1.2 Local Asymptotics of a Painlevé equation

The first Painlevé equation

$$\mathrm{P_I}: \quad y'' = 6\,y^2 + t, \quad y = y(t) \tag{1.13}$$

or in system form, with $y_1 = y$, $y_2 = y'$:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ 6\,y_1^2 + t \end{pmatrix} \tag{1.14}$$

has a t-dependent Hamiltonian

$$H = \frac{1}{2}y_2^2 - 2\,y_1^3 - t\,y_1. \tag{1.15}$$

Exercise 1.2.1. *Show that the system* (1.14) *is equivalent to Hamiltonian's equations of motion*

$$\begin{aligned} \frac{dy_1}{dt} &= \frac{\partial H}{\partial y_2} \\ \frac{dy_2}{dt} &= -\frac{\partial H}{\partial y_1} \end{aligned}$$

with H defined by Equation (1.15).

To study the solutions of $\mathrm{P_I}$ as $|x| \to \infty$, we transform it to Boutroux's coordinates

$$y_1 = t^{1/2}\,u_1(z) \tag{1.16a}$$

$$y_2 = t^{3/4}\,u_2(z) \tag{1.16b}$$

$$z = \frac{4}{5}\,t^{5/4} \tag{1.16c}$$

This transformation converts (1.14) to

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6\,u_1^2 + 1 \end{pmatrix} - \frac{1}{(5z)}\begin{pmatrix} 2\,u_1 \\ 3\,u_2 \end{pmatrix} \tag{1.17}$$

The Hamiltonian H is now replaced by

$$E := \frac{u_2^2}{2} - 2\,u_1^3 - u_1 \tag{1.18}$$

Exercise 1.2.2. *Show that*

$$\frac{dE}{dz} = -\frac{1}{(5z)}\,(6\,E + 4\,u_1)\,. \tag{1.19}$$

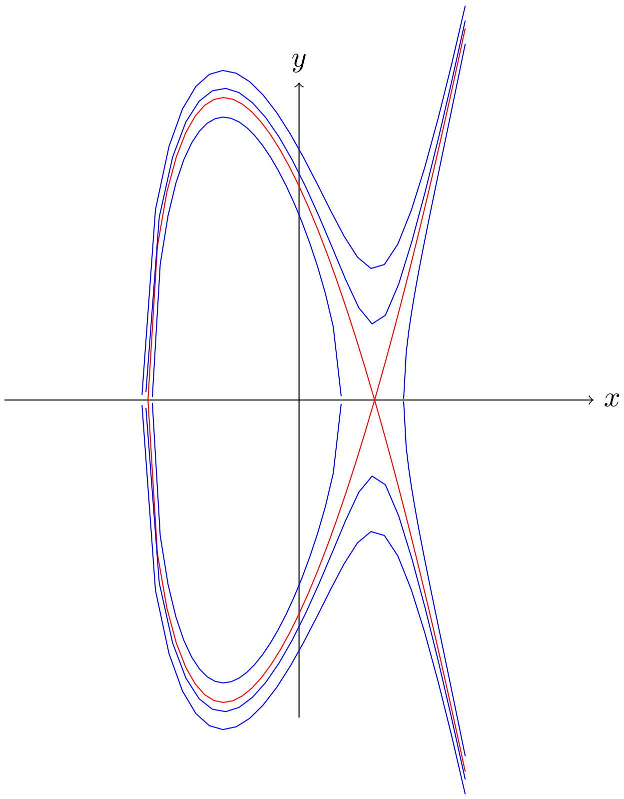

Figure 1.6: Level curves of $f = y^2 - 4\,x^3 + 12\,x$

This result implies that E approaches a constant value as $|z| \to \infty$. But, it is important to remember that this may not be a uniform limit for all rays of approach to infinity.

By definition, the values of E provide level curves of the cubic function

$$f(x, y) = y^2 - 4\,x^3 + g_2\,x \tag{1.20}$$

for $g_2 = -2$.

Example 1.2.1. *Consider $g_2 = 12$ and $c \geq 0$, then the level curves $f = c$ are displayed in Figure 1.6.*

Consider an initial point $z_0 > 0$, where (finite) initial values of E and u_1 are given, say $E = E_0$, $u_1 = u_0$. Then the standard theorems of existence and uniqueness show that there exists a unique solution of the system (1.17). (Usually, these standard theorems require initial values of u_1, u_2 to be given. However, by the implicit value theorem, we assume u_2 is given by E_0.)

The initial value E_0 picks out one of the level curves of f described above and the initial value u_1 provides a starting point on such a curve. As z

approaches infinity, Equation (1.19) shows that E varies slowly and so the solution moves on a slowly varying family of such level curves.

Suppose $E = E_0$ stayed constant as $z \to \infty$. The product of this level curve and the z-axis would be a complex cylinder. Since for $\mathrm{P_I}$, E changes slowly, we can visualise the change in the solution as though we are following a path on a cylinder that slowly distorts as z changes.

The continuation of such a solution becomes problematic at points where these level curves intersect, because the analytic continuation of the solution is no longer well defined in $\mathbb{C}$. Such intersection points are called *base points* and the question of how to separate the curves in order to understand how to continue the parametrizing function is completely understood in the theory of complex algebraic curves. In the next chapter, we show how to overcome this problem by using techniques from the theory of resolution of singularities.