GEOMETRY AND ASYMPTOTICS

by

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Chapter 1

Algebraic Curves

1.1 Motivation

Given a constant parameter g_2, consider the ordinary differential equation (ODE)

$$w'' = 6\,w^2 - \frac{g_2}{2}, \tag{1.1}$$

where w is a function of $t \in \mathbb{C}$ and primes denote derivatives with respect to t.

Multiplying Equation (1.1) by w' and integrating once, we obtain

$$w'^2 = 4\,w^3 - g_2\,w - g_3 \tag{1.2}$$

where g_3 is another constant parameter. Integrating once more, by separation of variables, we obtain the well known solutions:

$$w(t) = \wp\big(t - t_0; g_2, g_3\big) \tag{1.3}$$

which are functions of two arbitrary parameters t_0 and g_3.

Here, $\wp$ is the Weierstrass elliptic function, a doubly periodic, meromorphic function of order 2, which has a double pole at the origin. The equivalent notation $\wp(t) = \wp\big(t; g_2, g_3\big)$ is often used for conciseness, when the dependence on g_2 and g_3 is assumed. Below, we use the fact that it is an even function, i.e., $\wp(-t) = \wp(t)$. (For further information, see a reference on the theory of analytic functions of one complex variable, such as Ahlfors [1].)

Equation (1.2) defines a curve

$$y^2 = 4\,x^3 - g_2\,x - g_3 \tag{1.4}$$

called an elliptic curve (or Weierstrass' cubic curve), which is parameterised by

$$x = w(t), \quad y = w'(t),$$

where $w(t)$ is given by Equation (1.3).

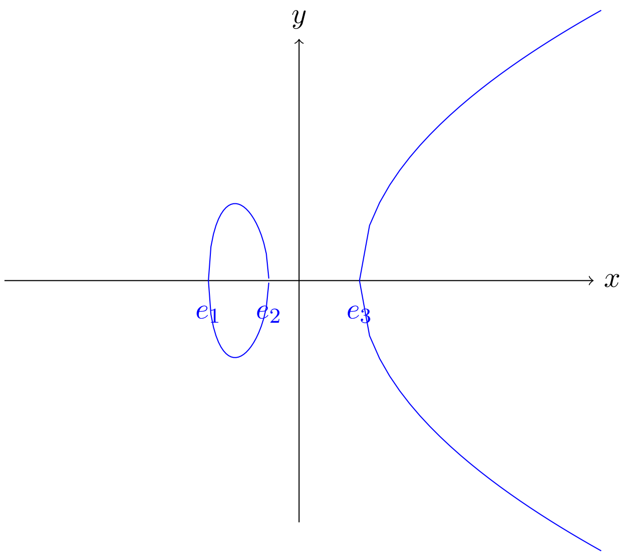

Figure 1.1: Weierstrass cubic curve

Let the roots of the cubic on the right of (1.4) be e_1, e_2, e_3. If they are real, assume without loss of generality that $e_1 \le e_2 \le e_3$. In the real case, the graph of y as a function of x, given by (1.4) for generic values of e_i, $i \in \{1, 2, 3\}$, is shown in Figure 1.1.

But solutions of the ODE (1.1) vary as its accompanying initial data vary. Such initial data determine the values of g_3 and t_0, i.e., the values of e_1, e_2, e_3 and a starting point on the corresponding curve, such as the one in Figure 1.1. The values of g_3 give a family of level curves of the polynomial

$$f(x, y) = y^2 - 4\,x^3 + g_2\,x \tag{1.5}$$

The collection of corresponding curves, a subset of which is depicted in Figure 1.2, is called a *pencil* of curves.

As g_3 varies, two of the roots e_1, e_2, e_3 may coincide. An example is given below.

Example 1.1.1. *Take $g_2 = 2$, $g_3 = -\big(2/3\big)^{3/2}$ and transform variables in Equation (1.4) to*

$$x = \frac{\xi}{\sqrt{6}}, \quad y = \left(\frac{2}{3^3}\right)^{1/4} \eta$$

Then the curve becomes

$$\eta^2 = (\xi - 1)^2\,(\xi + 2)$$

whose graph is depicted in Figure 1.3.

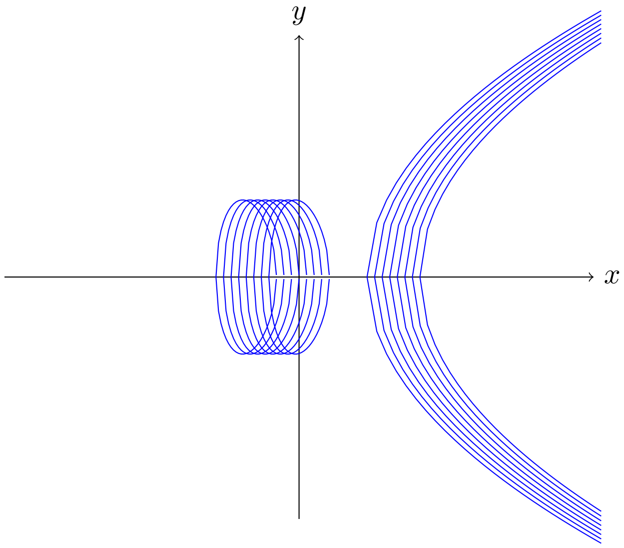

Figure 1.2: A pencil of Weierstrass cubic curves

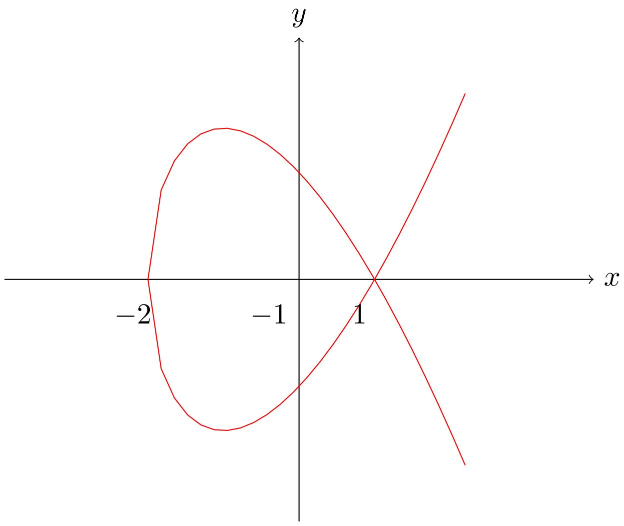

Figure 1.3: Singular Weierstrass cubic curve

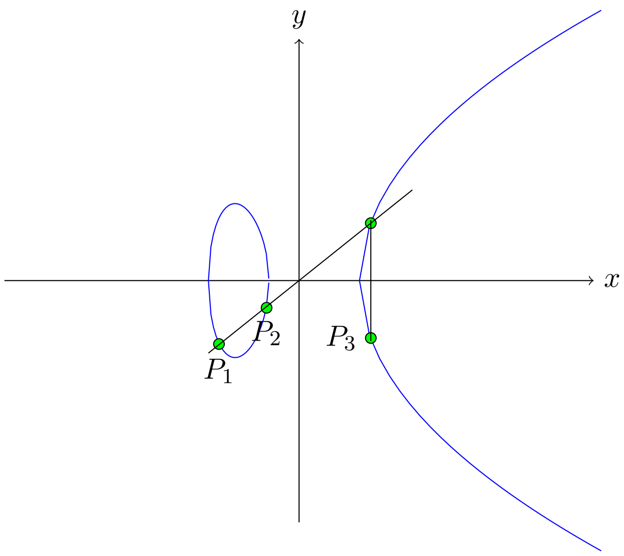

Figure 1.4: Addition on Weierstrass cubic curve

The Weierstrass elliptic function $\wp(t-t_0)$ parametrizes the curve (1.4) as a function of a continuous variable t. But, there is also a discrete mapping that parametrizes this curve as a function of a discrete variable n. Geometrically, this mapping is given by taking two distinct points P_1 and P_2 on the curve and finding a third point P_3 also on the curve constructed as follows.

Take the straight line passing through P_1 and P_2. (We assume below that the x coordinates of these points are distinct[1].) As we show below, this line must intersect with the curve again. Take the resulting point of intersection and reflect this point across the x-axis to obtain P_3. This construction is depicted graphically in Figure 1.4.

We provide an analytic proof here that the image of this mapping can be expressed rationally in terms of the coordinates of P_1 and P_2. Let $2\,\omega_1$ and $2\,\omega_2$ be the (smallest) periods of $\wp(t)$. (By the definition of $\wp(t)$, ω_1 and $i\,\omega_2$ are real.) Denote the fundamental period parallelogram with vertices at the origin, $2\,\omega_1$, $2\,\omega_2$ and $2\,(\omega_1+\omega_2)$ by Π. The integer linear combinations of $2\,\omega_1$ and $2\,\omega_2$ generate a lattice L in the complex plane. A typical such L and Π is drawn in Figure 1.5.

Choose $t_1,\, t_2 \in \mathbb{C}$ but not in L and assume $t_1 \neq t_2 \mod L$. Let $a,\, b \in \mathbb{C}$

[1] P_3 can also be constructed when P_1 and P_2 have the same x-coordinate. But, in this case, the line containing these points is vertical and P_3 will lie at infinity.

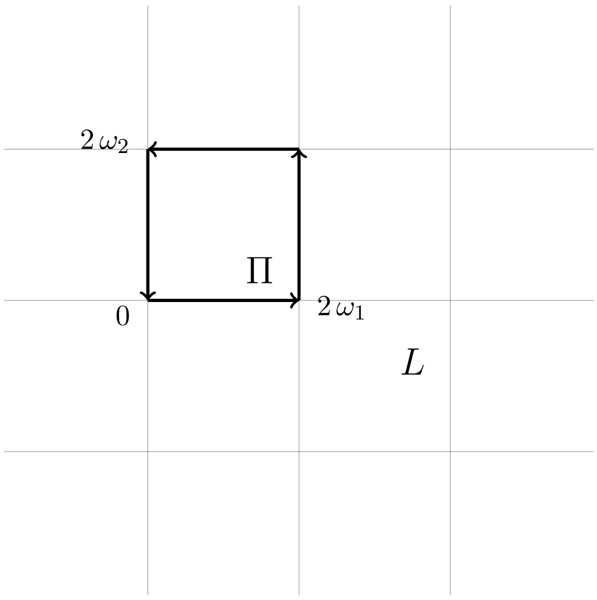

Figure 1.5: A period lattice

such that

$$\begin{aligned}\wp'(t_1) &= a\,\wp(t_1) + b\\ \wp'(t_2) &= a\,\wp(t_2) + b\end{aligned}$$

That is, $y = a\,x + b$ is the line through $P_i = \big(\wp(t_i), \wp'(t_i)\big)$, $i = 1, 2$.

For any elliptic function $F(t)$ with period lattice L and a fundamental period parallelogram Π, we have

$$\frac{1}{2\,\pi\,i}\oint_{\Pi} t\,\frac{F'(t)}{F(t)}\,dt = \sum_i (z_i - p_i) = 0$$

by Cauchy's residue theorem, where z_i and p_i are respectively zeroes and poles of F in Π. We take

$$F(t) = \wp'(t) - \,a\,\wp(t) - \,b$$

which is an elliptic function of order 3, with a triple pole at the origin. So if t_1, t_2 are zeroes of $F(t)$, then (because the pole is located at the origin), a third zero must exist at $t_3 = -\,(t_1 + t_2)$ modulo L. So we have

$$\wp'(t_3) = a\,\wp(t_3) + b.$$

Note that this shows that the straight line $y = ax + b$ must intersect the Weierstrass cubic curve (1.2) a third time.

At such an intersection between the curve given by (1.2) and the straight line $y = ax + b$, we also have

$$4\,x^3 - g_2\,x - g_3 - (a\,x + b)^2 = 0 \tag{1.6}$$

which has three roots given by $\wp(t_1)$, $\wp(t_2)$, $\wp(t_3)$. So we get

$$4\,(x-\,\wp(t_1))\,(x-\,\wp(t_2))\,(x-\,\wp(t_3))=0. \tag{1.7}$$

Comparing the coefficient of x^2 between Equations (1.6-1.7), we get

$$\wp(t_1)+\wp(t_2)+\wp(t_3)=\frac{a^2}{4} \tag{1.8}$$

But also, because a is the slope of the line through the two points $P_i = \big(\wp(t_1), \wp'(t_i)\big)$, $i = 1, 2$, we have

$$a=\frac{\wp'(t_1)-\wp'(t_2)}{\wp(t_1)-\wp(t_2)}. \tag{1.9}$$

Moreover, $\wp(t_3) = \wp\big(-(t_1+t_2)\big) = \wp(t_1+t_2)$ by the evenness of $\wp(t)$ and $b = \wp'(t_1) - a\,\wp(t_1)$. We find therefore from Equation (1.8) that

$$\wp(t_1+t_2)=-\wp(t_1)-\wp(t_2)+\frac{1}{4}\left(\frac{\wp'(t_1)-\wp'(t_2)}{\wp(t_1)-\wp(t_2)}\right)^2. \tag{1.10}$$

and

$$\begin{aligned} -\wp'(t_1+t_2) &= a\,\wp(t_1+t_2)+\wp'(t_1)-a\,\wp(t_1) \\ &= \wp'(t_1)+\frac{\wp'(t_1)-\wp'(t_2)}{\wp(t_1)-\wp(t_2)}\,(\wp(t_1+t_2)-\wp(t_1)) \end{aligned} \tag{1.11}$$

If we write $\overline{y} = \wp'(t_1+t_2)$, $y = \wp'(t_1)$, $y_0 = \wp'(t_2)$, $\overline{x} = \wp(t_1+t_2)$, $x = \wp(t_1)$, $x_0 = \wp(t_2)$, then these equations become

$$\left\{\begin{array}{ll} \overline{x} & = \frac{1}{4}\left(\dfrac{y-y_0}{x-x_0}\right)^2 - x - x_0 \\ \overline{y} & = -y-\left(\dfrac{y-y_0}{x-x_0}\right)(\overline{x}-x) \end{array}\right. \tag{1.12}$$

which provides a discrete mapping on the Weierstrass cubic curve.

Bibliography

[1] Lars V. Ahlfors, *Complex analysis. An introduction to the theory of analytic functions of one complex variable*, 3rd edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York, 1978.