1 The p-canonical basis of the anti-spherical Hecke module

In this talk we provide a brief introduction to the p-canonical basis of the anti-spherical Hecke module. Moreover, we explicitly calculate it in type $\tilde{A}_1$; the only case where it is explicitly known.

The structure of this talk is as follows:

(1) The Hecke algebra and its canonical bases;
(2) The anti-spherical Hecke module;
(3) Explicitly calculating the p-canonical basis in type $\tilde{A}_1$.

Notation

Throughout this talk we adopt the following notation:

- $\Bbbk$: an algebraically closed field of characteristic $p > 0$;
- $G_{\Bbbk}$: a simple, simply connected, algebraic group scheme over $\Bbbk$;
- $(R, X, R^{\vee}, X^{\vee})$: the root datum associated to $G_{\Bbbk}$;
- $T_{\Bbbk} \subset B_{\Bbbk} \subset G_{\Bbbk}$: a pinning of $G_{\Bbbk}$;
- $W_{\mathrm{f}} = N_G(T)/T$: the finite Weyl group;
- $W_{\mathrm{p}} = W_{\mathrm{f}} \ltimes p\mathbb{Z}R$: the p-dilated affine Weyl group;
- ${}^{\mathrm{f}}W_p$: the minimal left coset representatives;
- w_f: the longest element of W_{f}; and
- h: the Coxeter number.

The Hecke algebra and its canonical bases

Recall that the Hecke algebra H associated to the Coxeter system (W_p, S_p) is the associative, unital, $\mathbb{Z}[v^{\pm 1}]$-algebra generated by the symbols $\{\delta_w | w \in W_p\}$ subject to the relations:

$$\delta_w \delta_{w'} = \delta_{ww'} \qquad \text{if } \ell(w) + \ell(w') = \ell(ww')$$
$$(\delta_s + v)(\delta_s - v^{-1}) = 0 \qquad \text{for all } s \in S_{\mathrm{p}}$$

Matsumoto's lemma implies the symbols $\{\delta_w | w \in W_{\mathrm{p}}\}$ are well defined. Moreover, $\{\delta_w | w \in W_{\mathrm{p}}\}$ forms a basis of H called the standard basis. Each standard basis element is invertible.

There exists an involution $\overline{}$ on H given by:

$$\overline{\delta_w} = \delta_{w^{-1}}^{-1} \qquad \overline{v} = v^{-1}$$

By a theorem of Kazhdan and Lusztig, for each $w \in W_{\mathrm{p}}$, there exists a unique element $b_w \in H$ such that $\overline{b_w} = b_w$ and $b_w \in \delta_w + \sum_{u<w} \mathbb{Z}[v]\delta_u$. The set $\{b_w | w \in W_p\}$ form a basis of H called the canonical basis.

To motivate the definition of the p-canonical basis, we have to first categorify the presentation of the Hecke algebra given by the canonical basis.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Consider the groups $G^{\vee}(\mathcal{O}) \subset G^{\vee}(\mathcal{K})$. There is a natural map $G^{\vee}(\mathcal{O}) \to G^{\vee}(\mathbb{C})$ induced by $t \mapsto 0$. The Iwahori subgroup $I \subset G^{\vee}(\mathcal{O})$ is defined to be the pre-image of B under this map. The affine flag variety $\mathcal{F}l$ is the ind-projective ind-scheme whose $\mathbb{C}$-points may be identified with the space $G^{\vee}(\mathcal{K})/I$. It is a Kac-Moody flag variety, and thus admits a Bruhat decomposition:

$$\mathcal{F}l = \bigsqcup_{w \in W_p} \mathcal{F}l_w \qquad \text{where } \mathcal{F}l_w := IwI/I$$

The closure order is given by the Bruhat order on W_{p}. More precisely:

$$\overline{\mathcal{F}l_w} = \bigsqcup_{u \leq w} \mathcal{F}l_u$$

Each $\overline{\mathcal{F}l_w}$ is called an affine Schubert variety.

The category of I-equivariant perverse sheaves on the affine flag variety with coefficients in $\mathbb{C}$, $\mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})$, has simple objects $\{\mathbf{IC}_w \,|w \in W_{\mathrm{p}}\}$ called intersection cohomology sheaves. Each $\mathbf{IC}_w$ is supported on the affine Schubert variety $\overline{\mathcal{F}l_w}$. We can also take the convolution of I-equivariant perverse sheaves

$$* : \mathrm{Perv}_I(\mathcal{F}l, \mathbb{C}) \times \mathrm{Perv}_I(\mathcal{F}l, \mathbb{C}) \to \mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})$$

which endows $\mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})$ with the structure of a monoidal category.

The split Grothendieck ring of $\mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})$, denoted $[\mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})]_\oplus$, can be endowed with the structure of a $\mathbb{Z}[v^{\pm 1}]$-algebra where $v[\mathcal{F}] = [\mathcal{F}[1]]$. We then have an isomorphism of $Z[v^{\pm 1}]$-algebras:

$$\begin{aligned} [\mathrm{Perv}_I(\mathcal{F}l, \mathbb{C})]_\oplus &\xleftrightarrow{\sim} H, \\ [\mathbf{IC}_w] &\longmapsto b_w \\ [\mathcal{F}] &\longmapsto \sum_{u \in W_{\mathrm{p}}} \sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_w) v^{i-\ell(u)} \delta_u \end{aligned}$$

Thus the canonical basis can be interpreted geometrically as the characters of simple objects in the category of perverse sheaves on the affine flag variety.

The p-canonical basis is a generalisation of the canonical basis when the sheaf coefficients $\mathbb{C}$ are replaced by $\mathbb{k}$, a field of positive characteristic.

The category of I-equivariant parity sheaves on the the affine flag variety with coefficients in $\mathbb{k}$, $\mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})$, has indecomposable objects $\{\mathcal{E}_w | w \in W_{\mathrm{p}}\}$ called indecomposable parity sheaves. Each indecomposable parity sheaf $\mathcal{E}_w$ is the extension by zero of the constant sheaf $\underline{\mathbb{k}}_{\overline{\mathcal{F}l_w}}$. As before, there is a convolution of I-equivariant parity sheaves

$$* : \mathrm{Parity}_I(\mathcal{F}l, \mathbb{k}) \times \mathrm{Parity}_I(\mathcal{F}l, \mathbb{k}) \to \mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})$$

which endows $\mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})$ with the structure of a monoidal category.

The split Grothendieck ring of $\mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})$, denoted $[\mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})]_\oplus$, can be endowed with the structure of a $\mathbb{Z}[v^{\pm 1}]$-algebra where $v[\mathcal{F}] = [\mathcal{F}[1]]$. We then have an isomorphism of $Z[v^{\pm 1}]$-algebras:

$$\begin{aligned} [\mathrm{Parity}_I(\mathcal{F}l, \mathbb{k})]_\oplus &\xleftrightarrow{\sim} H, \\ [\mathcal{E}_w] &\longmapsto {}^p b_w \\ [\mathcal{F}] &\longmapsto \sum_{u \in W_{\mathrm{p}}} \sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_w) v^{i-\ell(u)} \delta_u \end{aligned}$$

The p-canonical basis is defined to be the character of the indecomposable parity sheaves on the affine flag variety.

The p-canonical basis satisfies the following properties:

(1) $\overline{{}^p b_w} = {}^p b_w$;
(2) ${}^p b_w = b_w + \sum_{u<w} {}^p a_{u,w} b_u$ with ${}^p a_{u,w} \in \mathbb{Z}[v^{\pm 1}]$ and ${}^p a_{u,w} = \overline{{}^p a_{u,w}}$;
(3) ${}^p b_w = b_w$ for $p \gg 0$.

The coefficients of the p-canonical basis have the following representation theoretic interpretation:

$$ {}^p b_{w_\mu, w_\lambda}(1) = \dim T_{\lambda,\mu} $$

where $T_{\lambda,\mu}$ is μ-weight space of the the indecomposable tilting module with highest λ, T_λ, $w_\lambda(0) = \lambda$, and $w_\mu(0) = \mu$.

Remarks.

(1) The character of the indecomposable parity sheaf depends only on the characteristic of $\mathbb{k}$, not on the field $\mathbb{k}$ itself.
(2) Whilst the canonical basis may be defined relative only to the Hecke algebra H, the p-canonical basis requires the addition data of a root system associated to H. For example, Jensen and Williamson show that the 2-canonical bases for the Hecke algebras of types $\tilde{C}_2$ and $\tilde{B}_2$ differ.
(3) The p-canonical basis is typically calculated using intersection forms and Elias-Williamson-Khovanov diagrammatics. When the associated Schubert variety is relatively nice (i.e. smooth/rationally smooth/low dimensional) the p-canonical basis can be determined using geometric techniques. It may also be calculated using the Braden-Macpherson algorithm.

The anti-spherical Hecke module

The quadratic relation $(\delta_s + v)(\delta_s - v^{-1}) = 0$ gives a morphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$ \begin{aligned} H &\longrightarrow \mathbb{Z}[v^{\pm}] \\ \delta_s &\longmapsto -v. \end{aligned} $$

For any parabolic subset of S_{p} containing s. In particular, if we take $S_{\mathrm{f}} \subset S_{\mathrm{p}}$ as the parabolic subset then the resulting H_{f}-module is denoted $\mathrm{sign}_{\mathrm{v}}$.

Inducing $\mathrm{sign}_{\mathrm{v}}$ to a representation of H produces the anti-spherical Hecke module N. Explicitly:

$$ N = \mathrm{sign}_{\mathrm{v}} \otimes_{H_{\mathrm{f}}} H $$

It is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{\nu_w := 1 \otimes \delta_w | w \in {}^f W_{\mathrm{p}}\}$ called the standard basis of N.

The Kazhdan-Lusztig involution extends to an involution of N in the following way:

$$ \overline{\nu_w} = 1 \otimes \overline{\delta_w}. $$

We analogously define the canonical basis of N to be the elements $\{d_w | w \in W_{\mathrm{a}}\}$ such that $\overline{d_w} = d_w$ and $d_w \in \nu_w + \sum_{u<w} v\mathbb{Z}[v]\nu_u v$.

The p-canonical basis of the anti-spherical Hecke module is then defined as:

$$ {}^p d_w := 1 \otimes {}^p b_w $$

for any $w \in {}^f W_{\mathrm{a}}$.

Remarks.

(1) There is a geometric interpretation of the p-canonical basis of the anti-spherical Hecke module in terms of Iwahori-Whittaker perverse sheaves on the affine grassmannian.
(2) Williamson and Lebidinsky have constructed a diagrammatic calculus which categorifies the anti-spherical Hecke module. In p-canonical basis then describes the graded Hom spaces between indecomposable objects.
(3) In general, the polynomials ${}^p d_{u,w}(v)$ are more less understood than the polynomials ${}^p b_{u,w}(v)$.

Calculations for SL_2

Recall that for SL_2 we have

- $W_{\mathrm{f}} \cong \langle s | s^2 = \mathrm{id} \rangle$;
- $W_{\mathrm{p}} \cong \langle s, t | s^2 = t^2 = \mathrm{id} \rangle$; and
- ${}^f W_{\mathrm{p}} = \{ w_l \in W_{\mathrm{p}} | \ell(w_l) = l \text{ and } s w_l > w_l \}$.

Moreover the Bruhat order is particularly simple. [INSERT PICTURE].

The canonical basis is particularly simple for SL_2. For any $w \in W_{\mathrm{p}}$ the Poincare polynomial of the Bruhat interval $[\mathrm{id}, w]$ is $1+2x+2x^2+\cdots+2x^{\ell(w)-1}+x^{\ell(w)}$. In particular it is palindromic. A result of Carell and Petersen implies the affine Schubert variety $\overline{\mathcal{F}l_w}$ is rationally smooth. Thus the canonical basis is

$$b_w = \sum_{u \leq w} v^{\ell(w)-\ell(u)} \delta_u.$$

It is then immediate that the canonical basis of the anti-spherical module is

$$d_{w_n} = \nu_{w_n} + v \nu_{w_{n-1}}$$

where w_{n-1} is taken to be 0 if $n = 0$.

Tilting modules for SL_2 can all be explicitly described. This allows an explicit description of the p-canonical bases of the Hecke algebra and the anti-spherical module of the Hecke algebra.

First, observe that $T_\lambda \cong \Delta_\lambda$ for $0 \leq \lambda \leq p-1$ by the linkage principle.

The T_λ where $p \leq \lambda \leq 2p-2$ are known to be the projective covers of the simple G_1-modules (where G_1 denotes the first Frobenius kernel of $G = SL_2$). The category $\operatorname{Rep} G_1$ is equivalent to $\operatorname{Rep} U_p(\mathfrak{sl}_2)$ where $U_p(\mathfrak{sl}_2)$ is the restricted Lie algebra of $\mathfrak{sl}_2$. Recall the restricted Lie algebra $\mathfrak{sl}_2$ is the Lie algebra $\mathfrak{sl}_2$ over a field $\Bbbk$ of characteristic p, endowed with a p-operation $(\cdot)^{[p]} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$. If we realise $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$ as the matrices

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $(\cdot)^{[p]}$ can be realised as the p-th power of each matrix:

$$f^{[p]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, \qquad h^{[p]} = \begin{bmatrix} 1^p & 0 \\ 0 & (-1)^p \end{bmatrix} = h, \qquad e^{[p]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Analogous to the universal enveloping algebra $U(\mathfrak{sl}_2)$ we have the restricted universal enveloping algebra $U_p(\mathfrak{sl}_2)$ which is the quotient $U(\mathfrak{sl}_2)/(x^p - x^{[p]})$. The Poincare-Birkhoff-Witt basis of $U(\mathfrak{sl}_2)$ descends to a basis of $U_p(\mathfrak{sl}_2)$ given by $\{f^i h^j e^k\}_{0 \leq i,j,k < p}$. It can be shown that

$$[T_\lambda] = [\Delta_\lambda] + [\Delta_{t \cdot \lambda}]$$

when $p \leq \lambda \leq 2p-2$.

Finally Donkin's tensor product theorem (for SL_2) states that if we write $\lambda = \lambda_0 + p\lambda_1$ where $p-1 \leq \lambda_0 \leq 2p-2$ and $\lambda_1 \in$ then:

$$T_\lambda = T_{\lambda_0} \otimes T_{\lambda_1}^{(1)}$$

where $(-)^{(1)} : \operatorname{Rep} G \to \operatorname{Rep} G$ denotes the Frobenius twist.

For SL_2 these suffice to inductively prove that for any fixed $\lambda = \sum_{i \geq 0} \lambda_i p^i \in X_+$ where $0 \leq \lambda_i \leq p-1$. Set $\lambda_{(k)} = \sum_{i \geq k} \lambda_i p^i$. Then

$$[T_\lambda] = \left(\prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot [\Delta_\lambda]$$

where the product acts on the left (i.e. $\prod_{k \geq 1} x_i = \ldots x_3 x_2 x_1$), and the action of W_a on $\operatorname{Rep} G$ is given by $w \cdot \Delta_\lambda = \Delta_{s \cdot \lambda}$.

Using explicit knowledge of the Weyl modules for SL_2 and the characterisation of the p-canonical basis of H in terms of tilting modules we find

$${}^p b_{w_\lambda} = \left(\prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot b_{w_\lambda}$$

and consequently

$${}^p d_{w_\lambda} = \left(\prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot d_{w_\lambda}$$