - \partial_\phi Y_\alpha \ \partial_\theta Y_\beta) Y_\gamma \, d\Omega \, . \tag{5.2}$$

From the complex conjugate property of the spherical harmonics (2.2),

$$A_{\alpha\beta\gamma^*} = \oint Y_\alpha Y_\beta Y_\gamma^* \, d\Omega = (-)^{m_\gamma} A^{m_\alpha m_\beta -m_\gamma}_{n_\alpha \ n_\beta \ n_\gamma} \, , \quad E_{\alpha\beta\gamma^*} = (-)^{m_\gamma} E^{m_\alpha m_\beta -m_\gamma}_{n_\alpha \ n_\beta \ n_\gamma} \, . \tag{5.3}$$

The integrals $A_{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma}$ have been evaluated (Adams 1900; James 1973) in closed form. In terms of $3j$-symbols,

$$A_{\alpha\beta\gamma} = 4\pi\Lambda(\alpha,\beta,\gamma) \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix} \tag{5.4}$$

and

$$E_{\alpha\beta\gamma} = -4\pi i\Lambda(\alpha,\beta,\gamma)\Delta(\alpha,\beta,\gamma) \begin{pmatrix} n_\alpha+1 & n_\beta+1 & n_\gamma+1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix} \, , \tag{5.5}$$

where

$$\Delta(\alpha,\beta,\gamma) = \sqrt{\frac{(n_\alpha+n_\beta+n_\gamma+2)(n_\alpha+n_\beta+n_\gamma+4)}{4(n_\alpha+n_\beta+n_\gamma+3)}} \sqrt{(n_\alpha+n_\beta-n_\gamma+1)(n_\gamma+n_\alpha-n_\beta+1)(n_\beta+n_\gamma-n_\alpha+1)} \, .$$

In this section dashes will denote radial derivatives, not perturbation fields.

5.1 The Momentum Equation

The spectral poloidal-toroidal spectral-interaction forms (5.6) and (5.8) of the momentum equation are derived. The toroidal momentum equation is derived by substituting (4.1), (4.3), (4.8) and (4.10) into (4.7) and summing over $n_{1\alpha}$ and $n_{1\beta}$. After lengthy calculations the spectral-interaction form of the toroidal momentum equation obtains,

$$\begin{aligned} &\rho\left(\frac{\partial}{\partial\tau} - \nu D_n\right) t_n^m - \rho 2\Omega p_n^{-1}(imt_n + C_n^m \partial_{n-1}^n s_{n-1}^m + C_{n+1}^m \partial_{n+1}^n s_{n+1}^m) + \frac{\rho r}{\sqrt{3}} \frac{d\Omega_n^m}{d\tau} = \\ &\sum_{\alpha,\beta} \{-\rho[(s_\alpha s_\beta t_n^m) + (s_\alpha t_\beta t_n^m) + (t_\alpha t_\beta t_n^m)] + \mu_0^{-1}[(S_\alpha S_\beta t_n^m) + (S_\alpha T_\beta t_n^m) + (T_\alpha T_\beta t_n^m)] + \rho\alpha_\Theta (U_\alpha^e \Theta_\beta t_n^m)\} \, , \end{aligned} \tag{5.6}$$

where the toroidal-poloidal spectral-interaction forms of the advection, Lorentz and buoyancy terms are given by

$$\begin{aligned} 4\pi p_\gamma r(s_\alpha s_\beta t_\gamma) &= -p_\beta (D_\alpha s_\alpha) s_\beta E_{\alpha\beta\gamma^*} \\ 8\pi p_\gamma r^2 (s_\alpha t_\beta t_\gamma) &= \{p_\beta(p_\beta - p_\alpha - p_\gamma)(rs_\alpha)' t_\beta - p_\alpha(p_\alpha - p_\beta - p_\gamma) s_\alpha (rt_\beta)'\} A_{\alpha\beta\gamma^*} \\ 4\pi p_\gamma r(t_\alpha t_\beta t_\gamma) &= p_\alpha t_\alpha t_\beta E_{\alpha\beta\gamma^*} \end{aligned}$$

$$
\begin{aligned}
4\pi p_\gamma r(S_\alpha S_\beta t_\gamma) &= -p_\beta (D_\alpha S_\alpha) S_\beta E_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (S_\alpha T_\beta t_\gamma) &= \{p_\beta(p_\beta - p_\alpha - p_\gamma)(rS_\alpha)' T_\beta - p_\alpha(p_\alpha - p_\beta - p_\gamma) S_\alpha (rT_\beta)'\} A_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r(T_\alpha T_\beta t_\gamma) &= p_\alpha T_\alpha T_\beta E_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r(U^e_\alpha \Theta_\beta t_\gamma) &= -U^e_\alpha \Theta_\beta E_{\alpha\beta\gamma^*} \,.
\end{aligned}
$$

The primes denote radial derivatives, $p_\alpha := n_\alpha(n_\alpha + 1)$ and

$$
C^m_n := (n^2 - 1)\sqrt{\frac{n^2 - m^2}{4n^2 - 1}} \,. \tag{5.7}
$$

The non-linear and other product interaction terms are given on the right side of (5.6) and the remaining linear terms on the left. There are no separate $(t_\alpha s_\beta t_\gamma)$ or $(T_\alpha S_\beta t_\gamma)$ interactions in (5.6) due to the summations over α and β.

In general, interaction terms may be modified using the double summation over α and β, the symmetry of $A_{\alpha\beta\gamma^*}$ in α and β or the anti-symmetry of $E_{\alpha\beta\gamma^*}$ and so are not unique. In particular, coefficients of $A_{\alpha\beta\gamma^*}$ or $E_{\alpha\beta\gamma^*}$, respectively, can be symmetrised or anti-symmetrised in α and β. These procedures often produce more complicated coefficients. The converse procedures of de-symmetrisation of an $A_{\alpha\beta\gamma^*}$-coefficient or de-anti-symmetrisation of an $E_{\alpha\beta\gamma^*}$-coefficient by the addition of an arbitrary anti-symmetric or symmetric function, respectively, can sometimes be more useful in practice.

To derive the spectral-interaction form of the poloidal momentum equation, equations (4.1), (4.3), (4.8) and (4.10) are substituted into (4.13) and the sums over $n_{1\alpha}$ and $n_{1\beta}$ evaluated. After lengthy calculations the spectral-interaction form of the poloidal momentum equation follows,

$$
\begin{aligned}
&\rho\left(\frac{\partial}{\partial\tau} - \nu D_n\right) D_n s^m_n - \rho 2\Omega p_n^{-1}(imD_n s^m_n - C^m_n \partial^n_{n-1} t^m_{n-1} - C^m_{n+1}\partial^n_{n+1} t^m_{n+1}) = \\
&\sum_{\alpha,\beta}\{-\rho[(s_\alpha s_\beta s^m_n) + (s_\alpha t_\beta s^m_n) + (t_\alpha t_\beta s^m_n)] + \mu_0^{-1}[(S_\alpha S_\beta s^m_n) + (S_\alpha T_\beta s^m_n) + (T_\alpha T_\beta s^m_n)] + \rho\alpha_\Theta (U^e_\alpha \Theta_\beta s^m_n)\} \,,
\end{aligned} \tag{5.8}
$$

where the toroidal-poloidal spectral-interaction forms of the advection, Lorentz and buoyancy terms are given by

$$
\begin{aligned}
8\pi p_\gamma r^2 (s_\alpha s_\beta s_\gamma) &= \{p_\gamma(p_\alpha + p_\beta - p_\gamma)(D_\alpha s_\alpha)(rs_\beta)' + p_\beta(p_\alpha - p_\beta + p_\gamma) r[(D_\alpha s_\alpha)s_\beta]'\} A_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r^3 (s_\alpha t_\beta s_\gamma) &= \{p_\gamma r^2 (D_\alpha s_\alpha) t_\beta + (p_\alpha + p_\beta + p_\gamma) s_\alpha t_\beta - (p_\alpha + p_\beta - p_\gamma)(rs_\alpha t'_\beta + rs'_\alpha t_\beta + r^2 s'_\alpha t'_\beta) \\
&\quad - p_\beta r^2 s''_\alpha t_\beta - p_\alpha r^2 s_\alpha t''_\beta\} E_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (t_\alpha t_\beta s_\gamma) &= \{p_\gamma(p_\alpha + p_\beta - p_\gamma)(rt_\alpha)' t_\beta + p_\alpha(-p_\alpha + p_\beta + p_\gamma) r(t_\alpha t_\beta)'\} A_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (S_\alpha S_\beta s_\gamma) &= \{p_\gamma(p_\alpha + p_\beta - p_\gamma)(D_\alpha S_\alpha)(rS_\beta)' + p_\beta(p_\alpha - p_\beta + p_\gamma) r[(D_\alpha S_\alpha)S_\beta]'\} A_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r^3 (S_\alpha T_\beta s_\gamma) &= \{p_\gamma r^2 (D_\alpha S_\alpha) T_\beta + (p_\alpha + p_\beta + p_\gamma) S_\alpha T_\beta - (p_\alpha + p_\beta - p_\gamma)(rS_\alpha T'_\beta + rS'_\alpha T_\beta + r^2 S'_\alpha T'_\beta) \\
&\quad - p_\alpha r^2 S''_\alpha T_\beta - p_\beta r^2 S_\alpha T''_\beta\} E_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (T_\alpha T_\beta s_\gamma) &= \{p_\gamma(p_\alpha + p_\beta - p_\gamma)(rT_\alpha)' T_\beta + p_\alpha(-p_\alpha + p_\beta + p_\gamma) r(T_\alpha T_\beta)'\} A_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r (U^e_\alpha \Theta_\beta s_\gamma) &= \{(p_\alpha - p_\beta + p_\gamma) U^e_\alpha \Theta'_\beta + (p_\alpha - p_\beta - p_\gamma) U^{e\,\prime}_\alpha \Theta_\beta\} A_{\alpha\beta\gamma^*} \,.
\end{aligned}
$$

The primes denote radial derivatives. There are no separate $(t_\alpha s_\beta s_\gamma)$ or $(T_\alpha S_\beta s_\gamma)$ interactions. Equations (5.6) and (5.8) are analogous to the equations derived by Bullard & Gellman (1954) from the magnetic induction equation for the magnetic toroidal and poloidal potentials. If the gravitational acceleration is spherically-symmetric, then the buoyancy term in the toroidal equation (5.6) vanishes and the buoyancy term in the poloidal equation (5.8) simplifies to

$$
(U^e_\alpha \Theta_\beta s_\gamma) = -\frac{g}{r}\Theta_\beta \delta^\beta_\gamma \delta^0_{n_\alpha} \delta^{m_\alpha}_0 \,, \tag{5.9}
$$

since $n_\alpha = 0$, $n_{1\alpha} = 1$, $m_\alpha = 0$, $g = \partial^1_0 U^{e,0}_0 = \partial_r U^{e,0}_0$ and $A_{0\beta\gamma^*} = 4\pi\delta_{\beta\gamma}$.

The highest radial derivatives of t^m_n in (5.6) and (5.8) are second-order, but those of s^m_n are second-order and fourth-order respectively.

5.2 The Magnetic Induction Equation

If (4.1) and (4.3) are substituted into (4.21) and (4.22), and $n_{1\alpha}$ and $n_{1\beta}$ are summed, the coupling integral reduces to either the Adams-Gaunt integral, (5.1) or the Elsasser integral (5.2). The well-known Bullard & Gellman (1954) equations are obtained, with minor modifications due to the different definition of toroidal-poloidal fields and different normalisation of the scalar spherical harmonics, after lengthy calculations:

$$\left(\frac{\partial}{\partial\tau} - \eta D_\gamma\right) S_\gamma = \sum_{\alpha,\beta} \{(s_\alpha S_\beta S_\gamma) + (s_\alpha T_\beta S_\gamma) + (t_\alpha S_\beta S_\gamma) + (t_\alpha T_\beta S_\gamma)\} \tag{5.10}$$

$$\left(\frac{\partial}{\partial\tau} - \eta D_\gamma\right) T_\gamma = \sum_{\alpha,\beta} \{(s_\alpha S_\beta T_\gamma) + (s_\alpha T_\beta T_\gamma) + (t_\alpha S_\beta T_\gamma) + (t_\alpha T_\beta T_\gamma)\}\,, \tag{5.11}$$

where now α and β are 2-indices and the toroidal-poloidal spectral-interaction terms are given by

$$\begin{aligned}
8\pi p_\gamma r^2 (s_\alpha S_\beta S_\gamma) &= \{-p_\alpha(-p_\alpha + p_\beta + p_\gamma) s_\alpha (rS_\beta)' + p_\beta(p_\alpha - p_\beta + p_\gamma)(rs_\alpha)' S_\beta\} A_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r (s_\alpha T_\beta S_\gamma) &= p_\alpha s_\alpha T_\beta E_{\alpha\beta\gamma^*} \\
4\pi p_\gamma r (t_\alpha S_\beta S_\gamma) &= p_\beta t_\alpha S_\beta E_{\alpha\beta\gamma^*} \\
(t_\alpha T_\beta S_\gamma) &= 0 \\
4\pi p_\gamma r^3 (s_\alpha S_\beta T_\gamma) &= \{(p_\alpha + p_\beta + p_\gamma) s_\alpha S_\beta - (p_\alpha + p_\beta - p_\gamma)(rs'_\alpha S_\beta + rs_\alpha S'_\beta + r^2 s'_\alpha S'_\beta) \\
&\quad - p_\alpha r^2 s_\alpha S''_\beta - p_\beta r^2 s''_\alpha S_\beta\} E_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (s_\alpha T_\beta T_\gamma) &= \{-p_\gamma(p_\alpha + p_\beta - p_\gamma)(s_\alpha T_\beta + rs'_\alpha T_\beta) + p_\alpha(p_\alpha - p_\beta - p_\gamma)(rs'_\alpha T_\beta + rs_\alpha T'_\beta)\} A_{\alpha\beta\gamma^*} \\
8\pi p_\gamma r^2 (t_\alpha S_\beta T_\gamma) &= \{p_\gamma(p_\alpha + p_\beta - p_\gamma)(t_\alpha S_\beta + rt_\alpha S'_\beta) - p_\beta(p_\beta - p_\alpha - p_\gamma)(rt'_\alpha S_\beta + rt_\alpha S'_\beta)\} A_{\alpha\beta\gamma^*} \\
4\pi r (t_\alpha T_\beta T_\gamma) &= t_\alpha T_\beta E_{\alpha\beta\gamma^*}\,.
\end{aligned}$$

The primes denote radial derivatives.

5.3 The Heat Equation

The spectral equations for the temperature can also be expressed in terms of the spherical harmonic coefficients of Θ and of the toroidal-poloidal potentials t_α, s_α, T_α and S_α of the velocity and the magnetic field. Substituting (4.2) and (4.3) into the heat equation (3.49), and summing over $n_{1\alpha}$ and $n_{1\beta}$, the coupling integral reduces to Adams-Gaunt integrals (5.1) and Elsasser integrals (5.2) giving,

$$\begin{aligned}
\rho c_p \left(\frac{\partial}{\partial\tau} - \kappa D_\gamma\right)\Theta_\gamma = Q_\gamma + \sum_{\alpha,\beta} \{&-\rho c_p[(s_\alpha \Theta_\beta \Theta_\gamma) + (t_\alpha \Theta_\beta \Theta_\gamma)] + \rho\nu[(s_\alpha s_\beta \Theta_\gamma) + (s_\alpha t_\beta \Theta_\gamma) \\
&+ (t_\alpha t_\beta \Theta_\gamma)] + [(S_\alpha S_\beta \Theta_\gamma) + (S_\alpha T_\beta \Theta_\gamma) + (T_\alpha S_\beta \Theta_\gamma) + (T_\alpha T_\beta \Theta_\gamma)]/\mu_0^2\sigma\}\,,
\end{aligned} \tag{5.12}$$

where the temperature-toroidal-poloidal spectral-interaction terms are given by

$$\begin{aligned}
4\pi r^2 (s_\alpha \Theta_\beta \Theta_\gamma) &= \{r p_\alpha s_\alpha \Theta'_\beta + \tfrac{1}{2}(p_\alpha + p_\beta - p_\gamma)(rs_\alpha)'\Theta_\beta\} A_{\alpha\beta\gamma^*} \\
4\pi r (t_\alpha \Theta_\beta \Theta_\gamma) &= -t_\alpha \Theta_\beta E_{\alpha\beta\gamma^*} \\
8\pi r^2 (s_\alpha s_\beta \Theta_\gamma) &= \{2p_\alpha p_\beta [r^2 D_\gamma (s_\alpha s_\beta / r^2) - 2s_\alpha (D_\beta s_\beta)] \\
&\quad + (p_\alpha + p_\beta - p_\gamma)[r^2 D_\gamma ((rs_\alpha)'(rs_\beta)'/r^2) - 2(rs_\alpha)'(rD_\beta s_\beta)' - r^2 (D_\alpha s_\alpha)(D_\beta s_\beta)]\} A_{\alpha\beta\gamma^*} \\
2\pi r (s_\alpha t_\beta \Theta_\gamma) &= \{rD_\gamma[(rs_\alpha)' t_\beta / r] - (rD_\alpha s_\alpha)' t_\beta - (rs_\alpha)'(D_\beta t_\beta) - (D_\alpha s_\alpha)(rt_\beta)'\} E_{\alpha\beta\gamma^*} \\
8\pi (t_\alpha t_\beta \Theta_\gamma) &= \{-2p_\alpha p_\beta t_\alpha t_\beta / r^2 + (p_\alpha + p_\beta - p_\gamma)[D_\gamma (t_\alpha t_\beta) - 2t_\alpha (D_\beta t_\beta) - (rt_\alpha)'(rt_\beta)'/r^2]\} A_{\alpha\beta\gamma^*} \\
8\pi (S_\alpha S_\beta \Theta_\gamma) &= (p_\alpha + p_\beta - p_\gamma)(D_\alpha S_\alpha)(D_\beta S_\beta) A_{\alpha\beta\gamma^*} \\
2\pi r (S_\alpha T_\beta \Theta_\gamma) &= (D_\alpha S_\alpha)(rT_\beta)' E_{\alpha\beta\gamma^*} \\
8\pi r^2 (T_\alpha T_\beta \Theta_\gamma) &= \{2p_\alpha p_\beta T_\alpha T_\beta + (p_\alpha + p_\beta - p_\gamma)(rT_\alpha)'(rT_\beta)'\} A_{\alpha\beta\gamma^*}
\end{aligned}$$

The dashes denote radial derivatives. The interactions $(t_\alpha s_\beta \Theta_\gamma)$ and $(T_\alpha S_\beta \Theta_\gamma)$ have been absorbed into $(s_\alpha t_\beta \Theta_\gamma)$ and $(S_\alpha T_\beta \Theta_\gamma)$ respectively.

6 Hybrid Spectral Forms of the Linearised Equations

The most important results of the paper for use in applications are the hybrid spectral forms of the linearised momentum, magnetic induction and heat equations derived in this section, namely (6.2)–(6.9), (6.11)–(6.14) and (6.15)–(6.19). In these hybrid spectral equations the basic state is described mathematically by the vector fields $\mathbf{v}_0$, $\boldsymbol{\omega}_0$, $\mathbf{B}_0$, $\mathbf{J}_0$, $\mathbf{q}_0 := \nabla\Theta_0$ and $\mathbf{g}_e$, but the perturbation state is given by scalar fields — the toroidal-poloidal potentials s', t', S', T' and the temperature Θ'. There may be a perturbed heat source Q'. The vector fields of the basic state are expanded in vector spherical harmonics and the perturbation fields in scalar spherical harmonics. The notation $v^0_\alpha = v^{0,m_\alpha}_{n_\alpha,n_{1\alpha}}$, etc, is adopted. Thus $\mathbf{v}_0 = \sum_\alpha v^0_\alpha \mathbf{Y}_\alpha$, etc. Spectral forms of the interaction terms are a hybrid of $\mathbf{Y}$-coefficients of the basic state and Y-coefficients of the toroidal-poloidal potentials and the temperature. In this section the z-axis coincides with the rotation axis.

The $\mathbf{Y}$-spectral form of the linearised momentum equation is obtained by linearising the $\mathbf{Y}$-spectral equation (3.12),

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_\gamma\right)v'_\gamma = -f_P(\gamma)\partial_\gamma P'_\gamma - \sum_{\alpha,\beta}\rho\alpha_\Theta g^e_\alpha \Theta'_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma) + \sum_{\alpha,\beta}(-\rho\omega^0_\alpha v'_\beta - \rho\omega'_\alpha v^0_\beta - \rho 2\Omega_\alpha v'_\beta + J'_\alpha B^0_\beta + J^0_\alpha B'_\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\,. \quad (6.1)$$

If toroidal-poloidal representations for both the basic state and the perturbation fields are substituted into the v'_γ-equation (6.1), proceeding as in Section 5 yields the linearised forms of the t_γ-equation (5.6) and the s_γ-equation (5.8), respectively. Instead, different hybrid spectral forms of the linearised t_γ-equation and s_γ-equation are derived by substituting toroidal-poloidal representations only for the perturbation magnetic, velocity, electric current and vorticity fields, not the basic state fields. The hybrid spectral equations are structurally more compact with greater redundancy and their use is therefore generally less error prone. The hybrid t'_γ-equation is

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_\gamma\right)t'_\gamma = \sum_{\substack{\alpha,\beta\\(n_{1\gamma}=n_\gamma)}}\{-\rho(\omega^0_\alpha v'_\beta v'_\gamma) + \rho(v^0_\alpha \omega'_\beta v'_\gamma) + (J^0_\alpha B'_\beta v'_\gamma) - (B^0_\beta J'_\alpha v'_\gamma) - \rho\alpha_\Theta(g^e_\alpha \Theta'_\beta v'_\gamma) - \rho 2(\Omega_\alpha v'_\beta v'_\gamma)\} \quad (6.2)$$

and the hybrid s'_γ-equation is

$$\rho\left(\frac{\partial}{\partial\tau} - \nu D_\gamma\right)D_\gamma s'_\gamma = \sum_{\substack{\alpha,\beta,n_{1\gamma}\\(n_{1\gamma}=n_\gamma\pm1)}}\{-\rho(\omega^0_\alpha v'_\beta v'_\gamma) + \rho(v^0_\alpha \omega'_\beta v'_\gamma) + (J^0_\alpha B'_\beta v'_\gamma) - (B^0_\beta J'_\alpha v'_\gamma) - \rho\alpha_\Theta(g^e_\alpha \Theta'_\beta v'_\gamma) - \rho 2(\Omega_\alpha v'_\beta v'_\gamma)\}\,. \quad (6.3)$$

The interaction terms in equations (6.2) and (6.3) are

$$(\omega^0_\alpha v'_\beta v'_\gamma) := e_v(\gamma) f_v(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\begin{cases}\partial^\gamma(\omega^0_\alpha \partial_\beta s'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1;\\ \partial^\gamma(\omega^0_\alpha t'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta;\\ \omega^0_\alpha \partial_\beta s'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1;\\ \omega^0_\alpha t'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta;\end{cases} \quad (6.4)$$

$$(v^0_\alpha \omega'_\beta v'_\gamma) := e_v(\gamma) f_\omega(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\begin{cases}\partial^\gamma(v^0_\alpha \partial_\beta t'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1;\\ \partial^\gamma(v^0_\alpha D_\beta s'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta;\\ v^0_\alpha \partial_\beta t'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1;\\ v^0_\alpha D_\beta s'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta;\end{cases} \quad (6.5)$$

$$(J^0_\alpha B'_\beta v'_\gamma) := e_v(\gamma) f_B(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)\begin{cases}\partial^\gamma(J^0_\alpha \partial_\beta S'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1;\\ \partial^\gamma(J^0_\alpha T'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta;\\ J^0_\alpha \partial_\beta S'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1;\\ J^0_\alpha T'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta;\end{cases} \quad (6.6)$$

$$(B^0_\alpha J'_\beta v'_\gamma) := e_v(\gamma) f_J(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \begin{cases} \partial^\gamma(B^0_\alpha \partial_\beta T'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1; \\ \partial^\gamma(B^0_\alpha D_\beta S'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta; \\ B^0_\alpha \partial_\beta T'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1; \\ B^0_\alpha D_\beta S'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta; \end{cases} \tag{6.7}$$

$$(\Omega_\alpha v'_\beta v'_\gamma) := e_v(\gamma) f_v(\beta)(\mathbf{Y}^0_{1,0} \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \begin{cases} \Omega \partial^\gamma \partial_\beta s'_\beta, & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1; \\ \Omega \partial^\gamma t'_\beta, & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta; \\ \Omega \partial_\beta s'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1; \\ \Omega t'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta. \end{cases} \tag{6.8}$$

For the general gravitational acceleration the spectral form is

$$(g^e_\alpha \Theta'_\beta v'_\gamma) := e_v(\gamma)(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma) \begin{cases} \partial^\gamma(g^e_\alpha \Theta'_\beta), & n_{1\gamma} = n_\gamma \pm 1; \\ g^e_\alpha \Theta'_\beta, & n_{1\gamma} = n_\gamma. \end{cases} \tag{6.9}$$

The buoyancy term for a spherically-symmetric gravitational acceleration does not have a special hybrid form. The Poincaré force has been omitted.

The **Y**-spectral form of the linearised magnetic vector potential equation is obtained by linearising the **Y**-spectral magnetic vector potential equation (3.34) to give

$$\left(\frac{\partial}{\partial \tau} - \eta D_\gamma\right) A'_\gamma = \sum_{\alpha,\beta} (v^0_\alpha B'_\beta - B^0_\alpha v'_\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) - (\nabla\{\Phi' + \eta \nabla \cdot \mathbf{A}'\})_\gamma . \tag{6.10}$$

As in the case of the momentum equation, substituting toroidal-poloidal representations for both the basic state and the perturbation fields into the A'_γ-equation (6.10) and proceeding as in Section 5 leads to the linearised form of the S_γ-equation (5.10) and the T_γ-equation (5.11), respectively. Hybrid spectral forms of the S'_γ-equation and the T'_γ-equation are obtained by substituting toroidal-poloidal representations only for the perturbation magnetic and velocity fields into (6.10). This yields the hybrid S'_γ- and T'_γ-equations,

$$\left(\frac{\partial}{\partial \tau} - \eta D_\gamma\right) S'_\gamma = \sum_{\substack{\alpha,\beta \\ (n_{1\gamma} = n_\gamma)}} \{(v^0_\alpha B'_\beta B'_\gamma) - (B^0_\alpha v'_\beta B'_\gamma)\} \tag{6.11}$$

$$\left(\frac{\partial}{\partial \tau} - \eta D_\gamma\right) T'_\gamma = \sum_{\substack{\alpha,\beta,n_{1\gamma} \\ (n_{1\gamma} = n_\gamma \pm 1)}} \{(v^0_\alpha B'_\beta B'_\gamma) - (B^0_\alpha v'_\beta B'_\gamma)\} . \tag{6.12}$$

The interaction terms in equations (6.11) and (6.12) are

$$(v^0_\alpha B'_\beta B'_\gamma) := e_B(\gamma) f_B(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \begin{cases} \partial^\gamma(v^0_\alpha \partial_\beta S'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1; \\ \partial^\gamma(v^0_\alpha T'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta; \\ v^0_\alpha \partial_\beta S'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1; \\ v^0_\alpha T'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta; \end{cases} \tag{6.13}$$

$$(B^0_\alpha v'_\beta B'_\gamma) := e_B(\gamma) f_v(\beta)(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \begin{cases} \partial^\gamma(B^0_\alpha \partial_\beta s'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta \pm 1; \\ \partial^\gamma(B^0_\alpha t'_\beta), & n_{1\gamma} = n_\gamma \pm 1, n_{1\beta} = n_\beta; \\ B^0_\alpha \partial_\beta s'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta \pm 1; \\ B^0_\alpha t'_\beta, & n_{1\gamma} = n_\gamma, n_{1\beta} = n_\beta. \end{cases} \tag{6.14}$$

Finally, the linearised form of the heat equation (3.49) is

$$\rho c_p \left(\frac{\partial}{\partial \tau} - \kappa D_\gamma\right) \Theta'_\gamma = Q'_\gamma + \sum_{\alpha,\beta} \{-\rho c_p (v^0_\alpha q'_\beta \Theta'_\gamma) - \rho c_p (q^0_\alpha v'_\beta \Theta'_\gamma) + \rho \nu (v^0_\alpha v'_\beta \Theta'_\gamma) + (J^0_\alpha J_\beta \Theta'_\gamma)/\mu_0 \sigma\} , \tag{6.15}$$

where the four interaction terms are

$$(v^0_\alpha q'_\beta \Theta'_\gamma) := f_q(\beta)(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma) \begin{cases} v^0_\alpha \partial_\beta \Theta'_\beta, & n_{1\beta} = n_\beta \pm 1; \\ 0, & n_{1\beta} = n_\beta; \end{cases} \tag{6.16}$$

$$(q^0_\alpha v'_\beta \Theta'_\gamma) := f_v(\beta)(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma) \begin{cases} q^0_\alpha \partial_\beta s'_\beta, & n_{1\beta} = n_\beta \pm 1; \\ q^0_\alpha t'_\beta, & n_{1\beta} = n_\beta; \end{cases} \tag{6.17}$$

$$(v^0_\alpha v'_\beta \Theta'_\gamma) := (\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma)\times$$
$$\begin{cases} 2f_v(\beta)[2(\partial_r v^0_\alpha)\partial_r \partial_\beta s'_\beta + (p_{1\alpha} + p_{1\beta} - p_\gamma) v^0_\alpha (\partial_\beta s'_\beta)/r^2] - f_\omega(\beta)\omega^0_\alpha \partial_\beta t'_\beta, & n_{1\beta} = n_\beta \pm 1; \\ 2f_v(\beta)[2(\partial_r v^0_\alpha)\partial_r t'_\beta + (p_{1\alpha} + p_{1\beta} - p_\gamma) v^0_\alpha\, t'_\beta/r^2] - f_\omega(\beta)\omega^0_\alpha D_\beta s'_\beta, & n_{1\beta} = n_\beta, \end{cases} \tag{6.18}$$

using (4.3) and (4.10); and

$$(J^0_\alpha J'_\beta \Theta'_\gamma) := 2f_J(\beta)(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma) \begin{cases} J^0_\alpha \partial_\beta T'_\beta, & n_{1\beta} = n_\beta \pm 1; \\ J^0_\alpha D_\beta S'_\beta, & n_{1\beta} = n_\beta; \end{cases} \tag{6.19}$$

where f_q is given following (3.51). The interaction terms in the toroidal-poloidal equations and the heat equation have the same structure: an equation factor, e_v, e_B or unity; a field factor, f_v, f_B, f_J, f_ω or f_q; a coupling integral; and a radial expression, which consists of coefficients of the basic state fields and the perturbation scalar fields, t', s', T', S' and Θ', and possibly their radial derivatives. There may also be a parameter depending on the scaling. The structure can be efficiently exploited in numerical implementations of the spectral equations.

7 Anelastic Spectral Equations

In the anelastic approximation the velocity satisfies $\nabla \cdot [\rho(r)\mathbf{v}] = 0$, the viscous force in the momentum equation is $\mathbf{F}_\nu := \nabla \cdot \{\rho\nu[\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}\mathbf{I}\nabla \cdot \mathbf{v}]\}$ and the viscous heating in the heat equation is $Q_\nu := \rho\nu\{\nabla\mathbf{v} : \nabla\mathbf{v} + \nabla\mathbf{v} : (\nabla\mathbf{v})^T - \frac{2}{3}(\nabla \cdot \mathbf{v})^2\}$. The first condition filters out sound waves. The density ρ varies only with spherical radius, except in the buoyancy term, where the density also depends affinely on the temperature. The density and the pressure are constant strictly only on the equipotential surfaces of the effective gravitational potential U_e, which are deformed from spherical surfaces by the rotation through the centripetal acceleration. Such deformation is neglected and the density ρ and the pressure depend only on the spherical radius r. The parameters ν, α_Θ and κ may depend on the pressure and hence the density. The velocity $\mathbf{v}$ could be used as a dependent field variable but the momentum density $\mathbf{u} := \rho\mathbf{v}$ is preferable, since it is solenoidal and hence possesses a toroidal-poloidal representation $\mathbf{u} = \mathbf{T}\{t\} + \mathbf{S}\{s\}$. The velocity itself is generally compressible, $\nabla \cdot \mathbf{v} = \nabla \cdot (\mathbf{u}/\rho) = u_r(1/\rho)'$. The radial dependence of the density implies that the vector spherical harmonic coefficients of $\mathbf{u}$ and $\mathbf{v}$ are simply related by $u_\alpha = \rho v_\alpha$ and that the anelastic spectral equations are no more strongly coupled in angle than the Boussinesq equations. Spectral equations can be derived for a more general density with lateral variations, but they are not considered herein.

7.1 The Momentum Equation

Since $\rho\nabla\mathbf{v} = \nabla\mathbf{u} - R'\mathbf{1}_r\mathbf{u}$, where $R := \ln\rho$, the viscous force can be written as

$$\mathbf{F}_\nu = \nu[\nabla^2\mathbf{u} - R'\partial_r\mathbf{u} - (R'' + 3R'/r)\mathbf{u} - (R'/r)'u_r\mathbf{r}] + \nabla\nu \cdot [\nabla\mathbf{u} - R'\mathbf{1}_r\mathbf{u} - R'\mathbf{u}\mathbf{1}_r + (\nabla\mathbf{u})^T] + \nabla(\tfrac{2}{3}\nu R' u_r)\,. \tag{7.1}$$

It is assumed that the kinematic viscosity is independent of the density as in the case of dense gases (Chapman & Cowling 1970). Moreover, it may be argued that the laminar viscous force is only important in boundary layers thin compared to a typical length scale of ν, so variation of ν with density can be neglected if ν is small. Thus (7.1) reduces to

$$\mathbf{F}_\nu = \nu[\nabla^2\mathbf{u} - R'\partial_r\mathbf{u} - (R'' + 3R'/r)\mathbf{u} - (R'/r)'u_r\mathbf{r}] + \nabla(\tfrac{2}{3}\nu R' u_r)\,.$$

The last two terms can be expressed in terms of the $\mathbf{Y}$-vector spherical harmonic coefficients of $\mathbf{u}$ using $\mathbf{Y}^m_{n,n-1} \cdot \mathbf{1}_r = \sqrt{n/(2n+1)}\; Y^m_n$, $\mathbf{Y}^m_{n,n} \cdot \mathbf{1}_r = 0$ and $\mathbf{Y}^m_{n,n+1} \cdot \mathbf{1}_r = -\sqrt{(n+1)/(2n+1)}\; Y^m_n$, which are deduced from (2.7)–(2.9). Thus since $u_r = \sum_\alpha u_\alpha \mathbf{1}_r \cdot \mathbf{Y}_\alpha$,

$$(u_r)^m_n = \sqrt{\frac{n}{2n+1}} u^m_{n,n-1} - \sqrt{\frac{n+1}{2n+1}} u^m_{n,n+1}\,. \tag{7.2}$$

Using (3.27),

$$u_r\mathbf{1}_r = \sum_{n,m}(u_r)_n^m Y_n^m \mathbf{1}_r = \sum_{n,m}\left(\sqrt{\frac{n}{2n+1}}u_{n,n-1}^m - \sqrt{\frac{n+1}{2n+1}}u_{n,n+1}^m\right)\left(\sqrt{\frac{n}{2n+1}}\mathbf{Y}_{n,n-1}^m - \sqrt{\frac{n+1}{2n+1}}\mathbf{Y}_{n,n+1}^m\right)$$

Thus

$$(u_r\mathbf{1}_r)_{n,n_1}^m = \begin{cases} \frac{n}{2n+1}u_{n,n-1}^m - \frac{\sqrt{n(n+1)}}{2n+1}u_{n,n+1}^m\,, & n_1 = n-1 \\ 0\,, & n_1 = n \\ -\frac{\sqrt{n(n+1)}}{2n+1}u_{n,n-1}^m + \frac{n+1}{2n+1}u_{n,n+1}^m\,, & n_1 = n+1 \end{cases} \tag{7.3}$$

and

$$F_\gamma^\nu = \nu[D_{1\gamma}u_\gamma - R'\partial_r u_\gamma - (R'' + 3R'/r)u_\gamma - (R'/r)'(u_r\mathbf{r})_\gamma] + (\nabla\tfrac{2}{3}\nu R' u_r)_\gamma\,, \tag{7.4}$$

where $(u_r\mathbf{r})_\gamma$ is given by (7.3) and $(\nabla\frac{2}{3}\nu R' u_r)_\gamma$ by (7.2) and the gradient formula (2.13).

Replacing v_γ by u_γ/ρ in (3.12), the anelastic $\mathbf{Y}$-momentum equation becomes

$$\frac{\partial u_\gamma}{\partial\tau} - F_\gamma^\nu = -f_P(\gamma)\partial_\gamma P_\gamma + \sum_{\alpha,\beta}\{[-\omega_\alpha u_\beta - 2\Omega_\alpha u_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta](\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g_\alpha^e\Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,, \tag{7.5}$$

where F_γ^ν is given by (7.4) and ω_α is related to the vector spherical harmonic coefficients of u_α by equations (3.15)–(3.17) with v_{n,n_1}^m replaced by $u_{n,n_1}^m/\rho$.

From the representation $\mathbf{u} = \mathbf{T}\{t\} + \mathbf{S}\{s\}$ and by comparison with (4.1),

$$u_{n,n_1}^m = f_v(n,n_1)\begin{cases}\partial_n^{n_1}s_n^m\,, & \text{if } n_1 = n\pm 1; \\ t_n^m\,, & \text{if } n_1 = n.\end{cases}$$

Thus similarly to (4.6) the t_n^m-equation is $e_v(n,n)$ times the $u_{n,n}^m$-equation. From (7.4),

$$F_{n,n}^{\nu,m} = \nu f_v(n,n)[D_n t_n^m - R'\partial_r t_n^m - (R'' + 3R'/r)t_n^m]\,, \tag{7.6}$$

and the compact anelastic t_γ-equation is

$$\frac{\partial t_\gamma}{\partial\tau} - \nu(D_\gamma - R'\partial_r - R'' - 3R'/r)t_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}=n_\gamma}} e_v(\gamma)\{(-\omega_\alpha u_\beta - 2\Omega_\alpha u_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta)(\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g_\alpha^e\Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,. \tag{7.7}$$

Analogously to (4.12) the anelastic s_n^m-equation is obtained by applying $e_v(n,n-1)\partial_{n-1}^n$ to the $u_{n,n-1}^m$-equation, $e_v(n,n+1)\partial_{n+1}^n$ to the $u_{n,n+1}^m$-equation and adding. Now $(ru_r)_n^m = n(n+1)s_n^m$, so by (3.27), $(u_r\mathbf{r})_{n,n-1}^m = n(n+1)\sqrt{n/(2n+1)}\,s_n^m$ and $(u_r\mathbf{r})_{n,n+1}^m = -n(n+1)\sqrt{(n+1)/(2n+1)}\,s_n^m$. Thus the viscous term is

$$\begin{aligned} e_v(n,n-1)\partial_{n-1}^n F_{n,n-1}^{\nu,m} + e_v(n,n+1)\partial_{n+1}^n F_{n,n+1}^{\nu,m} = \\ \nu D_n D_n s_n^m - (n+1)/(2n+1)\partial_{n-1}^n\nu[R'\partial_r\partial_n^{n-1}s_n^m + (R''+3R'/r)\partial_n^{n-1}s_n^m + (R'/r)'ns_n^m] \\ - n/(2n+1)\partial_{n+1}^n\nu[R'\partial_r\partial_n^{n+1}s_n^m + (R''+3R'/r)\partial_n^{n+1}s_n^m - (R'/r)'(n+1)s_n^m] \end{aligned}$$

and the compact anelastic s_n^m-equation is

$$\frac{\partial(D_\gamma s_\gamma)}{\partial\tau} - \nu\left[D_\gamma D_\gamma - R'\partial_r D_\gamma - \left(\frac{3R'}{r} + 2R''\right)D_\gamma - [R^{(3)} + 2(R'/r)']\partial_r - \frac{R^{(3)} + 2(R'/r)'}{r}\right]s_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}=n_\gamma\pm 1}} e_v(\gamma)\{[-\omega_\alpha u_\beta - 2\Omega_\alpha u_\beta - \rho(d\Omega_\alpha/d\tau)r_\beta + J_\alpha B_\beta](\mathbf{Y}_\alpha\times\mathbf{Y}_\beta, \mathbf{Y}_\gamma) - \rho\alpha_\Theta g_\alpha^e\Theta_\beta(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)\}\,. \tag{7.8}$$

In terms of the toroidal and poloidal potentials of $\mathbf{u}$, the vorticity is $\boldsymbol{\omega} = \nabla\times(\mathbf{u}/\rho) = \mathbf{S}\{t/\rho\} + \mathbf{T}\{-(\nabla^2 s)/\rho - (1/\rho)'\partial_r(rs)/r\}$. Hence the toroidal and poloidal potentials of the vorticity $\boldsymbol{\omega}$ are $-(\nabla^2 s)/\rho -$

$(1/\rho)'\partial_r(rs)/r$ and t/ρ, respectively. The only anelastic toroidal-poloidal interactions which differ from (5.6) and (5.8) arise from $\boldsymbol{\omega}\times\mathbf{u}$ and are thus given by replacing t_α by t_α/ρ and $D_\alpha s_\alpha$ by $(D_\alpha s_\alpha)/\rho+(1/\rho)'\partial_r(rs_\alpha)/r$ in the spectral interaction terms of $\boldsymbol{\omega}\times\mathbf{u}$. These anelastic interactions, indicated by a subscript a, are

$$\begin{aligned}
4\pi p_\gamma r\rho(s_\alpha s_\beta t_\gamma)_a &= -p_\beta[D_\alpha s_\alpha - R'(rs_\alpha)'/r]s_\beta E_{\alpha\beta\gamma^*}\\
(s_\alpha t_\beta t_\gamma)_a &= 0\\
8\pi p_\gamma r^2\rho(t_\alpha s_\beta t_\gamma)_a &= \{p_\alpha(p_\alpha-p_\beta-p_\gamma)t_\alpha(rs_\beta)' - p_\beta(p_\beta-p_\alpha-p_\gamma)\rho(rt_\alpha/\rho)'s_\beta\}A_{\alpha\beta\gamma^*}\\
4\pi p_\gamma r\rho(t_\alpha t_\beta t_\gamma)_a &= p_\alpha t_\alpha t_\beta E_{\alpha\beta\gamma^*}\\
8\pi p_\gamma r^2\rho(s_\alpha s_\beta s_\gamma)_a &= \{p_\gamma(p_\alpha+p_\beta-p_\gamma)[D_\alpha s_\alpha - R'(rs_\alpha)'/r]s_\beta + [rD_\alpha s_\alpha - R'(rs_\alpha)']s'_\beta)\\
&\quad + p_\beta(p_\alpha-p_\beta+p_\gamma)(r\rho\{[D_\alpha s_\alpha - R'(rs_\alpha)'/r]/\rho\}'s_\beta + [rD_\alpha s_\alpha - R'(rs_\alpha)']s'_\beta)\}A_{\alpha\beta\gamma^*}\\
4\pi r\rho(s_\alpha t_\beta s_\gamma)_a &= [D_\alpha s_\alpha - R'(rs_\alpha)'/r]t_\beta E_{\alpha\beta\gamma^*}\\
4\pi p_\gamma r^3\rho(t_\alpha s_\beta s_\gamma)_a &= \{-(p_\alpha+p_\beta+p_\gamma)t_\alpha s_\beta + (p_\alpha+p_\beta-p_\gamma)[r\rho(t_\alpha/\rho)'s_\beta + rt_\alpha s'_\beta + r^2\rho(t_\alpha/\rho)'s'_\beta]\\
&\quad + p_\alpha r^2 t_\alpha s''_\beta + p_\beta r^2\rho(t_\alpha/\rho)''s_\beta\}E_{\alpha\beta\gamma^*}\\
8\pi p_\gamma r^2\rho(t_\alpha t_\beta s_\gamma)_a &= \{p_\gamma(p_\alpha+p_\beta-p_\gamma)[t_\alpha t_\beta + r\rho(t_\alpha/\rho)'t_\beta] + p_\alpha(-p_\alpha+p_\beta+p_\gamma)[r\rho(t_\alpha/\rho)'t_\beta + rt_\alpha t'_\beta]\}A_{\alpha\beta\gamma^*}\,.
\end{aligned}$$

7.2 The Magnetic Induction Equation

The anelastic forms of the $\mathbf{Y}$-magnetic vector potential equation, the $\mathbf{Y}$-magnetic induction equation and the compact toroidal-poloidal induction equations are obtained from (3.34) or (3.35)–(3.37), (3.42)–(3.44), (4.21) and (4.22) by replacing v_α with u_α/ρ.

There are additional terms in the anelastic toroidal-poloidal spectral-interaction forms of the induction equation. The toroidal-poloidal representations for $\mathbf{u}$ and $\mathbf{B}$ imply $\mathbf{v}\times\mathbf{B} = \mathbf{u}\times\mathbf{B}/\rho = (\mathbf{T}\{t\}+\mathbf{S}\{s\})\times(\mathbf{T}\{T\}+\mathbf{S}\{S\})/\rho = \mathbf{T}\{t/\rho\}\times(\mathbf{T}\{T\}+\mathbf{S}\{S\})+\mathbf{S}\{s\}\times\mathbf{T}\{T/\rho\}+\mathbf{S}\{s\}\times\mathbf{S}\{S\}/\rho$. The first term implies that the anelastic terms, $(t_\alpha T_\beta T_\gamma)_a$, $(t_\alpha T_\beta S_\gamma)_a$, $(t_\alpha S_\beta T_\gamma)_a$ and $(t_\alpha S_\beta S_\gamma)_a$, are obtained from the corresponding Boussinesq terms by replacing t_α by t_α/ρ. The second term implies that $(s_\alpha T_\beta T_\gamma)_a = (s_\alpha T_\beta/\rho T_\gamma) = (s_\alpha T_\beta T_\gamma)/\rho$ and $(s_\alpha T_\beta S_\gamma)_a = (s_\alpha T_\beta/\rho S_\gamma) = (s_\alpha T_\beta S_\gamma)/\rho$. Consistent with these results is equation (4.21), which implies that the anelastic spectral-interaction form of the poloidal equation is simply (4.21) with the right side divided by ρ. Thus $(s_\alpha S_\beta S_\gamma)_a = (s_\alpha S_\beta S_\gamma)/\rho$. The remaining interaction contains an additional term,

$$(s_\alpha S_\beta T_\gamma)_a = (s_\alpha S_\beta T_\gamma)/\rho - (1/\rho)'\{p_\alpha s_\alpha(rS_\beta)' + p_\beta S_\beta(rs_\alpha)'\}E_{\alpha\beta\gamma^*}/4\pi r^2 p_\gamma\,.$$

7.3 The Heat Equation

The anelastic form of the viscous dissipation term in (1.3) is

$$Q_\nu = \tfrac{1}{2}\rho\nu\sum_{i,j}(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v})^2 = \rho\nu\{\nabla^2\mathbf{v}^2 - 2\mathbf{v}\cdot\nabla^2\mathbf{v} - \boldsymbol{\omega}^2 - \tfrac{2}{3}(\nabla\cdot\mathbf{v})^2\} \tag{7.9}$$

$$= \nu/\rho\{\nabla^2\mathbf{u}^2 - 2\mathbf{u}\cdot\nabla^2\mathbf{u} - \rho^2\boldsymbol{\omega}^2\} - 2\nu R'\{\partial_r(\mathbf{u}^2/\rho) - \tfrac{1}{3}u_r^2(1/\rho)'\}\,. \tag{7.10}$$

Thus from (7.9) the vector spherical harmonic form of the heat equation (3.49) becomes

$$\rho c_p\left(\frac{\partial}{\partial\tau} - \kappa D_\gamma\right)\Theta_\gamma = Q_\gamma + \sum_{\alpha,\beta}\{-c_p u_\alpha q_\beta + \rho\nu[2\partial_r(u_\alpha/\rho)\partial_r(u_\beta/\rho) + (p_{1\alpha}+p_{1\beta}-p_\gamma)u_\alpha u_\beta/(r^2\rho^2) - \omega_\alpha\omega_\beta]$$
$$+ J_\alpha J_\beta/\sigma\}(\mathbf{Y}_\alpha\cdot\mathbf{Y}_\beta, Y_\gamma) - \tfrac{2}{3}\rho\nu\{(1/\rho)'\}^2\sum_{\alpha,\beta}u_{r\alpha}u_{r\beta}(Y_\alpha Y_\beta, Y_\gamma)\,. \tag{7.11}$$

The spectral-interaction form of (7.11) is (5.12) with the following anelastic interaction expansions in place of the corresponding incompressible interactions,

$$\begin{aligned}
8\pi r^2\rho^2(s_\alpha s_\beta\Theta_\gamma)_a &= \{2p_\alpha p_\beta[r^2 D_\gamma(s_\alpha s_\beta/r^2) - 2s_\alpha(D_\beta s_\beta) - 2r^2\rho R'(s_\alpha s_\beta/r^2\rho)' - \tfrac{1}{3}(R')^2 s_\alpha s_\beta]\\
&\quad + (p_\alpha+p_\beta-p_\gamma)[r^2 D_\gamma((rs_\alpha)'(rs_\beta)'/r^2) - 2(rs_\alpha)'(rD_\beta s_\beta)'\\
&\quad - 2r^2\rho R'((rs_\alpha)'(rs_\beta)'/r^2\rho)' - r^2(D_\alpha s_\alpha - R'\partial_r(rs_\alpha)/r)(D_\beta s_\beta - R'\partial_r(rs_\beta)/r)]\}A_{\alpha\beta\gamma^*}\\
2\pi r\rho^2(s_\alpha t_\beta\Theta_\gamma)_a &= \{rD_\gamma[(rs_\alpha)'t_\beta/r] - (rD_\alpha s_\alpha)'t_\beta - (rs_\alpha)'(D_\beta t_\beta) - 2r\rho R'[(rs_\alpha)'t_\beta/r\rho]'
\end{aligned}$$

$$- \rho[D_\alpha s_\alpha - R'\partial_r(rs_\alpha)/r](rt_\beta/\rho)'\}E_{\alpha\beta\gamma^*}$$

$$8\pi\rho^2(t_\alpha t_\beta \Theta_\gamma)_a = \{-2p_\alpha p_\beta t_\alpha t_\beta/r^2 + (p_\alpha + p_\beta - p_\gamma)[D_\gamma(t_\alpha t_\beta) - 2t_\alpha(D_\beta t_\beta) - \rho^2(rt_\alpha/\rho)'(rt_\beta/\rho)'/r^2 - 2\rho R'(t_\alpha t_\beta/\rho)']\}A_{\alpha\beta\gamma^*}$$

and $(s_\alpha \Theta_\beta \Theta_\gamma)_a = (s_\alpha \Theta_\beta \Theta_\gamma)/\rho$, $(t_\alpha \Theta_\beta \Theta_\gamma)_a = (t_\alpha \Theta_\beta \Theta_\gamma)/\rho$. These follow from (7.10) by replacing t_α by t_α/ρ and $D_\alpha s_\alpha$ by $(D_\alpha s_\alpha)/\rho + (1/\rho)'\partial_r(rs_\alpha)/r$ in the incompressible ω^2 interactions of (5.12), and leaving the other terms unchanged.

8 Conclusions

The most important results of the paper in terms of applications are the hybrid toroidal-poloidal spectral forms (6.2), (6.3), (6.11), (6.12) and (6.15) of the linearised momentum, magnetic induction and heat equations, and the anelastic extensions of §7. There are effectively only three distinct coupling integrals, $(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)$, $(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)$ and $(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma)$. Although the third is simply related to the second by (3.51), it is simpler to implement the coupling integrals separately. There are only six types of radial expression containing derivatives, which occur in (6.4)–(6.9), (6.13)–(6.14) and (6.16)–(6.18), namely $f_0\partial_\gamma f'$, $\partial^\gamma(f_0 f')$, $\partial^\gamma(f_0\partial_\beta f')$, $f_0 D_\gamma f'$, $\partial^\gamma(f_0 D_\beta f')$ and $D_\gamma D_\gamma f'$. These include $f_0\partial_r f'$, $f_0\partial_r\partial_\beta f'$. Here f_0 indicates a known radial expression of the basic state and f' denotes a coefficient of one of the scalar perturbation fields s', t', S', T', Θ'. The hybrid spectral equations are in general much easier and less error prone to implement numerically than the toroidal-poloidal spectral interaction form of the equations due the greatly reduced number of terms over the linearised spectral interaction equations. Only three subroutines, one for each coupling integral, are needed to implement the angular dependence of all 12 product terms. Further, even though there are 29 different radial expressions, radial discretisation can be accomplished by a single subroutine, which discretises just the six distinct types of radial expression. These equations have been numerically implemented in Ivers and Phillips (2003) for an axisymmetric basic state using second-order finite-differences for the radial discretisation.

The other main results of the paper are the $\mathbf{Y}$-vector spherical harmonic momentum equation (3.12), magnetic induction equation (3.42)–(3.44), magnetic vector potential equation (3.34) and heat equation (3.49), the compact toroidal-poloidal momentum equation (4.7), (4.13) and magnetic induction equation (4.21), (4.22).

The hybrid spectral equations derived herein can be used with viscous and thermal anisotropic diffusion. Spectral expansions have been derived in Phillips and Ivers (2000) for general diffusion tensors. Toroidal-poloidal spectral-interaction expansions have been derived for special cases of rapidly-rotating and strong field anisotropic diffusion Phillips and Ivers (2001, 2003), which together with the spectral-interaction expansions of other linear terms derived herein, are useful in time-stepping dynamically consistent dynamo codes (Ivers 2003).

Acknowledgements

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