THE SPECTRUM OF TRAVELLING WAVE SOLUTIONS TO THE SINE-GORDON EQUATION

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Abstract. We investigate the spectrum of the linear operator coming from the sine-Gordon equation linearized about a travelling kink-wave solution. Using various geometric techniques as well as some elementary methods from ODE theory, we find that the point spectrum of such an operator is purely imaginary provided the wave speed c of the travelling wave is not ± 1. We then compute the essential spectrum of the same operator.

1. Introduction

The sine-Gordon equation:

$$u_{tt} = u_{xx} + \sin u \tag{1}$$

has applications in many areas of physics and mathematics. It can be used to model magnetic flux propagation in long Josephson junctions: two ideal superconductors separated by a thin insulating layer [SCR76], [DDvGV03]. It can be thought of a model for mechanical vibrations of the so-called 'ribbon pendulum' – the continuum limit of a line of pendula each coupled to their nearest neighbor via Hooke's law [BM07]. In biology, it has found applications in modeling the transcription and denaturation in DNA molecules [Sal91]. Further, it can be used to model propagation of a crystal dislocation, Bloch wall motion of magnetic crystals, propagation of "splay-wave" along a lipid membrane, and pseudo-spherical surfaces to name a few others (see [SCM73] and the references therein).

In [BM07], the Cauchy problem for the sine-Gordon equation in laboratory coordinates was studied using inverse scattering techniques. These techniques were later expanded to other types of stationary underlying waves in [BJ09]. We consider the spectrum of the linearized operator of the sine-Gordon equation when the underlying wave is a travelling wave. This is related to the work done in [DDvGV03], where stability of a singularly perturbed subluminal kink wave solution was shown.

Travelling wave solutions to the sine-Gordon equation for which the quantity $c^2 - 1 < 0$ are called *subluminal* waves. When $c^2 - 1 > 0$ they are called *superluminal* waves. Primarily, stability analysis of travelling waves has been concerned with subluminal waves. To the best of the authors' knowledge, all known stability results concerning superluminal travelling waves are either directly or indirectly related to a result found in [Sco69] (for examples see, [BS71], [Whi99], and [SCM73], and the references therein). One of the initial motivations of this paper was that the analysis found in [Sco69] appears incomplete, and does not seem to fully resolve the question of stability of travelling kink wave solutions in either the subluminal or superluminal case.

In this paper, we take a more geometric, dynamical systems approach. Kink-wave solutions correspond to critical points of a nonlinear dynamical system on an appropriate Hilbert space. We then consider the linearized operator about such a critical point corresponding to a travelling wave solution to equation (1). In particular we use a crossing form calculation in order to show the absence of real point spectrum of the linearized operator. The crossing form is inspired from techniques useful in integrable equations and Hamiltonian systems – the Maslov index. The Maslov index has a long history as a tool for determining stability in Hamiltonian and integrable systems, dating back to Maslov [Mas72] and Arnold [Arn67], with many contributions from numerous others continuing through the present day. In this paper we primarily reference the work of Robbin and Salamon [RS93], however there is much more in the literature, see for example, the references in the book by Abbondandolo [Abb01] for a more comprehensive list. Because we are in the case of such low spatial dimensions, we do not require the full machinery of the Maslov index as such; however, we briefly include a short survey of the underlying ideas in section 1.2 for completeness.

The distinction between subluminal and superluminal waves is important and breaks up the proof of the main theorem into two main cases. We will first tackle the case of subluminal waves and show the lack of real, and then complex eigenvalues. Before continuing with the superluminal case, we compute the essential spectrum of the linearized operator in both the subluminal and superluminal cases. The reason for this is that the essential spectrum in the superluminal case plays a role in organizing the calculations necessary for showing the absence of point spectrum. Specifically we find an ellipse of essential spectrum off the imaginary axis, but intersecting it. It turns out that outside of this ellipse and away from the imaginary axis, decay estimates show that there can be no eigenfunctions of the linearized operator, while inside of it, these decay estimates are not sufficient to show the absence of point spectrum.

We then return to the superluminal case and show that the point spectrum is empty in this case as well. The instability in the superluminal case is therefore due to the presence of essential spectrum. This gives a mathematical justification for the avoidance of using the Lorentz transformation. The aim of the Lorentz boost is to reduce the travelling wave problem to a steady-state problem; however, to show the correspondence requires checking the decay rates of the candidate eigenfunctions. To check this carefully in the subluminal case becomes cumbersome, and does not appear to be easier than the direct approach we take. Moreover, it does not reveal the different structure between the sub and superluminal cases. In the superluminal case the direct approach we take is necessary to reveal the exact nature of the instability, namely the presence of essential spectrum in the right half plane.

1.1. **The Eigenvalue Problem.** We consider solutions to (1) of the form $v(x+ct,t)$ where c is the (positive) speed of the traveling wave. Making the change of variable $z = x + ct$ and substituting into (1) gives

$$c^2 v_{zz} + 2c v_{zt} + v_{tt} = v_{zz} + \sin v. \tag{2}$$

Definition 1. *A* travelling wave *solution will be a t independent solution $v(z)$ to (2).*

Thus it solves the (nonlinear) pendulum equation:

$$(c^2 - 1) v_{zz} = \sin v. \tag{3}$$

Definition 2. *A* kink wave *solution is a travelling wave solution to (2) that is also a heteroclinic orbit in the phase plane of (3).*

Throughout this paper we will be following the setup as outlined in [BJ89]. For further details see the references therein. As outlined in [BJ89], we view equation (2) as a dynamical system on the space of functions $H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$\begin{pmatrix} v \\ v_t \end{pmatrix}_t = \begin{pmatrix} v_t \\ \sin v - (c^2-1)v_{zz} - 2cv_{zt} \end{pmatrix} \tag{4}$$

and we see that a kink wave is a critical point of such a dynamical system. Linearizing equation (2) about a kink wave solution v gives:

$$(c^2-1)\varphi_{zz} + 2c\varphi_{zt} + \varphi_{tt} = (\cos v)\varphi. \tag{5}$$

Setting $\varphi_t = \psi$ we can rewrite (5) as:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t = \begin{pmatrix} \psi \\ \cos v\varphi - (c^2-1)\varphi_{zz} - 2c\psi_z \end{pmatrix} := \mathcal{L}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{6}$$

We wish to consider the (temporal) eigenvalue problem of equation (6). Letting λ be the eigenvalue parameter, we obtain:

$$\mathcal{L}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ (\cos v)\varphi - (c^2-1)\varphi_{zz} - 2c\psi_z \end{pmatrix} = \lambda\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{7}$$

This leads to the following eigenvalue condition:

Definition 3. *A pair of functions $(\varphi,\psi) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is an eigenvector of the linearized operator $\mathcal{L}$ with eigenvalue λ if (7) is satisfied.*

We note that the pair $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ solving (7) is equivalent to φ satisfying

$$(c^2-1)\varphi_{zz} + 2c\lambda\varphi_z + (\lambda^2 - \cos v)\varphi = 0, \tag{8}$$

while the condition that $\varphi \in H^1(\mathbb{R})$ implies $\lim\limits_{z\to\pm\infty} \varphi(z) = 0$. We are thus led to an equivalent, alternative definition of an eigenvalue:

Definition 4. *A function $\varphi \in H^1(\mathbb{R})$ is an eigenfunction of the linearized operator $\mathcal{L}$ with eigenvalue λ if φ solves*

$$(c^2-1)\varphi_{zz} + 2c\lambda\varphi_z + (\lambda^2 - \cos v)\varphi = 0, \tag{9}$$

and $\lim\limits_{z\to\pm\infty} \varphi(z) = 0$.

This is primarily the definition we have in mind throughout this paper. We are now ready to state the main theorem.

Theorem 5. *There are no eigenvalues of the operator $\mathcal{L}$, the linearized sine-Gordon operator about a kink-wave solution, off the imaginary axis, provided that $c \neq \pm 1$.*

1.2. **The Crossing Form.** Note: In this paper we will use the description of the Maslov index and crossing form as in [RS93].

Let the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denote the standard symplectic structure on $\mathbb{R}^2$. A line ℓ passing through the origin is considered Lagrangian in the sense that for any $v_1, v_2 \in \ell$ the inner product of v_1 with Jv_2, $< v_1, Jv_2 >= 0$. Let $\ell(t)$ be a curve of lines in $\mathbb{R}^2$. If $\ell(t)$ can be written as the span of the 2×1 matrix $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we will call the functions $x(t)$ and $y(t)$ a *frame* for $\ell(t)$.

Alternatively we can view $\ell(t)$ as a curve in $\mathbb{R}P^1 \approx S^1$, and if $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a frame of ℓ, then $\ell(t) = [x(t) : y(t)]$. Now let $\mathbf{a} = [a_1 : a_2]$ be a fixed line in $\mathbb{R}^2$. Suppose that $\ell(t) : [t_0, t_n] \to \mathbb{R}P^1$, and $\ell(t) = \mathbf{a}$ at $t_1, t_2, \dots t_{n-1}$ with $t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$. Suppose further that we have $\dot{\ell}(t_i) \neq (0,0)$ for all t_i. When this last condition is satisfied, we will call the crossing *regular*. We define the *crossing form*, $\Gamma(\ell(t), \mathbf{a}, t_i)$ of $\ell(t)$, with respect to $\mathbf{a}$, at a regular crossing as:

$$\Gamma(\ell(t), \mathbf{a}, t_i) = x(t_i)\dot{y}(t_i) - y(t_i)\dot{x}(t_i). \tag{10}$$

For a curve $\ell(t)$ in $\mathbb{R}P^1$ as above, we define the *Maslov index*, $\mu(\ell(t), \mathbf{a})$, as:

$$\mu(\ell(t), \mathbf{a}) = \sum_{t_i} \text{sign } (\Gamma(\ell(t), \mathbf{a}). \tag{11}$$

The Maslov index defined above is a signed count of the number of times that $\ell(t)$, viewed as a curve in S^1 crosses the point $\mathbf{a}$.

Remark 1. We remark that because we are interested in travelling waves with only one spatial dimension, all of the relevant calculations will take place in $\Lambda(1)$, the space of one dimensional Lagrangian planes (lines) in $\mathbb{R}^2$. This space, as noted above, is naturally symplectically equivalent to $\mathbb{R}P^1 \approx S^1$, and as such the Maslov index of a curve in this case will simply be its homotopy class in $\pi_1(S^1)$. In this sense, the term "Maslov index" is a bit strong, and what is really being tracked is the winding number of the curve $\ell(z)$. This is determined in the following manner: the crossing form can be viewed as a means of determining the direction that $\ell(z)$ is travelling when it crosses the point (subspace) $\mathbf{a}$, and the Maslov index is just a signed count of such crossings. In [RS93] it is shown that such a count is well-defined up to homotopy type for a large class of continuous curves, and moreover generalizes well to higher dimensional spaces of Lagrangian planes.

We are now ready to move on to the proof of theorem 5. We proceed by breaking the proof up into two cases based on whether the wave speed has norm less (subluminal) or greater (superluminal) than one. In each case we show that there are no eigenvalues λ to (7) with non-zero real part.

2. Subluminal kink waves

In this section, the quantity $c^2 - 1 < 0$. We will focus primarily on the kink wave that satisfies the following boundary conditions, though the analysis that follows will apply to all subluminal kink waves. Let $v(z)$ be a solution to (3) satisfying:

$$\lim_{z \to -\infty} v(z) = -\pi \text{ and } \lim_{z \to \infty} v(z) = \pi \tag{12}$$

2.1. **Real Eigenvalues.**

The idea now is to reformulate the earlier eigenvalue definitions for a $\lambda \in \mathbb{R}$ in an equivalent, geometric way. Setting $\mathbf{w} = (w_1, w_2)$, with $w_1 = \varphi$, $w_2 = \varphi_z$ and $' = \frac{d}{dz}$, we have the following system of ODE, equivalent to (9):

$$\mathbf{w}' = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{\cos v - \lambda^2}{c^2 - 1} & \frac{-2c\lambda}{c^2-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =: A(\lambda, z)\mathbf{w}. \tag{13}$$

Also we set

$$A(\lambda) := \lim_{z\to\pm\infty} A(\lambda, z) = \begin{pmatrix} 0 & 1 \\ \frac{-1-\lambda^2}{c^2-1} & \frac{-2c\lambda}{c^2-1} \end{pmatrix}. \tag{14}$$

We remark that $A(\lambda)$ has an unstable and a stable subspace which we will denote by ξ^u and ξ^s respectively. Now, as $c^2-1<0$, the matrix

$$A(\lambda, z) \to A(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{-1-\lambda^2}{c^2-1} & \frac{-2c\lambda}{c^2-1} \end{pmatrix}$$

has eigenvalues γ_u and γ_s corresponding to the unstable and stable subspaces. Since we primarily have in mind the case when λ is real and positive, we can compute in this case explicit formulae for the stable and unstable eigenvalues:

$$\gamma_u = \frac{-c\lambda - \sqrt{\lambda^2-(c^2-1)}}{c^2-1}\ \text{, and } \gamma_s = \frac{-c\lambda + \sqrt{\lambda^2-(c^2-1)}}{c^2-1}, \tag{15}$$

and $(c^2-1)\gamma_{u,s}^2+2c\lambda\gamma_{u,s}+(\lambda^2+1)=0$. Moreover, using these values for $\gamma^{s,u}$, it is straightforward to verify that:

$$\xi^u =< \begin{pmatrix} 1 \\ \frac{\lambda^2+1}{-c\lambda+\sqrt{\lambda^2-(c^2-1)}} \end{pmatrix} >\ \text{, and } \xi^s =< \begin{pmatrix} 1 \\ \frac{\lambda^2+1}{-c\lambda-\sqrt{\lambda^2-(c^2-1)}} \end{pmatrix} > . \tag{16}$$

where $< \begin{pmatrix} a \\ b \end{pmatrix} >$ denotes the linear space spanned by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. (In the case that one considers negative real eigenvalues, the signs in front of the radicals in the expressions in (15) and (16) are exchanged).

Lemma 6 (Geometric version of the eigenvalue condition)**.** *Given that* $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ *solves (13)*

$$\lim_{z\to\pm\infty} w_1 = 0 \tag{17}$$

is equivalent to

$$\lim_{z\to-\infty} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \xi^u,\ \textit{and} \lim_{z\to\infty} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \xi^s\ \textit{in } \mathbb{R}P^1 \tag{18}$$

Proof. We compactify the extended phase plane of (13) by introducing a new variable $z = z(\tau) = \tan(\frac{\pi\tau}{2})$, where $\tau \in (-1,1)$. Now the extended system of ODE's becomes the (autonomous)

$$\begin{aligned} \mathbf{w}' &= A(\tau,\lambda)\mathbf{w} \\ \tau' &= \tfrac{2}{\pi}\cos^2\tfrac{\pi\tau}{2} \end{aligned} \tag{19}$$

Next we note that if we view the space $\mathbb{R}^2\times(-1,1)$ as a plane bundle over the segment $[-1,1]$, the linearity of (13) means that linear subspaces of the fibres are preserved, so the flow defined by (19) on $\mathbb{R}^2\times(-1,1)$ induces a well defined flow on the cylinder $S^1\times(-1,1)$. Moreover, this flow can be continuously extended to a flow on $S^1\times[-1,1]$. Finally we note that on this cylinder, the points $(\xi^u,-1)$ and $(\xi^s,1)$ are fixed points of the induced flow, with a one dimensional unstable and stable manifold respectively. The uniqueness of the stable and unstable manifolds concludes the proof of the lemma. ∎

To summarize: we have redefined the original eigenvalue condition as the existence of a heteroclinic orbit joining $(\xi^u, -1)$ and $(\xi^s, 1)$, where we are viewing $(\xi^s, 1)$ and $(\xi^u, -1)$ as steady states of the flow induced by (9) on the cylinder $S^1 \times [-1, 1]$.

For each fixed λ, let $\ell(z)$ be the set of lines in $\mathbb{R}^2$ that tend to the unstable subspace of A at $-\infty$, under the flow of (13) (equivalently (9)) , that is:

$$\ell(z) = \left\{ < \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} > \mid \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{ solves (9) and } \to \xi^u, \text{ as } z \to -\infty \right\} \tag{20}$$

As explained in section 1.2, the lines $\ell(z)$ are Lagrangian, and in fact the representation $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a frame for the line ℓ, so we can define the crossing form relative to the subspace ξ^s, $\Gamma(\ell(z), \xi^s, \lambda, z)$ as $\Gamma(\ell(z), \xi^s, \lambda, z) = w_1 w_2' - w_2 w_1'$. Substituting as in equation (13) gives

$$\Gamma(\ell(z), \xi^s, \lambda, z) = \frac{\cos v - \lambda^2}{c^2 - 1} w_1^2 - \frac{2c\lambda}{c^2 - 1} w_1 w_2 - w_2^2 \tag{21}$$

Lemma 7. *The crossing form of $\ell(z)$, relative to the stable subspace at infinity, $\Gamma(\ell(z), \xi^s, \lambda, z)$ defined above is independent of λ.*

Proof. To evaluate the crossing form on ξ^s, we use the fact that on ξ^s we have that $w_2 = \frac{\lambda^2+1}{-c\lambda + \sqrt{\lambda^2 - (c^2-1)}} w_1 = \frac{(\lambda^2+1)}{\gamma_s (c^2-1)} w_1$. This gives

$$\begin{aligned}
\Gamma &= \omega_1^2 \left[\frac{\cos v - \lambda^2}{c^2 - 1} - \frac{2c\lambda}{c^2 - 1} \frac{(\lambda^2 + 1)}{\gamma_s (c^2 - 1)} - \left(\frac{(\lambda^2 + 1)}{\gamma_s (c^2 - 1)} \right)^2 \right] \\
&= \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{(\cos v - \lambda^2)\gamma_s^2 (c^2 - 1) - 2c\lambda(\lambda^2 + 1)\gamma_s - (\lambda^2 + 1)^2}{\gamma_s^2 (c^2 - 1)} \right] \\
&= \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{\cos v \gamma_s^2 (c^2 - 1) - (\lambda^2)\gamma_s^2 (c^2 - 1) - 2c\lambda(\lambda^2 + 1)\gamma_s - (\lambda^2 + 1)^2}{\gamma_s^2 (c^2 - 1)} \right] \\
&= \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{\cos v \gamma_s^2 (c^2 - 1) - (\lambda^2)\gamma_s^2 (c^2 - 1) - 2c\lambda(\lambda^2 + 1)\gamma_s - (\lambda^2 + 1)^2}{\gamma_s^2 (c^2 - 1)} \right] \\
&\quad + \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{-\gamma_s^2 (c^2 - 1) + \gamma_s^2 (c^2 - 1)}{\gamma_s^2 (c^2 - 1)} \right] \\
&= \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{(\cos v + 1)\gamma_s^2 (c^2 - 1) - (\lambda^2 + 1)\big((c^2 - 1)\gamma_s^2 + 2c\lambda\gamma_s + (\lambda^2 + 1)\big)}{\gamma_s^2 (c^2 - 1)} \right] \\
&= \frac{\omega_1^2}{(c^2 - 1)} \left[\frac{(\cos v + 1)\gamma_s^2 (c^2 - 1)}{\gamma_s^2 (c^2 - 1)} \right]
\end{aligned}$$

because $(c^2 - 1)\gamma_s^2 - 2c\lambda\gamma_s + (\lambda^2 + 1) = 0$. Thus we have that:

$$\Gamma(\ell(z), \xi^s, \lambda, z) = \frac{(\cos v + 1)\omega_1^2}{(c^2 - 1)}. \tag{22}$$

Moreover, since everything takes place inside $\mathbb{R}P^1 \approx S^1$, as long as $\omega_1 \neq 0$, we can choose $\omega_1 = 1$. But $\omega_1 = k\gamma^s$ for some non-zero constant k, and $\omega_1 = 0$ would imply that $\gamma^s = 0$ which would imply that $\gamma^s\gamma^u = 0$. However we have that

$$\gamma^s\gamma^u = \frac{\lambda^2+1}{c^2-1} \neq 0 \tag{23}$$

as we are assuming that λ is real. Thus we have that at a crossing,

$$\Gamma(\ell(z), \xi^s, \lambda, z) = \frac{(\cos v + 1)}{(c^2-1)} \tag{24}$$

Which shows that Γ is independant of λ. $\square$

We next observe that only regular crossings occur.

Lemma 8. *If z is such that $\ell(z) = \xi^s$, then $\ell'(z) \neq 0$.*

Proof. Because $v(z)$ corresponds to a heteroclinic orbit in the phase plane of the pendulum equation (3), where $c^2 - 1 < 0$, equation (22) is clearly non-zero, except at $z = \pm\infty$. At a crossing we have that $\ell(z) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \xi^s$. Moreover in $\mathbb{R}P^1$ we can represent $\ell(z)$ by the function $f(z,\lambda) = \frac{w_2}{w_1}$, which will be well defined provided $\omega_1 \neq 0$ in a neighborhood of a crossing. We showed this was the case for real λ in the proof of lemma 7. Thus if $\ell'(z) = 0$ we have that $\frac{\partial f}{\partial z} = 0$. But we can see that at a crossing:

$$f(z,\lambda) = \frac{\lambda^2+1}{-c\lambda + \sqrt{\lambda^2 - (c^2-1)}}, \tag{25}$$

and that

$$\frac{\partial}{\partial z} f(z,\lambda) = \frac{\Gamma(\ell(z), \xi^s, \lambda, z)}{w_1^2} \neq 0. \tag{26}$$

And we see that only regular crossings can occur. $\square$

The function $f(z,\lambda)$ used in the proof of lemma 8, can be used to further characterize how $\ell(z)$ crosses ξ^s in $\mathbb{R}P^1$. Since $\dfrac{\partial}{\partial z} f(z,\lambda) \neq 0$ at a crossing, we can use the implicit function theorem to write the location of the crossing in the z variable as a function of the parameter λ, $z = z(\lambda)$, further because equation (22) is independant of λ, we can do this for all λ.

Evaluating the limits of $\ell(z)$ as $z \to \pm\infty$, and using lemmas 7 and 8, we see that the Maslov index of the curve of Lagrangian subspaces $\ell(z)$, with respect to the stable subspace at infinity is:

$$\mu(\ell(z), \xi^s, \lambda) = \sum_z \operatorname{sign} \Gamma(\ell(z), \xi^s, \lambda, z) \tag{27}$$

$$= -\#\text{ of crossings that occur.} \tag{28}$$

Now fix a λ and a location of a crossing $z_1(\lambda)$ say. Because the number of crossings tends to zero as λ increases, and because of the implicit function theorem assertion above, we have that for some finite λ_1, $\lim\limits_{\lambda\to\lambda_1} z_1(\lambda) = \infty$. But this is exactly the geometric reformulation of the eigenvalue condition. Moreover, the implicit function theorem implies that we have a unique value, $z_1(\lambda)$, for this crossing for each λ, and the locations of different crossings (say $z_2(\lambda)$) cannot intersect.

Thus as λ increases, the number of crossings will decrease monotonically in λ. Geometrically, this means that the line $\ell(z)$ viewed as a curve in S^1, can cross the point ξ^s in only one direction.

We have that an eigenvalue is a value of λ such that $\lim_{z\to-\infty} \ell(z) \to \xi^u$ and $\lim_{z\to\infty} \xi^s$, however when λ is not an eigenvalue, the flow induced by (9) on $S^1 \times [-1,1]$, is such that we have $\lim_{z\to\pm\infty} \ell(z) = \xi^u$. Since $\ell(z)$ can only cross the stable subspace in one direction (in z), this means that if the number of crossings (over all $z \in \mathbb{R}$) changes as we vary λ in an interval (λ_1, λ_2), then we must have an eigenvalue in that interval.

These last paragraphs are summarized in the following corollary.

Corollary 9. *If N_1 is the number of times $\ell(z)$ crosses ξ^s when $\lambda = \lambda_1$, for $z \in \mathbb{R}$ and N_2 is the number of times $\ell(z)$ crosses ξ^s when $\lambda = \lambda_2$, then $|N_1 - N_2|$ = the number of real eigenvalues of the linearized operator $\mathcal{L}$ in equation (6) in the interval (λ_1, λ_2).*

When $\lambda \gg 1$, the system in (13) tends to

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{-\lambda^2}{c^2-1} & \frac{-2c\lambda}{c^2-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2. \end{pmatrix} =: A(\lambda \gg 1) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{29}$$

which has eigenvalues $\gamma_\pm = \frac{-\lambda}{c\pm1}$. Which are real and as $c^2 - 1 < 0$, one is positive and one is negative. For large enough λ then, the number of crossings of $\ell(z)$ relative to ξ^s is the same as the number of crossings as for the solution of the constant coefficient equation (29) tending towards the unstable subspace of the matrix $A(\lambda \gg 1)$. Thus for large enough λ, the number of crossings of the unstable manifold $\ell(z)$ and ξ^s is equal to zero.

Lastly, we show that $\mu(\ell(z), \xi^s, 0, z) = 0$. This shows that there are no crossings when $\lambda = 0$ and so by corollary 9 there are no crossings for $\lambda \in (0, \infty)$. To see this observe that when $\lambda = 0$, equation (13) reduces to the equation of variations of the pendulum equation (3). Thus the unstable manifold of equation (13) is just the tangent line to the heteroclinic orbit, joining $(-\pi, 0)$ to $(\pi, 0)$ in the phase plane of (13). This orbit in the phase plane is given by:

$$v_z = \sqrt{2\frac{\cos v + 1}{1 - c^2}} \tag{30}$$

and the slope of the tangent line is given by

$$\frac{dv_z}{dv} = \frac{-\sin v}{\sqrt{(c^2-1)(2\cos v + 2)}}. \tag{31}$$

In order for a crossing to occur, the tangent line must be parallel to ξ^s. This means that

$$\frac{-\sin v}{\sqrt{(c^2-1)(2\cos v + 2)}} = \frac{-1}{\sqrt{-(c^2-1)}} \tag{32}$$

But this only happens when $v = \pm\pi$. But $v(z)$ is a heteroclinic orbit in the phase plane joining the critical points $(0, -\pi)$ and $(0, \pi)$. Thus there is no $z \in (-\infty, \infty)$ where $v(z) = \pm\pi$, so there can be no crossings. This shows that there are no real eigenvalues of the operator $\mathcal{L}$ for $\lambda \in (0, \infty)$ when the quantity $c^2 - 1 < 0$.

As as concluding remark we note that the crossing form argument works in the negative λ direction and that the asymptotic behaviour of the system in (13) for $\lambda \ll -1$ is the same as that for $\lambda \gg 1$. Thus we have no real eigenvalues $\lambda \neq 0$.

2.2. **Complex Eigenvalues.** We now return our attention to equation (9) to show that there are no complex eigenvalues $\lambda = p + iq$ where $p > 0$. Without loss of generality we can assume that $q > 0$ so $\lambda = re^{i\theta}$ where $r > 0$ and $\theta \in (0, \frac{\pi}{2})$. Rewriting (9) as:

$$w'' + \frac{2c\lambda}{c^2-1}w' + \frac{(\lambda^2 - \cos(v(z)))}{c^2-1}w = 0, \tag{33}$$

we make the substitution

$$\psi = we^{\left(\frac{c\lambda}{c^2-1}z\right)} \tag{34}$$

and re-write (33) as

$$\psi'' = \left(\frac{\cos v - \lambda^2}{c^2-1} + \frac{c^2\lambda^2}{(c^2-1)^2}\right)\psi \tag{35}$$

We wish to consider now complex valued solutions to (35). We let $\eta = \frac{\psi'}{\psi}$ and consider the induced flow on a chart of $\mathbb{C}P^1$, where $\psi \neq 0$:

$$\eta' = \frac{\psi''\psi - \psi'\psi'}{\psi^2} = \frac{\cos v - \lambda^2}{c^2-1} + \frac{c^2\lambda^2}{(c^2-1)^2} - \eta^2. \tag{36}$$

If we set $\eta = \alpha + i\beta$, and consider the imaginary part of the vector field of (36) on the real axis, that is those points where $\eta = \alpha$, we have

$$\beta'\big|_{\beta=0} = \frac{-2pq}{c^2-1} + \frac{2c^2pq}{(c^2-1)^2} = \frac{2pq}{(c^2-1)^2} > 0 \qquad \text{if } p, q > 0. \tag{37}$$

thus the flow of (36) on the real axis of $\mathbb{C}P^1$ is always pointing in the positive imaginary direction.

By considering $\frac{w_2}{w_1}$ from equation (13), we now interpret the eigenvalue conditions from definitions 3 and 4 for the eigenvalues that we are interested in.

Lemma 10. *An eigenvalue λ is a value of λ where there exists a heteroclinic orbit of the flow induced by (13), on (a chart of) $\mathbb{C}P^1$, i.e. λ an eigenvalue means that there is an orbit from $\lim_{z\to-\infty} \frac{w_2}{w_1} = \gamma_u$ to $\gamma_s = \lim_{z\to\infty} \frac{w_2}{w_1}$. Under the transformation given in (34) we have that*

$$\eta = \frac{w_2}{w_1} + \frac{c\lambda}{c^2-1}, \tag{38}$$

and so γ_u is mapped to η_u and γ_s is mapped to η_s where

$$\eta_u = \frac{-\sqrt{\lambda^2 - (c^2-1)}}{c^2-1}, \text{ and} \tag{39}$$

$$\eta_s = \frac{\sqrt{\lambda^2 - (c^2-1)}}{c^2-1}. \tag{40}$$

We claim that for $\lambda = p + iq$ with p and q both positive, that η_u has a positive imaginary part and η_s has a negative imaginary part. Thus as $\beta'\big|_{\beta=0} > 0$, from above, there can be no orbit joining the two critical points, and so no eigenvalues with positve real part. Thus all kink waves satisfying $c^2 - 1 < 0$ can have eigenvalues only on the imaginary axis.

To see that η_u has a positive imaginary part, write $\lambda = re^{i\theta}$ and $\eta_u = R_u e^{i\theta_u}$. Since we are assuming that λ has positive imaginary part $\theta \in (0, \frac{\pi}{2})$. By equation (39) we have that

$$\theta_u = \arctan\left(\frac{r^2\sin(2\theta)}{r^2\cos(2\theta) + (1-c^2)}\right). \tag{41}$$

Since θ_u will be the same for all points on the line $re^{i\theta_u}$, and since $c^2 - 1 < 0$, we can assume without loss of generality that $r^2 = (1-c^2)$. Thus substituting into equation (41) we have:

$$\theta_u = \arctan\left(\frac{\sin(2\theta)}{\cos(2\theta)+1}\right) = \arctan\left(\frac{2\sin\theta\cos\theta}{\cos^2(\theta) - \sin^2(\theta) + 1}\right) = \theta. \tag{42}$$

The exact same calculation shows that η_s has a negative imaginary part.

We have shown that there can be no heteroclinic orbit on the chart of $\mathbb{C}P^1$ parametrized by $\frac{w_2}{w_1}$. To see that on there can be no such orbit on the other chart, we have that off the origin, the charts are transformed into each other via $\eta \to \frac{1}{\eta}$ for $\zeta \in \mathbb{C}\backslash 0$. Thus $\eta_u \to \frac{1}{\eta_u}$ which will have a negative imaginary part. Likewise $\frac{1}{\eta_s}$ will have a positive imaginary part. If we write $\zeta = \frac{\psi}{\psi'} = \vartheta + i\varsigma$ and consider the flow on this chart induced by (9)

$$\zeta' = \frac{(\psi')^2 - \psi\psi''}{\psi'^2} = 1 - (\frac{\cos v - \lambda^2}{c^2-1} + \frac{c^2\lambda^2}{(c^2-1)^2})\zeta^2. \tag{43}$$

Now letting $\zeta = \sigma + i\tau$ and considering only the imaginary part of the flow on $\mathbb{C}$ given by (43) restricted to where $\tau = 0$ we have that:

$$\tau'\big|_{\tau=0} = \frac{-2pqc^2\sigma^2}{(c^2-1)^2} - \frac{2pq\sigma^2}{c^2-1} = -\frac{2pq\sigma^2}{(c^2-1)^2} < 0 \qquad\qquad \text{if } p, q > 0. \tag{44}$$

Thus the flow of (36) on the real axis of (this chart of) $\mathbb{C}P^1$ is always pointing in the negative imaginary direction, and so there can be no heteroclinic orbit connecting $\frac{1}{\eta_u}$ to $\frac{1}{\eta_s}$, and hence there are no eigenvalues λ with positive real part to equation (7).

The same argument can be run in for $\lambda = p + iq$, with p and q both negative to show that the flow on the real axis is pointing in the wrong direction to allow for an orbit joining η^u to η^s as well. Just as in the previous case, one makes a Liouville transformation and then tracks the location of the stable and unstable subspaces on both charts of $\mathbb{C}P^1$. Thus we conclude that there are no eigenvalues off the imaginary axis.

3. Essential Spectrum

In this section we compute the essential spectrum of the operator $\mathcal{L}$ in both the subluminal and superluminal cases. As stated earlier, the reason for not continuing directly to the superluminal case is that the essential spectrum in the superluminal case plays a role in organizing the calculations necessary to show the absence of point spectra.

Here, as in [BJ89], we consider the operator $\mathcal{L}$ given in equation (6), on $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. It is the infinitesimal generator of a C_0 semigroup which corresponds to the time evolution of the linearized sine-Gordon equation about a kink wave solution. Recall that $\mathcal{L}$ satisfies:

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}_t = \begin{pmatrix}\psi\\ \cos v\varphi - (c^2-1)\varphi_{zz} - 2c\psi_z\end{pmatrix} := \mathcal{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \tag{45}$$

where $v(z)$ is solution to the pendulum equation corresponding to a heteroclinic orbit in the phase-plane of equation (3).

As in [BJ89] one can apply results from [Paz83] to show that the operator $\mathcal{L}$ is a compact perturbation of the operators $\mathcal{L}_-$ in the subluminal case, and $\mathcal{L}_+$ in the superluminal case, both of which map $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ to itself, with:

$$\mathcal{L}_- \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \psi \\ -\varphi - (c^2-1)\varphi_{zz} - 2c\psi_z \end{pmatrix} \tag{46}$$

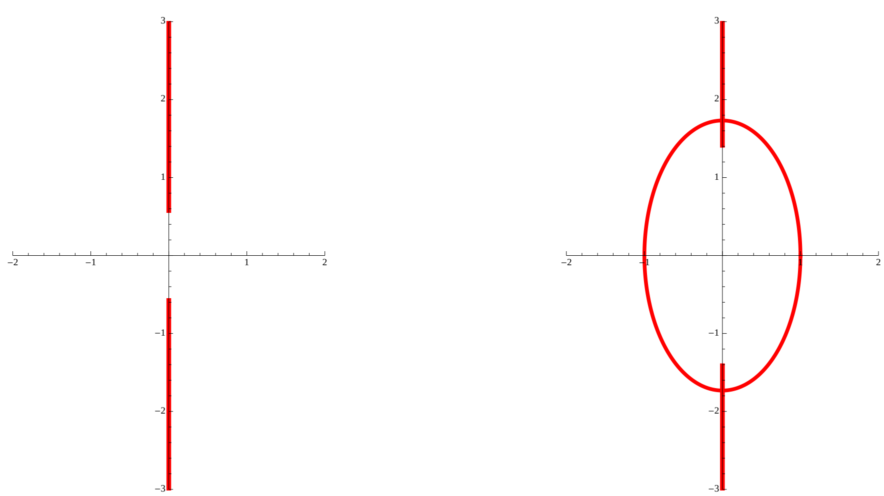
and

$$\mathcal{L}_+ \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \psi \\ \varphi - (c^2-1)\varphi_{zz} - 2c\psi_z \end{pmatrix}. \tag{47}$$

We can then compute the essential spectrum of $\mathcal{L}_\pm$ directly as:

$$\begin{aligned}
\sigma_{\text{ess}}(\mathcal{L}_-) &= \{\mu \in \mathbb{C} | (1 - (c^2-1)k^2) + 2cik\mu + \mu^2 = 0 \text{ for all values of } k \in \mathbb{R}\} && (48)\\
&= \{\mu \equiv i\beta | \beta \in (-\infty, -c] \cup [c, \infty)\} \text{ and,} && (49)\\
\sigma_{\text{ess}}(\mathcal{L}_+) &= \{\mu \in \mathbb{C} | -(1 + (c^2-1)k^2) + 2cik\mu + \mu^2 = 0 \text{ for all values of } k \in \mathbb{R}\} && (50)\\
&= \{\mu \equiv \alpha + i\beta | \alpha^2 + \frac{\beta^2}{c^2} = 1\} \cup \{\mu \equiv i\beta | \beta \in (-\infty, -\sqrt{c^2-1}] \cup [\sqrt{c^2-1}, \infty) && (51)
\end{aligned}$$

where the above α and β are real numbers. From here we see that in the subluminal case, the essential spectrum of $\mathcal{L}_-$ (and hence $\mathcal{L}$) lies entirely on the imaginary axis. In the superluminal case, the essential spectrum of $\mathcal{L}$ consists of an ellipse centered at the origin, intersecting the real axis at ± 1, and intersecting the imaginary axis at $\pm c$, together with a pair of lines which are contained in the imaginary axis, but extend past the vertices of the ellipse on the imaginary axis to the points $\pm i\sqrt{c^2-1}$ (see figure below).



(a) $c = \frac{1}{\sqrt{3}}$ (b) $c = \sqrt{3}$

FIGURE 1. The essential spectrum of the operator $\mathcal{L}$ in the subluminal 1(a) and superluminal 1(b) case.

4. Superluminal kink-waves

We now consider the superluminal eigenvalue problem. Recalling equation (7) we have:

$$\begin{pmatrix} \psi \\ (\cos v)\varphi - (c^2-1)\varphi_{zz} - 2c\psi_z \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{52}$$

We are interested in the values of $\lambda \in \mathbb{C}$ for which there are solutions to (52) that decay to zero as $z \to \pm\infty$. We first investigate the limiting cases $\lim_{z\to\pm\infty}$, which become the constant coefficient ODE

$$(1-c^2)\varphi'' - 2c\lambda\varphi - (\lambda^2+1)\varphi = 0 \tag{53}$$

which can easily be solved for any value of $\lambda \in \mathbb{C}$. The characteristic exponents $r_{1,2}$ are given by:

$$r_{1,2} = \frac{c\lambda \pm \sqrt{\lambda^2 + (1-c^2)}}{1-c^2} \tag{54}$$

We observe that for λ outside of the ellipse of essential spectrum and off the imaginary axis we have that the signs of the real parts of $r_{1,2}$ are equal. In fact they will be the opposite of the sign of the real part of λ. We have:

$$\text{sgn } (\Re(r_1)) = -\text{sgn } (\Re(2\lambda)) \qquad \text{and, sgn } (\Re(r_2)) = -\text{sgn } (\Re(\lambda + \frac{1}{\lambda}))$$

Thus we can conclude that in the case of $c^2 - 1 > 0$, we have that for the asymptotic cases when λ is outside the ellipse of essential spectrum described in section 3, as $z \to -\infty$, all eigenvalues have negative real part, and we conclude that there is no point spectrum outside the ellipse off the imaginary axis when $c^2 - 1$ is positive.

We next observe that for values λ inside the ellipse of essential spectrum but still off the imaginary axis, it is straightforward to check that the same crossing form argument and Riccati equation techniques as used in section 2 will hold. As we are only interested in whether or not the crossing form changes sign, it is apparent from the calculations in section 2 that the change in sign of $c^2 - 1$ will not affect the number of zeroes (or independence from the eigenvalue parameter) of the crossing form. Further we can make the appropriate Liouville transformation (as in (34)) and, by keeping track of the appropriate signs, determine the location of the stable and unstable subspaces of the transformed equation on $\mathbb{C}P^1$. Again it is straightforward to check that there can be no heteroclinic orbit on the charts of $\mathbb{C}P^1$. Thus we conclude that there can be no eigenvalues of the linearized operator $\mathcal{L}$, given in (7) with non-zero real part when $c^2 - 1$ is positive. This concludes the proof of theorem 5

References

[Abb01] A. Abbondandolo, *Morse theory for hamiltonian systems*, CRC research notes in mathematics, no. 425, Chapman and Hall, 2001.

[Arn67] V.I. Arnol'd, *Characteristic classes entering in quantization conditions*, Func. Anal. Appl. **1** (1967), 1–14.

[BJ89] P. Bates and C.K.R.T. Jones, *Invariant manifolds for semilinear partial differential equations*, Dynamics Reported, vol. 2, John Wiley & Sons, 1989.

[BJ09] J. C. Bronski and M. A. Johnson, *Krein signatures for the faddeev-takhtajan eigenvalue problem*, Communications in Mathematical Physics **288** (2009), 821–846.

[BM07] R. Buckingham and P. Miller, *Exact solutions of the semiclassical non–characteristic Cauchy problems for the sine-Gordon equation*, Physica D **237** (2007), 2296–2341.

[BS71] F. Magee C. J. Barone, A. Esposito and A. Scott, *Theory and applications of the sine-Gordon equation*, Riv. Nuovo. Cimento **1** (1971), 227–267.

[DDvGV03] G. Derks, A. Doelman, S. A. van Gils, and T. Visser, *Travelling waves in a singularly perturbed sine-gordon equation*, Physica D **180** (2003), 40–70.

[Mas72] V. Maslov, *Theory of perturbations and asymptotic methods*, French translation of Russian original (1965) (1972).

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[RS93] J. Robbin and D. Salamon, *The Maslov index for paths*, Toplogy **32** (1993), no. 4, 827 – 844.

[Sal91] M. Salerno, *Discrete model for DNA promoter dynamics*, Physical Review A **44** (1991), no. 8, 5292–5297.

[SCM73] A. Scott, F. Chu, and D. McLaughlin, *The soliton: a new concept in applied science*, Proc. of the IEEE **61** (1973), no. 10, 1443–1483.

[Sco69] A. Scott, *Waveform stability on a nonlinear klein-gordon equation*, Proc. Letters of the IEEE (1969).

[SCR76] A. Scott, F. Chu, and S. Reible, *Magnetic-flux propagation on a Josephson transmission line*, J. Applied Phys. **47** (1976), no. 7, 3272–3286.

[Whi99] G. B. Whitham, *Linear and nonlinear waves*, Wiley-Interscience Series, John Wiley & Sons, 1999.