Overview:

$$\begin{tikzcd}
 & G_{\mathbb{C},\mathbb{R}} \arrow[d] \arrow[rr, squiggly] & & \text{Transformations of the quantum plane / etc.} \\
\text{Lie Alg} & \mathfrak{g} \arrow[d, squiggly] & & \\
\text{UEA} & U(\mathfrak{g}) \arrow[r, squiggly] \arrow[d, squiggly] & U_q(\mathfrak{g}) \arrow[d, squiggly] & \text{Drinfeld - Jimbo Quantum Gp.} \\
\text{Kostant } \mathbb{Z}\text{-form} & U_{\mathbb{Z}}(\mathfrak{g}) \arrow[d, squiggly] & U_{\mathcal{A}}(\mathfrak{g}) \arrow[l, squiggly] \arrow[d, squiggly] & \text{Lusztig } \mathcal{A}\text{-form} \\
\text{Alg. Gp in } p \text{ char} & G_{\bar{k}} & U_\varepsilon(\mathfrak{g}) \arrow[l, squiggly] & \text{Quantum group at a root of unity.}
\end{tikzcd}$$

There are too many quantum groups at a root of unity.

$$\underset{\text{De Concini Kac Procesi}}{U_\varepsilon} \longrightarrow \underset{\text{small Quantum gp.}}{u_\varepsilon} \longrightarrow \underset{\text{Lusztig}}{U_\varepsilon}$$

Lie Algebras:

Serre's Thm:

Given an irreducible crystallographic root system R with simple roots $\{\alpha_i \in \Sigma\}$. Then the lie algebra generated by $\{e_i, f_i, h_i \mid 1 < i < n\}$ subject to

$$[h_i, h_j] = 0$$
$$[e_i, f_j] = \delta_{ij} h_i$$

$$[h_i, e_j] = \langle \alpha_j, \alpha_i^\vee \rangle e_j$$
$$[h_i, f_j] = -\langle \alpha_j, \alpha_i^\vee \rangle f_j$$

$$ad(e_i)^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(e_j) = 0.$$

$$ad(f_i)^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(f_j) = 0.$$

Universal Enveloping Algebra:

The UEA is the associative unital algebra generated by $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ subject to:

$$[x, y] = xy - yx.$$

$$[h_i, h_j] = 0$$
$$[e_i, f_j] = \delta_{ij} h_i$$

$$[h_i, e_j] = \langle \alpha_j, \alpha_i^\vee \rangle e_j$$
$$[h_i, f_j] = -\langle \alpha_j, \alpha_i^\vee \rangle f_j$$

$$\sum_{k=0}^{1-\langle\alpha_j,\alpha_i^\vee\rangle} (-1)^k \binom{1-\langle\alpha_j,\alpha_i^\vee\rangle}{k} e_i^{1-\langle\alpha_j,\alpha_i^\vee\rangle-k} e_j e_i^k = 0$$

$$\sum_{k=0}^{1-\langle\alpha_j,\alpha_i^\vee\rangle} (-1)^k \binom{1-\langle\alpha_j,\alpha_i^\vee\rangle}{k} f_i^{1-\langle\alpha_j,\alpha_i^\vee\rangle-k} f_j f_i^k = 0$$

Drinfeld - Jimbo Quantum Gp.

$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra generated by K_α K_α^{-1}, F_α, E_α where $\alpha \in \Sigma$

subject to the relations:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1,$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \qquad q_\alpha = q^{d_\alpha}.$$

$$\sum_{k=0}^{1-\langle\alpha_j,\alpha_i^\vee\rangle} (-1)^k \begin{bmatrix} 1-\langle\alpha_j,\alpha_i^\vee\rangle \\ k \end{bmatrix}_q E_i^{1-\langle\alpha_j,\alpha_i^\vee\rangle-k} E_j E_i^k = 0$$

$$\sum_{k=0}^{1-\langle\alpha_j,\alpha_i^\vee\rangle} (-1)^k \begin{bmatrix} 1-\langle\alpha_j,\alpha_i^\vee\rangle \\ k \end{bmatrix}_q F_i^{1-\langle\alpha_j,\alpha_i^\vee\rangle-k} F_j F_i^k = 0$$

$$K_\alpha E_\beta K_\beta^{-1} = q^{\langle\alpha,\beta^\vee\rangle} E_\beta$$
$$K_\alpha F_\beta K_\alpha^{-1} = q^{-\langle\alpha,\beta^\vee\rangle} F_\beta.$$

Quantum Integers: $n \in \mathbb{Z}$.

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]! = \begin{cases} 1 & \text{if } n=0 \\ [n]\cdots[1] & \text{if} > 0 \end{cases} \qquad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]![k]!}$$

Note : $[\mathrm{Rep}\, U_q(\mathfrak{sl}_2)] \xrightarrow{\sim}$ Ring of Quantum ints.

$$\Delta(1) \longmapsto [2].$$
$$\Delta(\lambda) \longmapsto [\lambda+1].$$

Important properties of $U_q(\mathfrak{g})$ and $\mathrm{Rep}\, U_q(\mathfrak{g})$.

1) $U_q(\mathfrak{g})$ admits a PBW basis.

2) Finite dimensional modules admit weight bases. where $m \in M$ is of weight

$$(\lambda, \sigma) \in X \times \mathrm{Hom}(\mathbb{Z}R, \{\pm 1\}) \text{ if}$$

$$K_\mu m = \sigma(\mu)\, q^{\langle \lambda, \alpha^\vee \rangle}. \qquad \mu = \sum a_i \alpha_i, \quad K_\mu = \prod K_{\alpha_i}^{a_i}$$

3) $\mathrm{Rep}\, U_q(\mathfrak{g})$ is a semi-simple abelian category. It decomposes as abelian cat into:

$$\mathrm{Rep}\, U_q(\mathfrak{g}) = \bigoplus_\sigma \mathrm{Rep}_\sigma\, U_q(\mathfrak{g})$$

where $\mathrm{Rep}_\sigma\, U_q(\mathfrak{g})$ is the full sub cat whose objects are (λ, σ) weight spaces.

4) We can construct Verma modules:

So if $U_q(\mathfrak{b}) = \operatorname{span}\{K_{\mathfrak{I}} E_{\mathfrak{I}} \mid \mathfrak{I} \subset R\}$,

Then:

$$Z(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b})} \mathbb{Q}(q)_\lambda$$

It has a BGG res:

$$\cdots \to \bigoplus_{s \in S} Z(s \cdot \lambda) \longrightarrow Z(\lambda)$$

with simple quotient $\Delta(\lambda)$.

5) The $\Delta(\lambda)$ form a complete set of simple objects of $\mathrm{Rep}_1 U_q(\mathfrak{g})$.
