When Applied Maths Collided with Algebra

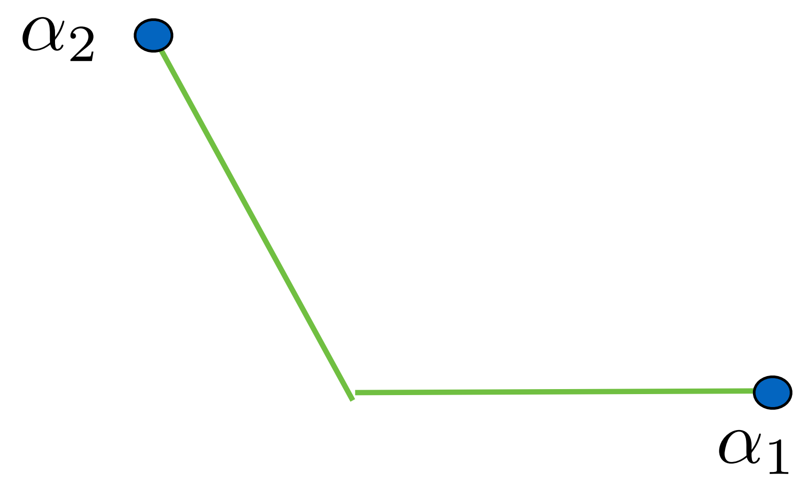
Nalini Joshi

@monsoon0

THE UNIVERSITY OF SYDNEY

Supported by the Australian Research Council

A Reflection



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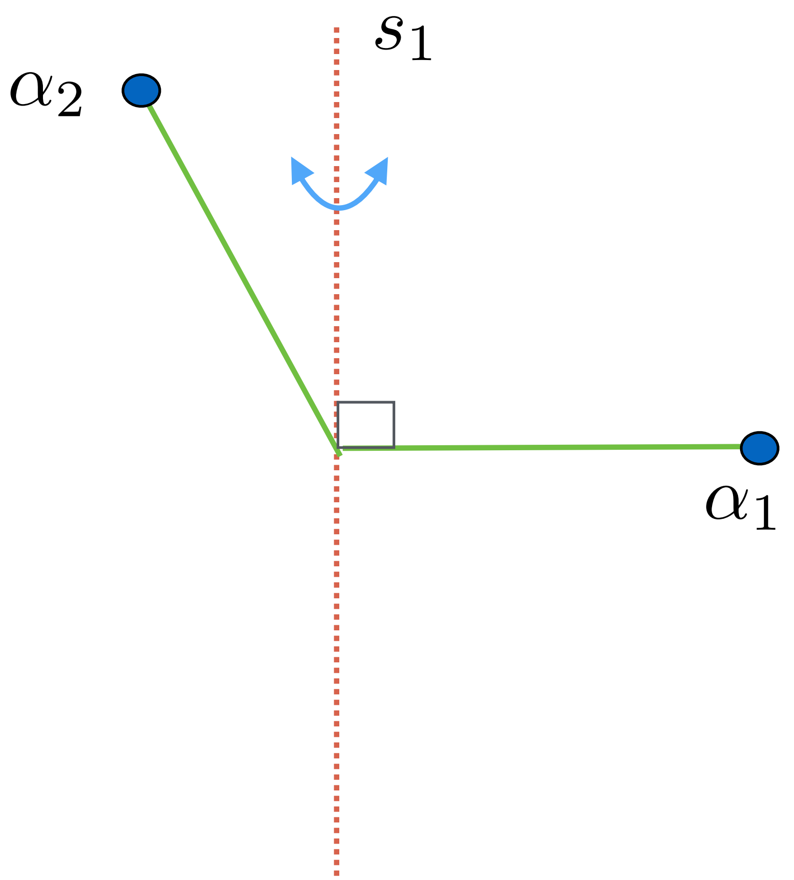

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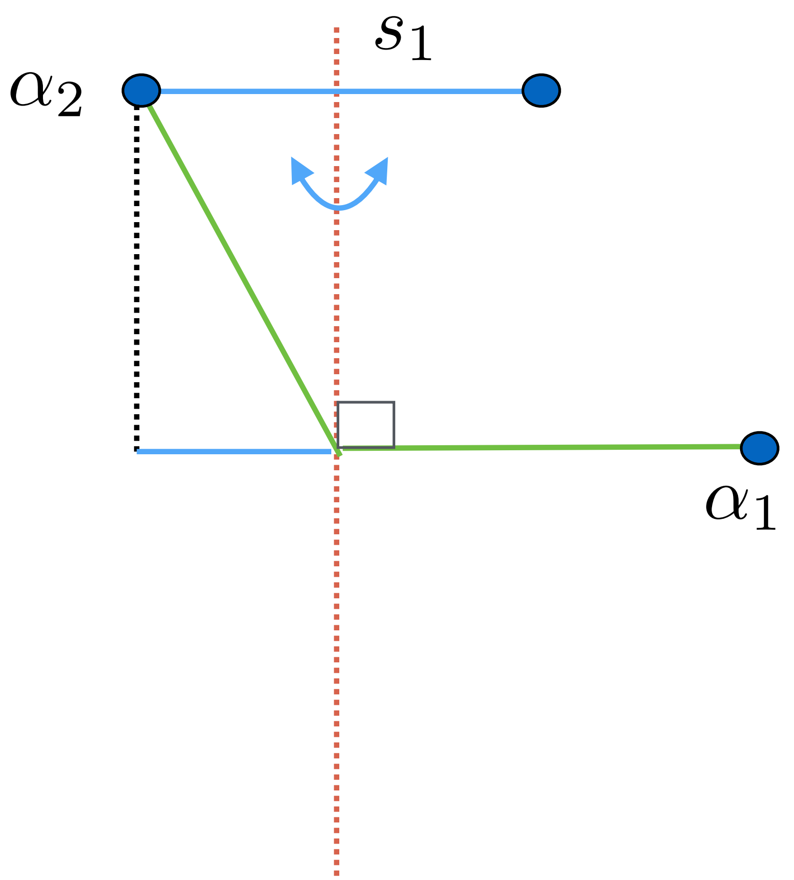

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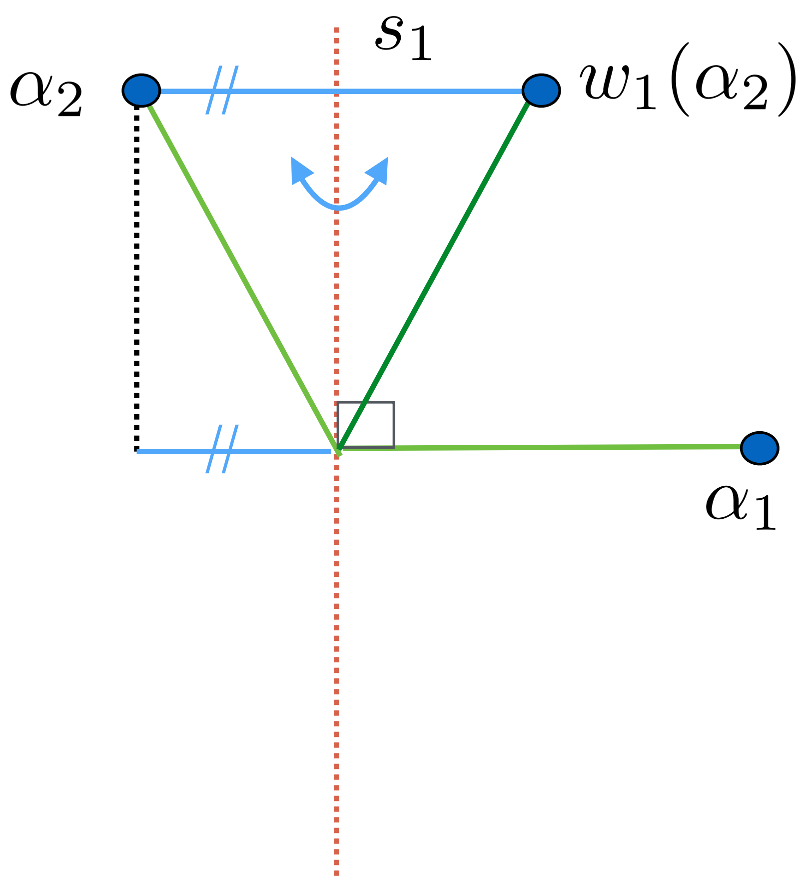

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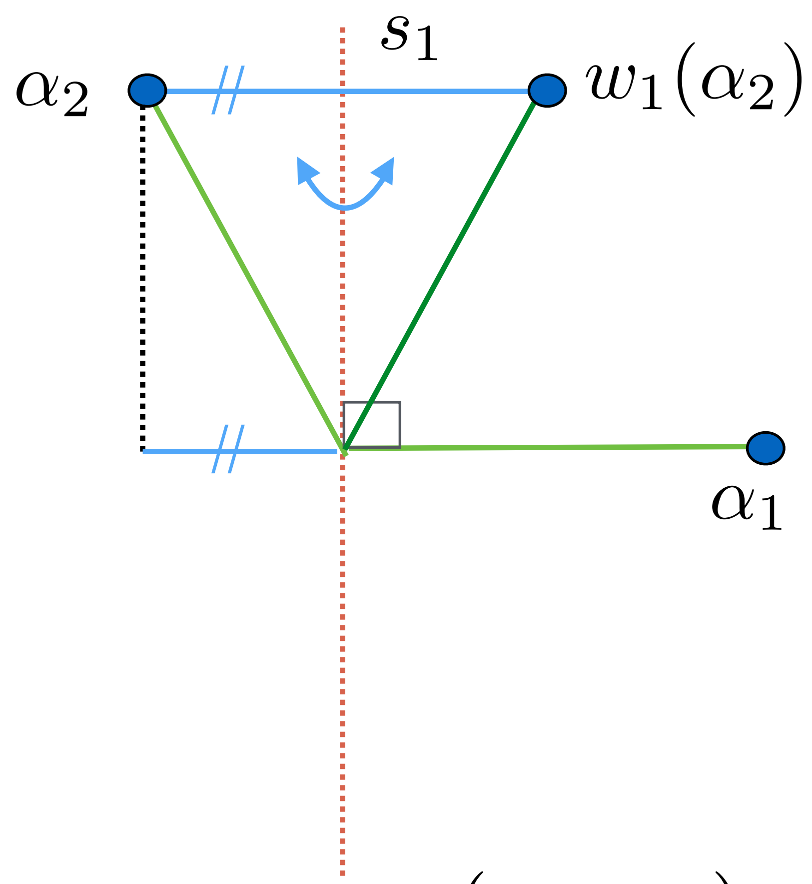
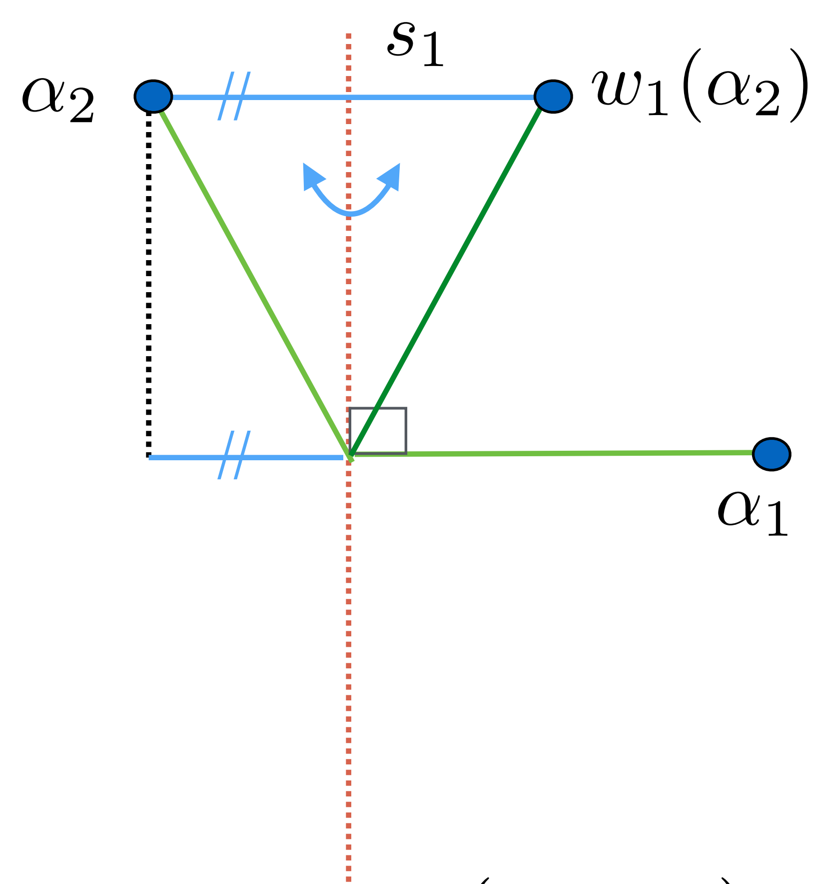


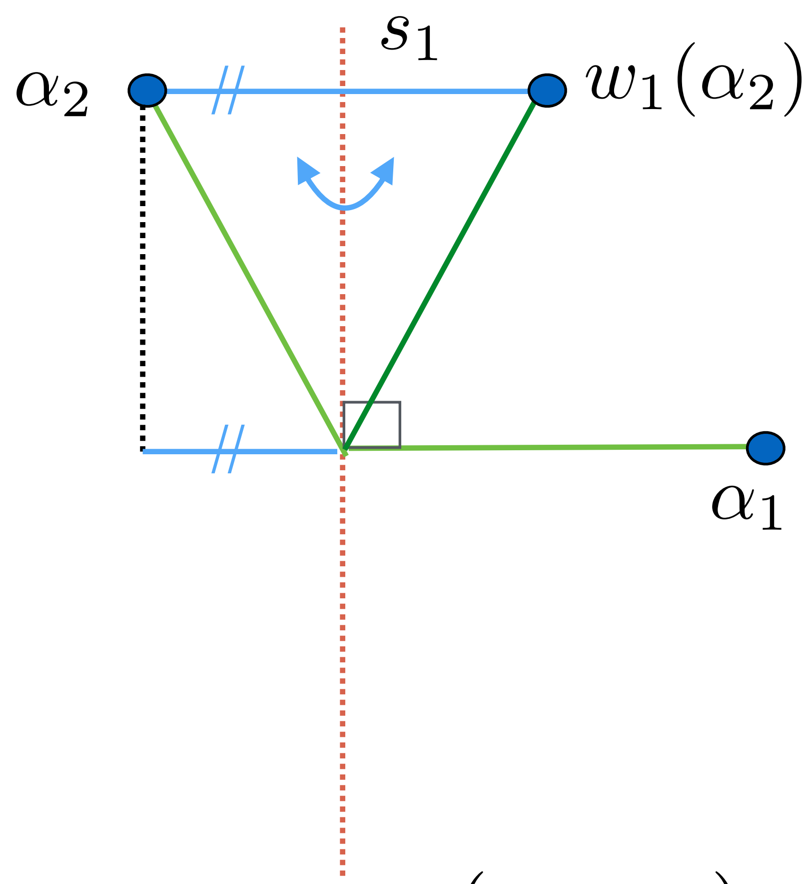
$$w_1(\alpha_2) = \alpha_2 - 2\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\,\alpha_1$$

A Reflection



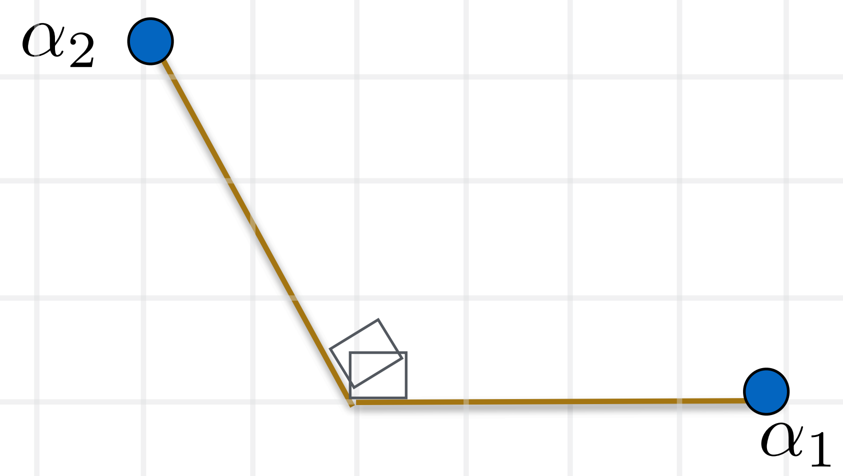
$$w_1(\alpha_2) = \alpha_2 - 2\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\alpha_1$$
$$= (-1, \sqrt{3}) + (2, 0)$$

A Reflection



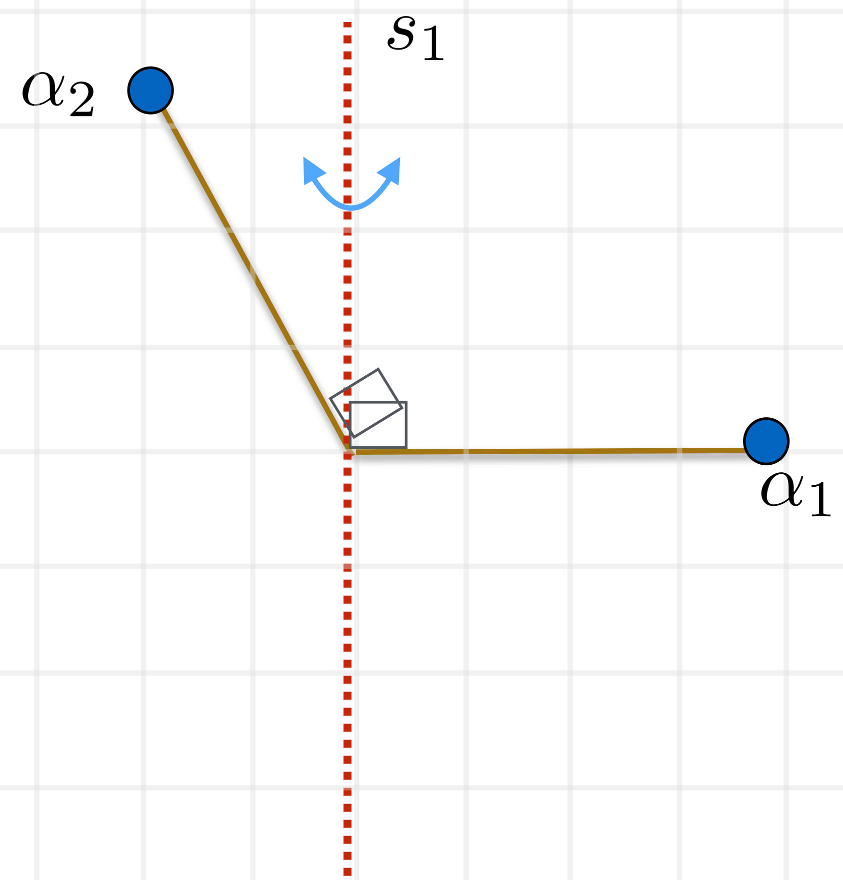
$$
\begin{aligned}
w_1(\alpha_2) &= \alpha_2 - 2\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\alpha_1 \\
&= (-1, \sqrt{3}) + (2, 0) \\
&= (1, \sqrt{3})
\end{aligned}
$$

Root System



α_1 and α_2 are “simple” roots

Root System



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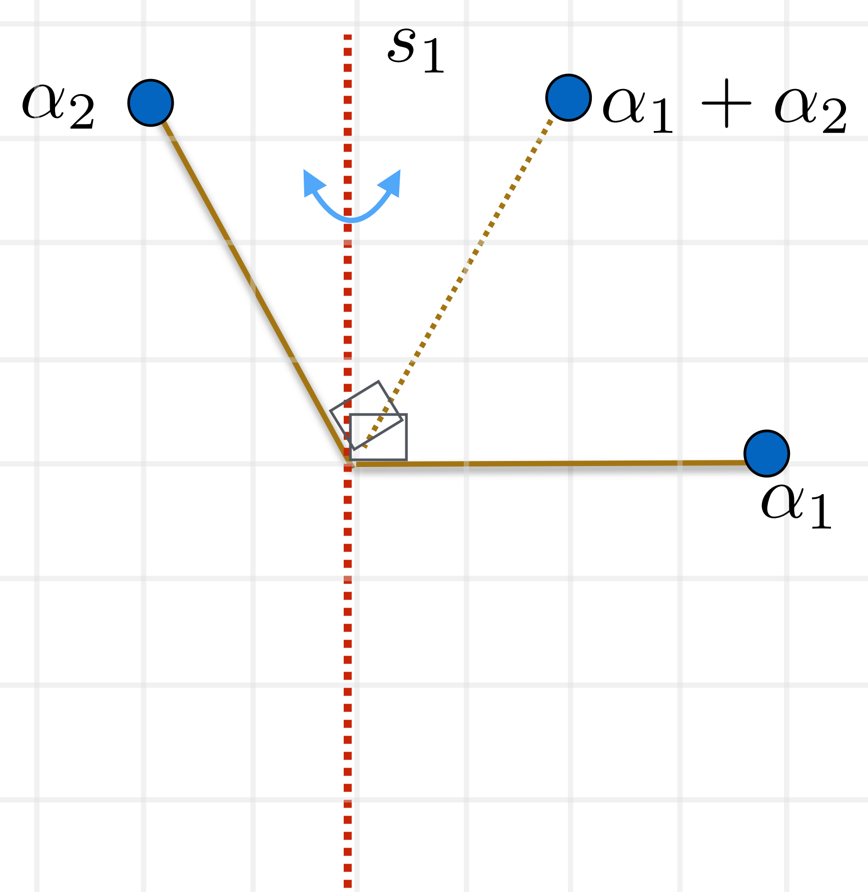
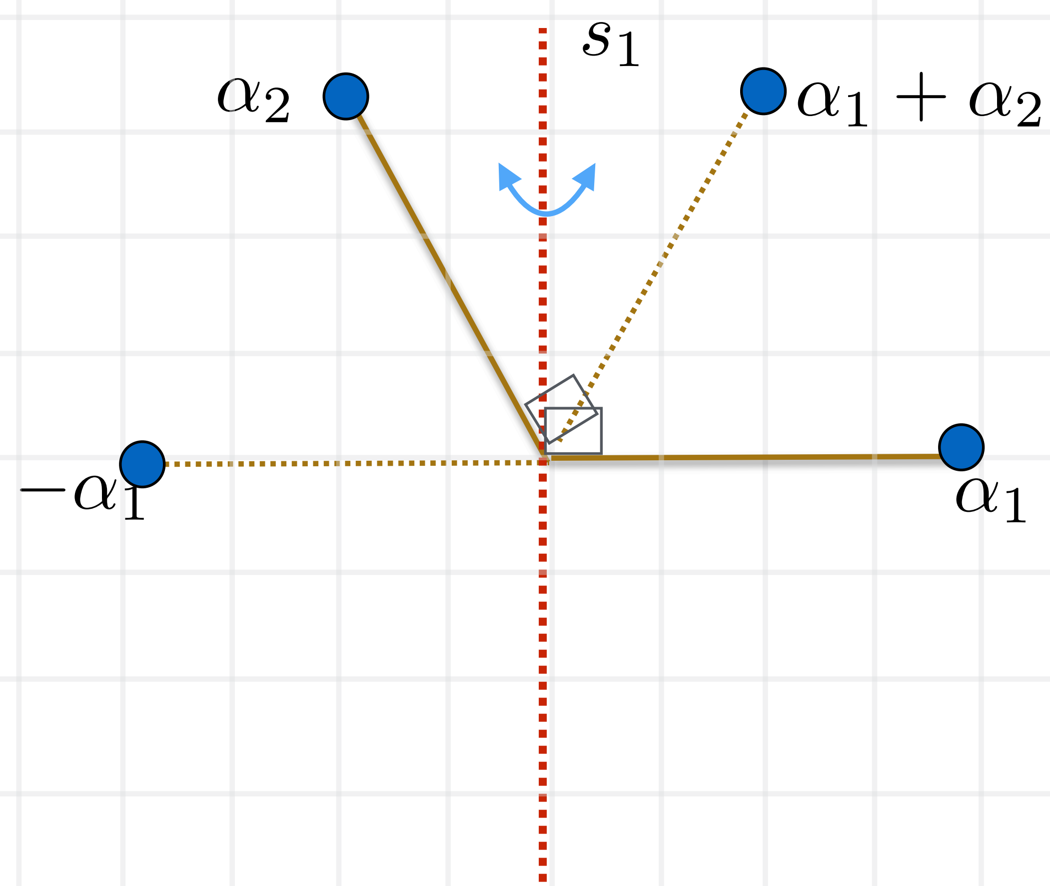


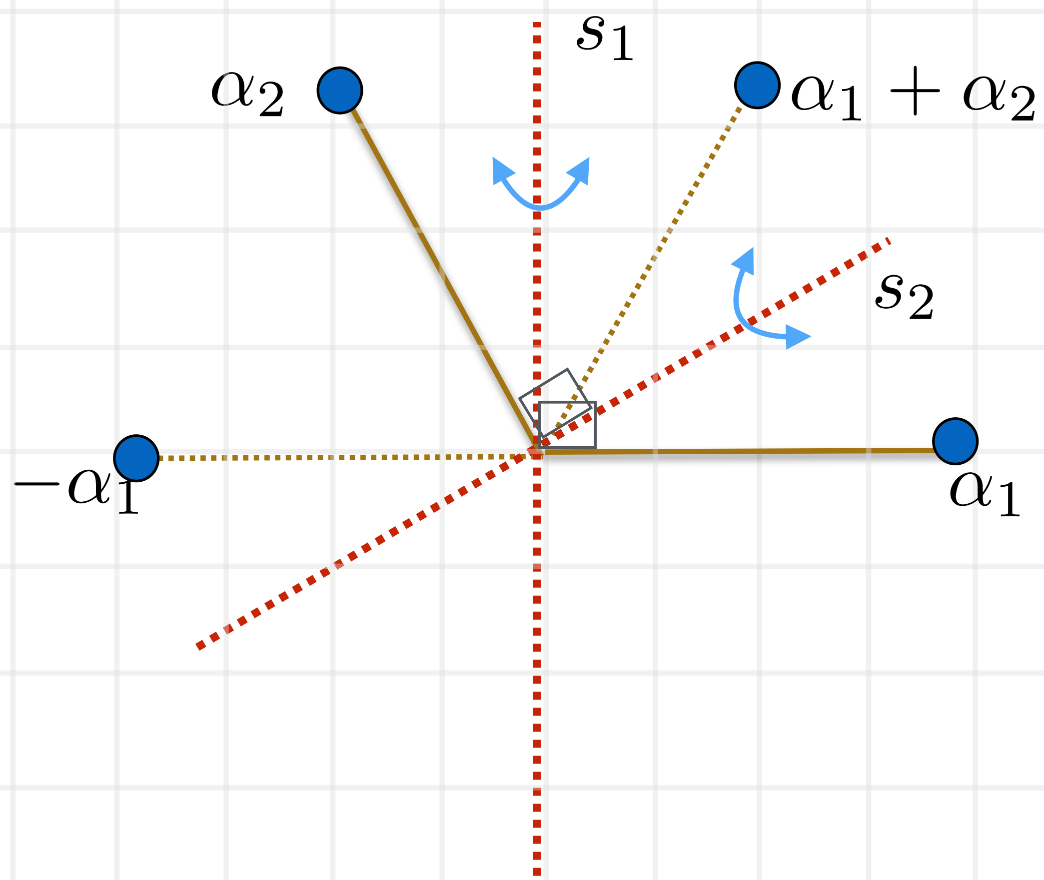
α_1 and α_2 are “simple” roots

Root System



α_1 and α_2 are "simple" roots

Root System



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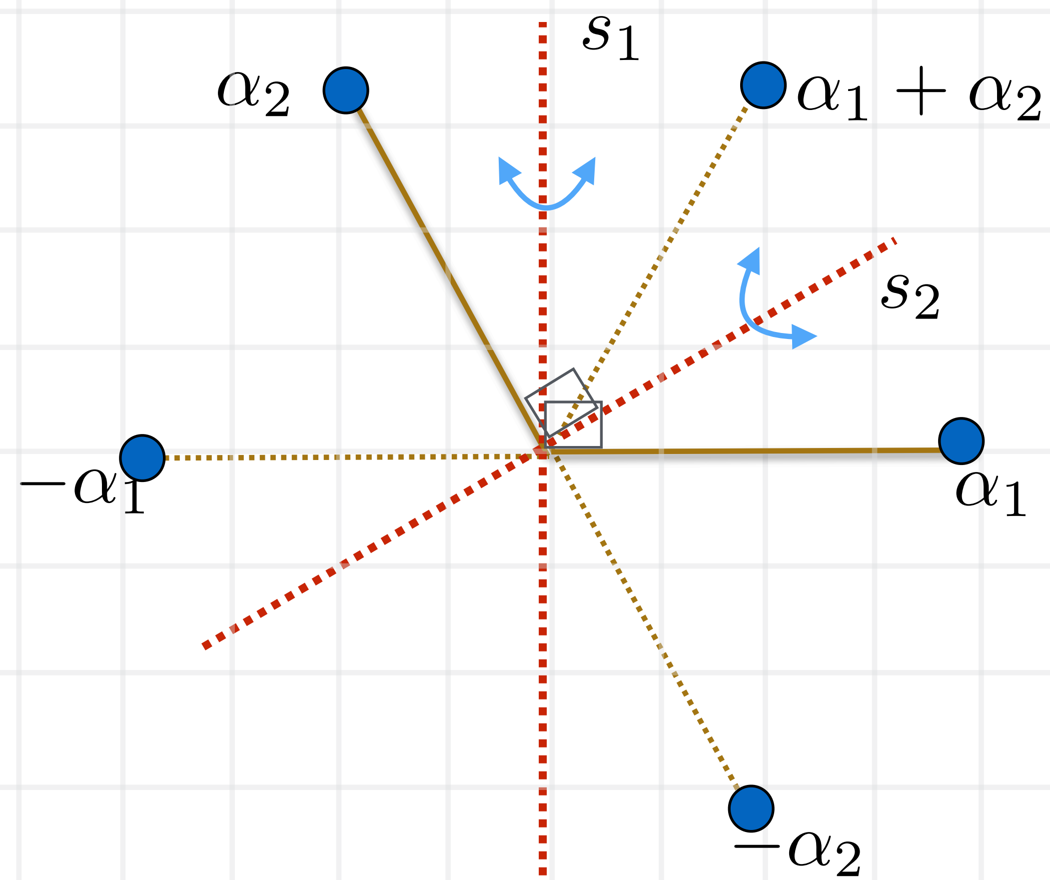
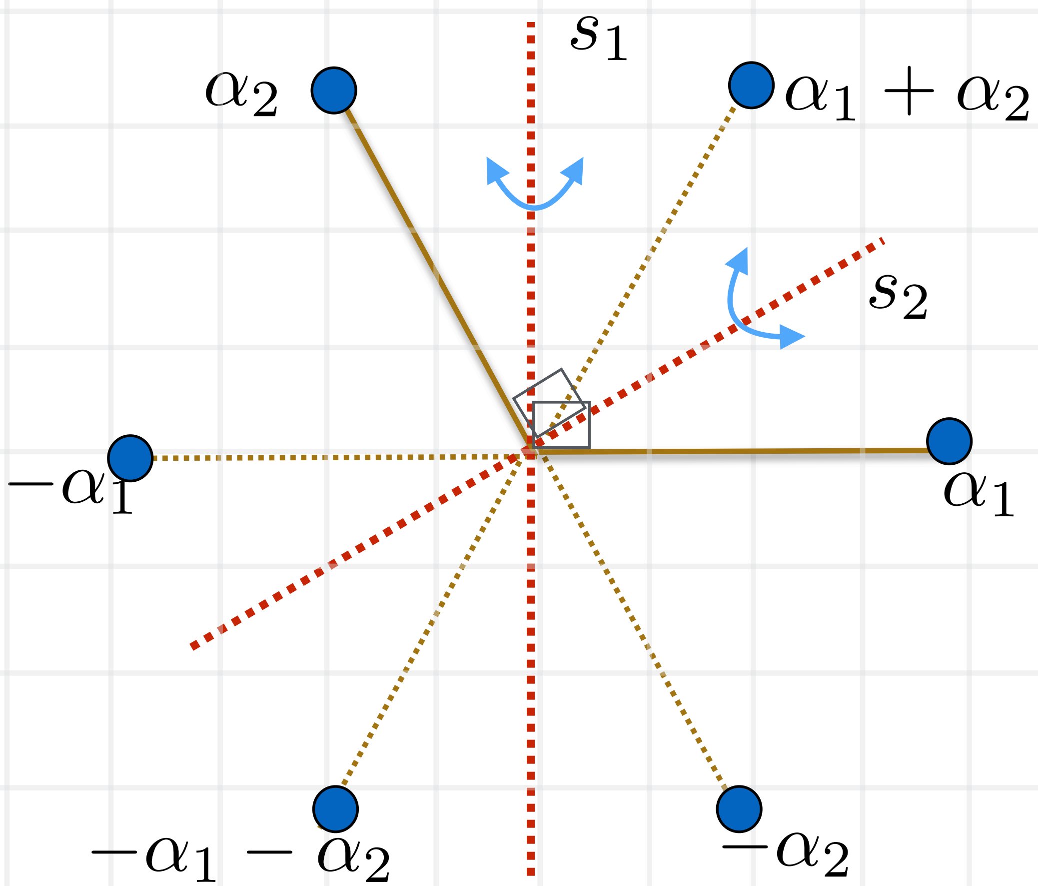
Root System



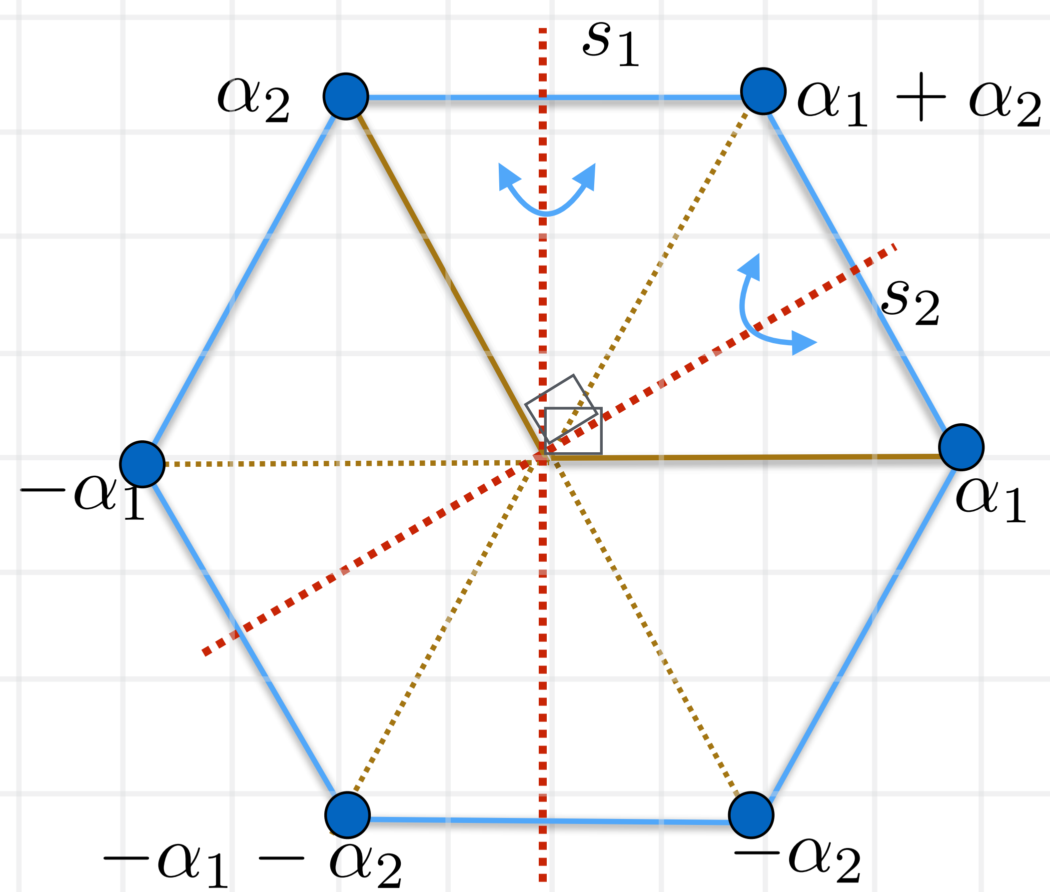
α_1 and α_2 are “simple” roots

Root System



α_1 and α_2 are “simple” roots

Root System



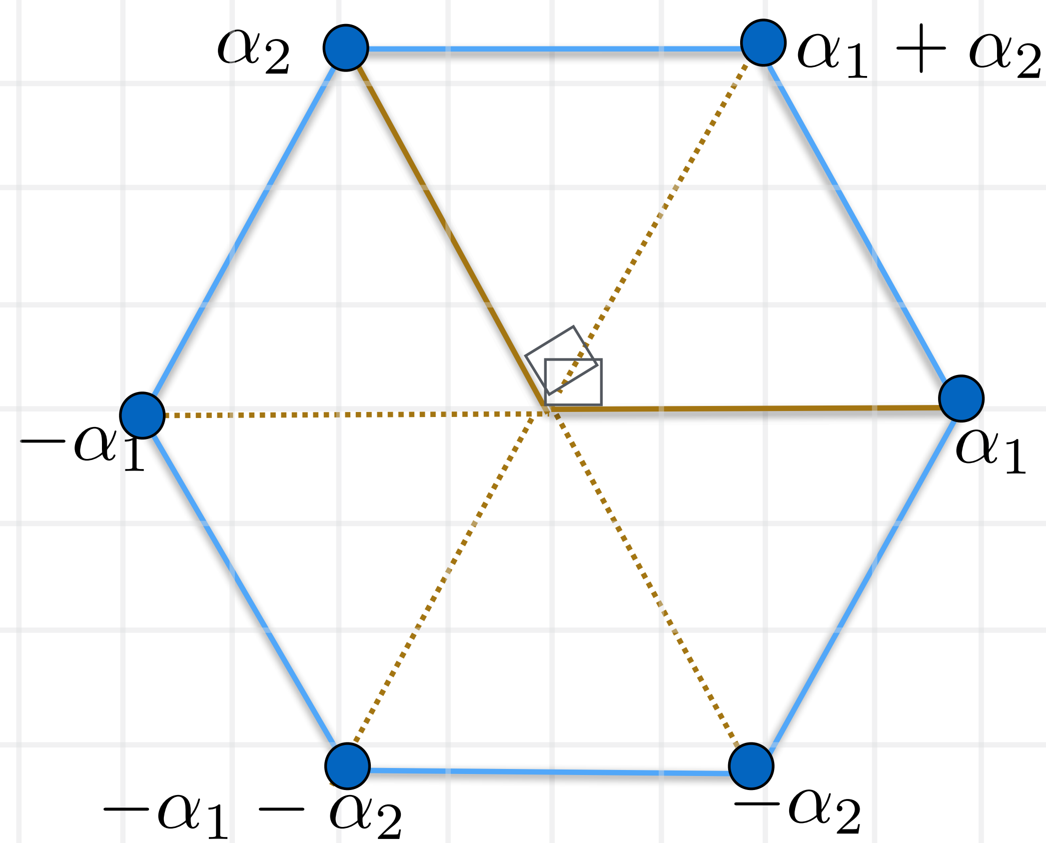
α_1 and α_2 are “simple” roots

Reflection Groups

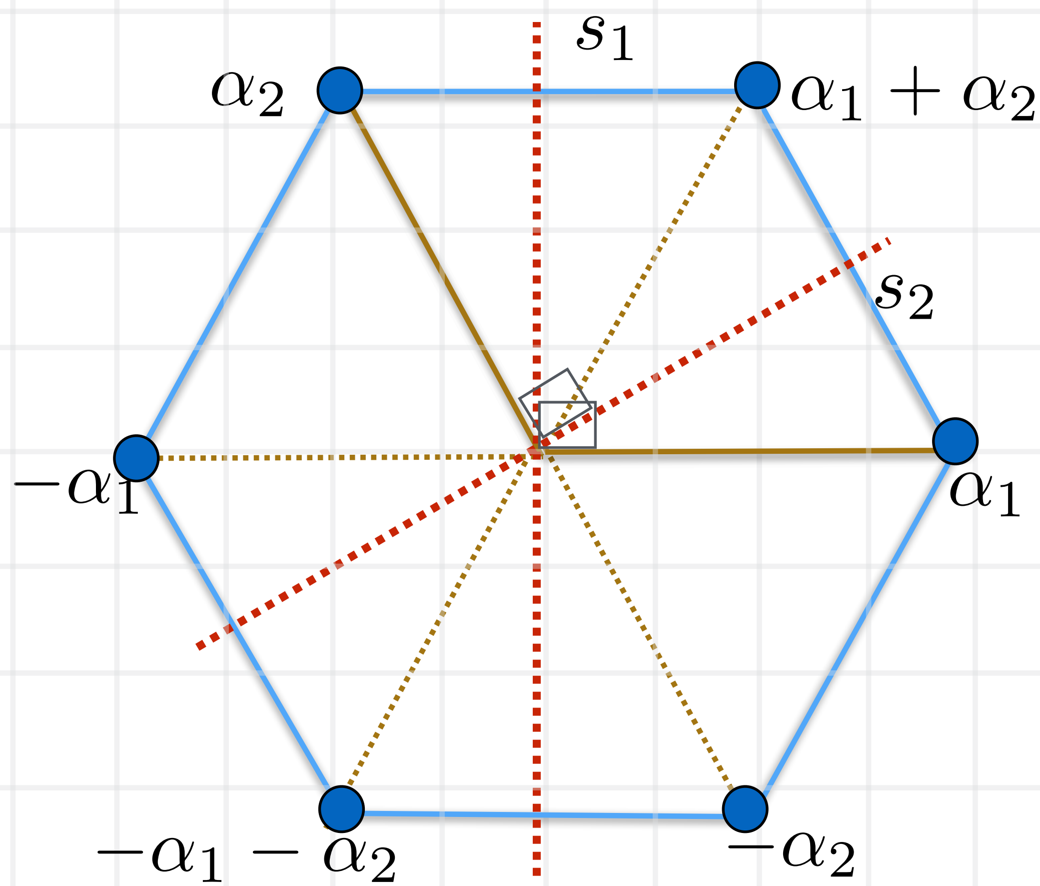
- Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$
- Reflections: $w_i(\alpha_j) = \alpha_j - \; 2\,\dfrac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\,\alpha_i$
- Co-roots: $\check{\alpha}_i = \; 2\,\dfrac{\alpha_i}{(\alpha_i, \alpha_i)}$
- Weights: $h_1, h_2, \ldots, h_n$

$$(h_i, \check{\alpha}_j) = \delta_{ij}$$

A_2

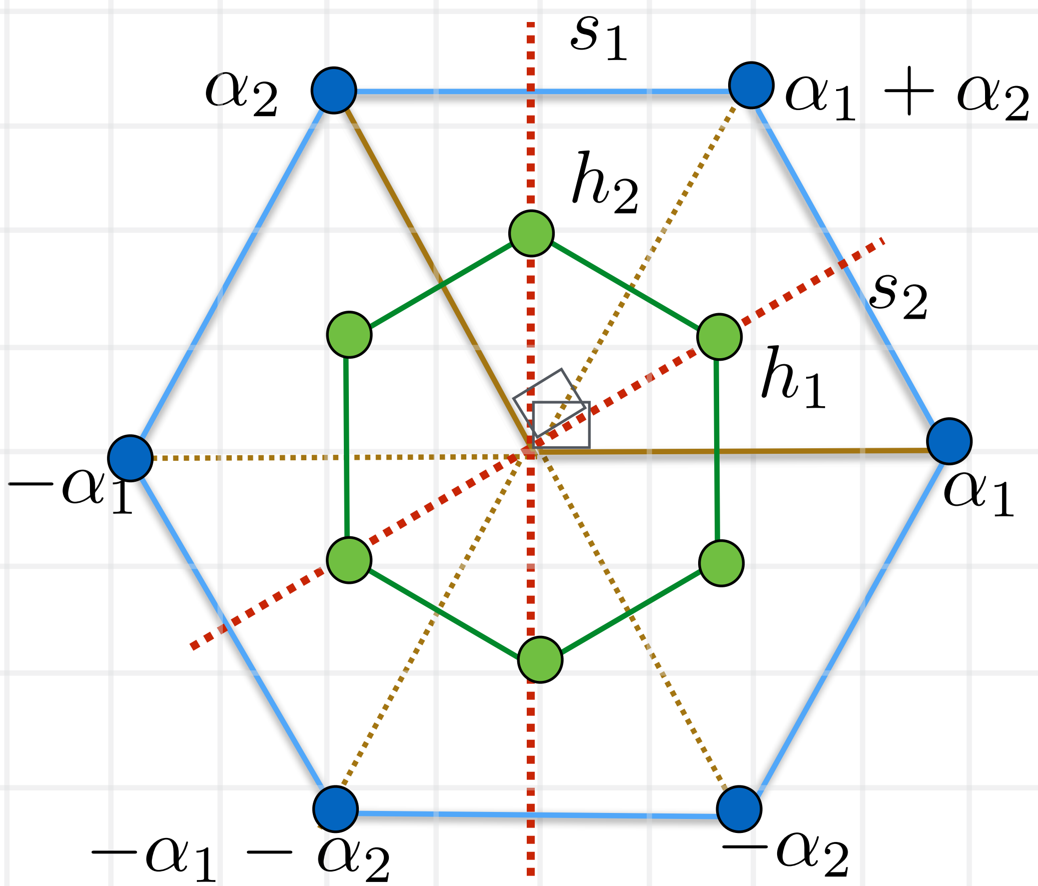


A_2



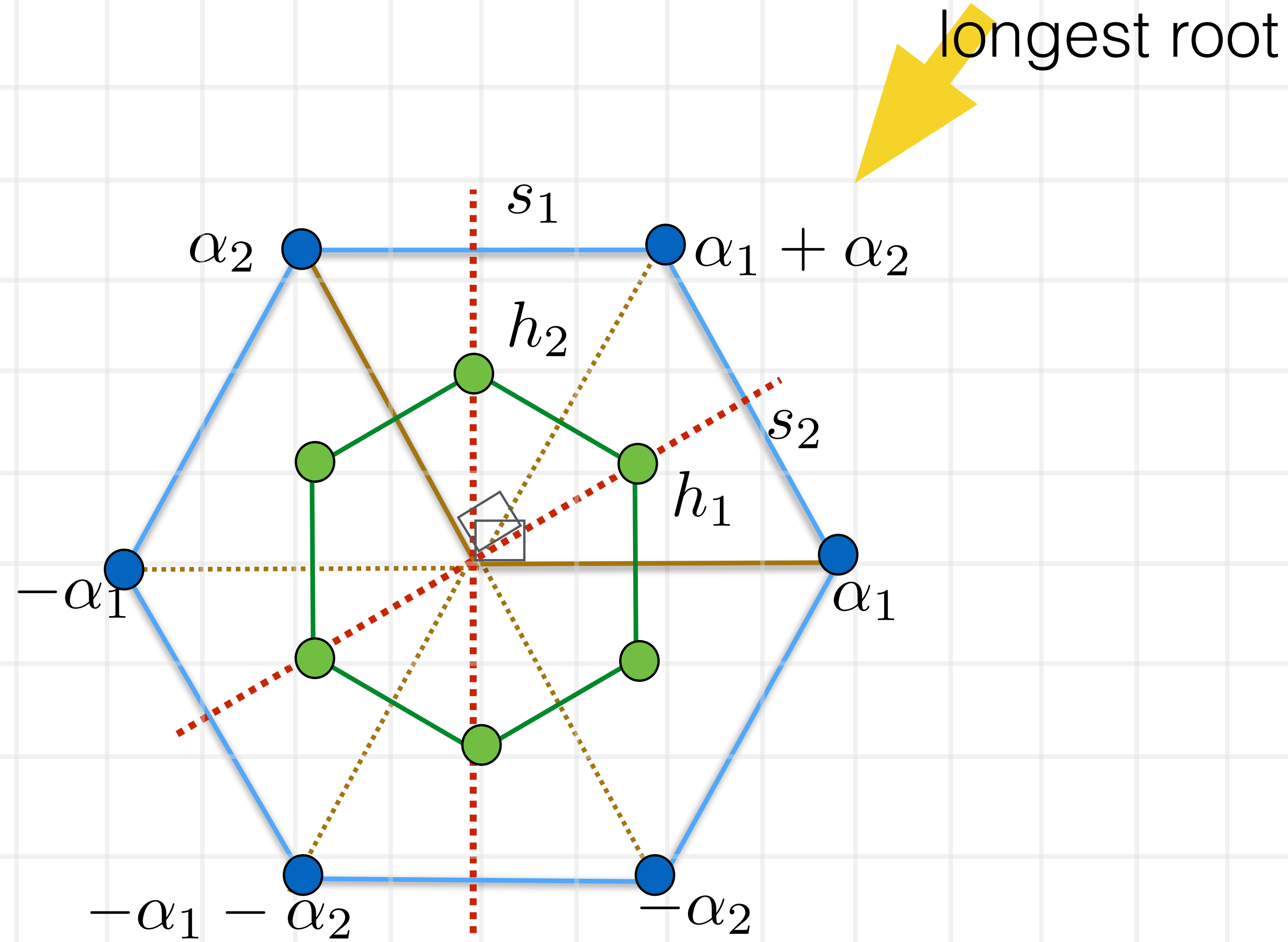
s_1
α_2
$\alpha_1 + \alpha_2$
s_2
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2

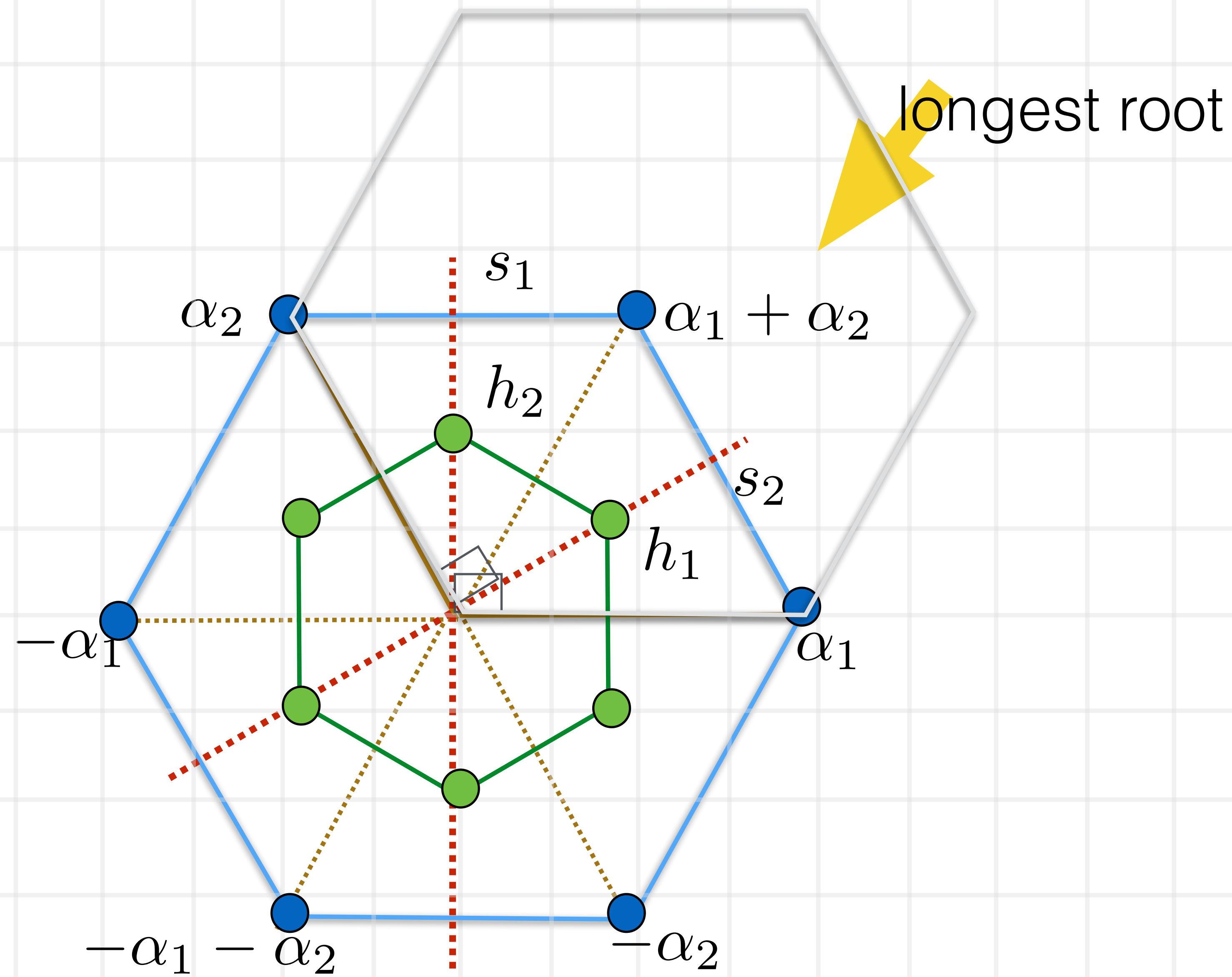


s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2


longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

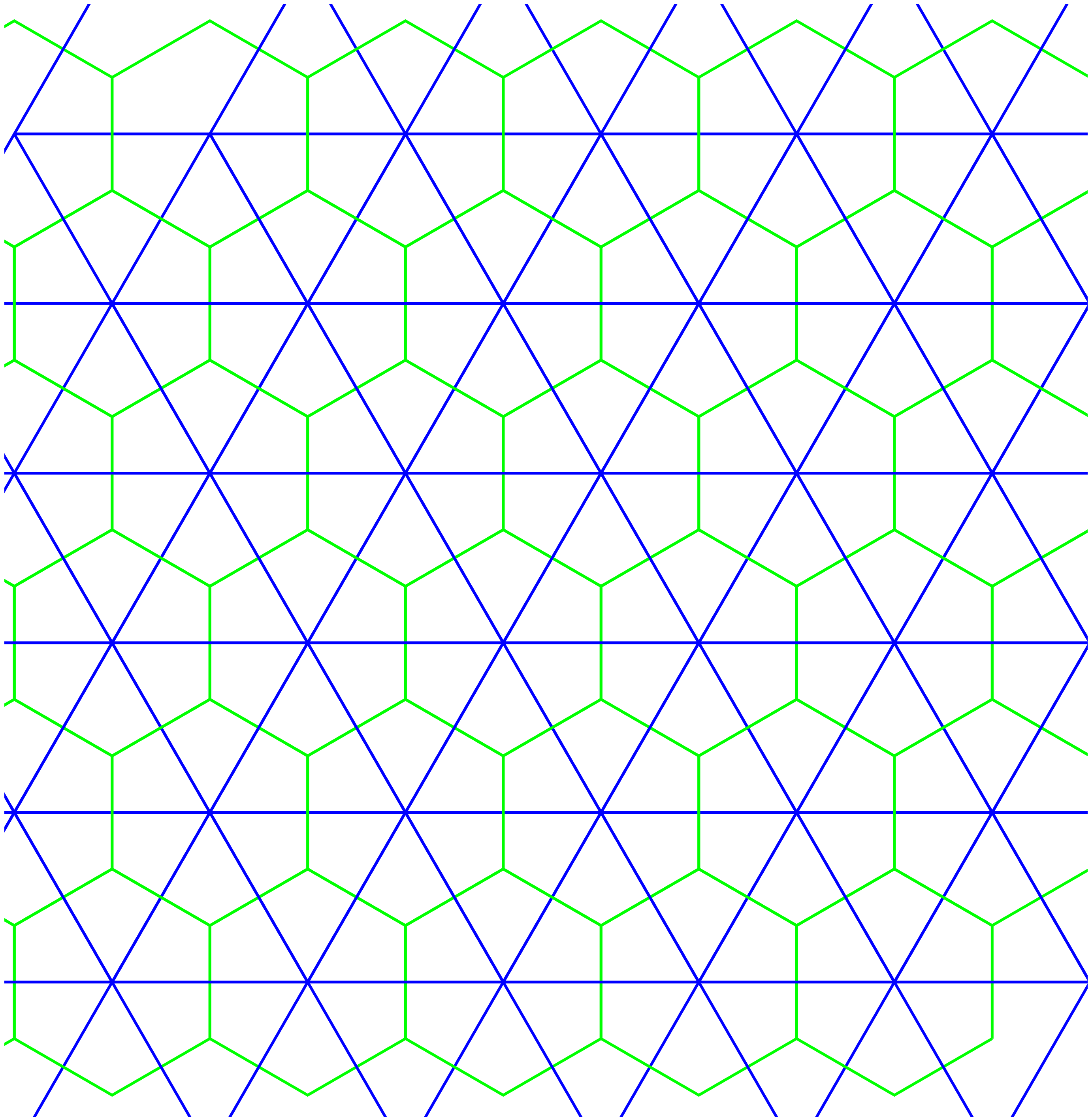
A_2



longest root
s_1
s_2
α_2
$\alpha_1 + \alpha_2$
h_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

Translations

$A_2^{(1)}$



Other Choices

Crystallographic property: $(\alpha_i, \check{\alpha}_j) \in \mathbb{Z}$

$$2\cos^2(\theta_{\alpha_1\alpha_2}) = n$$

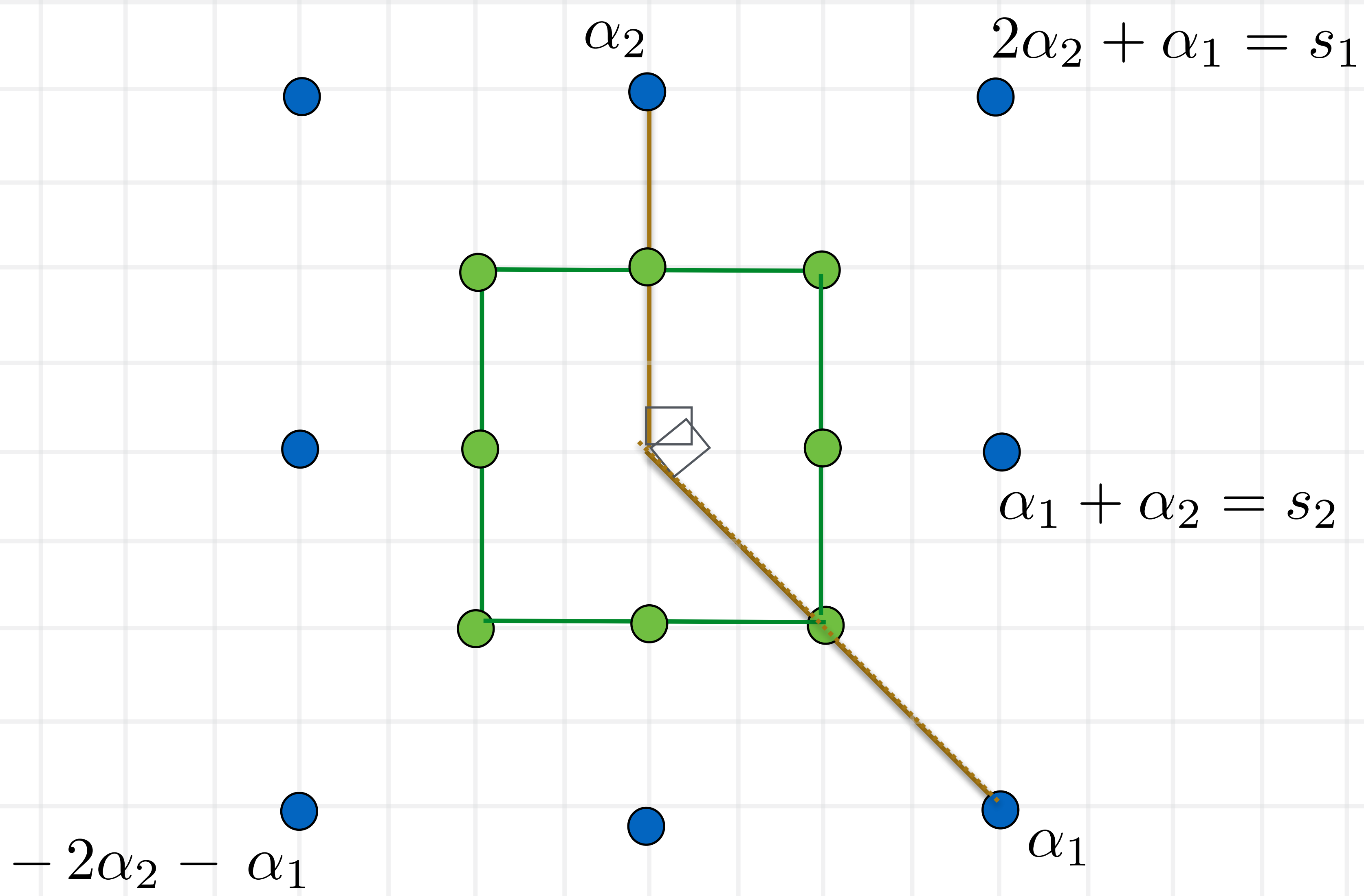
$$n = 0 \;\Rightarrow\; \theta_{\alpha_1\alpha_2} = 3\pi/2$$

$$n = 1 \;\Rightarrow\; \theta_{\alpha_1\alpha_2} = 2\pi/3$$

$$n = 2 \;\Rightarrow\; \theta_{\alpha_1\alpha_2} = 3\pi/4$$

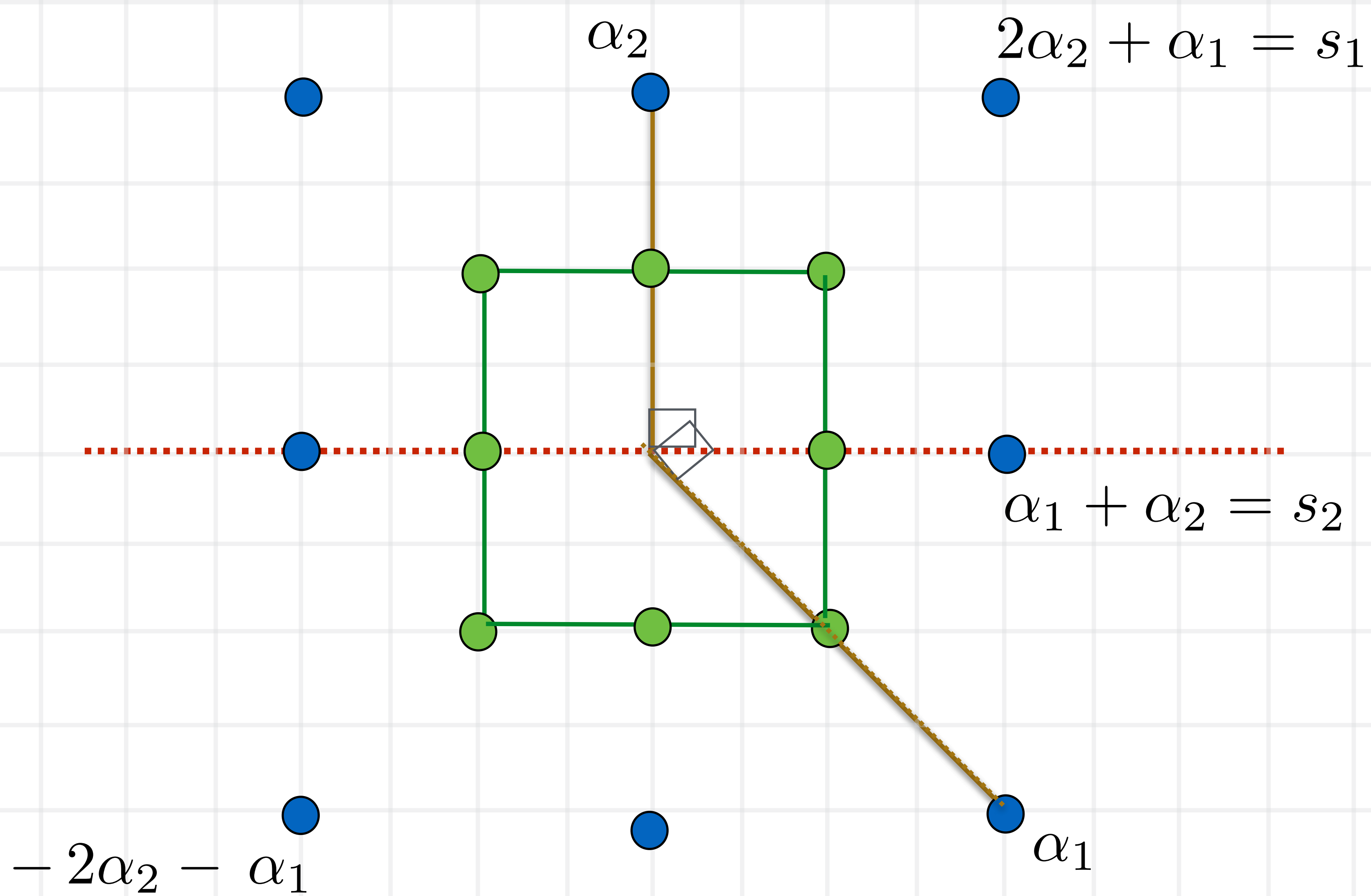
$$n = 3 \;\Rightarrow\; \theta_{\alpha_1\alpha_2} = 5\pi/6$$

B2

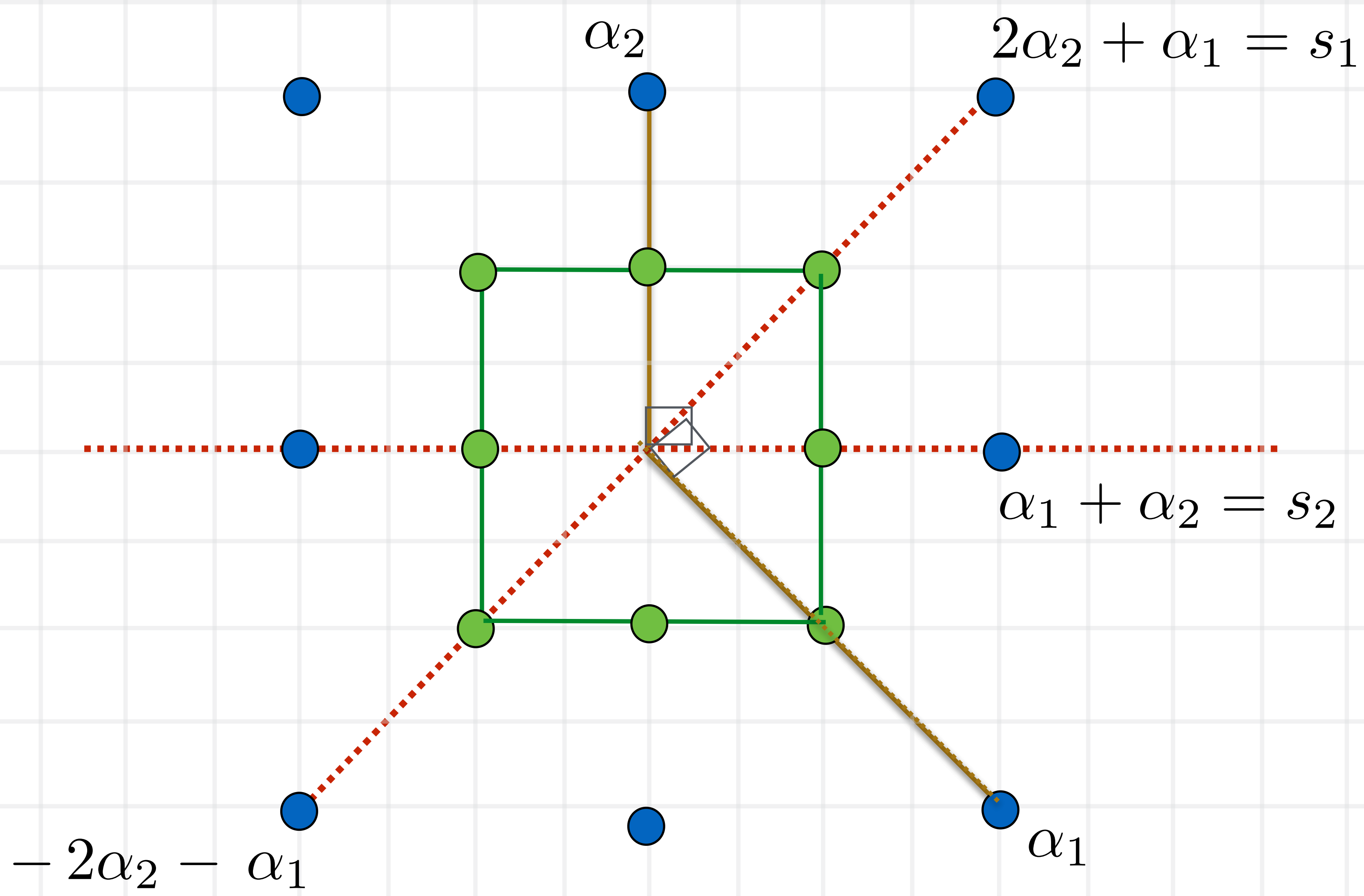


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B_2

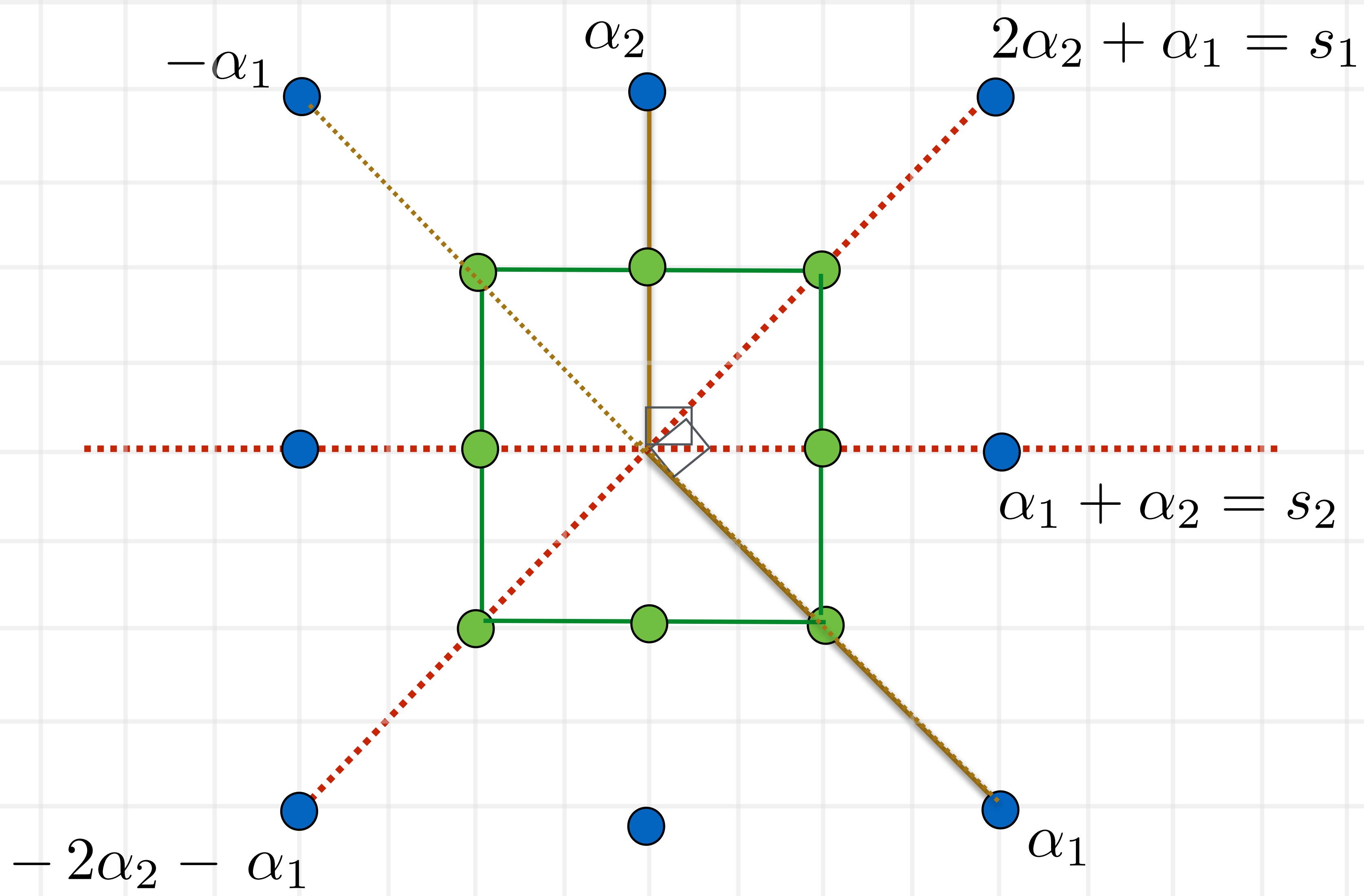

α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B_2

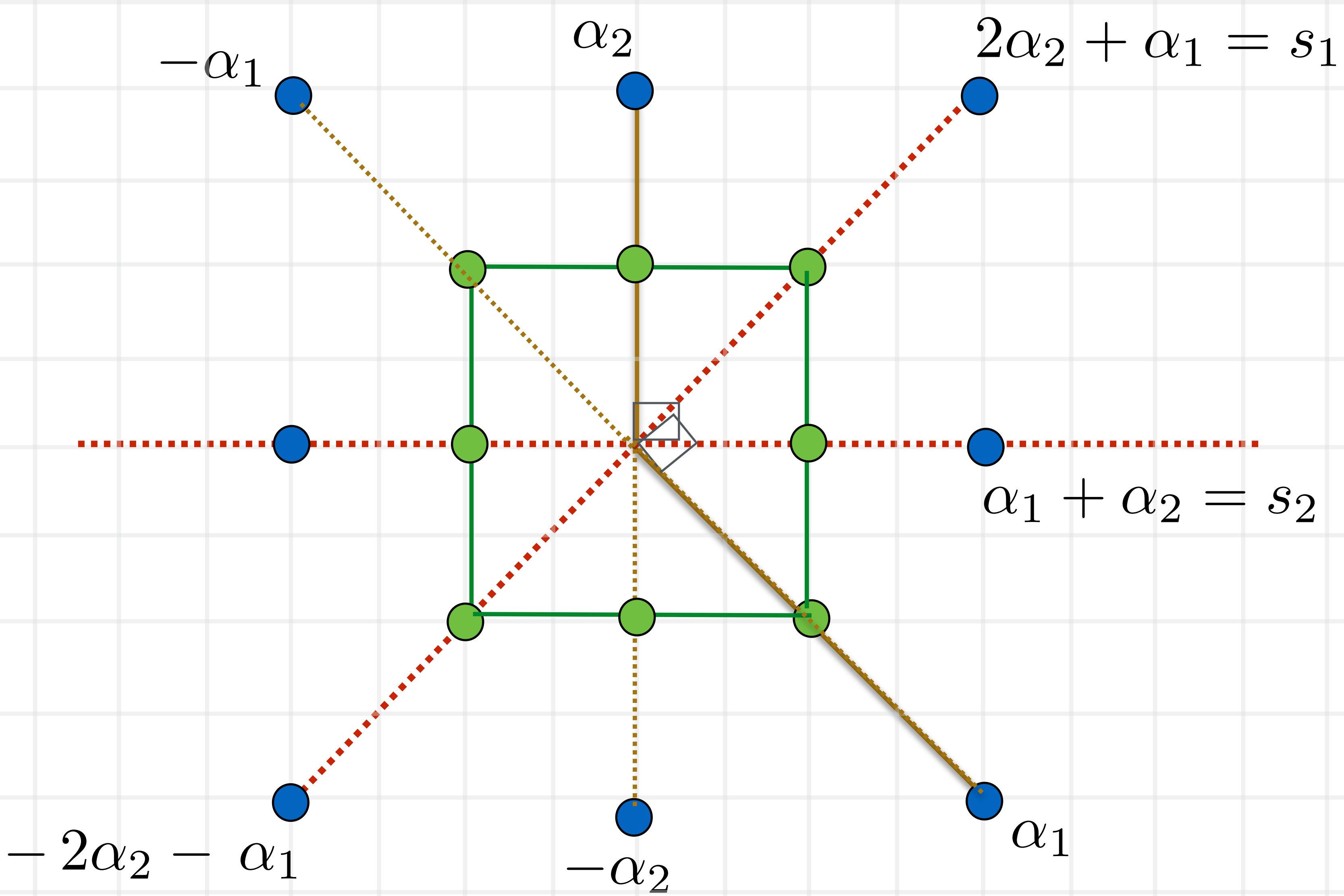


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B2

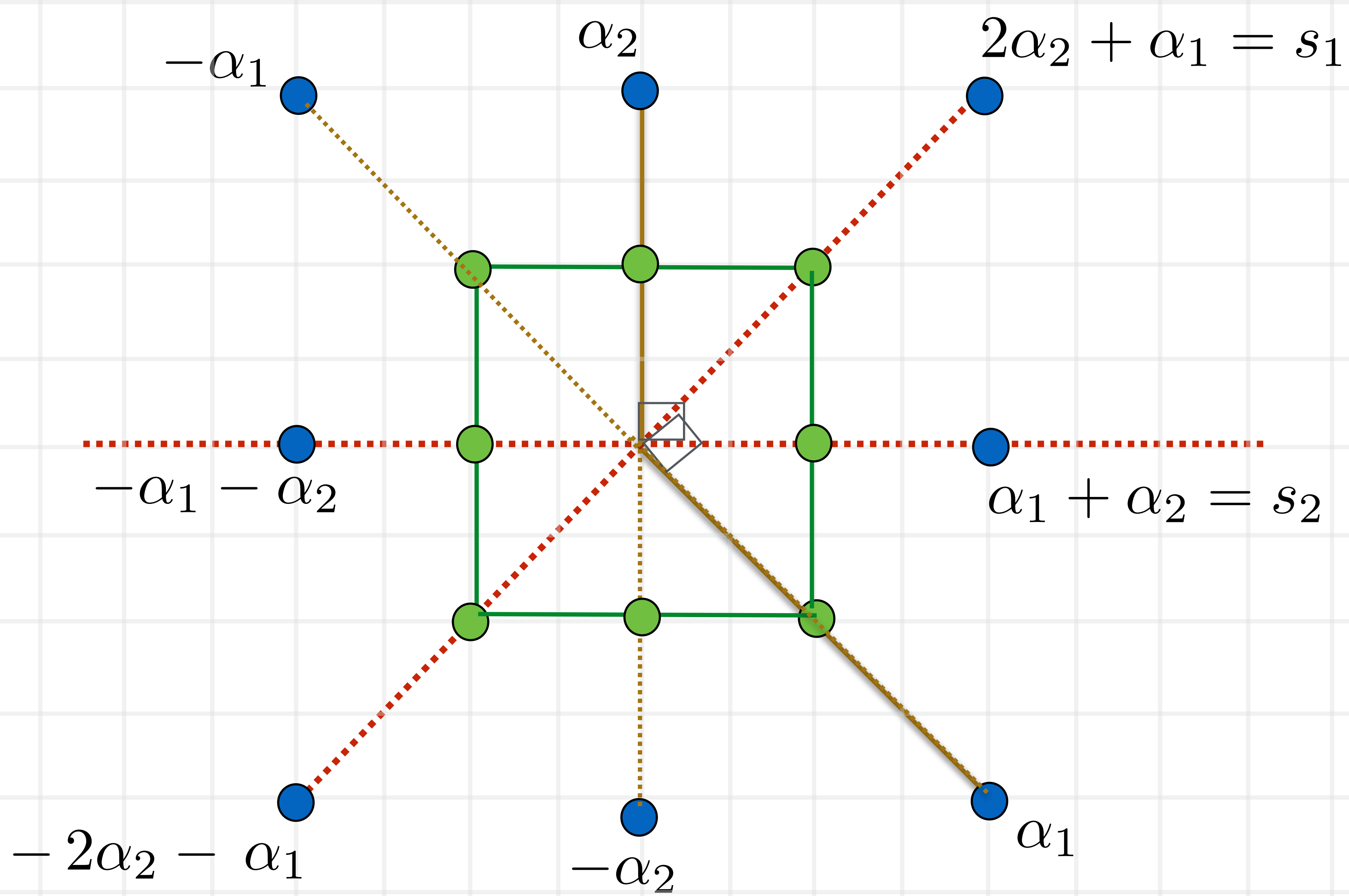

$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B_2

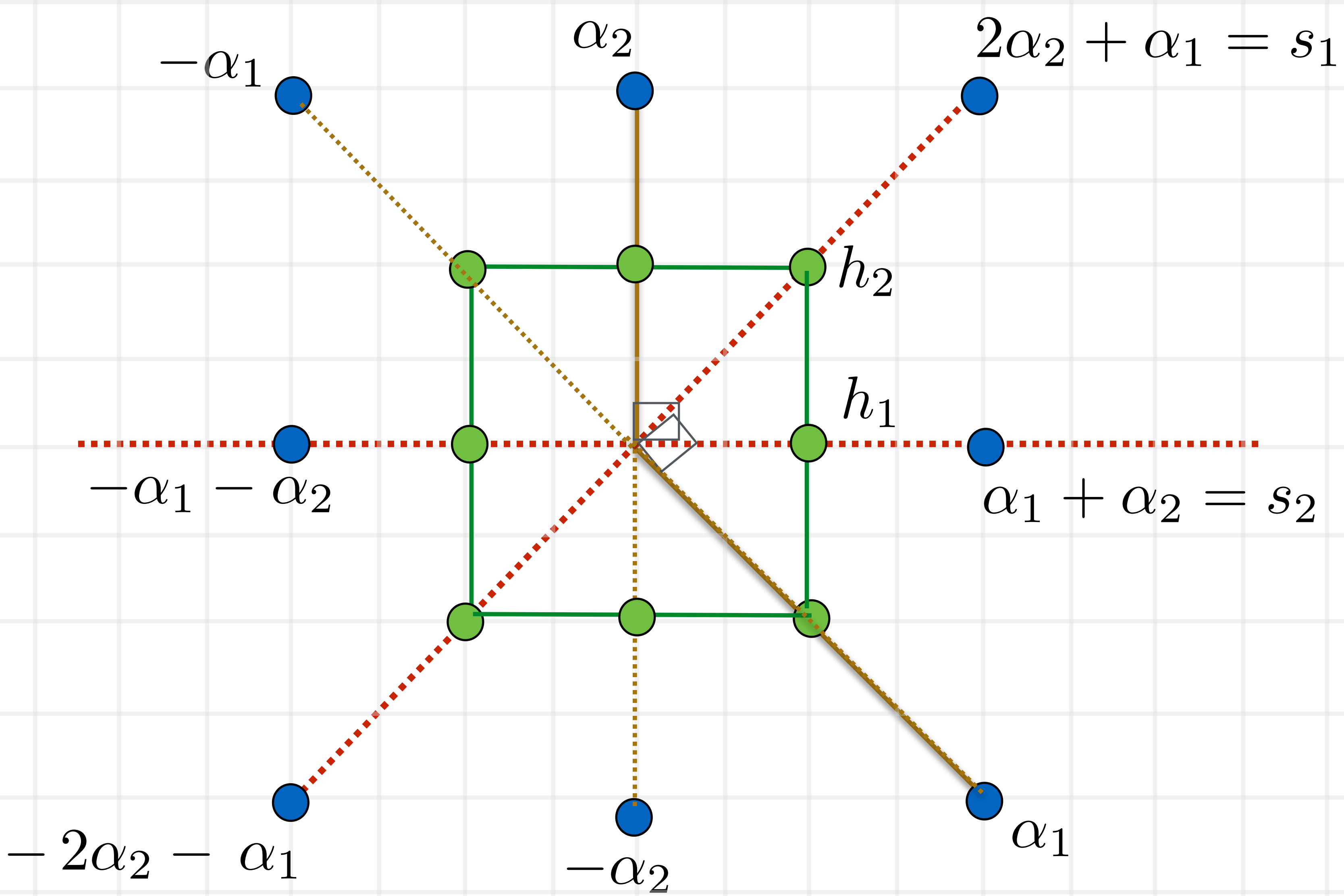


$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

B_2


$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$-\alpha_1 - \alpha_2$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

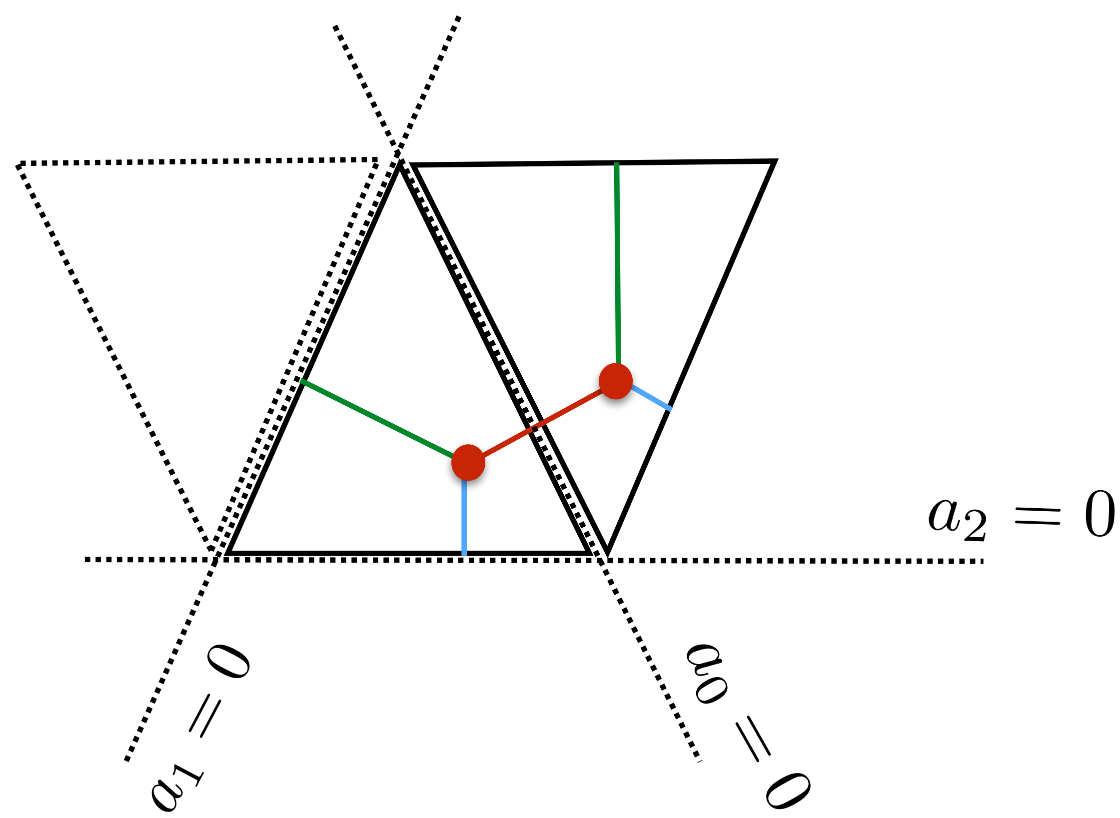
B2



α_2
$-\alpha_1$
$2\alpha_2 + \alpha_1 = s_1$
h_2
h_1
$-\alpha_1 - \alpha_2$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

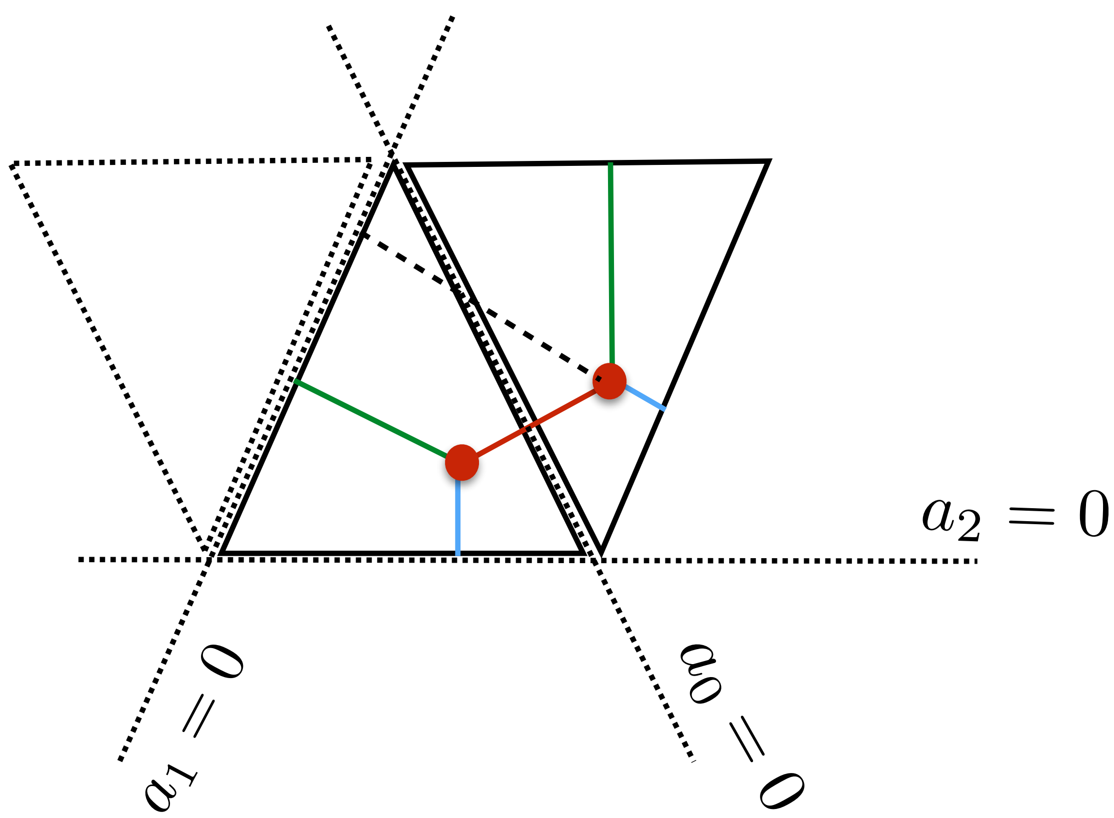
On the A_2 Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



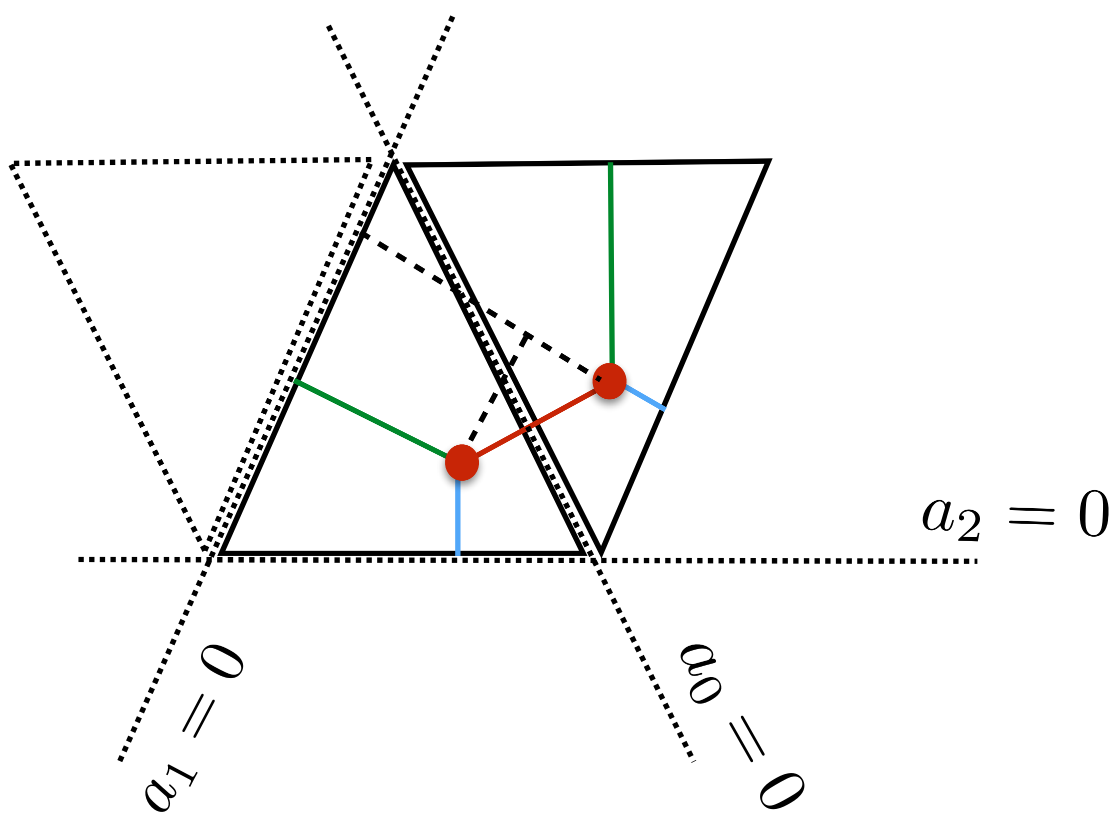
On the A_2 Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



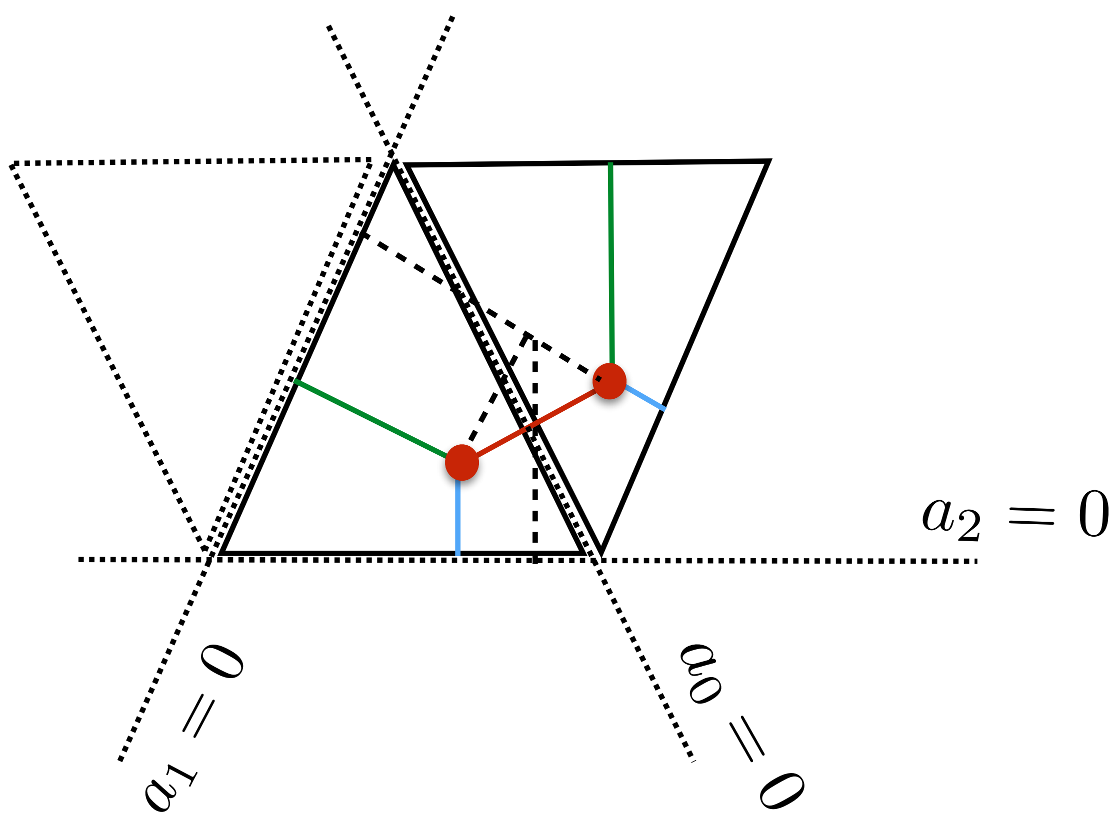
On the A_2 Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



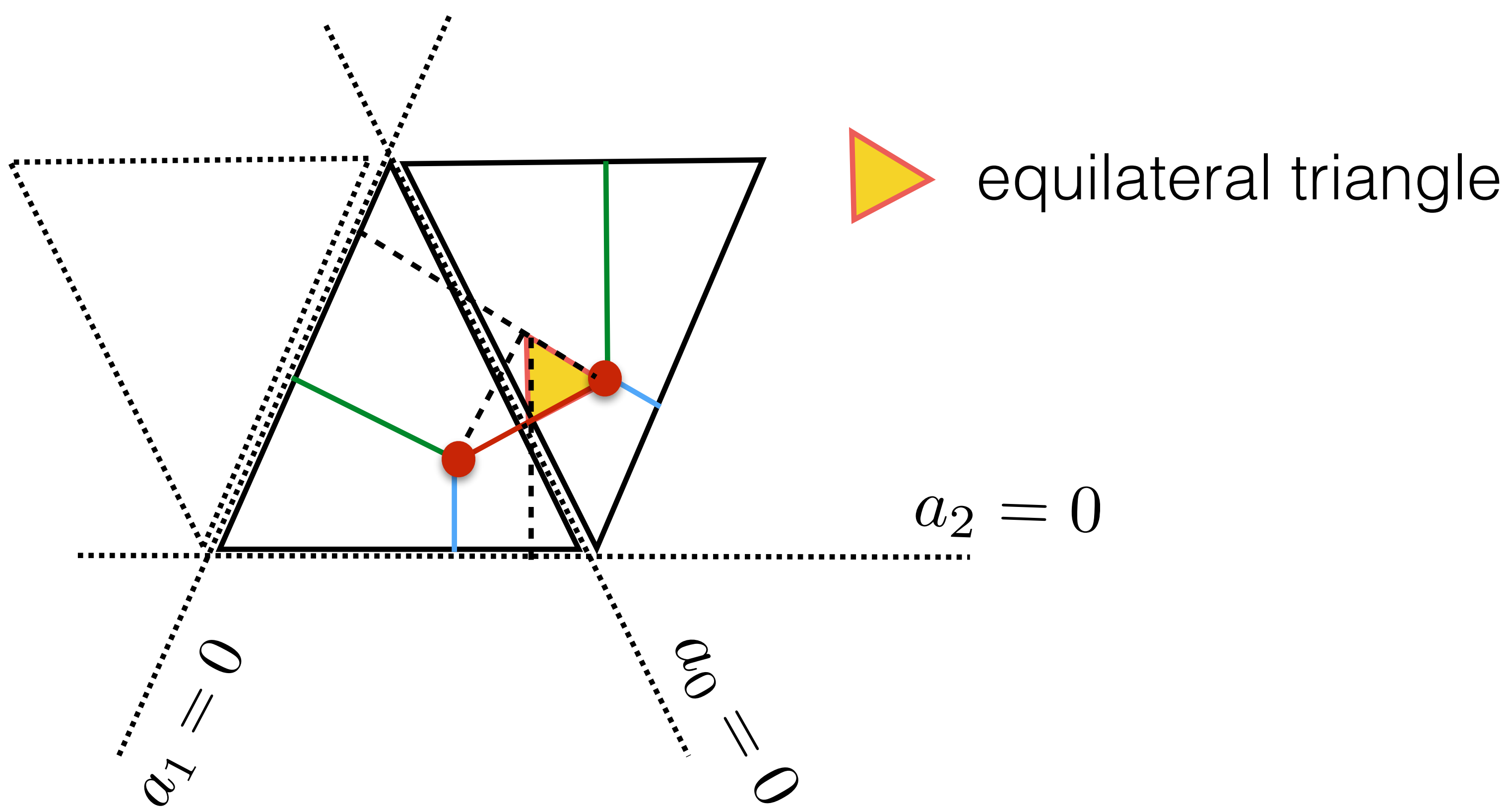
On the A$_2$ Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



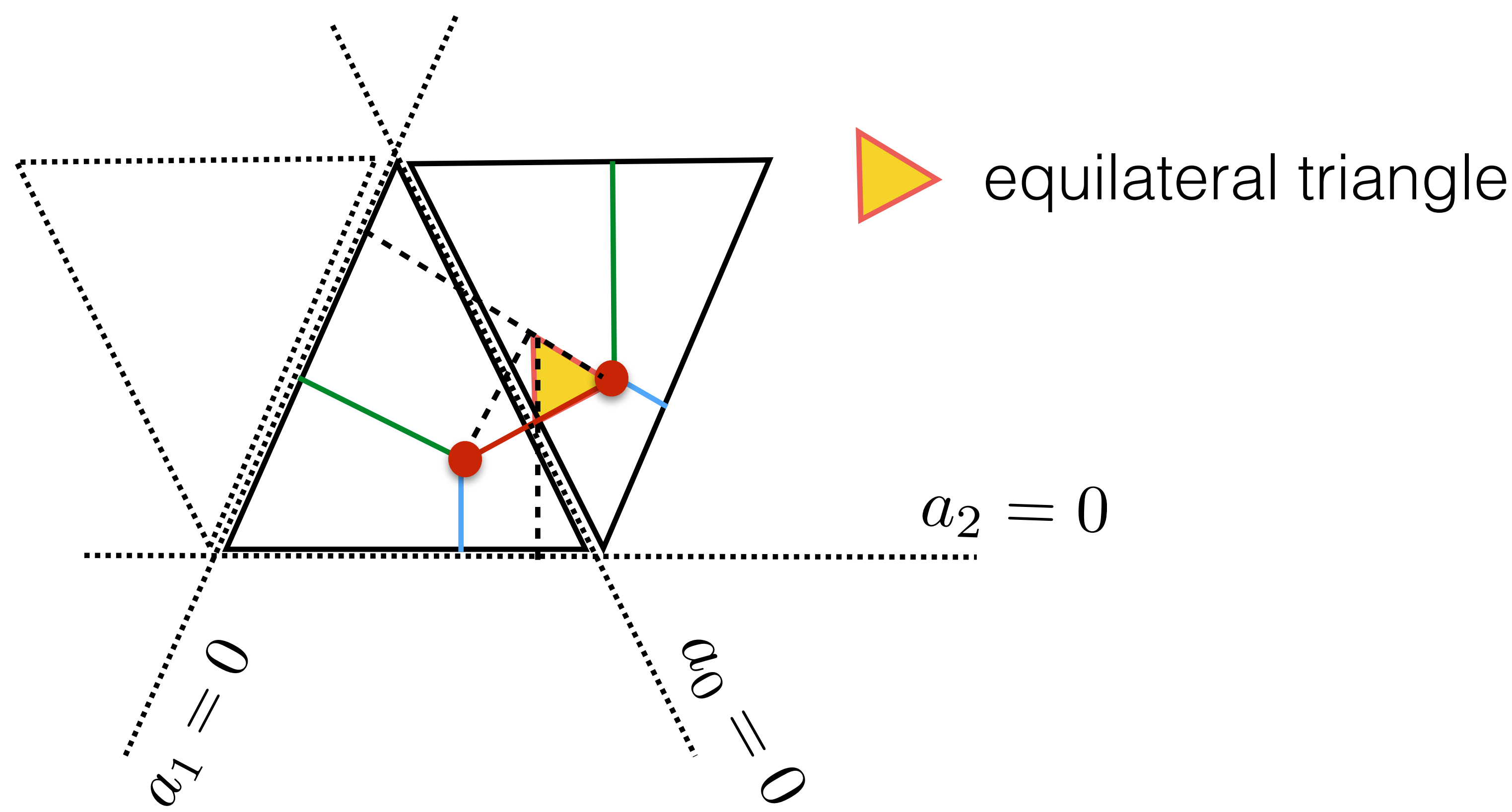
On the A_2 Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



On the A_2 Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



$$s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

Coxeter Relations

$$\widetilde{\mathcal{W}}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$$

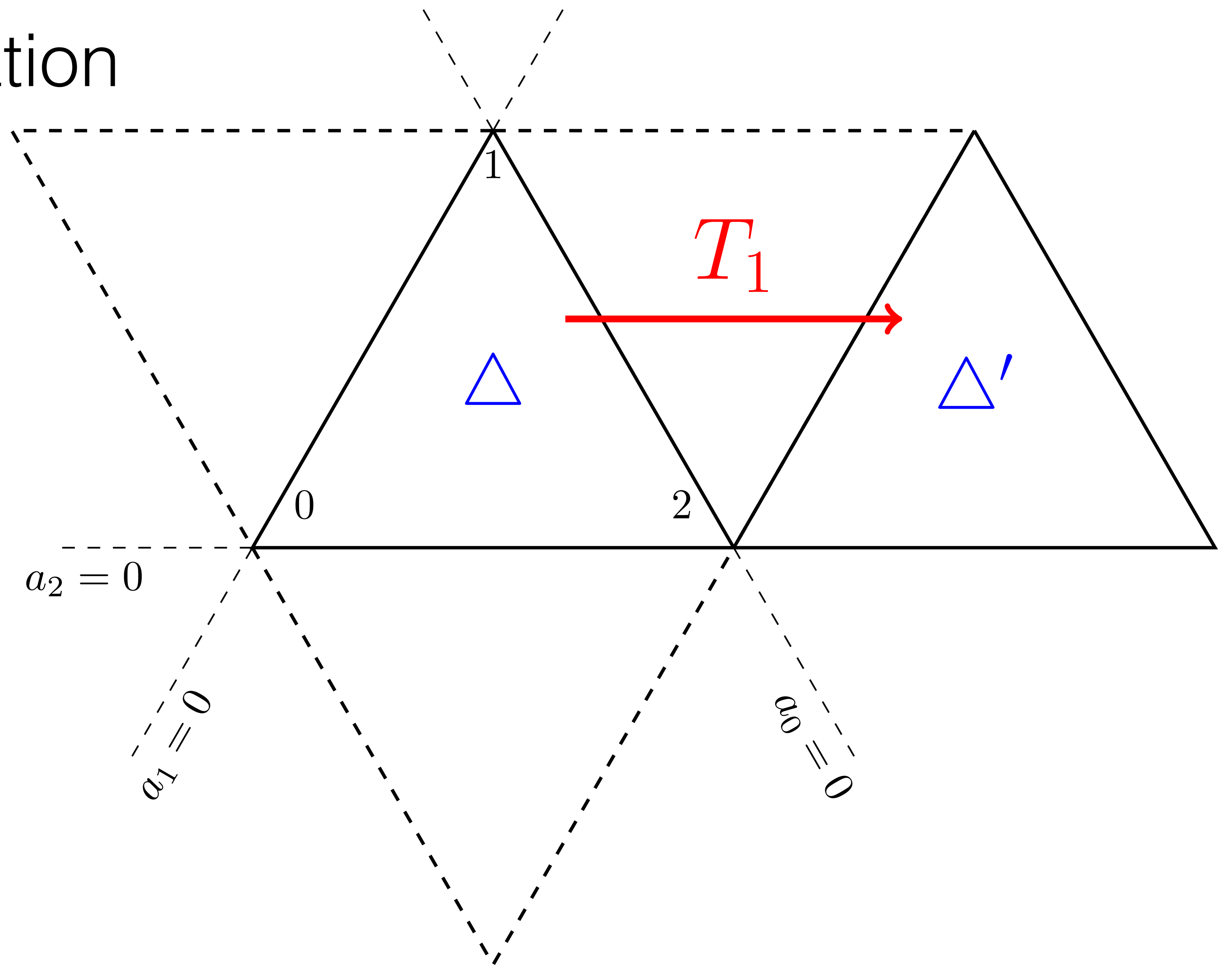
$$\left.\begin{array}{l} s_j^2 = 1 \\ (s_j\, s_{j+1})^3 = 1 \\ \pi\, s_j = s_{j+1}\, \pi \end{array}\right\} \quad j \in \mathbb{N} \bmod 3$$

π : diagram automorphism

$$\pi^3 = 1$$

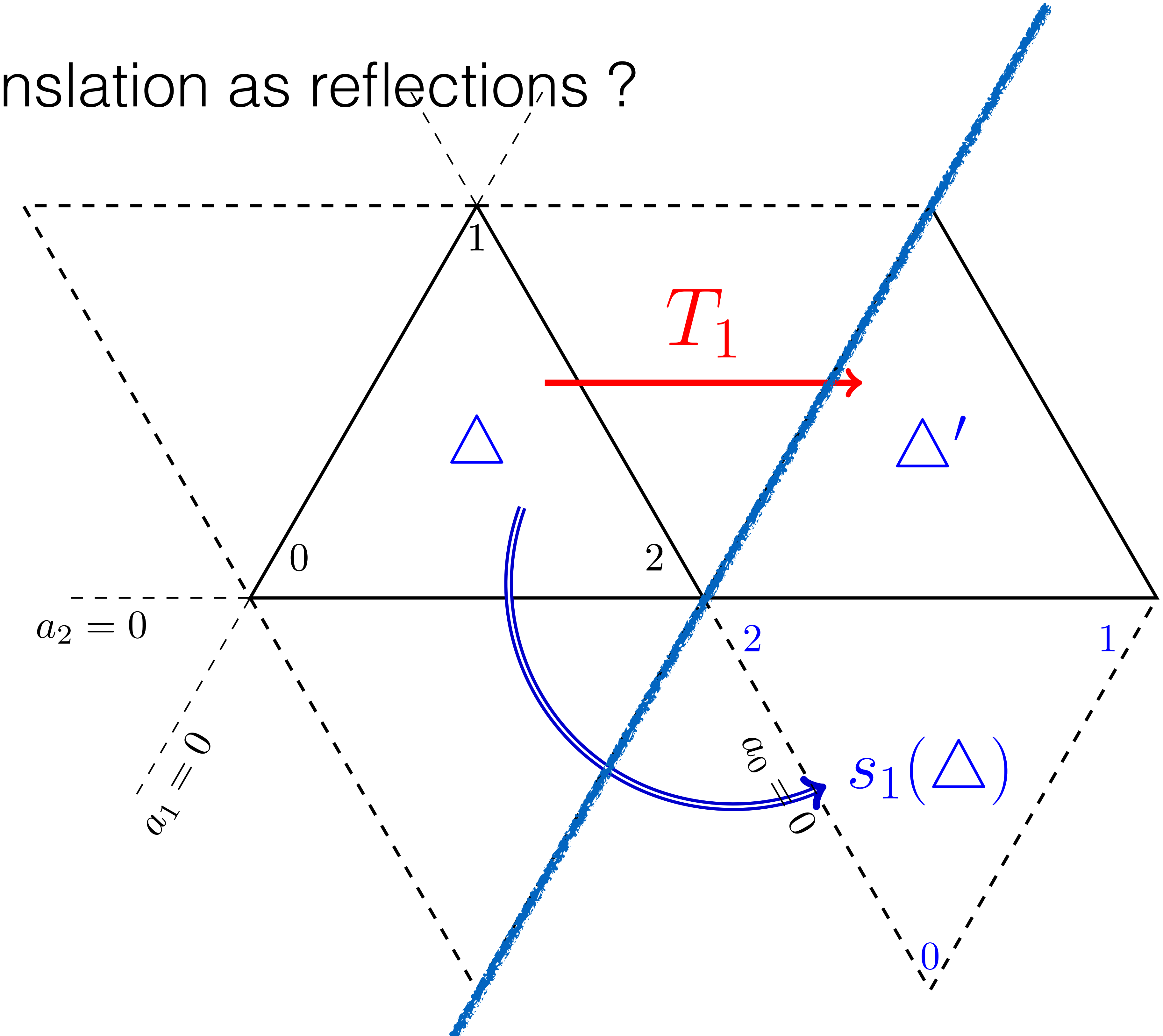
Discrete Dynamics I

- Translation



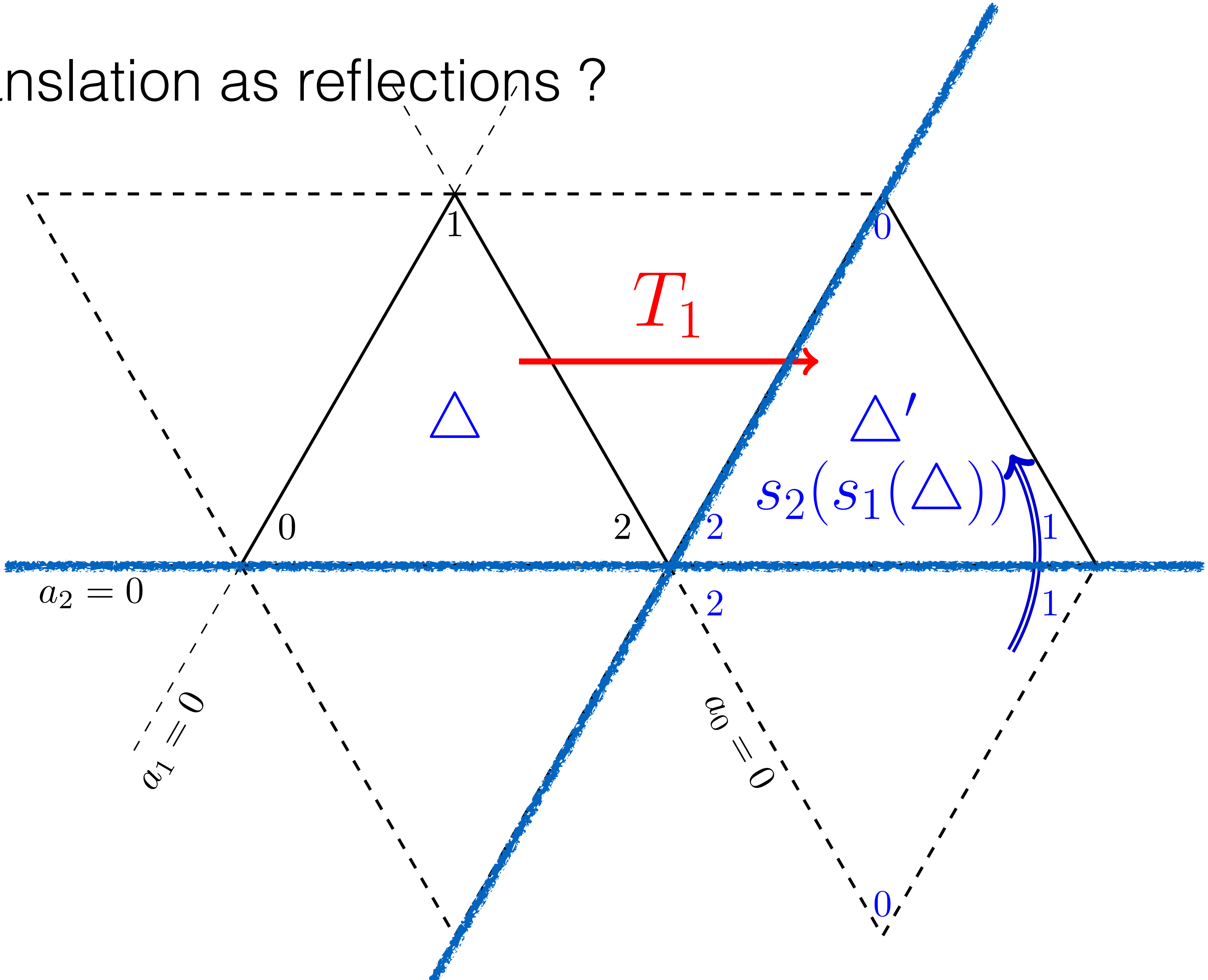
Discrete Dynamics II

- Translation as reflections ?



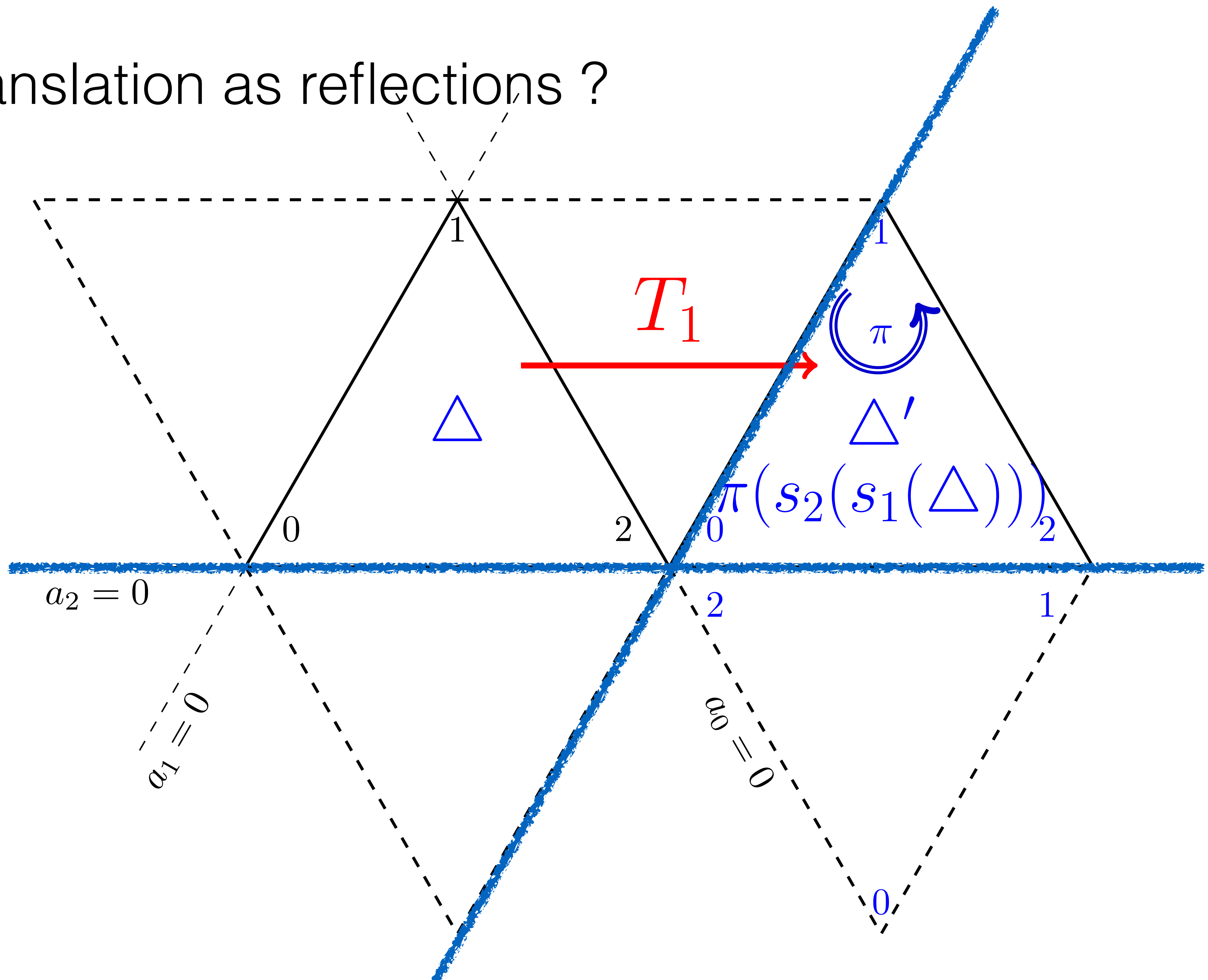
Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics III

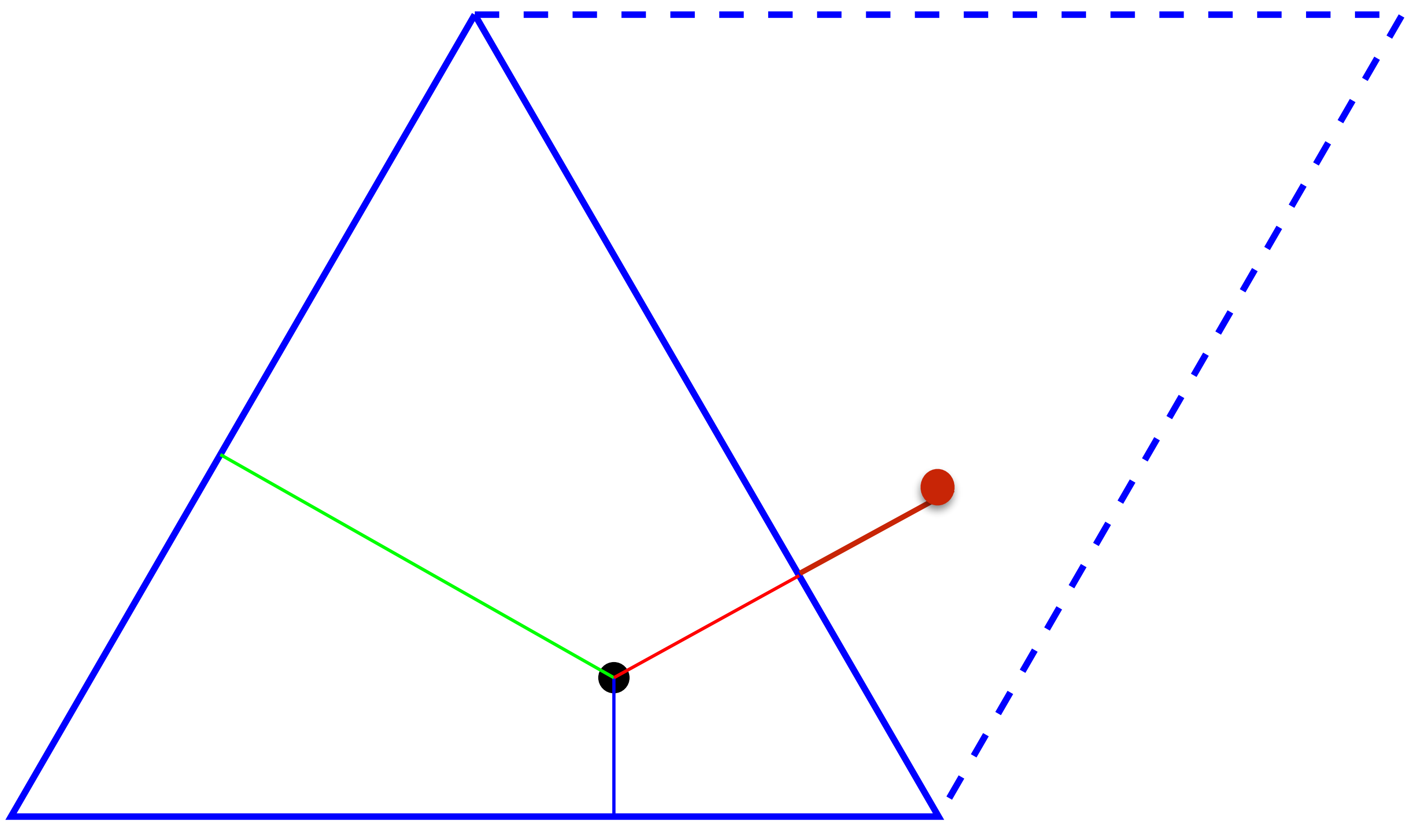
- Translations as reflections
  + diagram automorphism

$$T_1 = \pi \, s_2 \, s_1$$

$$T_2 = s_1 \, \pi \, s_2$$

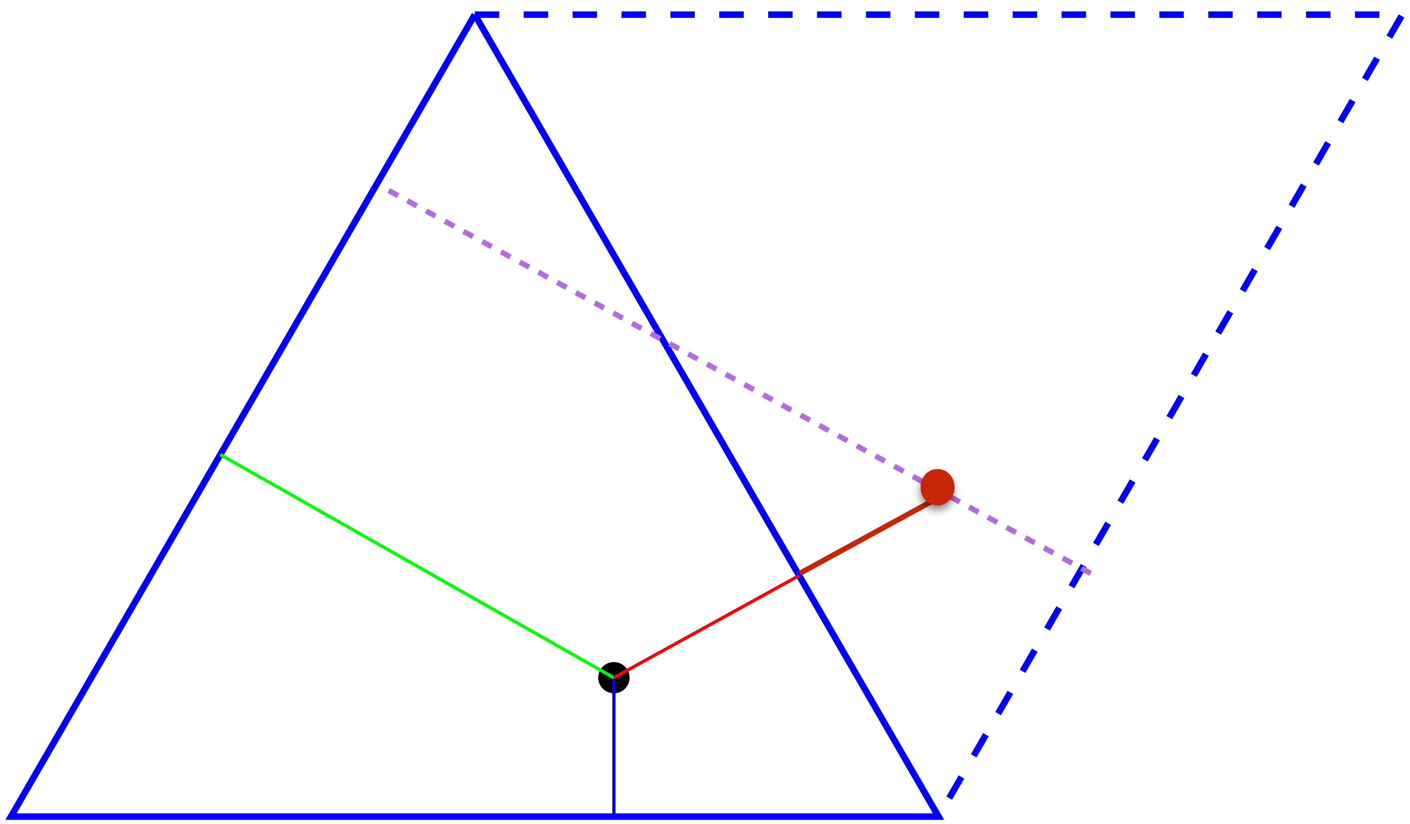
$$T_0 = s_2 \, s_1 \, \pi$$

Constancy of coordinates



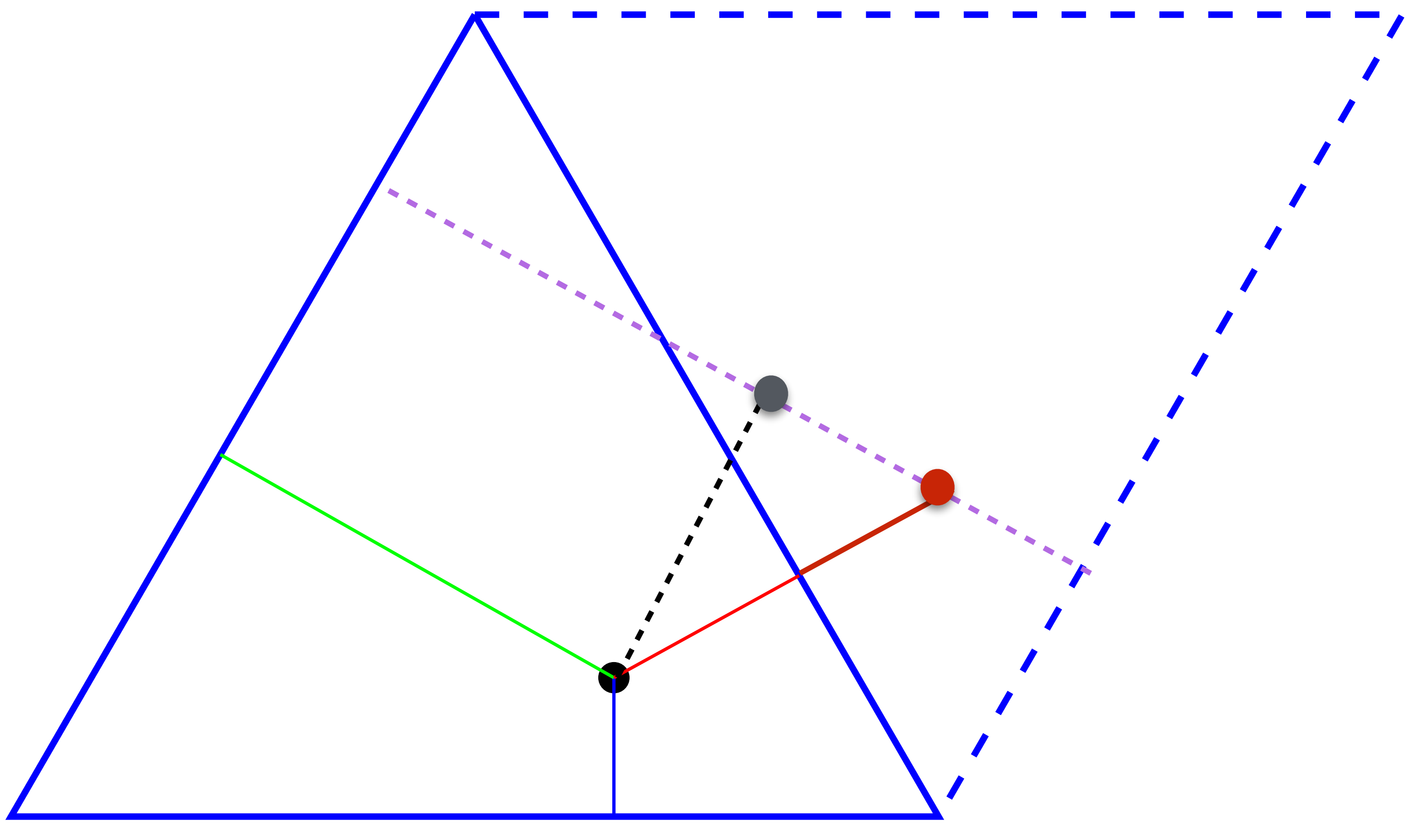
$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



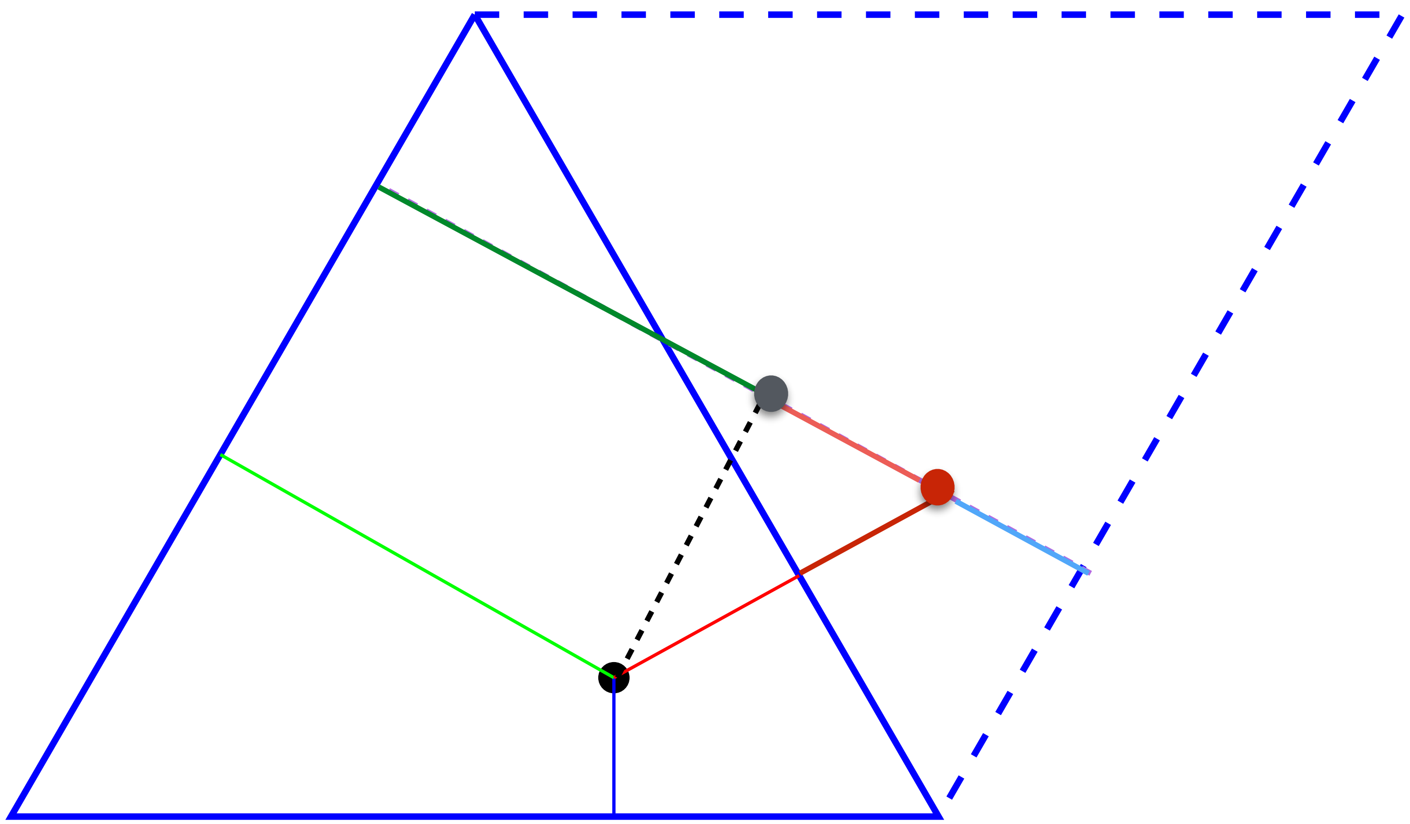
$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



$$a_0 + a_1 + a_2 = k$$

Translations

So we have

$$
\begin{aligned}
T_1(a_0) &= \pi\, s_2\, s_1(a_0) \\
&= \pi\, s_2\,(a_0 + a_1) \\
&= \pi\,(a_0 + a_1 + 2a_2) \\
&= a_1 + a_2 + 2\,a_0 = a_0 + k
\end{aligned}
$$

$\Rightarrow$

$$
T_1(a_0) = a_0 + k, \quad T_1(a_1) = a_1 - k, \quad T_1(a_2) = a_2
$$

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$\Rightarrow$

$$\begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

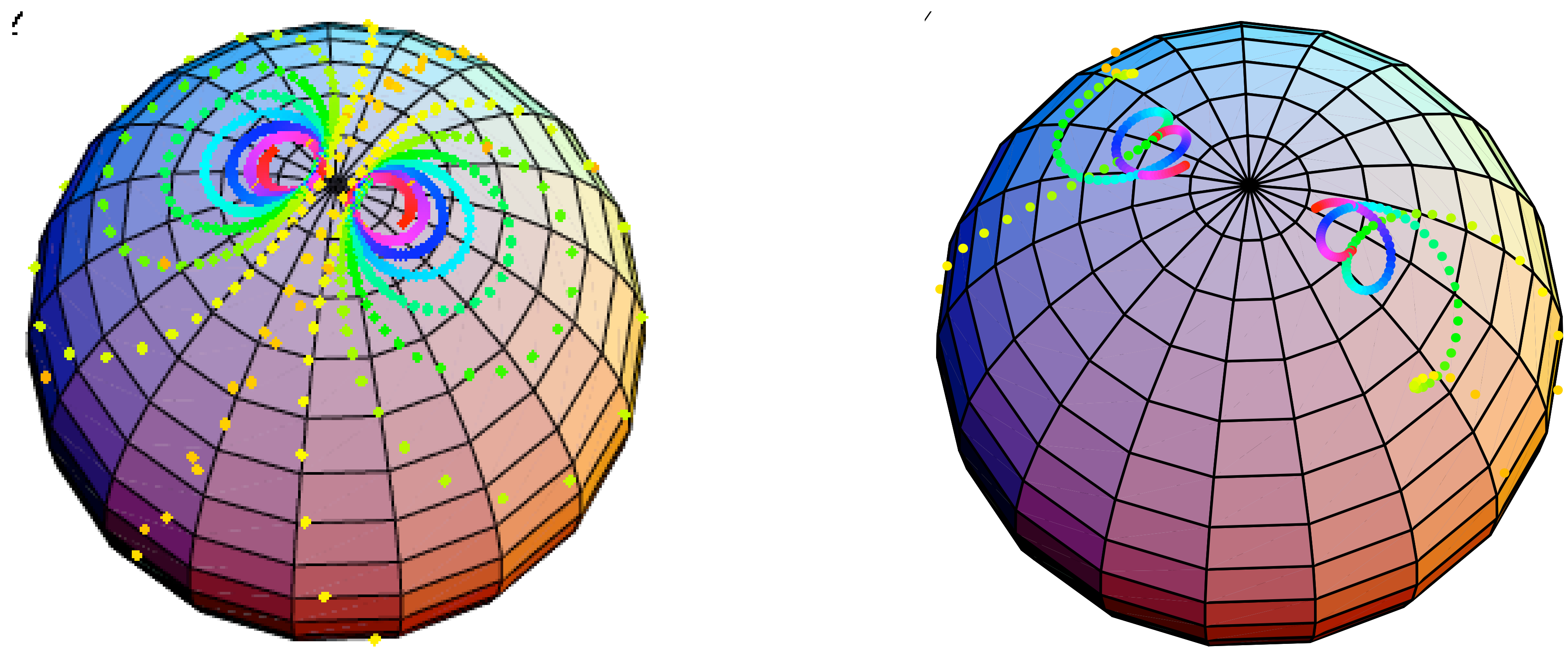
Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \quad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0 + n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1 - n}{u_n} \end{cases}$$

This is a discrete Painlevé equation, which arises in quantum gravity.

Solutions



Solution orbits of scalar dP1 on the Riemann sphere (where the north pole is infinity).

What does this have to do with applied mathematics?

Applications

- Electrical structures of interfaces in steady electrolysis *L. Bass, Trans Faraday Soc 60 (1964)1656–1663*
- Spin-spin correlation functions for the 2D Ising model *TT Wu, BM McCoy, CA Tracy, E Barouch Phys Rev B13 (1976) 316–374*
- Spherical electric probe in a continuum gas *PCT de Boer, GSS Ludford, Plasma Phys 17 (1975) 29–41*
- Cylindrical Waves in General Relativity *S Chandrashekar, Proc. R. Soc. Lond. A 408 (1986) 209–232*
- Non-perturbative 2D quantum gravity *Gross & Migdal PRL 64(1990) 127-130*
- Orthogonal polynomials with non-classical weight function *AP Magnus J. Comput Appl. Anal. 57 (1995) 215–237*
- Level spacing distributions and the Airy kernel *CA Tracy, H Widom CMP 159 (1994) 151–174*
- Spatially dependent ecological models: *J & Morrison Anal Appl 6 (2008) 371-381*
- Gradient catastrophe in fluids: *Dubrovin, Grava & Klein J. Nonlin. Sci 19 (2009) 57-94*

Summary

- New mathematical models of physics pose new questions for applied mathematics.
- Global dynamics of solutions can be found through geometry.
- It is currently the only analytic approach available in $\mathbb{C}$ for discrete equations.
- Tantalising questions remain open.