

# A Demazure character formula for the product monomial crystal

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# Statement of originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Signed:

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# Abstract

The product monomial crystal was defined by Kamnitzer, Tingley, Webster, Weekes, and Yacobi for any semisimple simply-laced Lie algebra and a multiset of parameters. The crystal is closely related to the representation theory of truncated shifted Yangians, a family of algebras quantising transversal slices to Schubert varieties in the affine Grassmannian. In this thesis we give a systematic study of the product monomial crystal using the novel tool of truncations, resulting in a Demazure-type character formula which is valid in any symmetric bipartite Kac-Moody type. We establish results on stability of the crystal, and use these and the character formula to show that in type  $A$  the product monomial crystal is the crystal of a generalised Schur module associated to a column-convex diagram, as defined by Magyar, Reiner, and Shimozono.



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# 1 Introduction

Let  $G$  be a simply-laced reductive group over the complex numbers  $\mathbb{C}$ , with  $\text{Rep}_{\mathbb{C}} G$  the category of its finite-dimensional algebraic representations. This category is semisimple with simple objects  $L(\lambda)$  indexed by dominant weights  $\lambda$ , and an interesting problem is to define ‘natural’ constructions of these representations for any dominant weight. This problem has been very fruitful, with three such constructions realising  $L(\lambda)$  as:

1. The space of sections  $\Gamma(G/B, \mathcal{L}_\lambda)$  of a line bundle  $\mathcal{L}_\lambda$  on the flag variety  $G/B$ ,
2. The cohomology of a Nakajima quiver variety associated to the pair  $(G, \lambda)$ , generalising a previous construction of Ginzburg using Springer fibres in the case of  $G = \text{GL}_n$ , and
3. The intersection homology  $IH(\text{Gr}^\lambda)$  inside the dual affine Grassmannian  $\text{Gr} = G^\vee((z))/G^\vee[[z]]$  of the spherical orbit  $\text{Gr}^\lambda$ .

The fact that these three realisations all give rise to  $L(\lambda)$  might be surprising, since the underlying geometric spaces are rather different, and consequently there has been some progress made to state relationships between these spaces. Throughout the papers [Kam+14; Kam+19a; Kam+19b] the authors investigate the relationship between the second and third realisations, establishing the fact that transverse slices in the dual affine Grassmannian are *symplectic dual* to Nakajima quiver varieties. Throughout their study they investigate a family of non-commutative deformations  $Y_\mu^\lambda(\mathbf{R})$  of the coordinate ring of the transverse slice  $\text{Gr}_\mu^\lambda$  depending on an integral set of parameters  $\mathbf{R}$ , and define a category  $\mathcal{O}(Y_\mu^\lambda(\mathbf{R}))$  of their representations. The sum  $\mathcal{V}(\lambda, \mathbf{R}) = \bigoplus_{\mu \leq \lambda} \mathcal{O}(Y_\mu^\lambda(\mathbf{R}))$  carries a categorical (Lie  $G$ )-action (in the sense of Chuang and Rouquier [CR04; Rou08]), making the complexified Grothendieck group  $V(\lambda, \mathbf{R}) = K_{\mathbb{C}}(\mathcal{V}(\lambda, \mathbf{R}))$  a representation of  $G$ .

This thesis concerns the representation  $V(\lambda, \mathbf{R})$  and its crystal  $\mathcal{M}(\lambda, \mathbf{R})$ , called the *product monomial crystal* after its embedding into Nakajima’s crystal of monomials. We give a novel new method of analysing the crystal  $\mathcal{M}(\lambda, \mathbf{R})$  by certain *global truncations*, which we use to give a *Demazure-type character formula* for the crystal, our first main result. Our second main result is specific to type A, where we show that the product monomial crystal is in fact the crystal of a previously-studied family of modules called the *generalised Schur modules* for column-convex diagrams, and give a bijection between the parameters  $\mathbf{R}$  defining the crystal and the column-convex diagrams defining the modules.

The column-convex diagrams include skew shapes as a special case, and so a corollary of our result is that for any skew Schur module there is a weight  $\lambda$  and a parameter multiset  $\mathbf{R}$  such that  $\mathcal{V}(\lambda, \mathbf{R})$  is a categorification of the skew Schur module. This is the first such categorification of a skew-Schur module known to the author.

Although the original setting started with a simply-laced reductive group  $G$ , the product monomial crystal makes sense for any symmetric bipartite Cartan type, and we prove all results in this generality. The only obstruction to our results being valid in arbitrary symmetrisable bipartite types is [Theorem 6.3.5](#), which is proved via the theory of Nakajima quiver varieties. If there were some alternative proof of this theorem valid in arbitrary symmetrisable bipartite type (and various computer experiments suggest it is true), the results within would automatically hold in that generality.

## 1.1 Structure

This thesis is divided into three parts. The first part (Chapters 2 to 5) is concerned with setting up notation and reviewing the theory of Cartan and root data, Kac-Moody algebras, quantum groups, crystals, characters, Demazure modules, and Demazure crystals. The second part (Chapters 6 to 8) examines the product monomial crystal in-depth, developing the novel concept of *truncations* of the crystal and proving our first main result: the character formula. The third part (Chapters 9 and 10) relates the product monomial crystal in Type A to the generalised Schur modules, proves our second main result: the crystal of a column-convex generalised Schur module is given by the product monomial crystal. A more detailed reader's guide is given below.

The first part begins with Chapter 2, a very brief overview of the notation used throughout to which the reader can refer. In Chapter 3 we review the notion of a Cartan datum  $(I, \cdot)$  and a root datum  $\Phi$  so as to put the representation theory of semisimple groups and reductive groups on equal footing with that of Kac-Moody algebras and their quantum analogues. We review the theory of crystals in Chapter 4, first reminding the reader of the original definition of a crystal base in terms of integrable modules over the quantum group, and then connecting it with that of an *abstract* crystal, which is much more common in the combinatorial-oriented literature. In Chapter 5 we define both Demazure modules and Demazure crystals, and state the main theorems concerning their characters.

In Chapter 6 we introduce Nakajima's crystal of monomials, and define the product monomial crystal as a particular subset of the monomial crystal. After establishing some basic notation and results about this crystal we move on to Chapter 7 where we introduce our truncations and give a character formula for each (Theorem 7.2.3), show that each truncation is a Demazure crystal (Theorem 7.3.7), and deduce a complete character formula for the product monomial crystal in finite type (Corollary 7.3.9). With this main result out of the way, we take a step backwards and introduce Nakajima quiver varieties in Chapter 8 so as to give proof of the fact that the product monomial subset is indeed a crystal (Theorem 6.3.5).

In the third part we focus mainly on type A phenomena. Chapter 9 reviews the definition of a Schur functor and Schur module for any arrangement of boxes on a grid (a straightforward generalisation of the well-known definition in the case when the arrangement is a Young diagram), along with the less well-known notion of a *flagged* Schur module. We show that the character of the product monomial crystal and the generalised Schur module agree when the  $n$  in  $GL_n$  is taken to be 'large enough'. In Chapter 10 we prove some stability results about the product monomial crystal, showing that when the data  $\mathbf{R}$  is held fixed and the underlying Cartan type increases in size the decomposition into highest weight crystals stabilises. Using this we can leverage the previous result to show that the product monomial crystal is the crystal of a generalised Schur module for all  $n$ , and as a consequence that  $\mathcal{V}(\lambda, \mathbf{R})$  categorifies a skew Schur module.

## 2 Notation

The blackboard bold letters  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  denote (as usual) the integers, rationals, reals, and complex numbers respectively. We further define the *natural numbers*  $\mathbb{N} = \{0, 1, 2, \dots\}$ , the *positive integers*  $\mathbb{P} = \{1, 2, 3, \dots\}$ , and for any natural number  $n \in \mathbb{N}$  we set  $[n] = \{1, 2, \dots, n\}$ . The *symmetric group* of permutations of the set  $[n]$  is denoted by  $\mathfrak{S}_n$ , and composition is written as composition of functions, so we have (using cycle notation) the equation  $(12)(23) = (123)$  inside  $\mathfrak{S}_3$ .

A *monoid* is the same thing as a group which may not have inverses, for instance  $(\mathbb{N}, +)$  is a monoid, but  $(\mathbb{P}, +)$  is not as it lacks an identity element. When  $R$  is a commutative ring (always assumed to be unital) and  $M$  is a monoid, we denote the unital *monoid algebra* by  $R[M]$ , also called the *group algebra* when  $M$  is a group. When  $M$  is commutative the algebra  $R[M]$  will often be written in *exponential notation*, where  $R[M]$  is free as an  $R$ -module on the basis  $\{e^m \mid m \in M\}$ , with multiplication on the basis given by  $e^m \cdot e^n = e^{m+n}$ . For example if  $M = (\mathbb{N}, +)$  then the monoid algebra  $R[M]$  is isomorphic to the polynomial ring  $R[x]$  under the map  $e^1 \mapsto x$ .

Just as a subset  $Y \subseteq X$  may be regarded as an indicator function  $Y : X \rightarrow \{0, 1\}$ , we define a *multiset* based in  $X$  to be a function  $\mathbf{R} : X \rightarrow \mathbb{N}$ . We will always denote multisets by boldface type, like  $\mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{Q}$ . The value of  $\mathbf{R}$  at the element  $x \in X$  is called the *multiplicity* of  $x$  in  $\mathbf{R}$ , written  $\mathbf{R}[x]$ . The *support* of a multiset is the subset  $\text{Supp } \mathbf{R} = \{x \in X \mid \mathbf{R}[x] > 0\} \subseteq X$ . A multiset with finite support is called a *finite multiset*. If  $X = \{x, y, z\}$  then the notation  $\mathbf{R} = \{x^2, y\}$  means that  $\mathbf{R}$  is a multiset based in  $X$ , where  $x$  has multiplicity 2,  $y$  has multiplicity 1, and  $z$  has multiplicity 0.

If both  $\mathbf{R}$  and  $\mathbf{S}$  are multisets based in  $X$ , then their *multiset union* is the function  $\mathbf{R} + \mathbf{S}$ . We say  $\mathbf{S}$  is a *sub-multiset* of  $\mathbf{R}$ , denoted  $\mathbf{S} \subseteq \mathbf{R}$ , if  $\mathbf{S}[x] \leq \mathbf{R}[x]$  for all  $x \in X$ , and in this case we define their *multiset difference* to be the function  $\mathbf{R} - \mathbf{S}$ . When taking sums or products over multisets, they should be taken with multiplicity. For instance, if  $f : X \rightarrow G$  is a function from  $X$  into an abelian group  $G$  (written multiplicatively), and  $\mathbf{R}$  is a finite multiset based in  $X$ , then the expression  $\prod_{x \in \mathbf{R}} f(x)$  means  $\prod_{x \in X} f(x)^{\mathbf{R}[x]}$ .

A *partial function*  $f : X \dashrightarrow Y$  between the sets  $X$  and  $Y$  is a function  $f : X \rightarrow Y \sqcup \{\perp\}$  where  $\perp$  is a special element not belonging to  $Y$ . Alternatively a partial function may be viewed as a function which is only defined on a subset of its domain. If  $f(x) = \perp$  we say that  $f$  is *undefined* at  $x$ , and the set  $\{x \in X \mid f(x) \in Y\}$  is called the *domain of definition* of  $f$ .

The remainder of the notation here is set up in [Chapter 3](#), but we leave it here as a quick reference guide.

The letter  $I$  will always refer to a *Cartan datum*, which is the data of a symmetrisable generalised *Cartan matrix*  $A_I = [a_{ij}]_{i,j \in I}$  together with a particular integral choice of symmetrising matrix  $B_I = [i \cdot j]_{i,j \in I}$ . This data determines the abstract *Weyl group*  $(W_I, S_I)$ , a Coxeter system with generators  $(s_i)_{i \in I}$ , and when  $W = W_I$  is finite we denote the *longest word* by  $w_I$ . The vertices  $i, j$  are *adjacent*, written  $i \sim j$ , if and only if  $a_{ij} < 0$ .

The letter  $\Phi$  will always refer to a *root datum* of type  $I$ , which for us means a choice of *weight lattice*  $X(\Phi)$  and *coweight lattice*  $X^\vee(\Phi)$  in a perfect pairing  $\langle -, - \rangle$ , along with *simple roots*  $(\alpha_i)_{i \in I}$  and *simple coroots*  $(\alpha_i^\vee)_{i \in I}$ . The additive monoid of *dominant weights* is denoted  $X(\Phi)_+$ . We will use sans-serif font to denote common root data, for example  $\text{GL}_n$  refers to an algebraic group, while  $\text{GL}_n$  refers to a particular root datum of type  $A_{n-1}$ .

A root datum determines a *Kac-Moody algebra*  $\mathfrak{g}(\Phi)$ , its *universal enveloping algebra*  $U(\Phi)$ , an associated *quantum group*  $U_q(\Phi)$ , and also a combinatorial category of  $\Phi$ -crystals. To each dominant weight  $\lambda$  we associate the highest-weight integrable representation  $L(\lambda)$  of the Kac-Moody algebra,  $L_q(\lambda)$  of the quantum group, and the crystal base  $\mathcal{B}(\lambda)$  of  $L_q(\lambda)$ . The modules  $L(\lambda)$  (resp  $L_q(\lambda)$ ) are the simple objects of the semisimple category  $\mathcal{O}^{\text{int}}(\Phi)$  (resp  $\mathcal{O}_q^{\text{int}}(\Phi)$ ), and we only ever use the notation  $L(\lambda)$ ,  $L_q(\lambda)$ , and  $\mathcal{B}(\lambda)$  when  $\lambda$  is dominant.



## 3 Lie Theory

Working with representations of semisimple groups, reductive groups, Kac-Moody algebras, or quantum groups combinatorially requires the introduction of a lot of notation — the Cartan matrix and Dynkin diagram, Weyl group, roots, coroots, and fundamental weights just to name a few. Since we need to introduce all of this notation anyway, we choose here to do it in a way which treats all of these cases uniformly, by introducing a *Cartan datum* and a *root datum* separately, from which we can build out a Kac-Moody algebra or quantum group (or Chevalley group scheme, although we do not cover that here). We briefly recall how the theory develops in the finite type case.

The finite-dimensional semisimple Lie algebras over  $\mathbb{C}$  (with fixed choice of splitting Cartan subalgebra) are classified up to isomorphism by their associated reduced *root systems*, in the sense of Bourbaki [BB02]. The classification of split reductive algebraic groups is more complicated, requiring not only the data of the underlying root system but also a *root datum*, an embedding of the root system into a pair of dual  $\mathbb{Z}$ -modules. We can further choose to *base* these data by choosing a Borel subgroup containing the split torus, in which case we need to only remember the dual  $\mathbb{Z}$ -modules along with the simple roots and simple coroots.

The combinatorial data we have chosen to go with is more or less an extension of the notion of a based root datum to arbitrary symmetrisable type, a synthesis of the definitions given in Part I of [Lus10] and Chapter 7 of [Mar18]. Such a datum is made of two parts: the ‘core’ comprises a *Cartan datum* (playing the role of a root system), a symmetrisable generalised Cartan matrix together with a choice of symmetrisation which makes the square length of each simple root a multiple of 2. The ‘realisation’ part of the datum is analogous to a based root datum in the reductive group sense: a pair of dual  $\mathbb{Z}$ -modules of finite rank, with a choice of simple roots and coroots whose pairing matrix is the generalised Cartan matrix, hence we call it a *root datum*. The ‘core’ determines the abstract Weyl group as well as the positive and negative parts of the associated Kac-Moody algebra and quantum group, whereas the ‘realisation’ part determines the maximal torus or Cartan subalgebra, as well as the full set of possible weights and coweights.

Many authors only treat Kac-Moody algebras or quantum groups using a specific realisation, rather than allowing it to vary. This does not affect much about the resulting representation theory, which mostly depends only on the underlying Cartan datum, but is often more convenient to permit more realisations. For example, while  $\mathfrak{sl}_n$  is a Kac-Moody algebra in the sense of [Kac90], the often-more-convenient  $\mathfrak{gl}_n$  is not. Another place we see the utility of allowing more general realisations is in Chapter 10 where we use arguments involving restriction to Levi subcrystals, since being able to have a Levi subgroup share the same set of weights as the larger subalgebra simplifies notation and arguments considerably.

These degree-zero modifications of Kac-Moody algebras are undoubtedly well-known to experts, but it is quite difficult for a non-expert to check what parts of the theory they actually affect. We assure the reader that as long as they make the assumption that the simple roots and coroots of a root datum are taken to be *linearly independent*, virtually all of the well-known theory of Kac-Moody algebras and quantum groups goes through without a hitch (and in finite type this is automatic). Conversely, if either of those sets is dependent the theory has to be modified significantly, for example by replacing weight spaces as simultaneous eigenspaces with weight spaces as abstract gradings on a vector space, and forgetting the partial order on weights (which is no longer defined). At points of this chapter we will remark on where these theories diverge, but after this chapter we will always take the linear independence, or *regularity*, assumption.

### 3.1 Cartan data

We begin by defining the ‘core’ of our Lie-theoretic data, a *Cartan datum*. Such a datum is equivalent to a symmetrisable generalised Cartan matrix, together with a choice of symmetrisation making the square length of each simple root a positive multiple of 2 (see [Remark 3.1.6](#)).

#### 3.1.1 Definition (*Cartan data*)

A *Cartan datum* is a pair  $(I, \cdot)$  of a finite set  $I$  together with a symmetric bilinear form  $(-\cdot-) : \mathbb{Z}[I] \times \mathbb{Z}[I] \rightarrow \mathbb{Z}$ , satisfying the two conditions

$$\begin{aligned} i \cdot i &\in \{2, 4, 6, \dots\} \text{ for all } i \in I, \text{ and} \\ 2 \frac{i \cdot j}{i \cdot i} &\in \{0, -1, -2, \dots\} \text{ for all } i \neq j. \end{aligned} \quad (3.1.2)$$

The  $I \times I$  matrix  $A_I = [a_{ij}]$  defined by  $a_{ij} = 2 \frac{i \cdot j}{i \cdot i}$  is called the *Cartan matrix* associated to  $(I, \cdot)$ . For each subset  $J \subseteq I$ , the *restricted datum* is  $(J, \cdot)$  with  $(-\cdot-)$  restricted to  $\mathbb{Z}[J] \times \mathbb{Z}[J]$ .

The simplest example of a Cartan datum is the empty datum  $A_0 = \emptyset$ , with the next simplest being the  $A_1$  datum  $I = \{i\}$  with the bilinear form  $i \cdot i = 2$ . A familiar example to most readers will be the type  $A_n$  Cartan datum, where  $I = \{1, \dots, n\}$  and the bilinear form is given by

$$i \cdot j = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1.3)$$

Since  $i \cdot i = 2$  for all  $i \in I$ , the Cartan matrix  $A_I$  is identical to the pairing matrix  $[i \cdot j]$ . For instance, if  $(I, \cdot)$  is the  $A_4$  Cartan datum we have

$$[i \cdot j] = [a_{ij}] = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}. \quad (3.1.4)$$

Define a binary relation on  $I$  by declaring  $i \sim j$  if and only if  $i \cdot j < 0$ , in which case we say that  $i$  is *adjacent* to  $j$ . This defines an undirected simple graph with vertex set  $I$ , which we will later upgrade to a *Coxeter graph* and a *Dynkin diagram*. In the case of  $A_4$ , the undirected graph is a path with four vertices:

$$\circ - \circ - \circ - \circ$$

Using the associated graph to a Cartan datum we can define the following adjectives.

#### 3.1.5 Definition (*Properties of Cartan data*)

Let  $(I, \cdot)$  be a Cartan datum. We say that the Cartan datum is:

1. *Symmetric* if  $i \cdot i = 2$  for all  $i \in I$ . (Equivalently,  $i \cdot j = a_{ij}$  for all  $i, j \in I$ ).
2. *Simply-laced* if it is symmetric, and  $i \cdot j \in \{0, -1\}$  for all  $i \neq j$ .
3. *Irreducible* if its associated graph is connected.
4. *Bipartite* if its associated graph is bipartite.
5. *Finite type* if the symmetric bilinear form  $(-\cdot-)$  is positive-definite over  $\mathbb{R}$ .
6. *Infinite type* if it is not finite type.
7. *Affine type* if it is irreducible, not of finite type, and the symmetric bilinear form  $(-\cdot-)$  is positive-semidefinite over  $\mathbb{R}$ .

It is easy to see that the  $A_n$  Cartan datum satisfies the first four properties above. In addition, some computations with determinants show that the bilinear form of  $A_n$  is positive-definite, and hence  $A_n$  is finite type.



If  $(\Delta \subseteq V)$  is a reduced root system in the sense of Bourbaki [BB02], then we can associate to it a Cartan datum  $(I_\Delta, \cdot_\Delta)$  in the following way. If  $\Delta$  is irreducible then there exists a unique inner product  $(-, -)$  on  $V$  which is Weyl-invariant, and such that the square length  $(\alpha, \alpha)$  of the shortest root is 2. By classification of irreducible root systems there are at most two lengths of roots. In the simply-laced cases  $A_n, D_n, E_6, E_7,$  and  $E_8$  every root is short, while in the cases  $B_n, C_n,$  and  $F_4$  the long roots have square length 4, and for  $G_2$  the long roots have square length 6. We may then choose a simple system  $\{\alpha_i \mid i \in I_\Delta\}$  of roots, and define the associated Cartan datum  $(I_\Delta, \cdot_\Delta)$  with  $i \cdot_\Delta j = (\alpha_i, \alpha_j)$ . If  $\Delta$  is reducible, then  $(I_\Delta, \cdot_\Delta)$  is defined by performing this process on each irreducible component.

In fact, by the classification of affine Cartan matrices (given in Chapter 4 of [Kac90]), we can see that *all* indecomposable finite type or affine type generalised Cartan matrices have only two square lengths, one of which can be taken to be 2, and the other will be 4, 6, or 8, with 8 occuring only in the case of  $A_2^{(2)}$ .

**3.1.6 Remark (Symmetrisability)**

In the usual definition of a Kac-Moody algebra (for example given in [Kac90]), a *generalised Cartan matrix* is an integral matrix  $A \in \text{Mat}_I(\mathbb{Z})$  satisfying the three conditions

1.  $a_{ii} = 2$  for all  $i$ ,
2.  $a_{ij} \leq 0$  for all  $i \neq j$ , and
3.  $a_{ij} = 0 \iff a_{ji} = 0$  for all  $i \neq j$ .

One then says that  $A$  is *symmetrisable* if there exists an invertible diagonal matrix  $D \in \text{Mat}_I(\mathbb{Q})$  and a symmetric matrix  $B \in \text{Mat}_I(\mathbb{Q})$  such that  $A = DB$ .

When  $(I, \cdot)$  is a Cartan datum its associated Cartan matrix  $A_I$  is always symmetrisable, with  $D_I$  having entries  $d_{ii} = \frac{2}{i \cdot j}$ , and  $B_I$  the matrix of the bilinear form  $(i \cdot j)$ . Conversely, when a generalised Cartan matrix  $A$  is symmetrisable in the sense of Kac where  $A = DB$ , it is possible to simultaneously re-scale  $D$  and  $B$  such that all diagonal entries of  $B$  are nonzero even integers, and by considering indecomposable components of  $A$  we may furthermore take those entries to be positive.

So a Cartan datum is precisely a symmetrisable generalised Cartan matrix  $A_I$  with a particular choice of integral  $B_I$ .

To conclude we remark that most definitions in this chapter rely only on the Cartan matrix  $A_I$  of a Cartan datum  $(I, \cdot)$ , with the only exception being the definition of the quantum enveloping algebra (which requires the integers  $i \cdot i$  rather than just the ratios  $a_{ij}$ ).

## 3.2 The Coxeter graph

Given a Cartan datum  $(I, \cdot)$  we can label each edge  $i \sim j$  of the associated graph by a number  $m_{ij} \in \{2, 3, 4, 6, \infty\}$ , producing what is called a *Coxeter graph*.

**3.2.1 Definition (Coxeter matrix and graph)**

The *Coxeter matrix* associated to the Cartan datum  $(I, \cdot)$  is the symmetric matrix  $(m_{ij})_{i,j \in I}$  with diagonal entries  $m_{ii} = 1$ , and off-diagonal entries  $m_{ij} \in \{2, 3, 4, 6, \infty\}$  determined by the value of  $a_{ij}a_{ji}$  as given in the following table:

$$\frac{a_{ij}a_{ji} = \frac{4(i \cdot j)^2}{(i \cdot i)(j \cdot j)} \mid \begin{matrix} \geq 4 & 3 & 2 & 1 & 0 \\ \infty & 6 & 4 & 3 & 2 \end{matrix}}{m_{ij}} \tag{3.2.2}$$

The *Coxeter graph* associated to the Cartan datum  $(I, \cdot)$  is the same undirected graph defined before with  $i \sim j$  iff  $a_{ij} < 0$ , where we also label the edge between  $i$  and  $j$  by  $m_{ij}$ . (Note that for  $i \neq j$  we have  $i \rightsquigarrow j \iff m_{ij} = 2$ ).

The table above can be remembered by the ‘equation’  $\cos^2\left(-\frac{\pi}{m_{ij}}\right) = \frac{(i \cdot j)^2}{(i \cdot i)(j \cdot j)}$ .

The irreducible finite type and affine type Cartan data follow the same classification of indecomposable generalised Cartan matrices (see Chapter 4 of [Kac90]), as each such Cartan matrix is symmetrisable, and proportional Cartan data determine identical Cartan matrices. If  $|I| = 0$  or  $|I| = 1$ , then the Cartan datum is necessarily finite type. If  $|I| = 2$ , then a Cartan datum is of the form

$$\begin{pmatrix} n & -k \\ -k & m \end{pmatrix} \quad \text{where } n, m \in \{2, 4, \dots\}, \quad k \in \mathbb{N}, \quad \text{and both } n, m \text{ divide } 2k. \quad (3.2.3)$$

Since  $n, m > 0$ , this matrix is positive-definite if and only if its determinant  $nm - k^2$  is positive, which is equivalent to  $4 \frac{k^2}{nm} = a_{ij}a_{ji} < 4$ , hence the following result.

**3.2.4 Lemma**

Let  $(I, \cdot)$  be a Cartan datum with Coxeter matrix  $(m_{ij})_{i,j \in I}$ . For each pair  $i \neq j$ , the Coxeter entry  $m_{ij}$  is finite if and only if the restricted rank-two Cartan datum  $(\{i, j\}, \cdot)$  is finite type.

Interestingly, out of all the finite and affine indecomposable Cartan matrices, the only occurrences of  $m_{ij} = \infty$  is for the untwisted affinisation  $A_1^{(1)}$  and twisted affinisation  $A_1^{(2)}$  of  $A_1$ .

**3.2.5 Example (The  $B_4$  Cartan datum)**

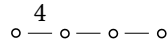
Taking the  $B_4$  type root system and performing the process outlined in the previous chapter, we get the  $B_4$  Cartan datum  $I = \{1, 2, 3, 4\}$  with the symmetric bilinear form and Cartan matrices

$$[i \cdot j] = \begin{pmatrix} 2 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 4 \end{pmatrix}, \quad [a_{ij}] = \begin{pmatrix} 2 & -2 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}. \quad (3.2.6)$$

The associated Coxeter matrix is

$$[m_{ij}] = \begin{pmatrix} 2 & 4 & & \\ 4 & 2 & 3 & \\ & 3 & 2 & 3 \\ & & 3 & 2 \end{pmatrix}, \quad (3.2.7)$$

which makes the associated Coxeter graph the labelled path on four vertices:



(When an edge has a weight  $m_{ij} = 3$ , it is typical to leave that edge unlabelled).

### 3.3 The Braid, Weyl, and Cactus groups

The Weyl group can be presented by generators-and-relations, independent of any realisation as a linear transformation on the weight or coweight lattice, hence the name *abstract Weyl group*.

**3.3.1 Definition (Abstract Weyl group)**

The *abstract Weyl group*  $W_I$  corresponding to a Cartan datum  $(I, \cdot)$  is the group generated by the elements  $S_I = \{s_i, i \in I\}$  subject to the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \in I, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ letters}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ letters}} && \text{for all } i \neq j \text{ with } m_{ij} < \infty. \end{aligned} \quad (3.3.2)$$

The second set of relations are called the *braid relations*. Note that both sets of relations can be compactly written as  $(s_i s_j)^{m_{ij}} = 1$  for all  $i, j \in I$  satisfying  $m_{ij} < \infty$ , presenting  $(W_I, S_I)$  as a *Coxeter system*.

We recall some properties of the Coxeter system  $(W_I, S_I)$ , referring the reader to [Hum90] for further details:

1. A *reduced expression* for  $w \in W_I$  is a word  $(i_1, \dots, i_k)$  with letters in  $I$  such that  $w = s_{i_1} \cdots s_{i_k}$ , and  $k$  is as small as possible.
2. The *length*  $l(w)$  is the length of any reduced expression for  $w$ .
3. The *Bruhat ordering* on  $W_I$  is  $u \leq w$  whenever  $u \in W$  can be obtained as a subexpression of a reduced expression of  $w$ .
4. The Bruhat ordering is compatible with the length function:  $u \leq w$  implies  $l(u) \leq l(w)$ .
5. The group  $W_I$  is finite if and only if  $(I, \cdot)$  is finite type.
6. If  $J \subseteq I$  is of finite type, then  $W_J \subseteq W_I$  is a finite group and contains a unique element  $w_J$  of maximum length, called the *longest word*. If  $I$  itself is finite type, we write  $w_I$  (rather than  $w$ , as is sometimes done) for the longest word.
7. If  $J \subseteq I$  is of finite type, then the automorphism  $\omega_J : W_J \rightarrow W_J$  defined by  $x \mapsto w_J x w_J$  induces a bijection on the set  $\{s_j \mid j \in J\}$  of generators. (This can be seen since  $l(w_J x) = l(w_J) - l(x)$  for any  $x \in W_J$ ). We also write  $\omega_J : J \rightarrow J$  for the induced bijection on the set  $J$ .

It can be shown that the Weyl group (or rather a finite covering on it) acts on any integrable representation of a Kac-Moody algebra. This fails in the case of quantum groups, and one has to look to the braid group instead to find an action on integrable representations.

### 3.3.3 Definition (Braid group)

The *Braid group*  $B_I$  corresponding to a Cartan datum  $(I, \cdot)$  the group generated by the elements  $\sigma_i, i \in I$  subject to the relations

$$\underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m_{ij} \text{ letters}} = \underbrace{\sigma_j \sigma_i \sigma_j \cdots}_{m_{ij} \text{ letters}} \quad \text{for all } i \neq j \text{ with } m_{ij} < \infty. \quad (3.3.4)$$

There is a natural surjection  $B_I \twoheadrightarrow W_I$  of the braid group onto the Weyl group, defined on generators by  $\sigma_i \mapsto s_i$ .

The map  $\omega_J : J \rightarrow J$  acts as an automorphism of the induced Coxeter subgraph of  $J$ , since conjugation preserves the order of a pair of simple generators. This automorphism can be used to define the *cactus group*. Here we follow the definition from Section 5 of [Bon16].

### 3.3.5 Definition (Cactus group)

The *cactus group*  $C_I$  corresponding to a Cartan datum  $(I, \cdot)$  is defined as the group generated by  $\tau_J$ , where  $J \subseteq I$  is a connected subgraph of finite type, modulo the relations

1.  $\tau_J^2 = 1$ ,
2.  $\tau_J \tau_K = \tau_K \tau_J$  if there are no edges between the induced subgraphs  $J$  and  $K$ , and
3.  $\tau_K \tau_J = \tau_J \tau_{\omega_J(K)}$ , whenever  $K \subseteq J$ .

There is a natural surjection  $C_I \twoheadrightarrow W_I$ , defined on the generators by  $\tau_J \mapsto w_J$ .

Of course for each disjoint  $J, K$  of finite type we have  $w_J^2 = 1$  and  $w_J w_K = w_K w_J$ , and if  $K \subseteq J$  then the longest word of  $\omega_J(K)$  is  $w_J w_K w_J$ . Hence  $C_I \twoheadrightarrow W_I$  is indeed a well-defined map of groups, surjective since the  $\tau_{\{i\}}$  are mapped to the simple generators  $s_i$ .

### 3.4 Kac-Moody root data

We now proceed to the second part of our Lie-theoretic data, the ‘realisation’ of the Cartan datum inside a pair of dual  $\mathbb{Z}$ -modules.

#### 3.4.1 Definition (Kac-Moody root data)

A *root datum* of Cartan type  $(I, \cdot)$  is a quintuple

$$\Phi = (X(\Phi), X^\vee(\Phi), \langle -, - \rangle, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I}), \text{ where} \quad (3.4.2)$$

1.  $X(\Phi)$  and  $X^\vee(\Phi)$  are free abelian groups of finite rank called the *weight lattice* and *coweight lattice*,
2.  $\langle -, - \rangle : X(\Phi) \times X^\vee(\Phi) \rightarrow \mathbb{Z}$  is a perfect pairing,
3. The *simple roots*  $\alpha_i \in X(\Phi)$  and *simple coroots*  $\alpha_i^\vee \in X^\vee(\Phi)$  are some pairwise distinct elements indexed by  $I$  such that for all  $i, j \in I$  we have  $\langle \alpha_j, \alpha_i^\vee \rangle = 2 \frac{ij}{i^2} = a_{ij}$  for all  $i, j \in I$ .

We will assume in every chapter except this one that a root datum is *regular*, meaning that both the simple roots and the simple coroots are linearly independent. This is always the case if  $I$  is finite type (Corollary 3.4.5).

For each  $J \subseteq I$  we can form the *restricted root datum*  $\Phi_J$  of Cartan type  $(J, \cdot)$ , by keeping the same weight lattice  $X(\Phi_J) = X(\Phi)$  and coweight lattice  $X^\vee(\Phi_J) = X^\vee(\Phi)$ , but remembering only those simple roots and coroots indexed by the set  $J$ . In the special case  $J = \{i\}$  for some  $i \in I$ , we will write  $\Phi_i = \Phi_{\{i\}}$  for this restriction.

A weight  $\lambda \in X(\Phi)$  is a *dominant weight* if  $\langle \lambda, \alpha_i^\vee \rangle \geq 0$  for all  $i \in I$ . The set  $X(\Phi)_+ \subseteq X(\Phi)$  of dominant weights form a monoid under addition. Provided the simple roots are linearly independent there is a partial order defined on  $X(\Phi)$  by  $\mu \leq \lambda \iff \lambda - \mu \in \sum_{i \in I} \mathbb{N}\alpha_i$ .

All definitions in this chapter can be made without the regularity assumption on  $\Phi$ , although the properties of the definitions are sometimes undesirable — for example,  $\leq$  would only be a preorder (lacking the partial order axiom that if  $a \leq b$  and  $b \leq a$ , then  $a = b$ ), the action of the Weyl group on the weight and coweight lattices may not be faithful, and the set of dominant weights turns out to be rather useless unless the coroots are linearly independent. The reason we do not rule these out completely from consideration is that they occasionally arise, for example by taking the derived subalgebra of an affine Kac-Moody algebra one arrives at a root datum which is not regular.

We remark that our definition of *root datum* differs from the term used in the literature on reductive algebraic groups. The two key differences are that our definition is only fixing a choice of *simple* roots and coroots (and is more similar to a *based* root datum for an algebraic group), and the second difference is that our definition permits infinite type root data. Both kinds of root data are used for the same purpose, as some combinatorial data identifying a group scheme, or Kac-Moody algebra, or quantum group. However, ours also builds in a choice of simple/positive system.

We assure the reader that in the familiar cases of semisimple or reductive algebraic groups, they need not worry about regularity.

#### 3.4.3 Lemma

If the bilinear form  $(I, \cdot)$  is nondegenerate, then any root datum  $\Phi$  of type  $(I, \cdot)$  is regular.

#### 3.4.4 Proof

Since  $(I, \cdot)$  is invertible as a matrix over  $\mathbb{Q}$ , then so is the Cartan matrix  $A_I$ . Thinking of the roots and coroots as linear maps  $\alpha^\vee : \mathbb{Z}[I] \rightarrow X^\vee(\Phi)$  and  $\alpha : \mathbb{Z}[I] \rightarrow X(\Phi)$ , we see that the Cartan matrix is the matrix of the bilinear form  $A(-, -) : \mathbb{Z}[I] \times \mathbb{Z}[I] \rightarrow \mathbb{Z}$  defined by  $A(i, j) = \langle \alpha_j, \alpha_i^\vee \rangle$ . Since the Cartan matrix is invertible, the bilinear form  $A$  is nondegenerate, and hence both of  $\alpha^\vee$  and  $\alpha$  must be injective.

In finite type the bilinear form is positive-definite (by definition) hence the Cartan matrix is invertible.

### 3.4.5 Corollary (*Finite-type root data are regular*)

If  $\Phi$  is a root datum for the finite-type Cartan datum  $(I, \cdot)$ , then  $\Phi$  is regular.

There is a category associated to a Cartan datum  $(I, \cdot)$ , where the objects are root data of type  $I$  and the morphisms are defined as follows:

### 3.4.6 Definition (*Morphism of root data*)

Let  $\Phi$  and  $\Psi$  both be root data of Cartan type  $(I, \cdot)$ . A *morphism*  $(f, g) : \Phi \rightarrow \Psi$  is a pair of adjoint maps

$$X^\vee(\Phi) \xrightarrow{f} X^\vee(\Psi), \quad X(\Phi) \xleftarrow{g} X(\Psi), \quad (3.4.7)$$

such that  $\langle \lambda, f(\mu) \rangle_\Psi = \langle g(\lambda), \mu \rangle_\Phi$  for all  $\mu \in X^\vee(\Phi)$  and  $\lambda \in X(\Psi)$ , and such that  $f(\alpha_i^\vee) = \alpha_i^\vee$  and  $g(\alpha_i) = \alpha_i$  for all  $i \in I$ . As adjoints under the perfect pairings, each of  $f$  and  $g$  uniquely determines the other.

The directions of these maps can be remembered by the fact that a morphism  $\Phi \rightarrow \Psi$  should induce morphisms  $\mathfrak{g}(\Phi) \rightarrow \mathfrak{g}(\Psi)$  of Kac-Moody algebras, or  $U_q(\Phi) \rightarrow U_q(\Psi)$  of quantum enveloping algebras, therefore it is the map on *coweights* which should go in the same direction as the map of root data, as the coweights become the Cartan subalgebra  $\mathfrak{h}(\Phi) = X^\vee(\Phi) \otimes \mathbb{C}$  or the group algebra  $U_q^0(\Phi) = \mathbb{Q}(q)[X^\vee(\Phi)]$  respectively.

The category of root data has initial and final objects, which we call simply-connected and adjoint root data respectively.

### 3.4.8 Example (*Simply-connected and adjoint root data.*)

Fix a Cartan datum  $(I, \cdot)$ . The *simply-connected root datum* of type  $(I, \cdot)$  is the root datum  $\Phi$  where  $X^\vee(\Phi)$  is the free  $\mathbb{Z}$ -module with basis  $(\alpha_i^\vee)_{i \in I}$ , and  $X(\Phi) = \text{Hom}_{\mathbb{Z}}(X^\vee(\Phi), \mathbb{Z})$  its dual, and the simple roots  $\alpha_i$  determined by the condition  $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ . The simply-connected root datum is cofree, coadjoint, cotorsion-free, and *initial* in the category of  $(I, \cdot)$ -root data.

The *adjoint root datum* of type  $(I, \cdot)$  is defined by interchanging the roles of simple roots and simple coroots in the above definition. The adjoint root datum is free, adjoint, torsion-free, and is *final* in the category of  $(I, \cdot)$ -root data.

For example, in the category of  $A_{n-1}$  root data, the root datum of  $\text{SL}_n$  is simply-connected and the root datum of  $\text{PGL}_n$  is adjoint. Simply-connected root data are particularly convenient to deal with, since the weight lattice  $X(\Phi)$  admits a basis of *fundamental weights*  $(\varpi_i)_{i \in I}$  dual to the coroots:  $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ . This is useful in finite type, but if  $(I, \cdot)$  is not finite type then the simply-connected root datum (and the adjoint datum) may not be regular, a fact which we will now demonstrate.

The Cartan datum for the affine type  $A_1^{(1)}$  (also called  $\widehat{\mathfrak{sl}}_2$ ) is  $I = \{0, 1\}$  with the bilinear form given by the matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (3.4.9)$$

hence the simply-connected root datum  $\Phi$  has  $X(\Phi) = \mathbb{Z}^2 = X^\vee(\Phi)$  with the linearly independent coroots  $\alpha_0^\vee = (1, 0)$  and  $\alpha_1^\vee = (0, 1)$ , but the linearly dependent roots  $\alpha_0 = (2, -2)$  and  $\alpha_1 = (-2, 2) = -\alpha_0$ . In order to get a regular root datum, the rank of  $X(\Phi)$  has to be taken to be 3 or more. In general, the rank of a *regular* root datum must always satisfy  $\text{rank } X(\Phi) \geq |I| + \text{corank } A_I$ .

The last piece of theory we will discuss is the action of the abstract Weyl group  $W_I$  on the weight and coweight lattices.

### 3.4.10 Lemma (*Weyl group action on weights and coweights*)

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ . Then the *simple reflections*  $r_i$  and  $r_i^\vee$  defined by

$$\begin{aligned} r_i : X(\Phi) &\rightarrow X(\Phi), & r_i^\vee : X^\vee(\Phi) &\rightarrow X^\vee(\Phi), \\ r_i(\lambda) &= \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, & r_i^\vee(\mu) &= \mu - \langle \alpha_i, \mu \rangle \alpha_i^\vee, \end{aligned} \quad (3.4.11)$$

satisfy the relations  $r_i^2 = 1$  and  $(r_i r_j)^{m_{ij}} = 1$ , and hence define an action of the Weyl group  $W_I(\Phi)$  on  $X(\Phi)$  and  $X^\vee(\Phi)$ . The pairing is  $W_I$ -invariant under this action, meaning  $\langle w\lambda, wv \rangle = \langle \lambda, v \rangle$  for all  $w \in W_I$ .

## 3.4.12 Proof

The property  $r_i^2 = 1$  follows easily from the definition together with the fact that  $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ . It is also straightforward to check that  $\langle r_i \lambda, r_i^\vee v \rangle = \langle \lambda, v \rangle$  for each  $i \in I$ , and so it remains to check that the  $r_i$  satisfy the braid relation  $(r_i r_j)^{m_{ij}} = 1$  for  $i \neq j$  and  $m_{ij} \neq \infty$ .

Define subspaces  $K_{ij}, R_{ij} \subseteq X_{\mathbb{R}}(\Phi)$  by

$$K_{ij} = \ker\langle -, \alpha_i^\vee \rangle \cap \ker\langle -, \alpha_j^\vee \rangle, \quad R_{ij} = \mathbb{R}\alpha_i + \mathbb{R}\alpha_j, \quad (3.4.13)$$

so that  $X_{\mathbb{R}}(\Phi) = K_{ij} \oplus R_{ij}$ . If  $\alpha_i$  and  $\alpha_j$  are linearly dependent with  $k_i \alpha_i = k_j \alpha_j$  for some nonzero integers  $k_i, k_j$ , then taking the pairing with  $\alpha_i^\vee$  gives that  $2k_i = k_j a_{ij}$ , and the pairing with  $\alpha_j^\vee$  gives that  $k_i a_{ji} = 2k_j$ . Multiplying through shows that  $a_{ij} a_{ji} = 4$  and hence  $m_{ij} = \infty$ , so there is nothing further to check in this case.

Assume now that  $\alpha_i$  and  $\alpha_j$  are linearly independent. Both  $r_i$  and  $r_j$  act by the identity on the vector space  $K_{ij}$ , and by the matrices

$$r_i|_{K_{ij}} = \begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}, \quad r_j|_{K_{ij}} = \begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix} \quad (3.4.14)$$

in the  $(\alpha_i, \alpha_j)$  basis. The product  $r_i r_j$  acts on  $K_{ij}$  by the matrix

$$r_i r_j|_{K_{ij}} = \begin{pmatrix} a_{ij} a_{ji} - 1 & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}, \quad (3.4.15)$$

which has characteristic polynomial  $\chi(t) = t^2 + (2 - a_{ij} a_{ji})t + 1$ . Since  $a_{ij} a_{ji} \leq 4$ , the polynomial has two (perhaps equal) complex conjugate roots  $\eta, \bar{\eta}$  on the unit circle, whose real part is  $\frac{a_{ij} a_{ji}}{2} - 1$ . Therefore:

1. If  $a_{ij} a_{ji} = 0$  then  $\eta = \bar{\eta} = -1$  and  $r_i r_j|_{K_{ij}}$  is the scalar matrix  $-1$ , hence  $(r_i r_j)^2 = 1$ .
2. If  $a_{ij} a_{ji} = 1$  then  $\eta, \bar{\eta}$  are rotations by  $\pm \frac{2\pi}{3}$ , hence  $(r_i r_j)^3 = 1$ .
3. If  $a_{ij} a_{ji} = 2$  then  $\eta, \bar{\eta}$  are rotations by  $\pm \frac{\pi}{2}$ , hence  $(r_i r_j)^4 = 1$ .
4. If  $a_{ij} a_{ji} = 3$  then  $\eta, \bar{\eta}$  are rotations by  $\pm \frac{\pi}{3}$ , hence  $(r_i r_j)^6 = 1$ .
5. If  $a_{ij} a_{ji} = 4$  then  $\eta = \bar{\eta} = 1$ , but  $r_i r_j|_{K_{ij}}$  is not a scalar matrix so  $r_i r_j$  is not diagonalisable and hence has infinite order.

Checking this table with the one appearing in [Definition 3.2.1](#), we that the map  $s_i \mapsto r_i$  is indeed a group homomorphism  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X(\Phi))$ . The same argument with roots and coroots swapped works to show that  $s_i \mapsto r_i^\vee$  is a group homomorphism  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X^\vee(\Phi))$ .

The representation  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X(\Phi))$  is faithful if and only if  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X^\vee(\Phi))$  is faithful, since they are dual representations. If the root datum  $\Phi$  is regular, then in fact both of these maps are isomorphisms. The proof above shows that if all simple roots are linearly independent, then the relations between  $r_i$  and  $r_j$  are precisely given by the Coxeter matrix, and in Chapter 1.3 of [\[Kum02\]](#) or 3.13 of [\[Kac90\]](#) it is further shown that when  $\Phi$  is regular the  $r_i$  satisfy the exchange condition, making them the generators of a Coxeter system. For example, when  $\Phi$  is the simply-connected root datum of affine type  $A_1^{(1)}$ , the representation  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X(\Phi))$  is faithful despite  $\Phi$  not being regular, due to the fact that the coroots are linearly independent.

However, without the assumption of regularity the representation  $W_I \rightarrow \text{Aut}_{\mathbb{Z}}(X(\Phi))$  may not be faithful. For example, there is a type  $A_1^{(1)}$  root datum of rank 1, defined by taking any two elements satisfying  $\langle \alpha_0, \alpha_0^\vee \rangle = 2$  and defining  $\alpha_1 = -\alpha_0$  and  $\alpha_1^\vee = -\alpha_0^\vee$ . In that case, both  $r_1$  and  $r_2$  act as multiplication by  $-1$  and hence we have  $r_1 r_2 = 1$ .

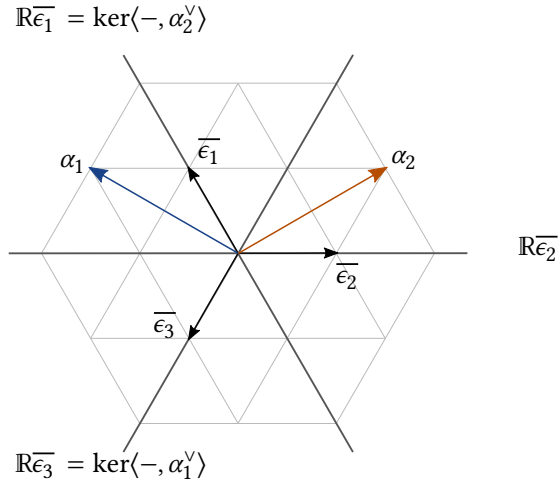
We conclude this section by giving a concrete example of the finite-type root datum  $SL_3$ .

### 3.4.16 Example (The root datum $SL_3$ )

The  $SL_3$  root datum is the simply-connected root datum associated to the Cartan type  $A_2$ . The standard way to construct this root datum is to take the weight lattice to be the quotient space  $X(SL_3) = \mathbb{Z}^3 / (1, 1, 1)$ , and the coweight lattice as the subspace  $X^\vee(SL_3) = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = 0\}$ , with the roots and coroots being given by

$$\begin{aligned} \alpha_1 &= \overline{(1, -1, 0)}, & \alpha_1^\vee &= (1, -1, 0) \\ \alpha_2 &= \overline{(0, 1, -1)}, & \alpha_2^\vee &= (0, 1, -1). \end{aligned} \quad (3.4.17)$$

Writing  $\overline{\epsilon}_i$  for the image of the  $i$ th coordinate vector in  $X(SL_3)$ , we can picture the realified weight lattice  $X_{\mathbb{R}}(SL_3)$  as follows.



The heavy black lines are the reflecting hyperplanes  $\ker\langle -, \alpha \rangle$  for some coroot  $\alpha^\vee$ , and the thin grey lines show the shifted hyperplanes  $\{x \in X_{\mathbb{R}}(SL_3) \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z}\}$  for some coroot  $\alpha^\vee$ . The points where the grey lines meet are the integral points  $X(SL_3)$ , the  $\mathbb{Z}$ -span of the  $\overline{\epsilon}_i$ .

### 3.5 Kac-Moody algebras

A root datum  $\Phi$  should define three related objects: a group scheme  $G(\Phi)$ , its Lie algebra  $\mathfrak{g}(\Phi)$ , and a quantum group  $U_q(\Phi)$ . For  $I$  of finite type,  $G(\Phi)$  would be the usual split reductive group scheme associated to the based root data  $\Phi$ , but attempting to define  $G(\Phi)$  for  $\Phi$  not of finite type would take us too far afield (in general it is an ind-scheme rather than a scheme), and we instead refer the interested reader to Chapters 7 and 8 of [Mar18]. Since we are working in characteristic zero however, the representation theory of the group  $G(\Phi)$  is reflected closely by its Lie algebra, which we *can* define in a straightforward way.

These algebras are called *Kac-Moody algebras* after their independent discoverers Victor Kac and Robert Moody. We don't intend to develop their theory here, instead we state definitions and results which can be found in [Kac90; Mar18; Kum02].

#### 3.5.1 Definition (Kac-Moody algebra)

Let  $\Phi$  be a root datum of Cartan type  $(I, \cdot)$ . The Kac-Moody algebra  $\mathfrak{g}(\Phi)$  is the Lie algebra over  $\mathbb{C}$  generated by the *Cartan subalgebra*  $\mathfrak{h} := X^\vee(\Phi) \otimes \mathbb{C}$  and the *Chevalley generators*  $e_i, f_i$  for  $i \in I$ , subject to the relations

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i \quad \text{for } h \in \mathfrak{h}, \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i \quad \text{for } h \in \mathfrak{h}, \\ [e_i, f_j] &= \delta_{ij} \alpha_i^\vee, \\ (\text{ad } e_i)^{1+|a_{ji}|} e_j &= 0, \quad \text{for } i \neq j, \\ (\text{ad } f_i)^{1+|a_{ji}|} f_j &= 0, \quad \text{for } i \neq j. \end{aligned} \tag{3.5.2}$$

The last two relations are called the *Serre relations*.

The *Chevalley involution* is the map of Lie algebras  $\omega : \mathfrak{g}(\Phi) \rightarrow \mathfrak{g}(\Phi)$  defined on generators by  $\omega(e_i) = -f_i$ ,  $\omega(f_i) = -e_i$  and  $\omega(h) = -h$ . Let  $\mathfrak{n}^+(\Phi)$  (resp.  $\mathfrak{n}^-(\Phi)$ ) be the subalgebra generated by the  $e_i$  (resp.  $f_i$ ), then the direct sum of vector spaces  $\mathfrak{g}(\Phi) = \mathfrak{n}^-(\Phi) \oplus \mathfrak{h}(\Phi) \oplus \mathfrak{n}^+(\Phi)$  is called the *triangular decomposition*.

The *universal enveloping algebra* is denoted  $U(\Phi)$ , and inherits a triangular decomposition  $U(\Phi) = U^-(\Phi) \otimes U^0(\Phi) \otimes U^+(\Phi)$  by the PBW theorem.

Our definition differs slightly from that of Kac [Kac90]. Firstly we permit an arbitrary realisation  $\Phi$  of the Cartan matrix, rather than taking one of smallest possible dimension  $|I| + \text{corank } A_I$ . As we have remarked previously, provided that we always work with regular root data then all the theory remains virtually the same. Secondly our definition of Cartan datum forces  $A_I$  to be symmetrisable, so we are always working with *symmetrisable* Kac-Moody algebras. Thirdly we have defined the algebra by the Serre relations, rather than a quotient of the algebra generated by the first four relations by a certain maximal ideal *containing* the Serre relations. The Gabber-Kac theorem (Theorem 9.11 of [Kac90]) states that when  $A_I$  is symmetrisable the Serre relations generate this ideal, so we have preferred to go with the explicit relations for our presentation. One should also note that the subalgebras  $\mathfrak{n}^+(\Phi)$  and  $\mathfrak{n}^-(\Phi)$  depend only on the underlying Cartan datum  $(I, \cdot)$  rather than on  $\Phi$ .

We define representations and their weights as usual, with one exception: since  $X(\Phi)$  is a  $\mathbb{Z}$ -module of finite rank, our definition of *weight* is what is usually called an *integral weight*. This is not a problem for us as we wish to restrict to integrable representations, whose weights pair integrally with the coroots automatically anyway.

#### 3.5.3 Definition (Representations of a Kac-Moody algebra)

A *representation* of  $\mathfrak{g}(\Phi)$  is a vector space  $V$  equipped with a Lie algebra homomorphism  $\rho_V : \mathfrak{g}(\Phi) \rightarrow \mathfrak{gl}(V)$ , or equivalently a left  $U(\Phi)$ -module. For each weight  $\lambda \in X(\Phi)$  the corresponding *weight space* of the representation  $V$  is  $V_\lambda = \{v \in V \mid hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}(\Phi)\}$ . A *weight vector* is a nonzero vector in a weight space. Weight spaces for distinct weights intersect trivially, and when  $V = \bigoplus_{\lambda \in X(\Phi)} V_\lambda$  then  $V$  is called a *weight module*. When a weight space  $V_\lambda$  is nonzero, we say that  $\lambda$  is a *weight* of  $V$ .

The category of weight representations is a monoidal category, with the monoidal structure and unit coming



| from the usual bialgebra structure on  $U(\Phi)$ .

The category of weight module of  $\mathfrak{g}(\Phi)$  is far too large for our purposes, so we will narrow the set of objects we are considering in two separate ways. The first narrowing is to only consider modules in *category*  $\mathcal{O}$ , meaning weight modules with finite-dimensional weight spaces, whose weights are ‘finitely bounded above’ in a precise sense written below. The second narrowing is to the *integrable* modules, which have good symmetry properties with respect to the Weyl group, and can be ‘integrated’ (in a precise sense) to a representation of a *Kac-Moody group*. When we take the intersection of these we get the category  $\mathcal{O}^{\text{int}}(\Phi)$ , which is a remarkably similar category to the finite-dimensional representations of a semisimple Lie algebra, and is the category we will primarily use for the rest of the thesis.

#### 3.5.4 Definition (Category $\mathcal{O}$ )

Let the category  $\mathcal{O}(\Phi)$  be the full subcategory of  $\mathfrak{g}(\Phi)$  modules  $V$  for which:

1.  $V$  is a weight module with finite-dimensional weight spaces, and
2. There exist finitely many weights  $\lambda_1, \dots, \lambda_s$  (depending on  $V$ ) such that if  $\mu$  is a weight of  $V$  then  $\mu \leq \lambda_k$  for some  $1 \leq k \leq s$ .

The first condition on Category  $\mathcal{O}$  means that we can equip it with a reflexive *dual*: let the *graded dual* of  $V$  be  $V^\omega = \bigoplus_{\lambda \in X(\Phi)} V_\lambda^*$ , with the action  $(x \cdot f)(v) = -f(\omega(x)v)$  where  $\omega$  is the Chevalley involution. This duality is reflexive (meaning  $(V^\omega)^\omega \cong V$ ) by the finite-dimensionality of weight spaces, and the fact that  $\omega$  is an involution. Composing with the Chevalley involution means we preserve the second property of  $\mathcal{O}$ : without it, the graded dual would have weights bounded below rather than above.

The second condition on Category  $\mathcal{O}$  means that the category is closed under taking tensor products. The tensor product of two arbitrary modules with finite-dimensional weight spaces may *not* have finite-dimensional weight spaces, since there could be infinitely many pairs of weights summing to the same weight in the product. The ‘bounded above’ condition on weights in  $\mathcal{O}(\Phi)$  implies that each sum must be finite, since for any two weights  $\mu, \lambda \in X(\Phi)$  the set  $\{v \mid \mu \leq v \leq \lambda\}$  is finite. One could compare this condition to the algebraist’s definition of formal Laurent series, being a sum of the form  $\sum_{i \in \mathbb{Z}} a_i t^i$  where  $a_i$  is eventually zero for  $i \ll 0$ : this boundedness condition means the product of two such series is defined.

#### 3.5.5 Warning

When  $\mathfrak{g}(\Phi)$  is a finite-dimensional semisimple Lie algebra, there is the similarly named “BGG Category  $\mathcal{O}$ ”, which we will write as  $\mathcal{O}^{\text{BGG}}(\Phi)$ . It is defined as all *finitely generated* weight modules which are locally  $\mathfrak{n}^+(\Phi)$ -finite, meaning that  $U^+(\Phi)v$  is finite-dimensional for all  $v \in V$  (Chapter 1.1 of [Hum08]). It is a consequence of these axioms that  $\mathcal{O}^{\text{BGG}}(\Phi)$  is a full subcategory of  $\mathcal{O}(\Phi)$ , but in general the containment is strict.

One way to see this strict containment is to notice that our category  $\mathcal{O}(\Phi)$  is closed under taking tensor products, while the BGG category is not. This can be seen even for  $\mathfrak{g}(\Phi) = \mathfrak{sl}_2$  by tensoring two Verma modules (objects of  $\mathcal{O}^{\text{BGG}}(\Phi)$ ) and noticing that the resulting module has weight spaces of unbounded dimension, and hence cannot be finitely generated.

One reason for introducing the category  $\mathcal{O}(\Phi)$  is to have a category of  $\mathfrak{g}(\Phi)$ -modules which is not ‘too large’, and still includes the *highest-weight modules*.

#### 3.5.6 Definition (Highest-weight modules)

Let  $V$  be a weight module. A *primitive vector* is a weight vector  $v_\lambda \in V_\lambda$  such that  $e_i v_\lambda = 0$  for all  $i \in I$ . If  $V$  is generated by a primitive vector  $v_\lambda$ , it is called a *highest-weight module* with highest-weight  $\lambda$ .

Amongst the highest-weight modules of highest weight  $\lambda$ , there is a *universal* such module  $M(\lambda)$  such that  $M(\lambda)$  surjects onto any other highest-weight module of highest weight  $\lambda$ . The module  $M(\lambda)$  is called a *Verma module*.

One can easily argue using the triangular decomposition of  $U(\Phi)$  that every highest-weight module is a member of  $\mathcal{O}(\Phi)$ . Conversely, every module in  $\mathcal{O}(\Phi)$  has a (possibly infinite) filtration by submodules such that successive quotients are highest-weight modules. We should remark at this point that the category  $\mathcal{O}(\Phi)$  is far from being semisimple.

We now briefly discuss our other narrowing of the category of weight modules.

**3.5.7 Definition** (*Integrable module*)

A  $\mathfrak{g}(\Phi)$ -module  $V$  is *integrable* if it is a weight module, and  $e_i$  and  $f_i$  act locally nilpotently for all  $i \in I$ , meaning that for any  $v \in V$  there exists an  $N \geq 0$  such that  $e_i^N v = 0 = f_i^N v$  for all  $i \in I$ .

The weight module condition and local nilpotency of the  $e_i$  and  $f_i$  is equivalent to  $V$  being a *locally finite*  $\mathfrak{g}(\Phi_i)$  weight module for each  $i \in I$ , meaning  $V$  is a union of finite-dimensional  $\mathfrak{g}(\Phi_i)$  weight modules. Each subalgebra  $\mathfrak{g}(\Phi_i) \subseteq \mathfrak{g}(\Phi)$  is essentially an  $\mathfrak{sl}_2$  with an enlarged toral subalgebra, and hence we can apply the theory of finite-dimensional  $\mathfrak{sl}_2$  representations to  $V$ . For example, the symmetry of weight spaces of finite-dimensional  $\mathfrak{sl}_2$  representations implies that  $\dim V_\lambda = \dim V_{s_i \lambda}$  for each  $i \in I$ . Since this works for all  $i$ , we get that  $\dim V_\lambda = \dim V_{w\lambda}$  for all  $w \in W_I$ .

The fact that  $V$  is locally finite as a  $\mathfrak{g}(\Phi_i)$ -module means that we can even define *exponentials* of the Chevalley generators. For each weight vector  $v \in V_\lambda$ , we have that  $\exp(h) \cdot v = \exp(\langle \lambda, h \rangle)v$  for all  $h \in \mathfrak{h}(\Phi)$ , and the local nilpotency means that  $\exp(e_i) \cdot v = \sum_{n \geq 0} e_i^n v / n!$  is a finite sum (and similarly for  $\exp(f_i)$ ), hence each generator of  $\mathfrak{g}(\Phi)$  can be exponentiated, leading to an action of the associated *Kac-Moody group* on  $V$ . We won't have use for Kac-Moody groups here, but this explains the terminology 'integrable'.

The category of integrable modules is not semisimple, and is distinct from  $\mathcal{O}(\Phi)$ . For example, the adjoint representation is always an integrable module, but not a member of  $\mathcal{O}(\Phi)$  unless  $(I, \cdot)$  is finite type. On the other hand the Verma module  $M(0) = U^-(\Phi)$  is not integrable, but clearly a highest-weight module generated by  $1 \in U^-(\Phi)$ . Remarkably, when we intersect category  $\mathcal{O}(\Phi)$  with the integrable modules, we get a semisimple category which behaves very much like the category of finite-dimensional representations of a semisimple Lie algebra.

**3.5.8 Theorem** (*Integrable highest weight modules and complete reducibility*)

The Verma module  $M(\lambda)$  admits a unique simple quotient  $L(\lambda)$ , which is integrable if and only if  $\lambda$  is a dominant<sup>1</sup> weight.

Let  $\mathcal{O}^{\text{int}}(\Phi)$  be the full subcategory of  $\mathcal{O}(\Phi)$  consisting of integrable modules. Then  $\mathcal{O}^{\text{int}}(\Phi)$  is semisimple, and the set  $\{L(\lambda) \mid \lambda \in X(\Phi)_+\}$  is a complete irredundant set of simple objects.

The category  $\mathcal{O}^{\text{int}}(\Phi)$  looks quite similar to the representations of a finite-dimensional semisimple algebra: it is semisimple, with simple objects indexed by dominant weights. Indeed if  $(I, \cdot)$  is finite type, then  $\mathcal{O}^{\text{int}}(\Phi)$  is *almost* equivalent to the category of finite-dimensional  $\mathfrak{g}(\Phi)$  representations, the difference being that a module in  $\mathcal{O}^{\text{int}}(\Phi)$  might be an infinite direct sum of finite-dimensional modules.

## 3.6 Quantum groups

The 'quantum group' is a mythical object  $\mathfrak{Q}(\Phi)$  associated to a root datum  $\Phi$ , to which one can nevertheless associate a Hopf algebra  $U_q(\Phi)$  over the field  $\mathbb{Q}(q)$  of rational functions called a *quantum enveloping algebra*. Although as far as the author knows there is no agreement upon this mythical object, there is a fair amount of agreement on  $U_q(\Phi)$ , and the fact that a  $U_q(\Phi)$ -module is called a 'representation of a quantum group'.

There is a similar definition of a category  $\mathcal{O}_q^{\text{int}}(\Phi)$  of  $U_q(\Phi)$ -modules, which recovers the category  $\mathcal{O}^{\text{int}}(\Phi)$  in the  $q \rightarrow 1$  limit. The most remarkable thing we get from representations of quantum groups is the existence of a *crystal base* for each integrable highest-weight module  $L_q(\lambda)$ , an amazing combinatorial basis behaving well under tensor products, a phenomenon which is not possible when just using the Kac-Moody algebra. As before, we do not develop the theory here but merely state definitions and results, referring the reader to [Jos95; Lus10; HK12].

<sup>1</sup>In the classic treatment one would usually say that  $L(\lambda)$  is integrable if and only if  $\lambda$  is a dominant *integral* weight, but since all weights for us are integral we have omitted this adjective.

We need to first introduce the quantum analogues of integers, factorials, and binomial coefficients. For any  $n \in \mathbb{Z}$ , define the *quantum integer*

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Q}(q). \quad (3.6.1)$$

A convenient way of remembering these numbers is that  $[0] = 0$ ,  $[1] = 1$ ,  $[2] = q + q^{-1}$ , and in general we have that for  $n \geq 1$  the quantum integer  $[n]$  is the character of the  $n$ -dimensional  $\mathfrak{sl}_2$  representation. Together with  $[-n] = -[n]$  this completely defines the quantum integers, for example

$$[3] = q^{-2} + 1 + q^2 \quad \text{and} \quad [-4] = -(q^3 + q + q^{-1} + q^{-3}). \quad (3.6.2)$$

We define the *quantum factorial* for  $n \in \mathbb{N}$  as a product of quantum integers:

$$[n]! = \prod_{m=1}^n [m], \quad (3.6.3)$$

and the *quantum binomial coefficient* for all  $n \in \mathbb{Z}$  and  $r \in \mathbb{N}$ :

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]! [n-r]!}. \quad (3.6.4)$$

The quantum integers, factorials, and binomial coefficients are all members of the subring  $\mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q)$  of Laurent polynomials with integer coefficients.

When dealing with Cartan data which is not simply-laced (meaning that  $i \cdot i > 2$  for some  $i \in I$ ), we need to introduce some scaled versions of these quantum numbers. Let  $q_i = q^{(i \cdot i)/2} \in \mathbb{Q}(q)$ , and let  $[n]_i$  be the quantum integer  $[n]$  composed with the substitution  $q \mapsto q_i$ . For example, when  $i \cdot i = 4$ , we have  $[2]_i = q^2 + q^{-2}$ . We can now introduce the quantum enveloping algebra associated to a root datum  $\Phi$ .

### 3.6.5 Definition (Quantum enveloping algebra)

The *quantum enveloping algebra* associated to  $\Phi$  is the unital associative algebra  $U_q(\Phi)$  over  $\mathbb{Q}(q)$  generated by the symbols  $E_i, F_i$  for  $i \in I$  and  $K_\mu$  for  $\mu \in X^\vee(\Phi)$ , subject to the following relations:

$$\begin{aligned} K_0 &\text{ is the unit,} \\ K_\mu K_\nu &= K_{\mu+\nu}, \\ K_\mu E_i &= q^{\langle \alpha_i, \mu \rangle} E_i K_\mu, \\ K_\mu F_i &= q^{-\langle \alpha_i, \mu \rangle} F_i K_\mu, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q_i - q_i^{-1}}, \end{aligned} \quad (3.6.6)$$

$$\sum_{a+b=1+|a_{ij}|} (-1)^a E_i^{[a]} E_j E_i^{[b]} = 0 \quad \text{for all } i \neq j,$$

$$\sum_{a+b=1+|a_{ij}|} (-1)^a F_i^{[a]} F_j F_i^{[b]} = 0 \quad \text{for all } i \neq j.$$

The notation  $E_i^{[n]}$  means the *quantum divided power*  $E_i^n / [n]_i!$  for any  $n \geq 0$ . As above, the symbol  $q_i$  means  $q^{(i \cdot i)/2}$ , and we set  $\tilde{K}_i = K_{(i \cdot i/2)\alpha_i^\vee}$ . When  $I$  is symmetric,  $q_i = q$  and  $\tilde{K}_i = K_{\alpha_i^\vee}$ , and each  $[n]_i$  is just the usual quantum integer  $[n]$ .

Our notation for  $U_q(\Phi)$  follows [Lus10], however our definition is slightly different. Rather than presenting  $U_q(\Phi)$  with the Serre relations, Lusztig instead defines a certain bilinear form on the free  $\mathbb{Q}(q)$ -algebra generated by the  $E_i$ , and takes the quotient by the radical of the bilinear form (and similarly for the  $F_i$ ). As in the case of Kac-Moody algebras, it is clear that the Serre relations vanish in this quotient, but it requires further careful work to show

that the Serre relations generate the radical. The statement that these two definitions are equivalent is Corollary 33.1.5 in [Lus10].

The scaled terms  $\tilde{K}_i$  are the correct analogue of the coroots  $\alpha_i^\vee \in \mathfrak{h}(\Phi)$  in the Kac-Moody case, including the quantum scaling  $q_i = q^{i/2}$ . They obey the relations

$$\tilde{K}_i E_j = q_i^{a_{ij}} E_j \tilde{K}_i \quad \text{and} \quad \tilde{K}_i F_j = q_i^{-a_{ij}} F_j \tilde{K}_i, \quad (3.6.7)$$

and in particular we have the ‘quantum  $\mathfrak{sl}_2$  relations’  $\tilde{K}_i E_i = q_i^2 E_i \tilde{K}_i$  and  $\tilde{K}_i F_i = q_i^{-2} F_i \tilde{K}_i$ . The action of  $\tilde{K}_i$  on a weight vector  $v_\lambda \in V_\lambda$  is  $\tilde{K}_i v_\lambda = q_i^{\langle \lambda, \alpha_i^\vee \rangle} v_\lambda$ , and hence the fourth relation above gives that  $(E_i F_i - F_i E_i) v_\lambda = [\langle \lambda, \alpha_i^\vee \rangle]_i v_\lambda$ , which is perhaps how the fourth relation should really be remembered.

In order to define trivial representations, tensor products, and duals, we give a Hopf algebra structure on  $U_q(\Phi)$ , following [Lus10].

**3.6.8 Definition** (*Hopf algebra structure on  $U_q(\Phi)$* )

The comultiplication on  $U_q(\Phi)$  is the unique extension of

$$\begin{aligned} \Delta(K_\mu) &= K_\mu \otimes K_\mu, \\ \Delta(E_i) &= E_i \otimes 1 + \tilde{K}_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i \end{aligned} \quad (3.6.9)$$

making  $\Delta : U_q(\Phi) \rightarrow U_q(\Phi) \otimes U_q(\Phi)$  a map of algebras, where the tensor product on the right is given the usual algebra structure  $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ . Notably, this comultiplication is not commutative.

The counit  $\epsilon : U_q(\Phi) \rightarrow \mathbb{Q}(q)$  is the unique map of algebras taking  $E_i, F_i$  to 0 and each  $K_\mu$  to 1.

The antipode  $S : U_q(\Phi) \rightarrow U_q(\Phi)$  is the unique anti-homomorphism of algebras such that

$$S(E_i) = -\tilde{K}_{-i} E_i, \quad S(F_i) = -F_i \tilde{K}_i, \quad S(K_\mu) = K_{-\mu}. \quad (3.6.10)$$

In direct analogy with the representations of a Kac-Moody algebra, we make the following definitions.

**3.6.11 Definition** (*Representations of a quantum group*)

A *representation* of  $U_q(\Phi)$  is a  $\mathbb{Q}(q)$ -vector space  $M$  equipped with an action making it a left  $U_q(\Phi)$ -module.

The  $\lambda$  *weight space* of  $M$  is  $M_\lambda = \{v \in M \mid K_\mu v = q^{\langle \lambda, \mu \rangle} v \text{ for all } \mu \in X^\vee(\Phi)\}$ , and when  $M = \bigoplus_{\lambda \in X(\Phi)} M_\lambda$  we say that  $M$  is a *weight module*.

The category  $\mathcal{O}_q(\Phi)$  consists of weight modules with finite-dimensional weight spaces, such that the set of weights is bounded above by some finite set  $\lambda_1, \dots, \lambda_s \in X(\Phi)$  depending on  $M$ . A weight module is *integrable* if  $U_q(\Phi_i)$  acts locally finitely, and the full subcategory of  $\mathcal{O}_q(\Phi)$  of integrable objects is denoted by  $\mathcal{O}_q^{\text{int}}(\Phi)$ .

We define primitive vectors, highest-weight modules, quantum Verma modules  $M_q(\lambda)$ , and their simple quotients  $L_q(\lambda)$  in direct analogy with the Kac-Moody case. The same complete reducibility theorem applies to  $\mathcal{O}_q^{\text{int}}(\Phi)$ , namely that every object is isomorphic to a possibly infinite direct sum of  $L_q(\lambda)$  for  $\lambda \in X(\Phi)_+$ .

**3.6.12 Remark**

Sometimes a more general notion of a weight is used, parametrised by a group homomorphism  $\sigma : X^\vee(\Phi) \rightarrow \{\pm 1\}$  together with a weight  $\lambda$ : a vector  $v \in M$  is of weight  $(\sigma, \lambda)$  if  $K_\mu v = \sigma(\mu) q^{\langle \lambda, \mu \rangle} v$  for all  $\mu \in X^\vee(\Phi)$ . These weights naturally arise in the study of finite-dimensional representations of  $U_q(\Phi)$ . The weight representations of a fixed type  $\sigma \in \text{Hom}(X^\vee(\Phi), \{\pm 1\})$  form an abelian subcategory, and all  $2^{\text{rank } \Phi}$  such subcategories are equivalent. We have chosen the subcategory corresponding to the trivial type  $\sigma(\mu) = 1$  (sometimes called type  $(1, \dots, 1)$  in the literature), which is also closed under taking tensor products and duals.

There is another class of integrable modules which we will come across in passing. The extremal weight modules were first defined by Kashiwara in Section 8 of [Kas94], and generalise highest-weight modules. They are used to

study integrable modules for example in [Kas02a], where it is shown that if  $\Phi$  is affine and  $\varpi_i \in X(\Phi)$  is a level-zero fundamental weight, the extremal weight module  $V_q(\varpi_i)$  is the affinisation of a finite-dimensional representation  $L_q(\varpi_i)$ .

### 3.6.13 Definition (*Extremal weight modules*)

A weight vector  $v \in V_\lambda$  of an integrable  $U_q(\Phi)$ -module  $V$  is called *i-extremal* if  $e_i v = 0$  or  $f_i v = 0$ . In this case, define  $S_i v = F_i^{[\langle \lambda, \alpha_i^\vee \rangle]} v$  or  $S_i v = E_i^{[-\langle \lambda, \alpha_i^\vee \rangle]} v$  respectively (“the” element at the opposite end of the  $i$ -string). The weight vector  $v$  is further called *extremal* if  $S_{i_1} \cdots S_{i_r} v$  is  $i$ -extremal for all  $i \in I$ , for all words  $(i_1, \dots, i_r)$ .

An integrable  $U_q(\Phi)$ -module  $V$  is called *extremal of weight  $\lambda$*  if it contains a vector  $v \in V_\lambda$  and there exist vectors  $(v_w)_{w \in W}$  such that  $v = v_e$  and

$$\begin{aligned} \text{if } \langle w\lambda, \alpha_i^\vee \rangle \geq 0, \text{ then } E_i v_w = 0 \text{ and } F_i^{[\langle w\lambda, \alpha_i^\vee \rangle]} v_w = v_{s_i w}, \\ \text{if } \langle w\lambda, \alpha_i^\vee \rangle \leq 0, \text{ then } F_i v_w = 0 \text{ and } E_i^{[-\langle w\lambda, \alpha_i^\vee \rangle]} v_w = v_{s_i w}. \end{aligned} \quad (3.6.14)$$

If such vectors  $(v_w)_{w \in W}$  exist then they are unique, and furthermore  $v_w \in V_{w\lambda}$ .

For each weight  $\lambda \in X(\Phi)$ , the *extremal weight module*  $V_q(\lambda)$  is the  $U_q(\Phi)$ -module generated by  $v_\lambda$  with the defining relation that  $v_\lambda$  is extremal of weight  $\lambda$ . For each  $w \in W$ , the map  $v_\lambda \mapsto S_{w^{-1}} v_{w\lambda}$  gives an isomorphism  $V(\lambda) \xrightarrow{\sim} V(w\lambda)$  of  $U_q(\Phi)$ -modules.

When  $\lambda$  is a dominant weight,  $V_q(\lambda) \cong L_q(\lambda)$  is the irreducible highest-weight module of highest-weight  $\lambda$ , and similarly if  $\lambda$  is antidominant, then  $V_q(\lambda)$  is the irreducible lowest-weight module of lowest-weight  $\lambda$ . When  $(I, \cdot)$  is finite-type, the extremal-weight modules  $V_q(\lambda)$  are finite-dimensional and irreducible. However, if  $(I, \cdot)$  is not finite-type, then not all extremal weight modules are members of  $\mathcal{O}_q^{\text{int}}(\Phi)$ .



## 4 Crystals

It was first shown by Kashiwara [Kas91] that each integrable highest-weight representation  $L_q(\lambda) \in \mathcal{O}_q^{\text{int}}(\Phi)$  admits a *crystal base*  $\mathcal{B}(\lambda)$ , which can be thought of (almost) as a certain weight basis of  $L_q(\lambda)$  enjoying very special properties. The set  $\mathcal{B}(\lambda)$  can be equipped with directed labelled edges coming from the ‘crystallisation’ of the quantum  $E_i$  and  $F_i$  operators (the *Kashiwara operators*), making  $\mathcal{B}(\lambda)$  into a connected graph with a unique *highest-weight element*  $b_\lambda$ . Remarkably, there is a *purely combinatorial* rule (stated purely in terms of just the directed graph structure) for forming the tensor product of two crystal bases, thus the theory of crystals allows us to reduce some linear algebraic problems into combinatorial problems. For example, the number of times  $L_q(\nu)$  appears in the tensor product  $L_q(\lambda) \otimes L_q(\mu)$  is equal to the number of times the connected component  $\mathcal{B}(\nu)$  appears inside the graph  $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ .

There are many more  $U_q(\Phi)$ -modules which admit crystal bases, for example the Verma modules (which are not integrable), and the extremal weight modules (which are integrable but not in general highest-weight). While the definition of crystal base (or at least of Kashiwara operators) needs to be modified slightly to include these different cases, there is a combinatorial category of  $\Phi$ -crystals into which all of the resulting crystal bases can be placed. These abstract crystals do not remember the original  $U_q(\Phi)$  representation but instead have some extra data attached: the raising and lowering statistics  $\varepsilon_i$  and  $\varphi_i$  which (very roughly) allow a  $\Phi$ -crystal to capture some non-semisimple behaviour, as one would expect from (say) the crystal of a Verma module.

The combinatorial  $\Phi$ -crystals are very convenient for working with crystals, as all of the data needed to compute tensor products, restrictions, morphisms and so on are right at our fingertips. However, the  $\Phi$ -crystals also include many objects which do *not* arise from any  $U_q(\Phi)$ -module, which is both a blessing and a curse: on one hand this flexibility allows the crystal base associated to a Demazure module (which is only a  $U_q^+(\Phi)$ -module) to be an honest  $\Phi$ -crystal, while on the other hand one has to work very hard to show that a given  $\Phi$ -crystal actually came from a  $U_q(\Phi)$ -module and is not some ‘virtual’ crystal. So while the combinatorial axioms are convenient for working *with* crystals, they are not so suited to *producing* useful crystals in the first place.

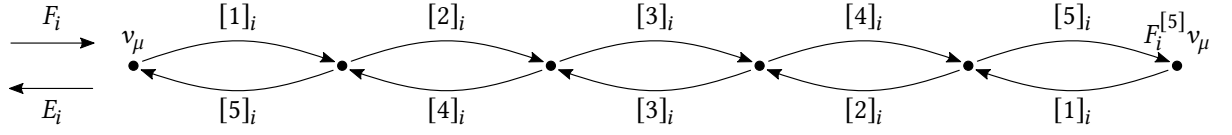
We begin this chapter by grounding ourselves in the theory of crystal bases of integrable highest-weight modules. Despite the fact we will really only use abstract crystals for the rest of the thesis, the author feels that without this grounding it is hard to see the explicit connection between bases of  $U_q(\Phi)$ -modules and crystals. Next we spend some time discussing the category of  $\Phi$ -crystals, giving many examples of crystals both coming and not coming from integrable  $U_q(\Phi)$ -representations. We then come to the *recognition theorems*, which are the methods by which we can show an abstract  $\Phi$ -crystal does indeed come from the category  $\mathcal{O}_q^{\text{int}}(\Phi)$ . Finally we describe some interesting actions of the Weyl group and cactus group on these crystals.

### 4.1 Crystal bases of integrable modules

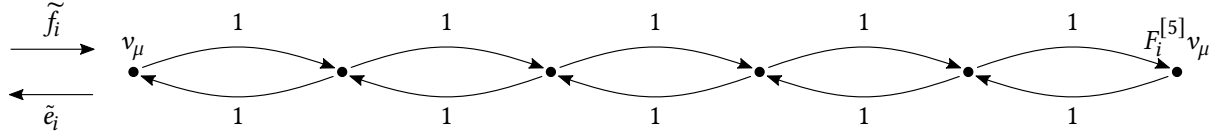
In this section we will give Kashiwara’s original [Kas91] definition of the crystal base of an integrable  $U_q(\Phi)$ -module  $V$ . A crystal base  $B$  is not quite a  $\mathbb{Q}(q)$ -basis of  $V$ , but a  $\mathbb{Q}$ -basis of a module obtained from a *crystal lattice*  $L \subseteq V$ . In order to define what properties  $L$  and  $B$  should satisfy, we first introduce the Kashiwara operators, which are  $\mathbb{Q}(q)$ -linear endomorphisms of  $V$ .

For any root datum  $\Phi$  of Cartan type  $I$  and a choice of vertex  $i \in I$ , we can consider the *restricted* root datum  $\Phi_{i\bar{i}} = \Phi_i$  having the same weight and coweight lattices as  $\Phi$ , but where we have forgotten all simple roots and coroots except for  $\alpha_i, \alpha_i^\vee$ . The representation theory of  $U_q(\Phi_i)$  looks very similar to the representation theory of  $U_q(\text{SL}_2)$ , just with a larger set of weights  $X(\Phi_i) = X(\Phi)$ . The finite-dimensional weight representations of

$U_q(\Phi_i)$  are semisimple, each isomorphic to an integrable highest-weight module  $L_q(\mu)$  for some  $i$ -dominant weight  $\mu \in X(\Phi_i)_+ = \{\mu \in X(\Phi) \mid \langle \mu, \alpha_i^\vee \rangle \geq 0\}$ . The irreducible  $U_q(\Phi_i)$ -module  $L_q(\mu)$  is generated by a highest-weight vector  $v_\mu$  of weight  $\mu$ , and has a *string basis*  $(v_\mu, F_i^{[1]}v_\mu, \dots, F_i^{[n]}v_\mu)$  where  $n = \langle \mu, \alpha_i^\vee \rangle$  and  $F_i^{[k]}$  are the quantum divided powers  $F_i^{[k]} = F_i^k / [k]_i!$ . Take for example any  $\mu \in X(\Phi)$  satisfying  $\langle \mu, \alpha_i^\vee \rangle = 5$ , then we can picture the actions of  $E_i$  and  $F_i$  on the string basis as a diagram:



The Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$  are defined to be the  $\mathbb{Q}(q)$ -linear right and left-shift operators on this basis:



For example, we would have

$$\tilde{f}_i(F_i^{[2]}v_\mu + F_i^{[5]}v_\mu) = F_i^{[3]}v_\mu + 0 \quad \text{and} \quad \tilde{e}_i(F_i^{[2]}v_\mu + F_i^{[5]}v_\mu) = F_i^{[1]}v_\mu + F_i^{[4]}v_\mu. \quad (4.1.1)$$

Suppose that  $V$  is an isotypic  $U_q(\Phi_i)$  module, having direct summands isomorphic to  $L_q(\mu)$ . We can define Kashiwara operators in a more coordinate-free way by restricting the operators  $F_i^{[1]}, \dots, F_i^{[n]}$  to the highest-weight subspace  $V_\mu$ , yielding isomorphisms  $V_\mu \rightarrow V_{\mu-\alpha_i}, \dots, V_\mu \rightarrow V_{\mu-n\alpha_i}$ . These chosen isomorphisms may be inverted and composed to get a chain of isomorphisms  $V_\mu \rightarrow V_{\mu-\alpha_i} \rightarrow \dots \rightarrow V_{\mu-n\alpha_i}$ , and along with the condition that  $\tilde{f}_i(V_{\mu-n\alpha_i}) = 0$  this chain is precisely the graded decomposition of the Kashiwara operator  $\tilde{f}_i$ . The operator  $\tilde{e}_i$  may be defined by inverting the chain of isomorphisms and adding the condition that  $\tilde{e}_i(V_\mu) = 0$ .

Now, suppose that  $V$  is an integrable  $U_q(\Phi)$ -module. Then for any  $i \in I$  the module  $V$  becomes a  $U_q(\Phi_i)$ -module by restriction, and the integrability condition gives that it decomposes into a direct sum of finite-dimensional  $U_q(\Phi_i)$ -modules. For each  $i$ -dominant  $\mu \in X(\Phi_i)_+$ , let  $V(i, \mu)$  be the  $\mu$ -isotypic component: the sum of all subrepresentations of  $V$  isomorphic to  $L_q(\mu)$  as  $U_q(\Phi_i)$ -modules. The Kashiwara operators  $\tilde{e}_i(v)$  and  $\tilde{f}_i(v)$  are defined on this isotypic component, hence we get the  $i$ -Kashiwara operators defined on the whole of  $V$  by taking the sum over the  $U_q(\Phi_i)$ -isotypic components<sup>1</sup> of  $V$ .

Let  $A \subseteq \mathbb{Q}(q)$  be the subring consisting of rational functions without a pole at  $q = 0$ . This is a discrete valuation ring: a principal ideal domain with unique nonzero maximal ideal  $qA$ . Its field of fractions is  $\mathbb{Q}(q)$ , and its residue field is  $A/qA \cong \mathbb{Q}$  via the isomorphism  $f + qA \mapsto f(0)$ . Recall that an  $A$ -lattice in a  $\mathbb{Q}(q)$ -vector space  $V$  is a free  $A$ -submodule  $L \subseteq V$  such that  $L \otimes_A \mathbb{Q}(q) \cong V$ , and that we have an isomorphism of  $\mathbb{Q}$ -vector spaces  $L \otimes_A \mathbb{Q} \cong L/qL$ .

Let  $V$  be an integrable  $U_q(\Phi)$ -module. An  $A$ -lattice  $L \subseteq V$  is called a *crystal lattice* if  $L$  is graded, meaning that  $L = \bigoplus_{\lambda \in X(\Phi)} L \cap V_\lambda$ , and lattice  $L$  is invariant under the Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$ . This ensures that the  $\mathbb{Q}$ -vector space is still graded into weight spaces, and that the Kashiwara operators descend to  $\mathbb{Q}$ -linear operators on the  $\mathbb{Q}$ -vector space  $L/qL$ . A pair  $(L, B)$  is called a *crystal basis* of  $V$  if:

1.  $L$  is a crystal lattice of  $V$ ,
2.  $B$  is a weight basis of  $L/qL$ ,

<sup>1</sup>If we were defining crystals of  $\mathcal{O}(\Phi)$ -modules rather than integrable modules, the definition of the Kashiwara operators would have to change slightly (but everything following this point would remain the same). We cannot quite use  $i$ -isotypic components in this case, since  $V$  may not be semisimple as a  $U_q(\Phi_i)$ -module, but one can nevertheless define an appropriate direct sum decomposition by ‘pulling down’  $i$ -highest weight vectors. See Section 3.5 of [Kas91] for details.



3.  $\tilde{e}_i B \subseteq B \cup \{0\}$  and  $\tilde{f}_i B \subseteq B \cup \{0\}$  for all  $i \in I$ , and
4. For any  $b, b' \in B$  and  $i \in I$  we have  $\tilde{e}_i(b) = b'$  if and only if  $b = \tilde{f}_i(b')$ .

It is *a priori* unclear whether crystal bases exist for integrable highest-weight modules when  $|I| > 1$ . However, some properties of crystal bases are already visible, for example if  $(L_1, B_1)$  and  $(L_2, B_2)$  are crystal bases of  $V_1$  and  $V_2$  respectively then  $(L_1 \oplus L_2, B_1 \sqcup B_2)$  is a crystal basis of  $V_1 \oplus V_2$ . Remarkably, something similar works for the tensor product:  $(L_1 \otimes_A L_2, B_1 \times B_2)$  is a crystal base for  $V_1 \otimes V_2$ . Furthermore, the action of the Kashiwara operators on  $B_1 \times B_2$  is given purely in terms of their actions on  $B_1$  and  $B_2$  individually, with a simple rule for which side to act on (see Definition 4.3.1 below). We could have attempted to formulate crystal bases for  $\mathfrak{g}(\Phi)$ -modules rather than  $U_q(\Phi)$ -modules, but we would have fallen flat attempting to get any nice behaviour out of the tensor product: something interesting is happening in the quantum world which makes this work. It is worth noting that just as the tensor product of  $U_q(\Phi)$ -modules is asymmetric, so is the tensor product of crystals.

The existence of crystal bases was shown by Kashiwara.

#### 4.1.2 Theorem (Existence of crystal bases in $\mathcal{O}_q^{\text{int}}(\Phi)$ )

(Theorem 2, [Kas91]). Let  $\Phi$  be a root datum of type  $(I, \cdot)$  and let  $\lambda \in X(\Phi)_+$  be a dominant weight. Then the integrable highest-weight module  $L_q(\lambda)$  admits a crystal base, constructed as follows. Let  $v_\lambda \in L_q(\lambda)$  be a highest-weight vector, and let  $\mathcal{F}v_\lambda$  be the set of vectors of the form  $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$ . Define the lattice  $\mathcal{L}(\lambda)$  to be the  $A$ -span of the set  $\mathcal{F}v_\lambda$ , and  $\mathcal{B}(\lambda)$  to be the image of  $\mathcal{F}v_\lambda$  in  $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$  with zero removed. Then  $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$  is a crystal base of  $L_q(\lambda)$ .

There is a little more discussion of the theory of crystal bases at the end of this chapter, but from now on we will largely focus on the more combinatorial aspects of crystals.

## 4.2 The category of crystals

The combinatorial nature of crystal bases suggests that they are governed by a combinatorially defined category. To the author's knowledge, the first appearance of this category was in [Kas93].

### 4.2.1 Definition (Abstract crystal)

Let  $\Phi$  be a root datum of Cartan type  $(I, \cdot)$ . An *abstract crystal* of type  $\Phi$ , or  $\Phi$ -*crystal*, is the data of

$$(B, \text{wt}, (\varepsilon_i)_{i \in I}, (\varphi_i)_{i \in I}, (e_i)_{i \in I}, (f_i)_{i \in I}), \quad (4.2.2)$$

where  $B$  is a set,  $\text{wt} : B \rightarrow X(\Phi)$  and  $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}$  are functions which we call the *raising* and *lowering statistics*, and  $e_i, f_i : B \dashrightarrow B$  are partial functions called the *crystal operators*, such that for all  $b, b' \in B$  and  $i \in I$  the following axioms hold:

1. *Balanced-strings*:  $\varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), \alpha_i^\vee \rangle$ ,
2. *Raising*: If  $e_i(b) \neq \perp$ , then

$$\text{wt}(e_i(b)) = \text{wt}(b) + \alpha_i, \quad \varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1, \quad \varphi_i(e_i(b)) = \varphi_i(b) + 1, \quad (4.2.3)$$

3. *Lowering*: If  $f_i(b) \neq \perp$ , then

$$\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i, \quad \varepsilon_i(f_i(b)) = \varepsilon_i(b) + 1, \quad \varphi_i(f_i(b)) = \varphi_i(b) - 1, \quad (4.2.4)$$

4. *Partial inverse*:  $e_i(b) = b'$  if and only if  $b = f_i(b')$ , and

5. *Infinity*: If  $\varphi_i(b) = -\infty$ , then  $e_i(b) = f_i(b) = \perp$ .

In this definition,  $\mathbb{Z} \sqcup \{-\infty\}$  is understood to have the additive structure where  $x + (-\infty) = -\infty$ .

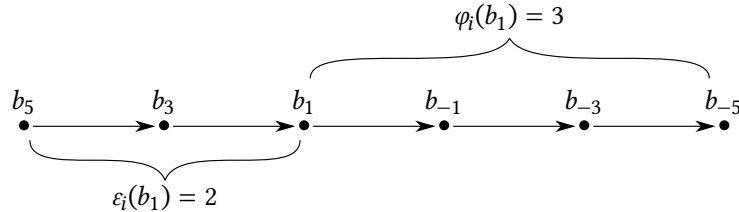
If  $(L, B)$  is the crystal base of an integrable  $U_q(\Phi)$ -module  $V$  as defined above, then we can put an abstract  $\Phi$ -crystal structure on the set  $B$  by taking the crystal  $e_i$  and  $f_i$  operators to be the Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$ , and the raising and lowering statistics to be  $\varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq \perp\}$  and  $\varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq \perp\}$ . In this way we get an abstract crystal  $\mathcal{B}(\lambda)$  for each integrable highest-weight module  $L_q(\lambda) \in \mathcal{O}_q^{\text{int}}(\Phi)$ .

### 4.2.5 Example (Root strings)

Let  $\Phi$  be the root datum  $\text{SL}_2$  with  $I = \{i\}$ , and  $X(\Phi) = \mathbb{Z}$  be the weight lattice, so that  $\langle 1, \alpha_i^\vee \rangle = 1$ . Fix a dominant weight  $n \geq 0$ , and define the crystal  $\mathcal{B}(n) = \{b_n, b_{n-2}, \dots, b_{-n+2}, b_{-n}\}$  as a set, with crystal operators

$$\begin{aligned} e_i(b_n) &= \perp, & f_i(b_{-n}) &= \perp, & \varepsilon_i(b_k) &= \frac{1}{2}(n-k), & \text{wt}(b_k) &= k. \\ e_i(b_k) &= b_{k+2}, & f_i(b_k) &= b_{k-2}, & \varphi_i(b_k) &= \frac{1}{2}(n+k). \end{aligned} \quad (4.2.6)$$

Then  $\mathcal{B}(n)$  is a  $\Phi$ -crystal, and it is obvious from the definition of crystal bases that it is the crystal arising from the integrable highest-weight module  $L_q(n)$ . It can be pictured by its *crystal graph*, shown below for  $\mathcal{B}(5)$ :



The raising statistic  $\varepsilon_i(b_1) = 2$  counts the number of edges above  $b_1$ , and  $\varphi_i(b_1) = 3$  is counting the number of edges below  $b_1$ .

If an abstract crystal comes from a crystal base of some integrable  $U_q(\Phi)$ -module, then (by definition) the raising and lowering statistics behave in the simple way shown in Example 4.2.5, counting the number of arrows above and below an element on a root string. This is called *seminormality*, sometimes called *normality* or *regularity* in the literature.

**4.2.7 Definition (Seminormality of crystals)**

A crystal is called

1. *Upper seminormal* if  $\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k(b) \neq \perp\}$  for all  $b \in B$ ,
2. *Lower seminormal* if  $\varphi_i(b) = \max\{k \geq 0 \mid f_i^k(b) \neq \perp\}$  for all  $b \in B$ , and
3. *Seminormal* if it is both upper and lower seminormal.

The balanced-strings axiom implies that if  $B$  is lower or upper seminormal then the statistics  $\varepsilon_i$  and  $\varphi_i$  take only integer values. Furthermore, together with the partial inverse axiom it means that the data of an upper-seminormal crystal  $B$  is completely determined by  $(B, \text{wt}, (e_i)_{i \in I})$ , and similarly the data of a lower-seminormal crystal is completely determined by  $(B, \text{wt}, (f_i)_{i \in I})$ . (One still needs to remember whether the crystal was upper or lower-seminormal).

We remark that there are many examples of abstract crystals which are *not* seminormal, for example those coming from Verma modules such as the crystal  $\mathcal{B}(\infty)$  shown in Example 4.2.12, or those that do not come from  $U_q(\Phi)$ -modules as all, such as the crystal  $T_\lambda$  of Example 4.2.11.

The ad-hoc illustration of the crystal  $\mathcal{B}(5)$  in Example 4.2.5 can be formalised into the notion of a *crystal graph*.

**4.2.8 Definition (Crystal graph)**

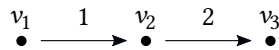
Let  $B$  be an abstract  $\Phi$ -crystal. The *crystal graph* associated to  $B$  is the directed edge-labelled graph with vertex set  $B$ , with an edge  $b \xrightarrow{i} b'$  labelled  $i$  if and only if  $f_i(b) = b'$ , or equivalently if and only if  $b = e_i(b')$ . If the underlying undirected graph of  $B$  is connected, we say that  $B$  is a *connected crystal*. A vertex with no incoming edges is called a *primitive element*. If a primitive element  $b_\lambda \in B_\lambda$  generates  $B$  under the  $f_i$ , then we say that  $B$  is a *highest-weight crystal* of weight  $\lambda$ .

The crystal operator  $e_i$ , the crystal operator  $f_i$ , and the crystal graph are all equivalent data. If the crystal is known to be upper or lower-seminormal, then the statistics  $\varepsilon_i$  and  $\varphi_i$  can also be inferred from the graph.

As an example for the reader to keep in mind, we present some  $SL_n$ -crystals.

**4.2.9 Example (Some  $SL_n$  crystals)**

The quantum group  $U_q(\text{GL}_n)$  has a representation on the  $\mathbb{Q}(q)$ -vector space  $V$  with basis  $v_1, \dots, v_n$  where  $E_i$  acts as the coordinate matrix  $v_{i+1} \mapsto v_i$ ,  $F_i$  acts as the coordinate matrix  $v_i \mapsto v_{i+1}$ , and the action of the  $K_\nu$  is determined by requiring that  $v_i$  is a weight vector of weight  $\varepsilon_i$ . We call  $V$  the *natural representation* of  $U_q(\text{GL}_n)$ , and in this special case the Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $V$  are equal to  $E_i$  and  $F_i$ , therefore setting  $B = \{v_1, \dots, v_n\}$  and  $L = AB$  makes  $(L, B)$  into a crystal base for  $V$ . For example, when  $\Phi = \text{GL}_3$  we can draw the crystal  $B$  as follows:

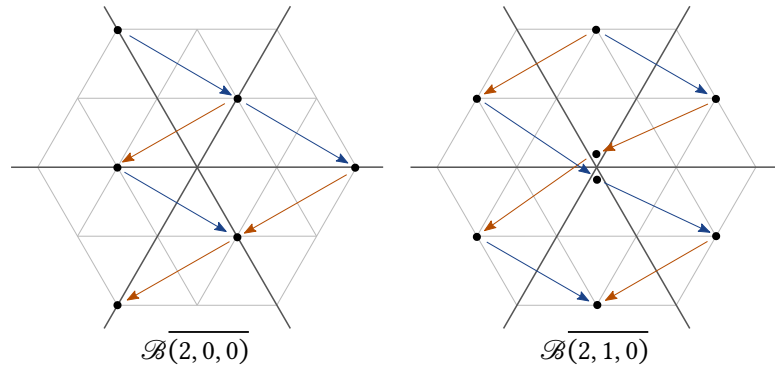


where the vector  $v_i$  is in weight  $\varepsilon_i$ . What we have drawn is exactly a *crystal graph* in the sense of Definition 4.2.8: we are saying that  $f_1(v_1) = v_2$  and  $f_2(v_2) = v_3$ , and  $f_i(v_j) = \perp$  everywhere else. In order to grasp the shape of more complicated graphs, we will often *colour* the set  $I$ , and then use those colours to label the edges. For example, if we let  $I = \{1, 2\}$  so that 1 is coloured *blue* and 2 is coloured *orange*, then we can draw the diagram more simply:



There is a morphism of  $A_2$ -root data  $SL_3 \rightarrow GL_3$ , and hence each  $GL_3$ -crystal becomes an  $SL_3$ -crystal by this restriction. Conveniently  $SL_3$  has a two-dimensional weight lattice (shown in Example 3.4.16), allowing

us to draw crystals on top of it and picture the weights of a crystal element by its position on the drawing. Below are examples of two  $\mathfrak{sl}_3$  representations, the one on the left corresponding to the dominant weight  $(2, 0, 0)$ , and the one on the right to  $\theta = (2, 1, 0)$ , the highest root, making  $\mathcal{B}(\theta)$  the crystal of the adjoint representation.



The diagrams above show the crystal graph explicitly, and the weights of elements by their position on the weight space. This is *almost* enough information to specify the crystal completely, and once we declare both of these crystals to be seminormal then the  $\varepsilon_i$  and  $\varphi_i$  statistics can also be inferred from the graph. We remark that in the crystal for the adjoint representation, the two vertices in the zero weight space are far apart in terms of the crystal graph, despite having equal weight.

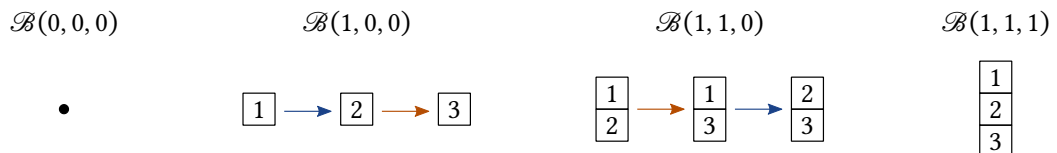
The vertex set of the crystals in the previous example are not explicitly given (merely shown in the graph). It is often the case in Lie theory that we are able to find bases of the modules  $L(\lambda)$  indexed by a set of combinatorial objects depending on  $\lambda$ . Perhaps the most famous example of this is of Young tableaux. When  $\lambda$  is a partition with at most  $n$  rows, it may be interpreted as a dominant weight of  $GL_n$ , and there is a basis of  $L(\lambda)$  made of *semistandard* Young tableaux: fillings of the Young diagram  $\lambda$  using the letters  $1, \dots, n$  such that the filling strictly increases down columns and weakly increases to the right. A lovely account of this can be found in [Ful96].

We might then expect the existence of a crystal  $\mathcal{B}(\lambda)$  with vertices indexed by such semistandard tableaux, and moreover we should be able to interpret the crystal operators  $e_i, f_i$  as partial functions from this set to itself. This is in fact the case, and furthermore we will see later on that one could use the theory of crystals to work the other way: starting only with a crystal structure on semistandard tableaux, it is possible to prove that the semistandard tableaux of shape  $\lambda$  *must* be a basis of  $L(\lambda)$ , using the tensor product of crystals and the notion of a *closed family* of crystals. But we are getting ahead of ourselves, let's see some more examples of crystals.

**4.2.10 Example (Crystals of tableaux)**

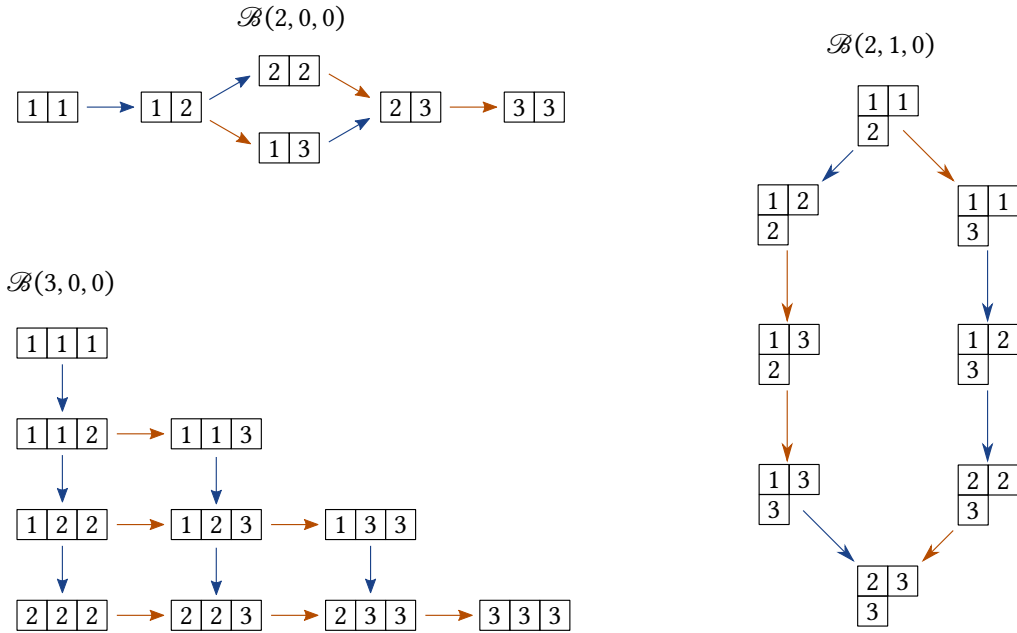
The dominant polynomial weights of  $GL_n$  are in bijection with the set of partitions of length at most  $n$ , and for a partition  $\lambda$  there exists a crystal basis  $\mathcal{B}(\lambda)$  of  $L_q(\lambda)$  in bijection with the set of semi-standard tableaux of shape  $n$  on the letters  $1, \dots, n$ . The crystal operators  $e_i$  and  $f_i$  can be given on tableaux explicitly by relatively simple rules (see Chapter 7 of [HK12], or [BS17]). The weight of a semistandard tableau is determined by its entries, with a number  $i$  contributing  $\varepsilon_i$  to the weight of the tableau.

Some examples for  $GL_3$  are shown below, in left-to-right order we have the trivial crystal, the crystal of the natural representation  $V$ , the crystal of the representation  $\Lambda^2(V)$ , and the crystal of the determinant representation  $\Lambda^3(V)$ .



The crystals  $\mathcal{B}(0, 0, 0)$  and  $\mathcal{B}(1, 1, 1)$  are identical except for their weights. Here are three more interesting

$GL_3$ -crystals for the reader to gaze upon.



We have a morphism of  $A_2$  root data  $SL_3 \rightarrow GL_3$ , and so we may view each of the crystals above as  $SL_3$  crystals by restriction. While all the above crystals are non-isomorphic as  $GL_3$  crystals, the crystals  $\mathcal{B}(0, 0, 0)$  and  $\mathcal{B}(1, 1, 1)$  are isomorphic as  $SL_3$  crystals, due to the restricted weights  $(0, 0, 0)$  and  $(1, 1, 1)$  being equal in  $X(SL_3)$ . The  $SL_3$  restrictions of  $\mathcal{B}(2, 0, 0)$  and  $\mathcal{B}(2, 1, 0)$  both appeared in the previous Example 4.2.9.

The crystals we have seen so far have all come from  $\mathcal{O}_q^{\text{int}}(\Phi)$  representations, however there are many  $\Phi$ -crystals which do not come from *any* representation of  $U_q(\Phi)$ , let alone integrable representations. For example, this next crystal could be thought of as a one-dimensional representation of  $U_q^0(\Phi)$  rather than  $U_q(\Phi)$ .

**4.2.11 Example (Character crystal)**

Given a weight  $\lambda \in X(\Phi)$ , let  $T_\lambda = \{t_\lambda\}$  with  $\text{wt}(t_\lambda) = \lambda \in X(\Phi)$ , the raising and lowering statistics  $\varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$ , and the crystal operators  $e_i(b_0) = \perp$  and  $f_i(b_0) = \perp$ .

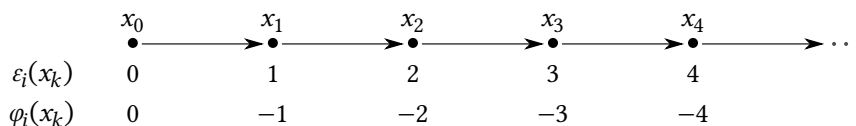
We could complain that the character crystal is a silly example of a crystal not coming from an  $\mathcal{O}_q^{\text{int}}(\Phi)$  module, since the raising and lowering statistics take the special value  $-\infty$ . However, even if they take integer values, we may have a crystal of a non-integrable module such as the  $\mathcal{B}(\infty)$  crystal coming from the Verma module  $M_q(0)$ .

**4.2.12 Example (The rank-one infinity crystal)**

Let  $\Phi$  be the root datum of  $SL_2$ . Define the set  $\mathcal{B}(\infty) = \{x_0, x_1, x_2, \dots\}$  with the operators

$$\begin{aligned} e_i(x_0) &= \perp, & f_i(x_k) &= b_{x+1}, & \varepsilon_i(x_k) &= k, & \text{wt}(x_k) &= -2k, \\ e_i(x_k) &= x_{k-1}, & \varphi_i(x_k) &= -k. \end{aligned} \tag{4.2.13}$$

Then  $\mathcal{B}(\infty)$  is a  $\Phi$ -crystal. Its crystal graph looks as follows, with the  $\varepsilon_i$  and  $\varphi_i$  statistics drawn underneath:



The example of the  $\mathcal{B}(\infty)$  crystal brings us to the definition of *morphisms* in the category of abstract  $\Phi$ -crystals, which are slightly tricky to define.

**4.2.14 Definition** (*Morphisms of crystals*)

A *morphism*  $B_1 \rightarrow B_2$  of  $\Phi$ -crystals is a partial function  $\psi : B_1 \dashrightarrow B_2$  such that

1.  $\psi$  commutes with  $\text{wt}$ ,  $\varepsilon_i$ , and  $\varphi_i$  on its domain of definition.
2. If both  $b, e_i(b) \in B_1$  are in the domain of definition of  $\psi$ , then  $\psi(e_i(b)) = e_i(\psi(b))$ .
3. If both  $b, f_i(b) \in B_1$  are in the domain of definition of  $\psi$ , then  $\psi(f_i(b)) = f_i(\psi(b))$ .

A crystal morphism is called *strict* if it commutes with all the  $e_i$  and  $f_i$ . It is called an *embedding* if it is defined on the whole of  $B_1$  and is an injective function. It is called an *isomorphism* if it is defined on the whole of  $B_1$  and is a bijective function.

A non-strict crystal morphism need not *quite* commute with the  $e_i$  and  $f_i$  crystal operators. For example, there is a crystal morphism  $\psi : \mathcal{B}(0) \rightarrow \mathcal{B}(\infty)$  by taking the unique element  $b_0 \in \mathcal{B}(0)$  of the trivial crystal to the element  $x_0 \in \mathcal{B}(\infty)$  of the same weight, which is an example of a *non-strict* embedding. However, if  $\psi$  is an isomorphism, or both  $B_1$  and  $B_2$  are seminormal, then  $\psi$  is automatically strict.

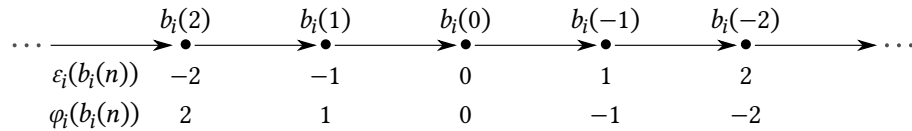
We will leave one final example here of a crystal coming from a  $U_q(\Phi)$  module which is neither highest-weight nor integrable, but is nevertheless occasionally used in abstract arguments.

**4.2.15 Example** (*The elementary crystal*)

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ . Fix a fixed vertex  $i \in I$ , define the *elementary crystal*  $\mathcal{B}_i = \{b_i(n) \mid n \in \mathbb{Z}\}$ , with structure

$$\begin{aligned} \text{wt}(b_i(n)) &= n\alpha_i & f_i(b_i(n)) &= b_i(n-1) \\ \varepsilon_i(b_i(n)) &= -n & \varphi_i(b_i(n)) &= n, \end{aligned} \tag{4.2.16}$$

with  $\varepsilon_j = \varphi_j = -\infty$  and  $e_j = f_j = \perp$  for  $j \neq i$ . The crystal graph is the following:



## 4.3 Tensor product of crystals

The category of abstract  $\Phi$ -crystals is equipped with a monoidal structure, the *tensor product of crystals*. There are two different definitions of the tensor product, each the reverse of the other. Here we use the “combinatorialist’s convention”, which plays the nicest with existing combinatorics such as the RSK algorithm.

### 4.3.1 Definition (*Tensor product of crystals*)

Let  $B$  and  $C$  be two abstract  $\Phi$ -crystals. The *tensor product*  $B \otimes C$  has underlying set the Cartesian product  $B \times C$ , with elements denoted by  $b \otimes c$  rather than  $(b, c)$ . We use the convention that  $b \otimes \perp = \perp = \perp \otimes c$ , and equip the tensor product with the following crystal structure:

$$\begin{aligned}
 \text{wt}(b \otimes c) &= \text{wt}(b) + \text{wt}(c) \\
 \varepsilon_i(b \otimes c) &= \max\{\varepsilon_i(c), \varepsilon_i(b) - \langle \text{wt}(c), \alpha_i^\vee \rangle\} \\
 \varphi_i(b \otimes c) &= \max\{\varphi_i(b), \varphi_i(c) + \langle \text{wt}(b), \alpha_i^\vee \rangle\} \\
 e_i(b \otimes c) &= \begin{cases} e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \\ b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c) \end{cases} \\
 f_i(b \otimes c) &= \begin{cases} f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c) \\ b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \end{cases}
 \end{aligned} \tag{4.3.2}$$

It is routine to verify that this defines the structure of an abstract  $\Phi$ -crystal on  $B \otimes C$ , and furthermore that if both  $B$  and  $C$  are both upper seminormal, then so is  $B \otimes C$  (and similarly for lower seminormal).

The remarkable thing about the crystal tensor product is if  $\mathcal{B}(\lambda)$  and  $\mathcal{B}(\mu)$  are the crystals of  $L_q(\lambda)$  and  $L_q(\mu)$  respectively, then  $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$  is the crystal of  $L_q(\lambda) \otimes L_q(\mu)$ . This means that if we can find combinatorial models for  $\mathcal{B}(\lambda)$  and  $\mathcal{B}(\mu)$  (such as the tableaux hinted at in [Example 4.2.10](#)), then we can compute the decomposition of  $L_q(\lambda) \otimes L_q(\mu)$  by just finding the connected components (or highest-weight vertices) of the graph  $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ .

We defined the crystal  $\mathcal{B}(\infty)$  when  $|I| = 1$  in [Example 4.2.12](#). For general  $\Phi$ , the crystal  $\mathcal{B}(\infty)$  is a kind of limit of the  $\mathcal{B}(\lambda)$ , which again plays the role of the Verma module  $M(0)$ . The tensor product  $T_\lambda \otimes \mathcal{B}(\infty)$  plays the role of the Verma module  $M(\lambda)$  in the setting of crystals.

### 4.3.3 Example (*Shadows of Verma modules*)

Let  $\Phi = \text{SL}_2$ . Taking the tensor product of  $T_\lambda$  with  $\mathcal{B}(\infty)$  gives an interesting crystal: the tensor product structure in [Definition 4.3.1](#) simplifies to give the following crystal structure on  $T_\lambda \otimes \mathcal{B}(\infty)$ :

$$\begin{aligned}
 \text{wt}(t_\lambda \otimes x_n) &= \lambda - n\alpha_i & e_i(t_\lambda \otimes x_n) &= t_\lambda \otimes e_i(x_n) & \varepsilon_i(t_\lambda \otimes x_n) &= n \\
 f_i(t_\lambda \otimes x_n) &= t_\lambda \otimes x_{n+1} & \varphi_i(t_\lambda \otimes x_n) &= \langle \alpha_i^\vee, \lambda \rangle - n.
 \end{aligned} \tag{4.3.4}$$

For example, if we suppose that  $\lambda = 3$  then we have the following picture of  $T_\lambda \otimes \mathcal{B}(\infty)$ :

$$\begin{array}{ccccccccc}
 & t_3 \otimes x_0 & & t_3 \otimes x_1 & & t_3 \otimes x_2 & & t_3 \otimes x_3 & & t_3 \otimes x_4 & & \dots \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 \varepsilon_i(t_3 \otimes x_k) & 0 & & 1 & & 2 & & 3 & & 4 & & \\
 \varphi_i(t_3 \otimes x_k) & 3 & & 2 & & 1 & & 0 & & -1 & & 
 \end{array}$$

Here we can see there is a non-strict crystal morphism  $\mathcal{B}(3) \hookrightarrow T_3 \otimes \mathcal{B}(\infty)$  (recall that crystal morphisms must commute with  $\text{wt}$ ,  $\varepsilon_i$ , and  $\varphi_i$ , but only need to commute with the  $e_i$  and  $f_i$  operators on their domain of definition).

## 4.4 Recognition theorems

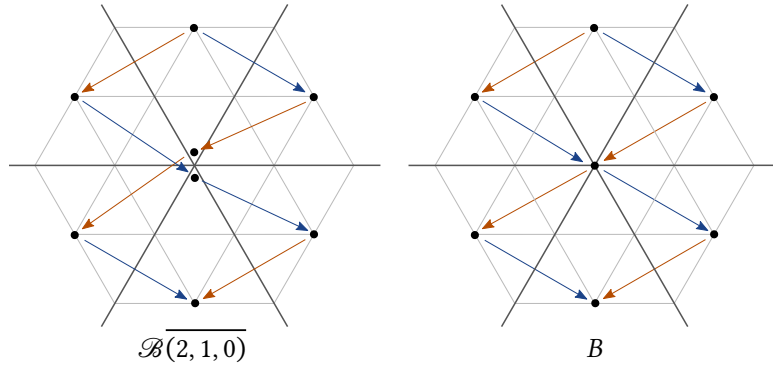
We know by now that the category of  $\Phi$ -crystals is far more general than just those corresponding to crystals of  $\mathcal{O}_q^{\text{int}}(\Phi)$  modules, which form a full monoidal subcategory. Some abstract  $\Phi$ -crystals we have seen so far which are *not* crystals of  $\mathcal{O}_q^{\text{int}}(\Phi)$ -modules are:

- The ‘Verma crystal’  $\mathcal{B}(\infty)$ , a  $\Phi$ -crystal coming from the category  $\mathcal{O}_q(\Phi)$ , but not from  $\mathcal{O}_q^{\text{int}}(\Phi)$ .
- The ‘principal series’ crystals  $\mathcal{B}_i$ , which come from a representation of  $U_q(\text{SL}_2)$  which is neither highest-weight nor integrable.
- The ‘character crystal’  $T_\lambda$ , which does not come from a  $U_q(\Phi)$ -representation at all.

The next counterexample in particular shows that the problem of classifying those abstract  $\Phi$ -crystals coming from  $\mathcal{O}_q^{\text{int}}(\Phi)$  is quite subtle.

### 4.4.1 Example

The following are both abstract seminormal  $\text{SL}_3$ -crystals, but only the one on the left comes from a  $U_q(\Phi)$ -module.



The question is: given an abstract  $\Phi$ -crystal  $B$ , how can one check that it is the crystal of a module from  $\mathcal{O}_q^{\text{int}}(\Phi)$ ? If the crystal is connected, one could hope to find a dominant weight  $\lambda$  and an existing model for  $\mathcal{B}(\lambda)$  and give a crystal isomorphism  $B \xrightarrow{\sim} \mathcal{B}(\lambda)$ . However this does not help to ‘bootstrap’ the theory, since one needs to start with an existing model for  $\mathcal{B}(\lambda)$ , and furthermore specifying an isomorphism is impossible if the exact decomposition of  $B$  into highest weights is unknown.

Fortunately, there are a number of *recognition theorems* available to us which we can use to check whether a  $\Phi$ -crystal  $B$  really does come from  $\mathcal{O}_q^{\text{int}}(\Phi)$ . The first of these is due to Kashiwara, allowing us to reduce the problem to Cartan data of rank 2. As with  $\mathfrak{g}(\Phi)$ -modules and  $U_q(\Phi)$ -modules, an element  $b$  of a crystal is called *primitive* if  $e_i(b) = \perp$  for all  $i \in I$ , and a primitive element is called *highest weight* if it generates  $B$ .

### 4.4.2 Theorem (Recognition by rank-2 restriction)

(Proposition 2.4.4 of [Kan+92]). Let  $B$  be a  $\Phi$ -crystal such that, for any subset  $J \subseteq I$  with at most two elements, any connected component of the restricted crystal  $B_J$  containing a  $J$ -primitive element is a crystal isomorphic to  $\mathcal{B}(\Phi_J, \lambda)$  for some  $J$ -dominant  $\lambda$ . Then any connected component of  $B$  containing a primitive element generates a subcrystal isomorphic to  $\mathcal{B}(\lambda)$ , for some dominant  $\lambda$ .

The above condition on subsets  $J \subseteq I$  with one element is simply checking that  $B$  is seminormal, so checking the rank-2 restrictions is where all the work is. This theorem is very frequently used, and the other recognition theorems below rely on this one. We remark that if  $B$  satisfies the above condition then only the connected components containing primitive vectors come from  $\mathcal{O}_q^{\text{int}}(\Phi)$ -modules, there may be other connected components which are not highest-weight.



Joseph [Jos95] has given a purely combinatorial method of checking whether a collection  $\mathcal{C} = \{C(\lambda) \mid \lambda \in X(\Phi)_+\}$  of candidate  $\Phi$ -crystals coincide with the  $\mathcal{B}(\lambda)$ . Clearly, each  $C(\lambda)$  should be a seminormal highest weight crystal of highest weight  $\lambda$ . We need only one more condition on  $\mathcal{C}$ , ensuring that it behaves well with tensor products.

#### 4.4.3 Definition (Closed families of crystals)

Let  $\mathcal{C} = \{C(\lambda) \mid \lambda \in X(\Phi)_+\}$  be a family of highest-weight seminormal crystals, where  $c_\lambda \in C(\lambda)$  is the highest-weight element of weight  $\lambda$ . We say that  $\mathcal{C}$  is a *closed family* if the element  $c_\lambda \otimes c_\mu \in C(\lambda) \otimes C(\mu)$  generates a strict subcrystal of highest weight  $\lambda + \mu$  isomorphic to  $C(\lambda + \mu)$ .

It is immediate from the definition of the tensor product that  $c_\lambda \otimes c_\mu$  will be a primitive element of  $C(\lambda) \otimes C(\mu)$  of the correct weight, but it is far from clear that the subset generated by  $c_\lambda \otimes c_\mu$  under the  $f_i$  operators is a strict subcrystal of  $C(\lambda) \otimes C(\mu)$ , or even that it is stable under the  $e_i$  operators.

#### 4.4.4 Theorem

(6.4.21 of [Jos95]). Let  $\mathcal{C} = \{C(\lambda) \mid \lambda \in X(\Phi)_+\}$  be a closed family of highest-weight seminormal crystals. Then  $C(\lambda) \cong \mathcal{B}(\lambda)$  for all  $\lambda \in X(\Phi)_+$ .

It is very easy to show that the  $\{\mathcal{B}(n) \mid n \geq 0\}$  constructed in Example 4.2.5 is a closed family of  $\mathrm{SL}_2$ -crystals, and with some work (and proper definitions of the  $e_i$  and  $f_i$  operators) it can be shown that the crystals of semistandard tableaux hinted at in Example 4.2.10 form a closed family of  $\mathrm{GL}_n$  crystals. Given a closed family of  $\Phi$ -crystals, we have a way to work with the crystals of  $\mathcal{O}_q^{\mathrm{int}}(\Phi)$ -modules without the need to introduce the quantum enveloping algebra and the theory of crystal bases.

It can be shown that the Lakshmibai-Seshadri paths, with the Littelmann root operators, form a closed family and hence give an explicit model for all the  $\mathcal{B}(\lambda)$  in any type [Lit94]. In type A, these paths are in bijection with Young tableaux, making the LS paths a generalisation of Young tableaux to any type.

There is another kind of recognition theorem due to Stembridge, when  $(I, \cdot)$  is simply-laced. In [Ste03], Stembridge gives necessary and sufficient conditions for a directed graph with edges coloured by  $I$  to be the crystal graph of a crystal from  $\mathcal{O}_q^{\mathrm{int}}(\Phi)$ . After showing that the Stembridge axioms are necessary and permit at most one connected graph per highest weight, it is shown that these connected graphs are isomorphic to the corresponding crystal graphs given by the Littelmann paths, and hence must be the crystals of highest-weight integrable modules. This description only works for simply-laced type, since the uniqueness of the graphs only holds in simply-laced type, but crystals in other types can be obtained through a standard technique known as *diagram folding*. The recent text [BS17] defines crystals this way, as either Stembridge crystals or folded Stembridge crystals, without recourse to quantum groups.

## 4.5 Weyl group action

We often talk about crystals which come from crystal bases of integrable  $U_q(\Phi)$ -modules. Throughout this section, we can use a slightly weaker property which appears in [Kas94].

#### 4.5.1 Definition

A  $\Phi$ -crystal  $B$  is called *finite-normal* if for any  $J \subseteq I$  of finite type, the restriction of  $B$  to  $\Phi_J$  is isomorphic (as  $\Phi_J$ -crystals) to the crystal base of an integrable  $U_q(\Phi_J)$ -module.

A finite-normal  $\Phi$ -crystal is automatically seminormal, and furthermore if  $(I, \cdot)$  is finite type then the finite-normality property is equivalent to being the crystal of a  $\mathcal{O}_q^{\mathrm{int}}(\Phi)$ -module.

For each  $J \subseteq I$  of finite type, a finite-normal crystal  $B$  decomposes into a disjoint union of finite  $\Phi_J$  crystals, each of the form  $\mathcal{B}(\Phi_J, \lambda)$  for some  $\lambda \in X(\Phi_J)_+$ . It was first shown by Kashiwara that one can exploit this finiteness to obtain an action of the Weyl group  $W_I$  on the whole crystal  $B$ .

**4.5.2 Theorem** (*Weyl action on finite-normal crystals*)

(Section 7 of [Kas94]). Let  $B$  be a finite-normal  $\Phi$ -crystal, and define for each  $i \in I$  a map  $c_i : B \rightarrow B$  by

$$c_i(b) = \begin{cases} f^{\langle \text{wt}(b), \alpha_i^\vee \rangle}(b) & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 0 \\ e^{-\langle \text{wt}(b), \alpha_i^\vee \rangle}(b) & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle \leq 0 \end{cases} \quad (4.5.3)$$

Then:

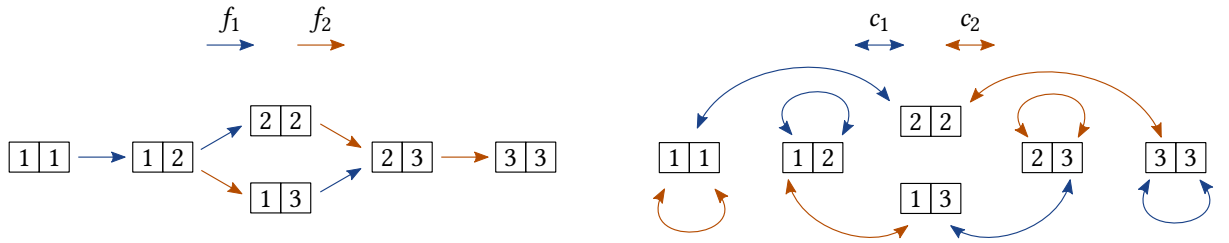
1. Each  $c_i$  is an involution:  $c_i^2 = \text{id}_B$ .
2. Each  $c_i$  acts by the reflection  $s_i : X(\Phi) \rightarrow X(\Phi)$  on the weight of an element:  $\text{wt}(c_i(b)) = s_i(\text{wt}(b))$ .
3. The  $c_i$  satisfy the braid relation  $(c_i c_j)^{m_{ij}} = 1$ .

Hence the Weyl group  $W_I$  acts on  $B$  via the map  $s_i \cdot b = c_i(b)$ .

Since  $B$  is a seminormal  $\Phi$ -crystal, its restriction to  $\Phi_i$  breaks up into a disjoint union of balanced  $i$ -strings of finite length, and the operator  $c_i$  acts by reversal on these strings.

**4.5.4 Example**

The following diagram shows the  $\text{GL}_3$  crystal of  $\mathcal{B}(2, 0, 0)$  from Example 4.2.10 on the left, with the computed actions of  $c_1$  and  $c_2$  on the right.



This makes the first two properties of Theorem 4.5.2 easy to verify: reversal is clearly an involution, and the balanced-strings axiom ensures that the reversal is taking place around zero. The third property is proved by reducing to the case where  $J \subseteq I$  is finite-type and has two elements, we refer the reader to Section 7 of [Kas94] for the full proof.

## 4.6 Cactus group action

The Weyl group action in Theorem 4.5.2 may be extended to an action of the cactus group on a finite-normal crystal, via some involutions  $c_J$  where  $c_{\{i\}} = c_i$  from before.

**4.6.1 Theorem**

Let  $B$  be a finite-normal  $\Phi$ -crystal, and let  $J \subseteq I$  be finite type and irreducible. Define a map  $c_J : B \rightarrow B$  to be the unique map preserving the connected components of the restricted crystal  $B_J$ , and satisfying

- (1)  $\text{wt}(c_J(b)) = w_J \cdot \text{wt}(b)$ ,
  - (2)  $e_j(c_J(b)) = c_J(f_{\omega_j(j)}(b))$  for all  $j \in J$ ,
  - (3)  $f_j(c_J(b)) = c_J(e_{\omega_j(j)}(b))$  for all  $j \in J$ .
- (4.6.2)

Then the maps  $c_J : B \rightarrow B$  satisfy the cactus relations:

1.  $c_J^2 = \text{id}_B$ ,
2.  $c_J c_K = c_K c_J$  if there are no edges between the vertices of  $J$  and  $K$ , and
3.  $c_J c_K = c_K c_{\omega_K(J)}$  if  $J \subseteq K$ .

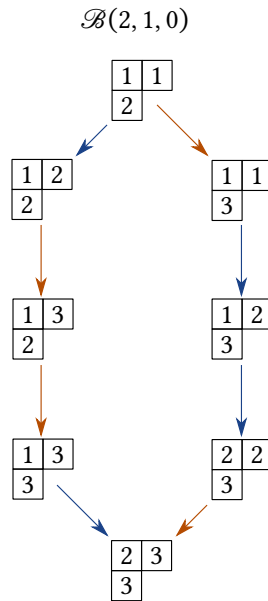
Furthermore, the definition of  $c_{\{i\}}$  agrees with the one given in Theorem 4.5.2. Hence there is an action of the cactus group  $C_I$  on  $B$  via the operators  $c_J$ , additionally satisfying the braid relations  $(c_i c_j)^{m_{ij}} = 1$ .

The quotient of the cactus group  $C_I$  by the braid relations  $(\tau_i \tau_j)^{m_{ij}}$  has been called the *reduced cactus group* (3.4 of [Hal20]).

We remark that it is straightforward to compute the involution  $c_J$  if one knows both the involution  $\omega_J : J \rightarrow J$  and the whole crystal graph  $B_J$ . Each element  $b$  of a connected component of  $B_J$  can be written (non-uniquely) as  $b = f_{j_1} \cdots f_{j_k} b_{\text{high}}$ , where  $b_{\text{high}}$  is the highest-weight element of the connected component and  $j_1, \dots, j_k \in J$ . The equality  $c_J(b_{\text{high}}) = b_{\text{low}}$  follows from (1), and applying condition (2) we get

$$\begin{aligned} c_J(b) &= c_J(f_{j_1} \cdots f_{j_k} b_{\text{high}}) \\ &= e_{\omega_J(j_1)} \cdots e_{\omega_J(j_k)} b_{\text{low}}. \end{aligned} \tag{4.6.3}$$

So if we get from the highest-weight element to  $b$  by following some arrows forwards, we get from the lowest-weight element to  $c_J(b)$  by following the *opposite arrows* (as defined by  $\omega_J$ ) backwards. Consider the  $\text{GL}_3$ -crystal  $\mathcal{B}(2, 1, 0)$  from earlier, which we show here:



In this case  $(I, \cdot) = A_2$ , and the involution  $\omega_I : I \rightarrow I$  swaps the blue and orange vertices 1 and 2 of the Dynkin diagram. So *opposite arrow* means the oppositely coloured arrow. Viewing  $\mathcal{B}(2, 1, 0)$  as a hexagon, the cactus involution  $c_I$  swaps opposite vertices. In particular, the two tableaux of weight  $(1, 1, 1)$  are swapped.

If  $\mathcal{B}(\lambda)$  is the  $\text{GL}_n$ -crystal of semistandard tableaux of shape  $\lambda$ , then the involution  $c_{\{1, \dots, i\}}$  acts as the well-known *Schützenberger involution* on the sub-tableaux containing only entries  $\{1, \dots, i + 1\}$ .

## 4.7 Addendum

Something that initially confused the author was the relationship between crystal bases and other bases of  $U_q(\Phi)$ , such as Lusztig's canonical basis. Some people like to say (informally) that they are 'the same', which doesn't make sense because they are not even objects of the same kind: the crystal basis of a representation  $V$  is a  $\mathbb{Q}$ -basis of a quotient lattice of  $V$ , while Lusztig's canonical basis is an honest  $\mathbb{Q}(q)$ -basis of the whole algebra  $U_q(\Phi)$ . Something that was also unclear was the connection between Littelmann's work on path models and Kashiwara's on crystals. We have written this short section to hopefully be a small guide to the early literature explaining these things.

The story of crystal bases of  $U_q(\Phi)$ -modules is due to Kashiwara. In [Kas90], he defined the notion of what we now call an *upper* crystal base of a  $U_q(\Phi)$ -module  $V$  to be a pair  $(L, B)$  of an  $A$ -lattice  $L \subseteq V$  together with a basis  $B$  of the  $\mathbb{Q}$ -vector space  $L/qL$ , satisfying some axioms. He proved the existence and uniqueness of these crystal

bases in the classical types  $A_n$ ,  $B_n$ ,  $C_n$ , and  $D_n$ . In [Kas91], he further defined a *lower crystal base* and showed the existence and uniqueness of these bases for the  $L(\lambda)$  and the negative half  $U_q^-(\Phi)$  of the quantised enveloping algebra using a huge inductive argument called the *grand loop*. He also defined a *global lower base*, defined on an integral form  $V_{\mathbb{Z}}$  over the integral algebra  $U_{\mathbb{Z}}(\Phi)$ , to be an integral basis  $\mathbf{B}$  which descends to a lower crystal basis in both the  $q \rightarrow 0$  and  $q \rightarrow \infty$  limits: for example, setting  $\mathcal{L} = \bigoplus_{b \in \mathbf{B}} \mathbb{Q}[q]b$  and  $B = \{b \pmod{q\mathcal{L}} \mid b \in \mathbf{B}\}$  should make  $(\mathcal{L}, B)$  a lower crystal base. A good self-contained account of *globalisation* can be found in Chapter 6.2 of [Jos95].

In [Lit95a], Littelmann applied Kashiwara's theory of crystal bases together with the theory of generalised Young tableaux (special cases of Lakshmibai-Seshadri paths) to give a combinatorial definition of the crystal graphs  $\mathcal{B}(\lambda)$  in the classical types  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ ,  $E_6$ , and  $G_2$ . He then used these graphs together with the crystal tensor product rule to give a short proof of the *generalised Littlewood-Richardson rule* for computing the decomposition multiplicities of  $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ . He also conjectured that the extension-of-strings operators on crystal graphs could be used to construct crystal bases for Demazure modules. In [Kas93], Kashiwara proved Littelmann's conjecture, giving a new proof of the Demazure character formula for symmetrisable Kac-Moody algebras. In this paper he also introduced the notion of an *abstract  $\Phi$ -crystal*, so that  $\mathcal{B}(\lambda)$  and  $\mathcal{B}(\infty)$  could be put on equal footing from a combinatorial point of view.

Around the same time that Kashiwara defined crystal bases, Lusztig defined canonical bases [Lus90]. Lusztig's canonical basis of  $U^-(\Phi)$  may be defined purely algebraically however its existence is shown using geometric methods, namely realising the multiplication in the quantum group  $U_q(\Phi)$  as a kind of convolution product on a Lusztig quiver variety. The canonical basis of Lusztig can be compared to the *global* (not crystal) basis of Kashiwara, and in fact the two have been shown to be equivalent [GL93].

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<sup>2</sup>Recall that  $A \subseteq \mathbb{Q}(q)$  is the subring of rational functions without a pole at  $q = 0$ , as in Section 4.1.

## 5 Demazure modules and crystals

Demazure modules are finite-dimensional subspaces of highest-weight representations: for each dominant weight  $\lambda \in X(\Phi)_+$  and Weyl group element  $w \in W_I$ , the Demazure module  $L_w(\lambda)$  is a certain  $U^+(\Phi)$ -stable subspace of the integrable highest-weight module  $L(\lambda)$ . The Demazure modules give an inductive method to study the  $L(\lambda)$ , by analysing the successive embeddings  $L_w(\lambda) \subseteq L_{s_i w}(\lambda)$  for the simple reflections  $s_i$ . In the case that  $I$  is finite type, a classic result due to Demazure is that the characters of these modules are related by the *Demazure operators*  $\pi_i$ , with the inductive step being  $\pi_i(\text{ch } L_w(\lambda)) = \text{ch } L_{s_i w}(\lambda)$ , leading to the *Demazure character formula*  $\text{ch } L_w(\lambda) = \pi_w(e^\lambda)$ .

There have been many proofs of the Demazure character formula when  $(I, \cdot)$  is not finite type, the most relevant to our work being the perspective of *Demazure crystals*. The theory of Demazure crystals was developed by Kashiwara and Littelmann, formalising the notion of a Demazure subcrystal  $\mathcal{B}_w(\lambda) \subseteq \mathcal{B}(\lambda)$  as the analogue of a Demazure module  $L_w(\lambda) \subseteq L(\lambda)$ , and showing that the Demazure operators  $\pi_i$  have combinatorial analogues on the level of crystals, the *extension of strings operators*  $\mathfrak{D}_i$ . Our analysis of the product monomial crystal in [Chapter 7](#) follows along the same lines, breaking the crystal up into small pieces related by the extension of strings operators.

We start this chapter with a digression into formal characters of representations and crystals, a topic we have not yet covered. The main point we want the reader to take away from this section is that the formal character gives a complete invariant of  $\mathcal{O}^{\text{int}}(\Phi)$ -modules: two representations in this category with the same formal character are isomorphic. After this, we give the definition of Demazure modules and Demazure crystals, along with their formal characters called *Demazure characters*. We follow Kashiwara's approach [[Kas93](#)] to the theory of Demazure crystals, the string property, and the extension-of-strings operators. Finally, we collect some history of Demazure modules and the Demazure character formula for the reader's interest.

### 5.1 Formal Characters

Suppose that  $\Phi$  is a root datum of finite type  $(I, \cdot)$ , and we are working in the category of finite-dimensional weight representations of  $\mathfrak{g}(\Phi)$ . To every finite-dimensional weight representation  $V$  we can associate its *formal character* recording the multiplicities of weight spaces:

$$\text{ch } V = \sum_{\lambda \in X(\Phi)} (\dim V_\lambda) e^\lambda \in \mathbb{Z}[X(\Phi)]. \quad (5.1.1)$$

Here  $\mathbb{Z}[X(\Phi)]$  is the group algebra of the free abelian group  $X(\Phi)$ , written multiplicatively so that  $e^\lambda e^\mu = e^{\lambda+\mu}$ . It is simple to verify that the characters of a direct sum add and the characters of a tensor product multiply, so we have  $\text{ch}(U \oplus V) = \text{ch } U + \text{ch } V$  and  $\text{ch}(U \otimes V) = (\text{ch } U)(\text{ch } V)$ . Furthermore, for any short exact sequence  $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$  of  $\mathfrak{g}(\Phi)$ -modules, we have  $\text{ch}(V) = \text{ch}(V') + \text{ch}(V'')$ , and therefore the characters of isomorphic modules are equal.

Characters give us useful invariants of modules, and in fact in the setting we are in (finite-dimensional representations over a field of characteristic zero) they give us *complete* invariants: a finite-dimensional weight representation  $V$  is determined up to isomorphism by its character  $\text{ch } V$ . We will explain why this is the case.

The Weyl group  $W = W_I$  acts on the space  $\mathbb{Z}[X(\Phi)]$  of formal characters by permuting the standard basis:  $w \cdot e^\lambda = e^{w\lambda}$ , where  $w\lambda$  is the usual action of the Weyl group on the weight lattice ([Lemma 3.4.10](#)). By considering the restriction of  $V$  to each rank-one algebra  $\mathfrak{g}(\Phi_i)$  and considering the classification of finite-dimensional  $\mathfrak{sl}_2$

modules, we see that  $V_\lambda$  and  $V_{s_i\lambda}$  are isomorphic vector spaces. Since this works for all  $i \in I$ , we have that  $\dim V_\lambda = \dim V_{w\lambda}$  for all  $w \in W$ , hence the character of a finite-dimensional representation  $V$  is a member of the subring  $\mathbb{Z}[X(\Phi)]^W$  of Weyl-invariant characters. In fact, the characters of the  $L(\lambda)$  for dominant  $\lambda$  form a *basis* for the subring of Weyl-invariant characters.

**5.1.2 Lemma** (*A basis of the Weyl-invariant character ring*)

Let  $\Phi$  be a root datum of finite type  $(I, \cdot)$ . Then the characters  $\{\text{ch } L(\lambda) \mid \lambda \in X(\Phi)_+\}$  of the highest-weight modules  $L(\lambda)$  form a basis of the invariant ring  $\mathbb{Z}[X(\Phi)]^W$ .

**5.1.3 Proof**

Since the Weyl orbit of a weight  $\lambda \in X(\Phi)$  intersects the dominant weights  $X(\Phi)_+$  exactly once, Weyl orbits can be parameterised by dominant weights. For each dominant weight  $\lambda$  define the orbit sum  $m_\lambda = \sum_{\mu \in W \cdot \lambda} e^\mu$ , then the set  $\{m_\lambda \mid \lambda \in X(\Phi)_+\}$  forms a basis of  $\mathbb{Z}[X(\Phi)]^W$ . Since the highest-weight space of  $L(\lambda)$  has dimension 1 and all other weights of  $L(\lambda)$  are lower than  $\lambda$  in the partial ordering, we have

$$\text{ch } L(\lambda) = m_\lambda + \sum_{\mu < \lambda} k_{\lambda, \mu} m_\mu \quad \text{for some } k_{\lambda, \mu} \in \mathbb{N}, \tag{5.1.4}$$

showing that  $\{\text{ch } L(\lambda) \mid \lambda \in X(\Phi)_+\}$  is an alternative basis of the invariant character ring<sup>1</sup>.

The category of finite-dimensional weight modules of  $\mathfrak{g}(\Phi)$  is semisimple because we are working over  $\mathbb{C}$ , hence the isomorphism class of  $V$  is determined by the decomposition multiplicities  $[L(\lambda) : V]$ , which are given exactly by the expression of the character of  $V$  in the basis of irreducible characters  $\text{ch } V = \sum_{\lambda \in X(\Phi)_+} [L(\lambda) : V] \text{ch } L(\lambda)$ . Hence the character gives a complete invariant.

We now consider the case when  $(I, \cdot)$  is of arbitrary type. For a module  $V \in \mathcal{O}(\Phi)$  the formal character Eq. (5.1.1) makes sense as a sum taking values in the *completed* group algebra  $\mathbb{Z}[[X(\Phi)]]$ , whose elements are infinite linear combinations of the  $e^\lambda$ . This completion is no longer an algebra since the product of two elements may result in each coefficient being given by an infinite, rather than finite, sum. However, if  $U, V \in \mathcal{O}(\Phi)$  then the product  $(\text{ch } U)(\text{ch } V)$  makes sense by the ‘weights bounded above’ condition on Category  $\mathcal{O}(\Phi)$  (Definition 3.5.3), and is equal to  $\text{ch}(U \otimes V)$ . The Weyl group still acts on the completed algebra, and the characters of *integrable* modules are Weyl-invariant, but we can no longer parametrise Weyl orbits in  $X(\Phi)$  by dominant weights since not every orbit meets the dominant chamber. However, the characters of the integrable highest-weight modules  $L(\lambda)$  are linearly independent, since if we have a sum  $\sum_{\lambda \in X(\Phi)_+} a_\lambda \text{ch } L(\lambda) = 0$  we must have  $a_\lambda = 0$  for any maximal  $\lambda$ . Therefore the characters of modules in category  $\mathcal{O}^{\text{int}}(\Phi)$  still give a complete invariant by semisimplicity, however we cannot package this up quite as nicely as in the finite case.

Modules over quantum groups and abstract  $\Phi$ -crystals also have formal characters. In the case of abstract crystals, provided that the ‘weight spaces’  $B_\lambda = \{b \in B \mid \text{wt}(b) = \lambda\}$  are finite we can define

$$\text{ch } B = \sum_{\lambda \in X(\Phi)} |B_\lambda| e^\lambda \in \mathbb{Z}[[X(\Phi)]], \tag{5.1.5}$$

in exact analogy with Eq. (5.1.1). (Again, if  $(I, \cdot)$  is not finite type then  $\text{ch } B$  may be valued in the completed group algebra rather than the group algebra). The discussion above shows that if  $B$  is the crystal of a module from  $\mathcal{O}_q^{\text{int}}(\Phi)$ , then  $B$  is determined up to isomorphism by  $\text{ch } B$ , and hence characters also give a complete invariant of such crystals.

<sup>1</sup>This fact would be immediate if  $(X(\Phi)_+, \leq)$  were a finite partially ordered set, since Eq. (5.1.4) would show that the endomorphism  $m_\lambda \mapsto \text{ch } L(\lambda)$  is an upper-triangular matrix (in any total order refining  $\leq$ ) with 1s along the diagonal, and hence has determinant 1. The analogous statement for infinite posets is not actually true in general, as one can see by considering the  $\mathbb{Z}$ -module  $\mathbb{Z}[\mathbb{Z}] = \text{span}_{\mathbb{Z}} \{\dots, e^{-1}, e^0, e^1, \dots\}$  and the linear map  $e^i \mapsto e^i - e^{i-1}$ . This satisfies the unitriangularity condition and is injective, but is not surjective since every element of the image has its sum of coefficients equal to zero. The technical condition needed here is that the poset  $(X(\Phi)_+, \leq)$  satisfies the property: every non-empty subset contains a minimal element (as in VI.3.4, Lemma 4 of [BB02]). Alternatively, and perhaps more intuitively, one could take some filtration of  $X(\Phi)_+$  with finite pieces and realise that if the map is an isomorphism in each piece, so must it be in the limit.

## 5.2 Demazure modules and the character formula

The integrable highest-weight module  $L(\lambda)$  is irreducible, and hence has no nontrivial  $U(\Phi)$ -submodules. However, it may also be viewed as a  $U^+(\Phi)$ -module, in which case it has many submodules. A particularly nice family of these submodules are the Demazure modules, each a finite-dimensional subspace of  $L(\lambda)$  parameterised by the Weyl group  $W$ .

### 5.2.1 Definition (Demazure module)

Fix an irreducible integrable highest-weight module  $L(\lambda) \in \mathcal{O}^{\text{int}}(\Phi)$ . The elements of the Weyl orbit  $W \cdot \lambda$  are called the *extremal weights* of  $L(\lambda)$ . The *Demazure module*  $L_w(\lambda) \subseteq L(\lambda)$  is defined as the  $U^+(\Phi)$ -orbit of the one-dimensional space  $L(\lambda)_{w\lambda}$  inside  $L(\lambda)$ . We say that  $L_w(\lambda)$  has *Demazure lowest weight*  $w\lambda$ .

For any dominant  $\lambda$ , the Demazure module  $L_e(\lambda)$  associated to the identity element  $e \in W$  is  $U^+(\Phi) \cdot V(\lambda)_\lambda = V(\lambda)_\lambda$ , the one-dimensional highest-weight space. At the other extreme, when  $I$  is finite-type and  $w_I \in W$  is the longest element,  $L(\lambda)_{w_I\lambda}$  is the lowest-weight space and hence the Demazure module  $L_{w_I}(\lambda)$  is equal to the whole module  $L(\lambda)$ . If  $I$  is not finite-type then there is no longest element of the Weyl group, but we can still realise the full representation  $L(\lambda)$  as a limit of the finite-dimensional  $L_w(\lambda)$ , since whenever  $x \leq y$  in the Bruhat order we have  $L_x(\lambda) \subseteq L_y(\lambda)$ .

For each  $i \in I$  define the  $\mathbb{Z}$ -linear *Demazure operator*  $\pi_i : \mathbb{Z}[X] \rightarrow \mathbb{Z}[X]$  by the formula

$$\pi_i(f) = \frac{f - e^{-\alpha_i}(s_i \cdot f)}{1 - e^{-\alpha_i}}. \quad (5.2.2)$$

This formula can be demystified a little by writing it out explicitly as a geometric series:

$$\pi_i(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda - \alpha_i} + \dots + e^{s_i(\lambda)} & \text{if } \langle \lambda, \alpha_i^\vee \rangle \geq 0, \\ 0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = -1, \\ -(e^{\lambda + \alpha_i} + e^{\lambda + 2\alpha_i} + \dots + e^{s_i(\lambda) - \alpha_i}) & \text{if } \langle \lambda, \alpha_i^\vee \rangle \leq -2. \end{cases} \quad (5.2.3)$$

It is straightforward to verify that if  $f \in \mathbb{Z}[X(\Phi)]$  is symmetric in  $s_i$ , meaning  $s_i f = f$ , then  $\pi_i(f) = f$ . Together with the property  $s_i \circ \pi_i = \pi_i$ , this implies that  $\pi_i^2 = \pi_i$  is a projector to the subspace  $\mathbb{Z}[X]^{s_i}$  of  $s_i$ -symmetric characters. It takes considerably more work (a case-by-case analysis of the cases  $m_{ij} \in \{2, 3, 4, 6\}$ ) to verify that the Demazure operators satisfy the *braid relations* (Eq. (3.3.2)), i.e. we have

$$\underbrace{\pi_i \pi_j \pi_i \dots}_{m_{ij} \text{ letters}} = \underbrace{\pi_j \pi_i \pi_j \dots}_{m_{ij} \text{ letters}}, \quad (5.2.4)$$

a fact we state but will not prove here. Since they do satisfy the braid relations, by Matsumoto's theorem  $\pi_w$  is well-defined for any  $w \in W$  by setting  $\pi_w = \pi_{s_1} \dots \pi_{s_k}$  where  $(s_1, \dots, s_k)$  is any reduced expression for  $w$  (see Remark 5.2.10 for an interpretation of this in terms of zero-Hecke representations). We can now state the *Demazure character formula*, originally due to Demazure [Dem74] in finite type and numerous others (see Section 5.4) in arbitrary type.

### 5.2.5 Theorem (Demazure character formula)

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ ,  $\lambda \in X(\Phi)_+$  a dominant weight, and  $w \in W$  an element of the Weyl group. Then the character of the Demazure module  $L_w(\lambda)$  is

$$\text{ch } L_w(\lambda) = \pi_w(e^\lambda). \quad (5.2.6)$$

Furthermore, if  $(I, \cdot)$  is finite type then  $\text{ch } L(\lambda) = \text{ch } L_{w_I}(\lambda) = \pi_{w_I}(e^\lambda)$ .

Each Demazure module is parametrised by a pair  $(\lambda, w)$  of a dominant weight  $\lambda \in X(\Phi)_+$  and a Weyl element  $w \in W$ , but this parametrisation has some redundancy. If  $\lambda$  is not a regular weight, then the stabiliser  $W_\lambda = \text{Stab}_W(\lambda)$  is nontrivial, and the Demazure submodules  $L_x(\lambda)$  and  $L_y(\lambda)$  are equal whenever  $x = y$  in the quotient  $W/W_\lambda$ .

When  $(I, \cdot)$  is finite type and we are working in the category of finite-dimensional  $\mathfrak{g}(\Phi)$ -modules, we can remove this redundancy by parametrising Demazure modules instead by their Demazure lowest weights.

**5.2.7 Lemma** (*The Demazure basis of the character ring*)

Let  $\Phi$  be a root datum of finite type  $(I, \cdot)$ , and let  $D(\mu)$  be the Demazure module of Demazure lowest weight  $\mu \in X(\Phi)$ . Then the formal characters  $\{\text{ch } D(\mu) \mid \mu \in X(\Phi)\}$  form a basis of the character ring  $\mathbb{Z}[X(\Phi)]$ .

**5.2.8 Proof**

The definition of  $D(\mu)$  is not ambiguous, since if we take any two Demazure modules of the same Demazure lowest weight  $\mu$ , they must be submodules of  $L(\lambda)$  where  $\lambda$  is that unique dominant weight in the orbit  $W \cdot \mu$ . We have  $\text{ch } D(\mu) = e^\mu + \sum_{\nu > \mu} d_{\mu,\nu} e^\nu$  for some  $d_{\mu,\nu} \in \mathbb{N}$ , and therefore the Demazure characters are linearly independent inside  $\mathbb{Z}[X(\Phi)]$  via a triangularity argument. The fact that they form a basis can be seen by taking a filtration of  $\mathbb{Z}[X(\Phi)]$  by suitable finite-rank spaces, such as sums supported only over the convex hull of the Weyl orbit of finitely many weights.

In the case that  $(I, \cdot)$  is not finite type, we need to be a little more careful about which weights we permit in the definition of  $D(\mu)$  as the ‘Demazure module of Demazure lowest weight  $\mu$ ’. Since we only want to be capturing Demazure submodules of the  $L(\lambda)$  for dominant  $\lambda$ , we need to ensure that  $\mu$  is in the Weyl orbit of some dominant weight. Define the *Tits cone* as the Weyl orbit of the dominant chamber:  $K(\Phi) = \bigcup_{w \in W} wX(\Phi)_+ \subseteq X(\Phi)$ . Then  $K(\Phi)$  is the whole of the weight lattice if and only if  $(I, \cdot)$  is finite type (Proposition 1.4.2 of [Kum02]), and so indeed outside of finite type there are weights  $\mu$  which are *not* in the Weyl orbit of any dominant weight. The dominant chamber  $X_{\mathbb{R}}(\Phi)_+$  is a fundamental domain for the action of  $W$  on  $K_{\mathbb{R}}(\Phi)$  however, and so for any  $\mu$  in  $K(\Phi)$  there is a unique dominant weight in its Weyl orbit. So we can speak of the Demazure module with Demazure lowest weight  $\mu$  provided that we limit ourselves to those weights  $\mu$  in the Tits cone.

**5.2.9 Lemma** (*Demazure characters of highest-weight modules are linearly independent*)

Let  $\Phi$  be a root datum of arbitrary type  $(I, \cdot)$ , and for any weight  $\mu$  in the Tits cone  $K(\Phi)$ , let  $D(\mu)$  be the Demazure module with lowest weight  $\mu$ . Then the formal characters  $\{\text{ch } D(\mu) \mid \mu \in K(\Phi)\}$  are linearly independent in the completed algebra  $\mathbb{Z}[[X(\Phi)]]$ .

We conclude this section with a discussion of the relations  $\pi_i^2 = \pi_i$  and the braid relations into a broader framework, that of *Hecke algebras*.

**5.2.10 Remark** (*Zero-Hecke actions*)

Let  $(W, S)$  be the Coxeter system associated to the Cartan datum  $(I, \cdot)$ , as defined in Definition 3.3.1. Given a commutative ring  $R$  and two parameters  $\lambda, \mu \in R$ , we may define the *Hecke algebra*  $\mathcal{H}(\lambda, \mu)$  to be the associative  $R$ -algebra generated by  $\{T_i \mid i \in I\}$ , with the two relations

$$\begin{aligned} \text{Quadratic relation: } T_i^2 &= \lambda T_i + \mu, \\ \text{Braid relation: } \underbrace{T_i T_j T_i \cdots}_{m_{ij}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij}}. \end{aligned} \quad (5.2.11)$$

For each  $w \in W$ , define  $T_w = T_{i_1} \cdots T_{i_r}$  for some reduced expression  $(i_1, \dots, i_r)$  of  $w$ . Since the  $T_i$  satisfy the braid relations, Matsumoto’s theorem implies that the  $T_w$  are independent of the choice of reduced expression. After much work (See Chapter 7 of [Hum90] for a full proof, or Exercise 2.23 in Chapter IV of [BB02] if you don’t want the fun spoiled), it turns out that  $\mathcal{H}(\lambda, \mu)$  is free as an  $R$ -module on the basis  $\{T_w \mid w \in W\}$ , with left multiplication by a generator  $T_i$  given by the rule

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ \lambda T_w + \mu T_{s_i w} & \text{if } \ell(s_i w) < \ell(w). \end{cases} \quad (5.2.12)$$

The choice of  $\lambda$  and  $\mu$  (the choice of quadratic relation) gives different algebras:

1. The quadratic relation  $T_i^2 = 1$  makes  $\mathcal{H}(0, 1)$  isomorphic to the group algebra  $RW$ .
2. The quadratic relation  $T_i^2 = 0$  makes  $\mathcal{H}(0, 0)$  an algebra called the *nil-Hecke ring*. The cell ordering on this ring with respect to the standard basis  $\{T_w \mid w \in W\}$  is precisely the Bruhat ordering.
3. The quadratic relation  $T_i^2 = T_i$  makes  $\mathcal{H}(1, 0)$  an algebra called the *zero-Hecke ring*.



The Demazure operators we have just defined satisfy the braid relation and the quadratic relation  $\pi_i^2 = \pi_i$ , and hence they define an action of the zero-Hecke ring (defined over  $R = \mathbb{Z}$ ) on the space  $\mathbb{Z}[X(\Phi)]$  of formal characters by sending  $T_i$  to  $\pi_i$ . This is useful to know, since we re-use the multiplication rule above to see that

$$\pi_i \pi_w = \begin{cases} \pi_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ \pi_w & \text{if } \ell(s_i w) < \ell(w). \end{cases} \quad (5.2.13)$$

We will eventually make use of this to argue that our character formula for truncations of the product monomial crystal gives a formula for the whole crystal in the case that  $(I, \cdot)$  is finite type.

### 5.3 Demazure crystals

When working with modules over the quantum group  $U_q(\Phi)$  we can define Demazure modules in the same way as [Definition 5.2.1](#): for any dominant  $\lambda \in X(\Phi)_+$  and Weyl group element  $w \in W_I$ , let  $L_{q,w}(\lambda) = U_q^+(\Phi) \cdot L_q(\lambda)_{w\lambda}$  be the orbit of the one-dimensional extremal weight space  $L_q(\lambda)_{w\lambda}$  under the positive half  $U_q^+(\Phi)$  of the quantum group. It was shown in [\[Kas93\]](#) that the quantum Demazure module admits a crystal base, which can be taken to be a subset of the crystal base  $\mathcal{B}(\lambda)$  of  $L_q(\lambda)$ . In order to specify this subset, we introduce the following operators.

**5.3.1 Definition** (*Extension of strings*)

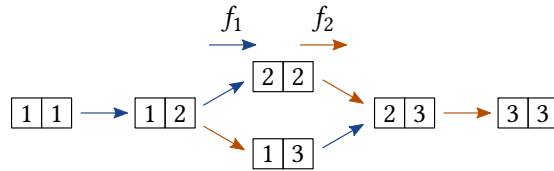
Let  $B$  be a  $\Phi$ -crystal. For each  $i \in I$ , define the *extension of  $i$ -strings* operator  $\mathfrak{D}_i$ , which takes a subset  $Z \subseteq B$  to the set

$$\mathfrak{D}_i Z = \bigcup_{n \geq 0} \{f_i^n(x) \mid x \in Z\} = \{b \in B \mid e_i^n(b) \in Z \text{ for some } n \in \mathbb{N}\}. \tag{5.3.2}$$

These operators satisfy  $Z \subseteq \mathfrak{D}_i Z \subseteq B$ , and  $\mathfrak{D}_i \mathfrak{D}_i Z = \mathfrak{D}_i Z$ . It is unclear to the author whether they satisfy the braid relations on arbitrary subsets of the crystal, however they *do* braid when  $Z = \{b_\lambda\}$  the highest-weight element.

**5.3.3 Example**

The picture below is the  $GL_3$  crystal  $\mathcal{B}(2, 0, 0)$ , previously seen in [Examples 4.2.10](#) and [4.5.4](#). If the subset  $Z \subseteq \mathcal{B}(2, 0, 0)$  is the two middle elements  $Z = \{(2, 2), (1, 3)\}$ , then  $\mathfrak{D}_1(Z) = Z \cup \{(2, 3)\}$  and  $\mathfrak{D}_2(Z) = Z \cup \{(2, 3), (3, 3)\}$ .



We can now give the definition of the Demazure crystal  $\mathcal{B}_w(\lambda)$ .

**5.3.4 Definition** (*Demazure crystal*)

Fix a dominant weight  $\lambda$  and a Weyl group element  $w \in W$ . Let  $(i_1, \dots, i_r)$  be a reduced expression for  $w$ . The *Demazure crystal* is the set

$$\mathcal{B}_w(\lambda) = \mathfrak{D}_{i_1} \dots \mathfrak{D}_{i_r} \{b_\lambda\}, \tag{5.3.5}$$

equipped with the canonical upper-seminormal crystal structure restricted from  $\mathcal{B}(\lambda)$ .

An interesting difference is that while the quantum Demazure module  $L_{q,w}(\lambda)$  is defined from the bottom upwards, by taking a lower weight and orbiting it under the positive half  $U_q^+(\Phi)$  of the quantum group, the Demazure crystal is defined from the *top down*, starting with a highest weight and taking just enough extensions of strings in the correct order. Something that might help the reader bridge this mental gap is [Corollary 3.2.2](#) from [\[Kas93\]](#): if  $(i_1, \dots, i_r)$  is a reduced expression for  $w$ , then

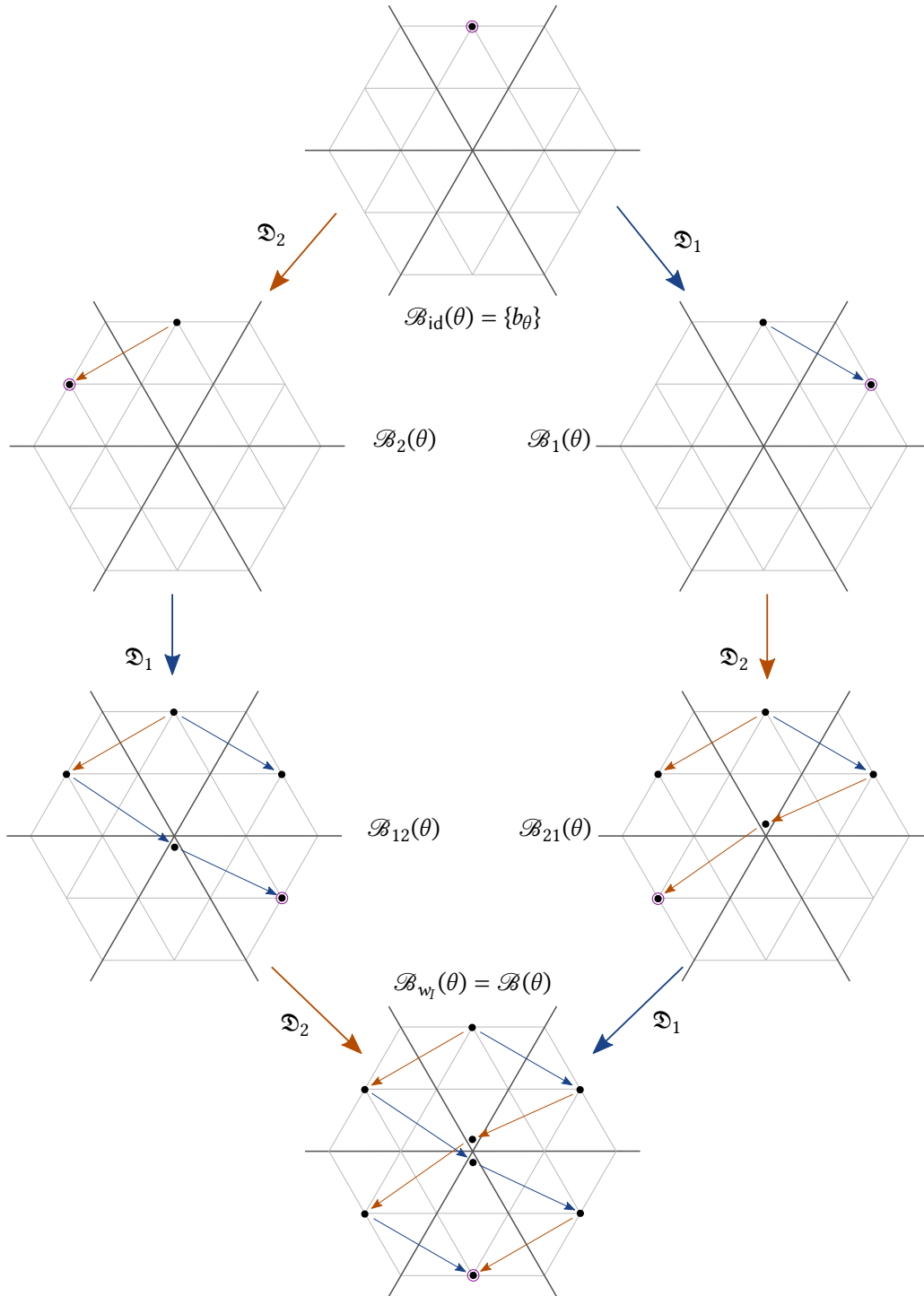
$$L_{q,w}(\lambda) = \sum_{k_1, \dots, k_r \geq 0} \mathbb{Q}(q) F_{i_1}^{k_1} \dots F_{i_r}^{k_r} v_\lambda, \tag{5.3.6}$$

where  $v_\lambda \in L_q(\lambda)$  is a highest-weight vector. This shows that the quantum Demazure module  $L_{q,w}(\lambda)$  can also be generated from the top downwards by ‘extending root strings’.

**5.3.7 Example** (*An example of Demazure crystals*)

Let  $\Phi$  be the root datum of  $SL_3$ , of Cartan type  $A_2$ . The highest root is  $\theta = \alpha_1 + \alpha_2 = \omega_1 + \omega_2$ , and  $L(\theta)$  is the adjoint representation of the semisimple Lie algebra  $\mathfrak{g}(\Phi) = \mathfrak{sl}_3$ . The Weyl group  $W = \langle s, t \mid s^2 = t^2 = 1, sts = tst \rangle$  has six elements, and since  $\theta$  is a regular weight there are six distinct Demazure subcrystals of  $\mathcal{B}(\theta)$ .

Each of these Demazure subcrystals  $\mathcal{B}_w(\theta)$  are shown in the following diagram, with the Demazure lowest weight element  $w \cdot b_\theta$  marked with a double circle (recall the action of  $W$  on  $\mathcal{B}(\theta)$  from Theorem 4.5.2).



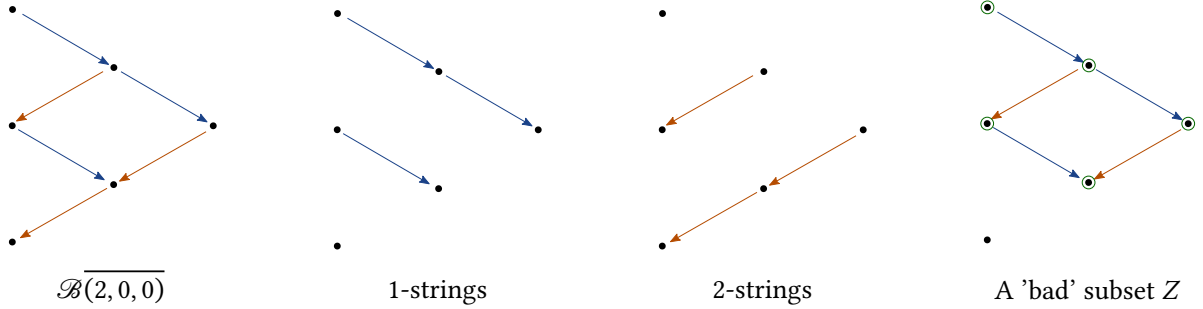
In the two crystals  $\mathcal{B}_{12}(\theta)$  and  $\mathcal{B}_{21}(\theta)$  there is an element which cannot be reached by following arrows backwards from the Demazure lowest weight element. Hence Demazure crystals really do need to be defined from the top down, rather than from the bottom up.

Kashiwara gave an alternative proof of the Demazure character formula, using an interesting equivariance property of the extension-of-strings operator  $\mathfrak{D}_i$  and the Demazure operator  $\pi_i$ . Before we introduce this equivariance, we need to state what we mean by the *string property*, a definition first introduced in [Kas93].

**5.3.8 Definition** (*String property*)

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ , and let  $B$  be a seminormal  $\Phi$ -crystal. We say that a subset  $Z \subseteq B$  has the *string property* if for any  $i$ -string  $S \subseteq B$ , either  $S \cap Z = S$ ,  $S \cap Z = \emptyset$ , or  $S \cap Z = S^{\text{top}}$ , where  $S^{\text{top}} \in S$  is that unique element at the top of the  $i$ -string, satisfying  $e_i(S^{\text{top}}) = \perp$ .

We have encountered  $i$ -strings before, when defining the action of the Weyl group on a crystal (Theorem 4.5.2). Shown below is the  $\text{SL}_3$ -crystal  $\mathcal{B}(2, 0, 0)$ , its 1-strings, its 2-strings, and a certain subset  $Z \subseteq \mathcal{B}(2, 0, 0)$  of cardinality 5 indicated by the circled points. The subset  $Z$  does *not* satisfy the string property, since the 2-string with 3 elements intersects  $Z$  in two elements.

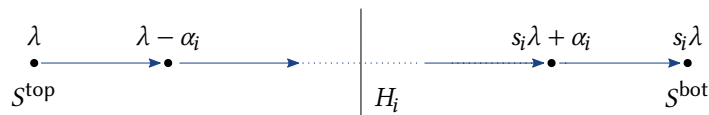


On the other hand, all of the Demazure crystals shown in Example 5.3.7 satisfy the string property – this turns out to be a general property of Demazure crystals, though is quite difficult to prove, and indeed is the ‘deepest’ result in Kashiwara’s proof of the Demazure character formula. We will not reproduce this proof here as it requires appealing to the theory of global bases and the crystal  $\mathcal{B}(\infty)$  of the Verma module  $U_q^-(\Phi)$ , we merely state the result.

**5.3.9 Theorem** (*Demazure crystals satisfy the string property*)

(Proposition 3.3.5 of [Kas93]). Let  $\Phi$  be a root datum of type  $(I, \cdot)$ ,  $\lambda$  a dominant weight, and  $w \in W$  a Weyl group element. The Demazure crystal  $\mathcal{B}_w(\lambda)$  satisfies the string property.

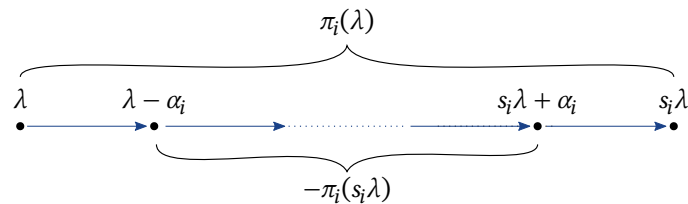
The last ingredient in Kashiwara’s proof is relating the extension of strings operator  $\mathfrak{D}_i$  to the Demazure operator  $\pi_i$ . Consider a seminormal  $\Phi$ -crystal  $B$  and any  $i$ -string  $S \subseteq B$ . By seminormality the string  $S$  is finite, having some  $i$ -highest element  $S^{\text{top}}$  and some  $i$ -lowest element  $S^{\text{bot}}$ . The weight  $\lambda$  of the  $i$ -highest element  $S^{\text{top}}$  must be  $i$ -dominant, meaning  $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ , and the balanced-strings axiom implies that the weight of the  $i$ -lowest element  $S^{\text{bot}}$  is  $s_i \lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ . The situation is pictured below: the reader should imagine that this is plotted inside the weight lattice  $X(\Phi)$ , with the reflecting hyperplane  $H_i = \ker \langle -, \alpha_i^\vee \rangle$  straight down the centre.



Now we want to consider the action of the Demazure operator on the character  $\text{ch } S$ . Recall from Eq. (5.2.3) that we can treat the Demazure operator as a geometric series. Since  $\lambda$  is  $i$ -dominant and  $s_i \lambda$  is  $i$ -antidominant, we have

$$\begin{aligned} \pi_i(\lambda) &= e^\lambda + e^{\lambda - \alpha_i} + \dots + e^{s_i \lambda} = \text{ch } S \\ \pi_i(s_i \lambda) &= -\left( e^{\lambda - \alpha_i} + \dots + e^{s_i \lambda + \alpha_i} \right) = -\pi_i(e^{\lambda - \alpha_i}). \end{aligned} \tag{5.3.10}$$

These two series can be pictured as follows:



We can see now that  $\pi_i(\text{ch } S) = \text{ch } S$ , by noting that almost every weight contributed by  $\pi_i(e^\lambda)$  is later removed by  $\pi_i(e^{s_i\lambda})$ , leaving  $\pi_i(e^\lambda + e^{s_i\lambda}) = e^\lambda + e^{s_i\lambda}$ . Working inwards from there shows that  $\pi_i(\text{ch } S) = \text{ch } S$ , or we could have appealed to the general fact that  $\pi_i$  acts as the identity on  $s_i$ -invariant characters. The key observation is that  $\text{ch } S = \pi_i(e^\lambda)$ . We are now ready to state and prove the equivariance property.

### 5.3.11 Lemma

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ , and  $B$  a seminormal  $\Phi$ -crystal. If  $Z \subseteq B$  is a subset with the string property, then  $\text{ch } \mathfrak{D}_i(Z) = \pi_i(\text{ch } Z)$  for all  $i \in I$ .

### 5.3.12 Proof

Fix an  $i \in I$ . By the additivity of characters on one hand, and the fact that  $\mathfrak{D}_i$  only depends on the  $i$ -strings on the other, it suffices to show that the theorem holds on each intersection  $Z \cap S$  with the  $i$ -strings  $S \subseteq B$ . Since  $Z$  satisfies the string property, for each  $i$ -string  $S \subseteq B$  there are only three cases to check:

1. If  $S \cap Z = \emptyset$ , then the theorem holds since  $0 = 0$ .
2. If  $S \cap Z = \{S^{\text{top}}\}$ , let  $\lambda = \text{wt } S^{\text{top}}$ . Then  $\mathfrak{D}_i(S \cap Z) = S$ , and by the above discussion we have  $\text{ch } S = \pi_i(e^\lambda) = \pi_i(\text{ch } S^{\text{top}})$ , so the theorem holds.
3. If  $S \cap Z = S$ , then  $\mathfrak{D}_i(S) = S$  and hence the theorem holds.

It seems that this theorem would immediately imply the Demazure character formula, but one really does need to prove [Theorem 5.3.9](#) somehow. It is *not* true in general that if  $Z$  satisfies the string property then  $\mathfrak{D}_i(Z)$  does also — see Section 13 of [\[BS17\]](#) for a counterexample. However, it is now clear to see that the Demazure character formula is a direct consequence of [Theorem 5.3.9](#) and [Lemma 5.3.11](#), and the fact that  $\mathcal{B}_w(\lambda)$  is a crystal base of  $L_{q,w}(\lambda)$ .

## 5.4 History: The Demazure character formula

Demazure modules were historically considered in a different way to how they have been presented above. A root datum  $\Phi$  of finite type  $(I, \cdot)$  determines a reductive group scheme  $G = G(\Phi)$  equipped with a pinning  $T \subseteq B \subseteq G$ , defined as a scheme over  $\mathbb{Z}$  and hence over any field  $k$  via base change. The quotient of the group  $G$  by its Borel subgroup  $B$  is called the *flag variety*, a projective scheme. The category of  $G$ -equivariant line bundles on  $G/B$  is equivalent to the one-dimensional representations of  $B$ , which are seen (via a factorisation  $B = TU$  into a product of the torus  $T$ , and the unipotent group  $U$  which must act trivially) to simply be the weights  $X(\Phi)$ . For each weight  $\lambda \in X(\Phi)$  there is a line bundle  $\mathcal{L}_\lambda$  over  $G/B$ , whose vector space of sections  $\Gamma(G/B, \mathcal{L}_\lambda)$  is nonzero precisely when  $\lambda$  is dominant.

When the base field  $k = \mathbb{C}$  and  $\lambda \in X(\Phi)_+$  is dominant, the vector space  $\Gamma(G/B, \mathcal{L}_\lambda)$  of sections is precisely the highest-weight module  $L(\lambda)$  we have been considering thus far, with the  $G$ -action canonically given by  $G$ -equivariance of  $\mathcal{L}_\lambda$ . Where the Demazure modules enter the picture is by considering the restriction of the line bundle  $\mathcal{L}_\lambda$  to the *Schubert variety*  $X(w) = \overline{B\dot{w}B}$ , the closure of the Bruhat cell  $B\dot{w}B$ . (In the setting of reductive group schemes, the Weyl group may be realised as the quotient  $W := N_G(T)/T$  of the normaliser of the maximal torus  $T$  inside  $G$ . We use the notation  $\dot{w} \in G$  to denote any lift of the Weyl group element  $w \in W$  back to  $G$ ). The space of sections  $\Gamma(X(w), \mathcal{L}_\lambda)$  is the Demazure module  $L_w(\lambda)$ , which is a  $B$ -module, but not a  $G$ -module unless  $X(w) = G$ , i.e.  $w = w_I$  the longest element.

A formula for the characters of the Demazure modules (in the setting above, with  $G$  semisimple and  $k = \mathbb{C}$ ) was first given by Demazure [\[Dem74\]](#). Later on in 1983, Victor Kac was attempting to generalise this formula to infinite root systems and found that the proof contained a gap, in an argument due to Verma. This created a ‘spate’ of correct proofs in the following few years [\[Jos95\]](#), in which many new techniques were developed.

Mehta and Ramanathan [\[MR85\]](#) introduced the notion of a *Frobenius split* variety, for which the higher cohomologies of ample line bundles vanish, and showed that the Schubert varieties are Frobenius split. Ramanan and Ramanathan [\[RR85\]](#) and Seshadri proved the projective normality of Schubert varieties, and improved the result of Mehta and Ramanathan to include *effective* line bundles. This work culminated with Andersen [\[And85\]](#) also

contributing a proof of the Demazure character formula, valid over reductive algebraic groups in any characteristic.

This problem also kicked off the systematic study of  $B$ -modules, a good survey of which is [KI93]. Joseph [Jos85; Jos86] defined functors (now called *Joseph functors*)  $H_w : \text{Rep } B \rightarrow \text{Rep } B$  taking a  $B$ -module  $M$  to the space of sections  $\Gamma(X(w), \mathcal{L}(M))$  of the vector bundle  $\mathcal{L}(M)$  over the Schubert variety  $X(w) \subseteq G/B$ , and was able to prove Demazure's original result for  $\lambda$  sufficiently large. Later, both Kumar [Kum87] and Mathieu [Mat88] independently proved the Demazure character formula, this time in the setting of an arbitrary Kac-Moody algebra and an integrable highest-weight module.

Our passage through this chapter has been following the much later work of both Littelmann and Kashiwara, from the perspective of crystals. Littelmann was involved in the development of *path models* for representations, bases indexed by certain piecewise-linear paths through the weight lattice, with root operators (crystal operators) given by operations on paths, and a positive combinatorial rule for the tensor product multiplicities given in terms of paths [Lit95b; Lit90]. It became clear that these paths could in fact index *crystal* bases, with the tensor product rule immediately following from the crystal tensor product, a fact which was shown in [Lit95a]. In this paper, Littelmann conjectured the Demazure character formula for Demazure crystals, which was later proven by Kashiwara in [Kas93].

## 6 Monomial crystals

In this chapter we introduce the *monomial crystal*  $\mathcal{M}(\Phi)$  associated to a root datum  $\Phi$ , first defined by Nakajima. The monomial crystal is very large, and inside of it we can find infinitely many copies of  $\mathcal{B}(\lambda)$  for any dominant weight  $\lambda$ , as well as more exotic crystals in the case where  $(I, \cdot)$  is not finite type. The monomial crystal has been used to study extremal weight crystals of affine algebras, each connected component being isomorphic to a subcrystal of such an extremal weight crystal.

As the underlying set of the monomial crystal  $\mathcal{M}(\Phi)$  is an abelian group, the group operation being multiplication of monomials, it is possible to form a monomial-wise product (rather than crystal tensor product) of subcrystals. If we are careful about the kind of subcrystals we start with, the resulting set of products is again a subcrystal of  $\mathcal{M}(\Phi)$  called the *product monomial crystal*. The proof we know of this fact is quite involved and only works when  $(I, \cdot)$  is simply-laced, as it goes via the geometry of Nakajima quiver varieties. We defer this proof to [Chapter 8](#).

In this chapter we first state the definition of the monomial crystal and give a feel for how it works, before moving on to define the product monomial crystal and start introducing terminology which will help us analyse it.

### 6.1 Nakajima's Monomial Crystal

The monomial crystal is originally due to Nakajima, first appearing in Section 3 of [\[Nak02\]](#) and Section 3 of [\[Kas02b\]](#). A later definition appearing in [\[HN06\]](#) is a straightforward modification of the original to make the crystal make sense for an arbitrary root datum  $\Phi$  rather than a simply connected one. We adopt this later definition since it will make the arguments in [Chapter 10](#) a lot more pleasant, allowing us to work with weights of  $\mathrm{GL}_n$  rather than  $\mathrm{SL}_n$ , but we will point out the simplification in simply-connected type. The other thing to notice about this definition is that we require the Cartan datum  $(I, \cdot)$  to be bipartite: the crystal structure does not work otherwise (see [Example 6.1.10](#) for a counterexample).

#### 6.1.1 Definition (*The monomial crystal*)

Let  $(I, \cdot)$  be a bipartite Cartan datum with a fixed two-colouring  $\zeta : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ . For a Kac-Moody root datum  $\Phi$  of type  $(I, \cdot)$ , let  $A(\Phi)$  be the product  $X(\Phi) \times \mathbb{Z}[I \times \mathbb{Z}]$  of abelian groups, each written multiplicatively so that a typical element  $p \in A(\Phi)$  is of the form

$$p = e^{\omega(p)} \cdot \prod_{(i,c) \in I \times \mathbb{Z}} y_{i,c}^{p[i,c]} \quad \text{for some } \omega(p) \in X \text{ and } p[i,c] \in \mathbb{Z}, \quad (6.1.2)$$

where  $p[i,c] \neq 0$  for only finitely many  $(i,c) \in I \times \mathbb{Z}$ . For each  $(i,k) \in I \times \mathbb{Z}$ , define the auxiliary monomial

$$z_{i,k} = e^{\alpha_i} \cdot y_{i,k} \cdot y_{i,k+2} \cdot \prod_{j \neq i} y_{j,k+1}^{a_{ji}}. \quad (6.1.3)$$

Let  $\mathcal{M}(\Phi) \subseteq A(\Phi)$  be the subgroup defined by the two conditions

$$\langle \omega(p), \alpha_i^\vee \rangle = \sum_{l \in \mathbb{Z}} p[i,l] \quad \text{for all } i \in I, \quad (6.1.4)$$

$$p[i,c] = 0 \text{ if } c \not\equiv \zeta(i) \pmod{2}. \quad (6.1.5)$$

For each monomial  $p \in \mathcal{M}(\Phi)$ , define

1.  $\varphi_i^k(p) = \sum_{l \geq k} p[l, i]$ , the *upper column sum*.
2.  $\varphi_i(p) = \max_k \varphi_i^k(p)$ , the *largest upper column sum*.
3.  $\varepsilon_i^k(p) = -\sum_{l \leq k} p[l, i]$ , the *negated lower column sum*.
4.  $\varepsilon_i(p) = \max_k \varepsilon_i^k(p)$ , the *largest negated lower column sum*.
5.  $n_{f,i}(p) = \max\{k \in \mathbb{Z} \mid \varphi_i^k(p) = \varphi_i(p)\}$ , the largest  $k$  maximising the upper column sum  $\varphi_i^k(p)$ .
6.  $n_{e,i}(p) = \min\{k \in \mathbb{Z} \mid \varepsilon_i^k(p) = \varepsilon_i(p)\}$ , the smallest  $k$  maximising the negated lower column sum  $\varepsilon_i^k(p)$ .

Note that  $n_{f,i}(p)$  is undefined if  $\varphi_i(p) = 0$ , and  $n_{e,i}(p)$  is undefined if  $\varepsilon_i(p) = 0$ . We set  $\text{wt}(p) = \omega(p)$ , and define the crystal operators

$$e_i(p) = \begin{cases} 0 & \text{if } \varepsilon_i(p) = 0 \\ pz_{i, n_{e,i}(p)} & \text{otherwise,} \end{cases} \quad (6.1.6)$$

$$f_i(p) = \begin{cases} 0 & \text{if } \varphi_i(p) = 0 \\ pz_{i, n_{f,i}(p)-2}^{-1} & \text{otherwise.} \end{cases} \quad (6.1.7)$$

The *monomial crystal* is the set  $\mathcal{M}(\Phi)$ , equipped with the crystal structure  $(\text{wt}, e_i, f_i, \varepsilon_i, \varphi_i)$  given above.

The definition can be simplified a little in the case that  $\Phi$  is simply-connected and of finite type, by erasing the  $e^\lambda$  term in each monomial and taking  $\text{wt}(y_{i,k})$  to be the fundamental weight  $\omega_i$ . When allowing arbitrary root data, we instead enforce the condition that in any monomial  $e^\lambda \cdot z$ , the number of  $y_{i,\bullet}$  appearing in  $z$  must be equal to  $\langle \lambda, \alpha_i^\vee \rangle$  – this is condition (1) above. The parity condition (2) is required in order to satisfy the partial inverse axiom of a crystal, and relies on the existence of a two-colouring  $\zeta : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

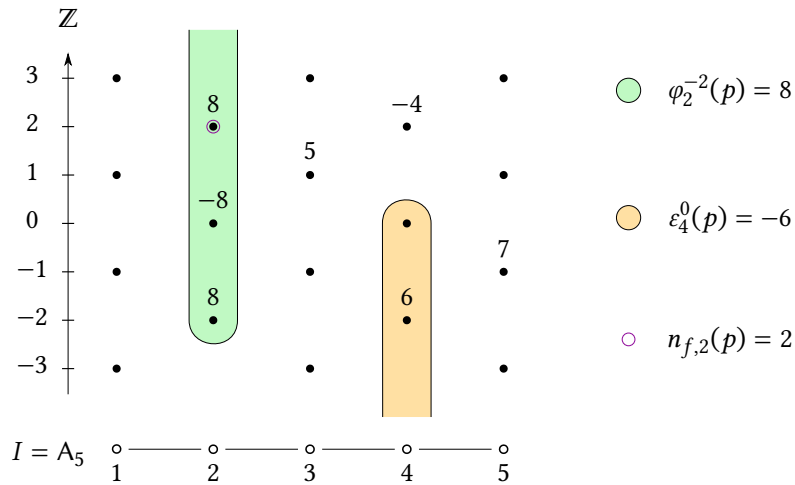
The definition of  $\mathcal{M}(\Phi)$  can be understood pictorially. Let  $I \dot{\times} \mathbb{Z} \subseteq I \times \mathbb{Z}$  denote the set of parity-respecting pairs<sup>1</sup>

$$I \dot{\times} \mathbb{Z} = \{(i, c) \in I \times \mathbb{Z} \mid \zeta(i) = c \text{ in } \mathbb{Z}/2\mathbb{Z}\}. \quad (6.1.8)$$

Forgetting the ‘tag’  $e^\lambda$  for a moment, a monomial  $p \in \mathcal{M}(\Phi)$  can be thought of as a finitely-supported function  $p : I \dot{\times} \mathbb{Z} \rightarrow \mathbb{Z}$ . We can draw the set  $I \dot{\times} \mathbb{Z}$  in the plane and label it with the values  $p[i, c]$ . The monomial

$$p = e^\lambda \cdot y_{2,-2}^8 \cdot y_{2,0}^{-8} \cdot y_{2,2}^8 \cdot y_{3,1}^5 \cdot y_{4,-2}^6 \cdot y_{4,2}^{-4} \cdot y_{5,-1}^7 \quad (6.1.9)$$

in Cartan type  $A_5 = \{1, 2, 3, 4, 5\}$  is pictured below, with example computations of the upper column sum  $\varphi_2^{-2}(p)$  and the negated lower column sum  $\varepsilon_4^0(p)$ , and the largest integer  $n_{f,2}(p)$  maximising the upper column sum  $\varphi_2^{\bullet}(p)$ .

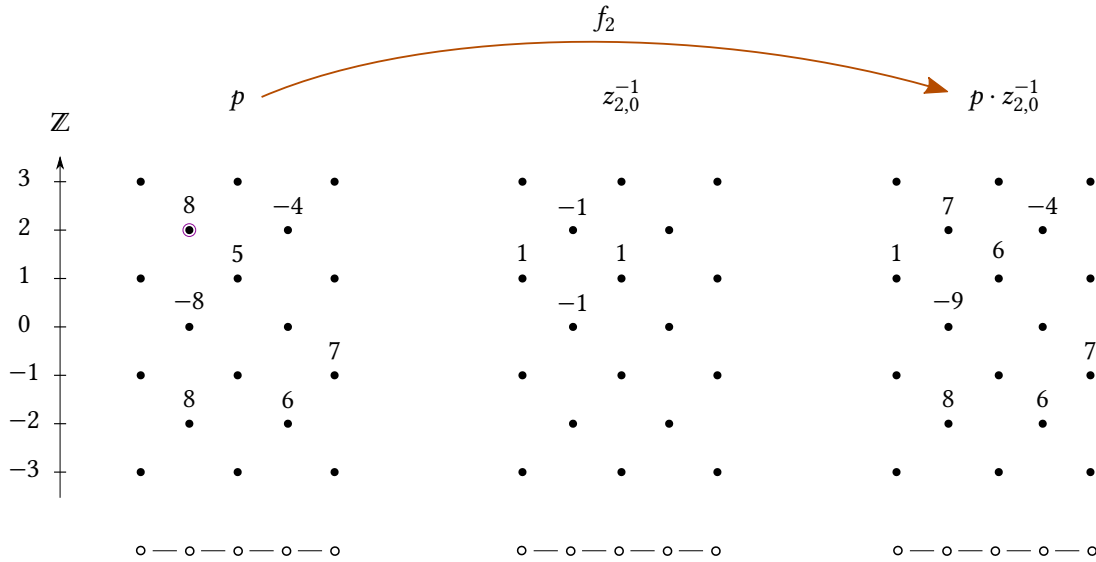


<sup>1</sup>The definition of  $I \dot{\times} \mathbb{Z}$  really depends on the choice of 2-colouring  $\zeta$ .



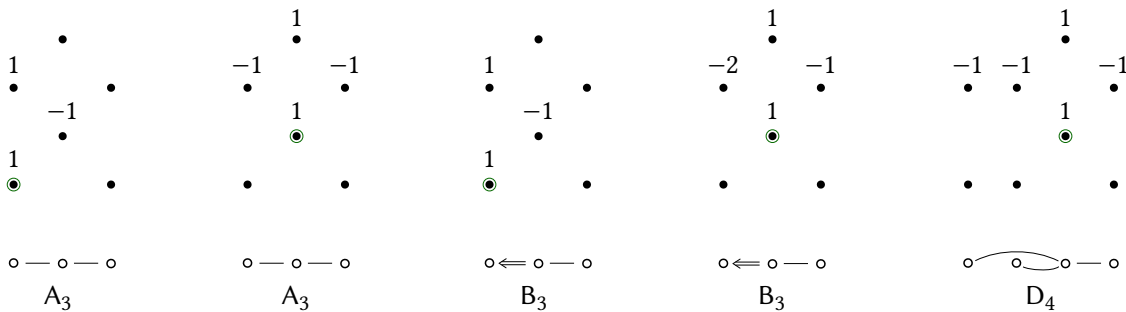
This *monomial diagram* captures the  $y_{i,c}$  terms but not  $\lambda$ , but due to the weight constraint Eq. (6.1.4) all of the inner products  $\langle \lambda, \alpha_i^\vee \rangle$  are determined by the  $y_{i,c}$ . Even though we have not yet specified a root datum  $\Phi$  we can be sure that  $\lambda$  must satisfy  $\langle \lambda, \alpha_4^\vee \rangle = 6 - 4 = 2$ . If  $\Phi$  is the simply-connected datum  $\Phi = \text{SL}_6$  then we must have  $\lambda = 8\omega_2 + 5\omega_3 + 2\omega_4 + 7\omega_5$  as a sum of fundamental weights.

Next, we look at the action of the crystal operator  $f_2$  on our monomial  $p$ . The maximum value of the upper column sum  $\varphi_2^k(p)$  in column 2 is achieved at  $\varphi_2^{-2}(p) = \varphi_2^2(p) = 8$ , hence  $n_{f_2}(p) = \max\{2, -2\} = 2$  as indicated on the diagram. By Eq. (6.2.4) the action of the crystal operator  $f_2$  on the monomial  $p$  is multiplication by  $z_{2,0}^{-1}$ , so we have  $f_2(p) = p \cdot z_{2,0}^{-1}$ . Recalling the definition of the auxiliary monomial  $z_{i,k}$  from Eq. (6.1.3), we can picture this multiplication as follows:



The reader could now check that the  $e_i$  operator will act at the correct spot, so that we have  $e_i(f_i(p)) = p$ . Also remember that we have excluded the weight  $\text{wt}(p) = \lambda$  from our diagrams, but one can simply imagine that each monomial is a pair consisting of a compatible weight  $\lambda$  and a diagram determined by the  $y_{i,k}$ , and remember that  $z_{i,k}$  has weight  $\alpha_i$ .

We present a few more examples of the  $z_{i,k}$  monomials in types  $A_3$ ,  $B_3$ , and  $D_4$ . In each picture the auxiliary monomial  $z_{i,k}$  is shown with the point  $(i, k)$  circled in green. It is clear that  $z_{i,k}$  has a straightforward interpretation in terms of the Dynkin diagram.



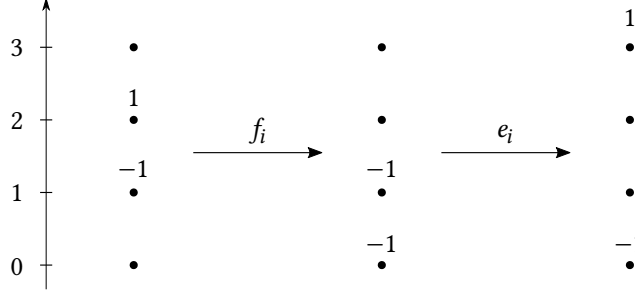
We now resume our formal discussion of the monomial crystal. We first point out that without the parity condition Eq. (6.1.5), the set  $\mathcal{M}(\Phi)$  would not be a  $\Phi$ -crystal as it would fail the partial inverse axiom. Therefore we really do need to have the Cartan datum  $(I, \cdot)$  bipartite.

**6.1.10 Example** (*Necessity of parity condition*)

This counterexample appears in Example 3.3 of [Kas02b]. Let  $\Phi$  a root datum of Cartan type  $(I, \cdot) = A_1$ . Here  $I = \{i\}$  is a single vertex, and the monomial  $z_{i,k}$  takes the simple form  $z_{i,k} = e^{\alpha_i} z_{i,k} z_{i,k+1}$ . Consider the monomial  $p = e^0 \cdot y_{i,2} \cdot y_{i,1}^{-1}$  acted on first by  $f_i$  and then by  $e_i$ :

$$p = e^0 \cdot y_{i,2} \cdot y_{i,1}^{-1} \xrightarrow{f_i} e^{-\alpha_i} \cdot y_{i,1}^{-1} \cdot y_{i,0}^{-1} \xrightarrow{e_i} e^0 \cdot y_{i,3} \cdot y_{i,0}^{-1}. \quad (6.1.11)$$

We find that even though  $f_i$  is defined on  $p$ , we have  $e_i(f_i(p)) \neq p$  which is a violation of the partial inverse axiom of a crystal. In terms of a monomial diagram, the sequence above appears as:



(The above picture is not a crystal graph, since we would usually omit the  $e_i$  from a crystal graph for the very reason that  $e_i$  and  $f_i$  are inverse partial functions).

We now check directly that  $\mathcal{M}(\Phi)$  is an abstract seminormal  $\Phi$ -crystal, paying close attention to the partial inverse axiom in light of the counterexample above.

**6.1.12 Theorem**

The set  $\mathcal{M}(\Phi)$  together with the maps  $\text{wt}, \varepsilon_i, \varphi_i, e_i, f_i$  for  $i \in I$  is an abstract seminormal  $\Phi$ -crystal.

**6.1.13 Proof**

We first show that  $\mathcal{M}(\Phi)$  is an abstract crystal, by checking the axioms.

1. *Balanced-strings*: we have the property that  $\varepsilon_i^k(p) + \langle \text{wt}(p), \alpha_i^\vee \rangle = \varphi_i^{k+1}(p)$ . Then

$$\varepsilon_i(p) + \langle \text{wt}(p), \alpha_i^\vee \rangle = \max_{k \in \mathbb{Z}} \left( \varepsilon_i^k(p) + \langle \text{wt}(p), \alpha_i^\vee \rangle \right) = \max_{k \in \mathbb{Z}} \varphi_i^{k+1}(p) = \varphi_i(p). \quad (6.1.14)$$

2. *Raising operators*: Suppose that  $e_i(p) \neq 0$ . Set  $v = n_{e_i}(p)$ , so by definition we have

$$\varepsilon_i^k(p) < \varepsilon_i^v(p) \geq \varepsilon_i^l(p) \quad \text{for all } k < v < l. \quad (6.1.15)$$

Since  $e_i(p) = p \cdot z_{i,v} = p \cdot y_{i,v} \cdot y_{i,v+2} \cdot \prod_{j \neq i} y_{j,v+1}^{a_{ji}}$ , we have

$$\begin{aligned} \varepsilon_i^k(e_i(p)) &= \varepsilon_i^k(p), \\ \varepsilon_i^v(e_i(p)) &= \varepsilon_i^v(p) - 1, \\ \varepsilon_i^l(e_i(p)) &= \varepsilon_i^l(p) - 2, \quad \text{for all } k < v < l, \end{aligned} \quad (6.1.16)$$

and hence

$$\varepsilon_i^k(e_i(p)) \leq \varepsilon_i^v(e_i(p)) > \varepsilon_i^l(e_i(p)) \quad \text{for all } k < v < l. \quad (6.1.17)$$

Therefore

$$\varepsilon_i(e_i(p)) = \max_k \varepsilon_i^k(e_i(p)) = \varepsilon_i^v(e_i(p)) = \varepsilon_i(p) - 1.$$

The property  $\varphi_i(e_i(p)) = \varphi_i(p) + 1$  then follows from both this, the fact that  $\text{wt}(e_i(p)) = \text{wt}(p) + \alpha_i$ , and the balanced-strings property. We will also check the *partial inverse* property now: applying the identity  $\varepsilon_i^k(p) + \langle \text{wt}(p), \alpha_i^\vee \rangle = \varphi_i^{k+1}(p)$  to the above equation gives

$$\varphi_i^{k+1}(e_i(p)) \leq \varphi_i^{v+1}(e_i(p)) > \varphi_i^{l+1}(e_i(p)) \quad \text{for all } k < v < l. \quad (6.1.18)$$

Now we apply the parity condition, which implies that  $\varphi_i^k(p) = \varphi_i^{k+1}(p)$  for all  $p \in \mathcal{M}(\Phi)$  and integers  $k \in \mathbb{Z}$ . This implies

$$\varphi_i^{k+2}(e_i(p)) \leq \varphi_i^{v+2}(e_i(p)) > \varphi_i^{l+2}(e_i(p)) \quad \text{for all } k < v < l. \quad (6.1.19)$$

Hence  $n_{f_i}(e_i(p)) = v + 2$ , and  $f_i$  acts on  $e_i(p) = p \cdot z_{i,v}$  as multiplication by  $z_{i,v}^{-1}$ .

The proof for the lowering operators  $f_i$  is similar, and hence  $\mathcal{M}(\Phi)$  is an abstract crystal. Seminormality follows from the fact that the statistics  $\varepsilon_i$  and  $\varphi_i$  take values in  $\mathbb{N}$ , and that  $e_i(p) = \perp$  if and only if  $\varepsilon_i(p) = 0$  (and similarly for  $f_i(p)$  and  $\varphi_i(p)$ ).

We now know that  $\mathcal{M}(\Phi)$  is a seminormal  $\Phi$ -crystal, but we do not know if it is a disjoint union of  $\mathcal{O}_q^{\text{int}}(\Phi)$  crystals. The following theorem is due to Kashiwara.

#### 6.1.20 Theorem

The monomial crystal  $\mathcal{M}(\Phi)$  is finite-normal.

#### 6.1.21 Proof

The proof is given in Proposition 3.1 of [Kas02b]. There it is stated that we need to find a ‘good’ subset  $\mathcal{M}_{\text{good}} \subseteq \mathcal{M}$  satisfying certain properties: in our setup above, this is our subset  $\mathcal{M}(\Phi) \subseteq A(\Phi)$ . Condition (i) of a ‘good’ subset is that  $p[i, c] > 0$  should imply  $p[i, c] \geq 0$ , which is true because of the parity condition, and condition (ii) of a ‘good’ subset is that it is stable under the  $e_i$  and  $f_i$  operators, which we have shown in Theorem 6.1.12.

Hence if  $(I, \cdot)$  is finite type, then every connected component of  $\mathcal{M}(\Phi)$  is isomorphic to  $\mathcal{B}(\lambda)$  for some dominant weight  $\lambda$ . This is no longer the case if  $(I, \cdot)$  is not finite type: the subcrystal generated by a primitive element  $b_\lambda \in \mathcal{M}(\Phi)$  of some dominant weight  $\lambda$  is still isomorphic to  $\mathcal{B}(\lambda)$ , but there are other subcrystals of  $\mathcal{M}(\Phi)$  which are not of this form. A theorem of Hernandez and Nakajima classifies those remaining.

#### 6.1.22 Theorem

(Due to [HN06], Theorem 2.2). Let  $p \in \mathcal{M}(\Phi)$  be any monomial. Then the subcrystal of  $\mathcal{M}(\Phi)$  generated by  $p$  is isomorphic to some connected component of an extremal weight module.

As the product monomial crystal (our main crystal of study, introduced in Section 6.3) is assembled from subcrystals of  $\mathcal{M}(\Phi)$  generated by highest-weight monomials of dominant weight, we will not concern ourselves further with the study of extremal weight modules or extremal weight crystals.

## 6.2 A variation on the monomial crystal

Kashiwara gives a different crystal structure  $\mathcal{M}_c(\Phi)$  on a set of monomials in Section 4 of [Kas02b], defined for all Cartan data  $(I, \cdot)$  rather than just for bipartite type. The definition depends on a choice  $c = (c_{ij})_{i,j \in I}$  of integers satisfying  $c_{ij} + c_{ji} = 1$  for  $i \neq j$ . We introduce this crystal since it appears in the literature much more frequently than  $\mathcal{M}(\Phi)$ . The main results of this thesis pertain to the product monomial crystal defined as a subcrystal of  $\mathcal{M}(\Phi)$ , but in fact by Corollary 6.2.10 (an observation we believe to be novel) our results apply to similar subcrystals of  $\mathcal{M}_c(\Phi)$ .

### 6.2.1 Definition (A variation on the monomial crystal)

Let  $\Phi$  be a root datum of any type  $(I, \cdot)$ , and let  $A(\Phi)$  be the set of monomials defined in Definition 6.1.1. Choose a set  $(c_{ij})_{i,j \in I}$  of integers satisfying  $c_{ij} + c_{ji} = 1$  for all  $i \neq j$  (this function need only be defined on those pairs  $(i, j) \in I$  which are connected in the Dynkin diagram). Let  $\mathcal{M}_c(\Phi) \subseteq A(\Phi)$  be the subset of monomials satisfying the compatible weight condition Eq. (6.1.4) (this subset does not depend on the choice of  $c$ ). We will now define an abstract  $\Phi$ -crystal structure on the set  $\mathcal{M}_c(\Phi)$ .

Introduce the auxiliary monomial

$$a_{i,k} = e^{\alpha_i} \cdot y_{i,k} \cdot y_{i,k+1} \cdot \prod_{j \neq i} y_{j,k+c_{ji}}^{a_{ji}}. \quad (6.2.2)$$

For each monomial  $p \in \mathcal{M}_c(\Phi)$ , define

1.  $\bar{\varphi}_i^k(p) = \sum_{l \leq k} p[i, l]$ , the *lower column sum*.
2.  $\bar{\varphi}_i^k(p) = \max_k \bar{\varphi}_i^k(p)$ , the *largest lower column sum*.
3.  $\bar{\varepsilon}_i^k(p) = -\sum_{l > k} p[i, l]$ , the *negated strict upper column sum*.
4.  $\bar{\varepsilon}_i(p) = \max_k \bar{\varepsilon}_i^k(p)$ , the *largest negated strict upper column sum*.
5.  $\bar{n}_{f,i}(p) = \min\{k \in \mathbb{Z} \mid \bar{\varphi}_i^k(p) = \bar{\varphi}_i(p)\}$ , the smallest  $k$  maximising the lower column sum  $\bar{\varphi}_i^k(p)$ .
6.  $\bar{n}_{e,i}(p) = \max\{k \in \mathbb{Z} \mid \bar{\varepsilon}_i^k(p) = \bar{\varepsilon}_i(p)\}$ , the largest  $k$  maximising the negated strict upper column sum  $\bar{\varepsilon}_i^k(p)$ .

Note that  $\bar{n}_{f,i}(p)$  is undefined if  $\bar{\varphi}_i(p) = 0$ , and  $\bar{n}_{e,i}(p)$  is undefined if  $\bar{\varepsilon}_i(p) = 0$ . We set  $\text{wt}(p) = \omega(p)$ , and define the crystal operators

$$\bar{e}_i(p) = \begin{cases} 0 & \text{if } \bar{\varepsilon}_i(p) = 0 \\ p \cdot a_{i, \bar{n}_{e,i}(p)} & \text{otherwise,} \end{cases} \quad (6.2.3)$$

$$\bar{f}_i(p) = \begin{cases} 0 & \text{if } \bar{\varphi}_i(p) = 0 \\ p \cdot a_{i, \bar{n}_{f,i}(p)}^{-1} & \text{otherwise.} \end{cases} \quad (6.2.4)$$

The *varied monomial crystal* is the set  $\mathcal{M}_c(\Phi)$ , equipped with the crystal structure  $(\text{wt}, \bar{e}_i, \bar{f}_i, \bar{\varepsilon}_i, \bar{\varphi}_i)$  given above.

The statistics  $\bar{\varphi}_i^k$  and  $\bar{\varepsilon}_i^k$  are essentially (modulo an off-by-one shift on the  $\varepsilon$ ) the composition of  $y_{i,k} \mapsto y_{i,-k}$  with the previous statistics  $\varphi_i^k$  and  $\varepsilon_i^k$ . However, the auxiliary monomial  $a_{i,k}$  is quite different to  $z_{i,k}$ , and furthermore depends on the integers  $c_{ij}$  chosen. Nevertheless, one proves in a similar way to Theorem 6.1.12 that the varied monomial crystal  $\mathcal{M}_c(\Phi)$  is a seminormal abstract  $\Phi$ -crystal, and furthermore by Corollary 4.4 of [Kas02b] it is shown that  $\mathcal{M}_c(\Phi)$  is a finite-normal crystal, regardless of the integers  $c_{ij}$  chosen.

There is an action of the abelian group  $\mathbb{Z}^I$  on the set of possible parameters  $(c_{ij})_{i \sim j}$  satisfying  $c_{ij} + c_{ji} = 1$ , where  $m \in \mathbb{Z}^I$  acts by  $(m \cdot c)_{ij} = c_{ij} + m_i - m_j$ , and the isomorphism of abelian groups defined by  $y_{i,k} \mapsto y_{i,k+m_i}$  gives an isomorphism of crystals  $\mathcal{M}_c(\Phi) \xrightarrow{\sim} \mathcal{M}_{m \cdot c}(\Phi)$ . When the graph  $I$  is acyclic, the action of  $\mathbb{Z}^I$  on the parameters is transitive, and therefore in this case the isomorphism class of  $\mathcal{M}_c(\Phi)$  does not depend on  $c$  (this is stated in [Kas02b]).

The next lemma shows that if  $(I, \cdot)$  is bipartite, then depending on the two-colouring  $\zeta : I \rightarrow \mathbb{Z}/2\mathbb{Z}$  defining the monomial crystal  $\mathcal{M}(\Phi)$ , we may find  $c_{ij}$  such that there is an isomorphism of crystals  $\mathcal{M}_c(\Phi) \xrightarrow{\sim} \mathcal{M}(\Phi)$ .

### 6.2.5 Lemma (Isomorphism of monomial and varied monomial crystals)

Let  $\Phi$  be a root datum of bipartite Cartan type  $(I, \cdot)$ , and let  $\zeta : I \rightarrow \{0, 1\}$  be a 2-colouring defining the monomial crystal  $\mathcal{M}(\Phi)$ . Whenever  $i$  and  $j$  are connected in the Dynkin diagram set  $c_{ij} = \zeta(i)$ , then we have  $c_{ij} + c_{ji} = \zeta(i) + \zeta(j) = 1$  by the fact that  $\zeta$  is a 2-colouring. Hence the choice of  $(c_{ij})_{i,j \in I}$  defines a varied monomial crystal  $\mathcal{M}_c(\Phi)$ .

Define the map  $\Gamma : \mathcal{M}_c(\Phi) \rightarrow \mathcal{M}(\Phi)$  of abelian groups by sending the generator  $y_{i,k}$  to the generator  $\Gamma(y_{i,k}) = y_{i,-2k+\zeta(i)}$ , and leaving  $\Gamma(e^\lambda) = e^\lambda$ . Then  $\Gamma$  is an isomorphism of crystals.

In fact, the proof of Lemma 6.2.5 will be valid in greater generality.

### 6.2.6 Remark

The function  $\zeta$  can be taken to be any map  $\zeta : I \rightarrow \mathbb{Z}$  such that  $|\zeta(i) - \zeta(j)| = 1$  for connected  $i, j \in I$ , in which case taking the parity of each  $\zeta(i)$  makes  $I \xrightarrow{\zeta} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  a 2-colouring defining the monomial crystal  $\mathcal{M}(\Phi)$ . For such a  $\zeta$  we should set  $c_{ij} = \frac{\zeta(i) - \zeta(j) + 1}{2}$ . Then for connected  $i, j$  we have  $c_{ij} \in \{0, 1\}$  and  $c_{ij} + c_{ji} = 1$ , and if  $\zeta$  takes values in  $\{0, 1\}$  then this choice of  $c_{ij}$  agrees with the one defined above.

We remark that the proof is made a little hard to follow because of the reversal of the  $\mathbb{Z}$ -indices of the monomials, along with the fact that the  $\varepsilon_i^k$  statistics are off-by-one between the crystals. However, we wanted to leave the definition of  $\mathcal{M}(\Phi)$  as close to its definition in [Kam+19a] as possible, and likewise for  $\mathcal{M}_c(\Phi)$  and its definition in [Kas02b] (this will also help up match up yet another crystal appearing in Chapter 8), deciding instead to complicate the isomorphism a little. Before reading the proof, one might like to skip past it and see what this map looks like in terms of the monomial diagrams.

### 6.2.7 Proof

$\Gamma$  is an isomorphism of abelian groups, commuting with the weight functions. We have that for the  $\varphi$  statistics that  $\overline{\varphi}_i^k(p) = \varphi_i^{-2k+\zeta(i)}(\Gamma(p))$  and for the  $\varepsilon$  statistics that  $\overline{\varepsilon}_i^k(p) = \varepsilon_i^{-2k-2+\zeta(i)}(\Gamma(p))$ , showing that  $\Gamma$  commutes with the crystal raising and lowering statistics  $\varepsilon_i$  and  $\varphi_i$ . Furthermore we have that  $-2\overline{n}_{f,i}(p) + \zeta(i) = n_{f,i}(\Gamma(p))$  and  $-2\overline{n}_{e,i}(p) - 2 + \zeta(i) = n_{e,i}(\Gamma(p))$  whenever  $n_{f,i}$  and  $n_{e,i}$  are defined.

We now perform straightforward comparisons of the auxiliary monomials:

$$\begin{aligned} \Gamma(a_{i,k}) &= \Gamma\left(e^{\alpha_i} \cdot y_{i,k} \cdot y_{i,k+1} \cdot \prod_{i \neq j} y_{j,k+c_{ji}}^{a_{ji}}\right) \\ &= e^{\alpha_i} \cdot y_{i,-2k+\zeta(i)} \cdot y_{i,-2k-2+\zeta(i)} \cdot \prod_{j \neq i} y_{j,-2k-2c_{ji}+\zeta(j)}^{a_{ji}} \\ z_{i,-2k-2+\zeta(i)} &= e^{\alpha_i} \cdot y_{i,-2k-2+\zeta(i)} \cdot y_{i,-2k+\zeta(i)} \cdot \prod_{j \neq i} y_{j,-2k-1+\zeta(i)}^{a_{ji}}. \end{aligned} \tag{6.2.8}$$

These two monomials are equal if and only if  $\zeta(i) - 1 = \zeta(j) - 2c_{ji}$  for all connected pairs  $(i, j)$ , which is true by our definition of  $c_{ij}$ . Hence we have  $\Gamma(a_{i,k}) = z_{i,-2k-2+\zeta(i)}$ .

Now examining the rules for the crystal operators in  $\mathcal{M}_c(\Phi)$  and  $\mathcal{M}(\Phi)$ , we have

$$\begin{aligned} \overline{f}_i(p) &= p \cdot a_{i,\overline{n}_{f,i}}^{-1}(p) \iff f_i(\Gamma(p)) = \Gamma(p) \cdot z_{i,-2\overline{n}_{f,i}(p)-2+\zeta(i)}^{-1} = \Gamma(p) \cdot \Gamma(a_{i,\overline{n}_{f,i}}^{-1}(p)), \\ \overline{e}_i(p) &= p \cdot a_{i,\overline{n}_{e,i}}(p) \iff e_i(\Gamma(p)) = \Gamma(p) \cdot z_{i,-2\overline{n}_{e,i}(p)-2+\zeta(i)} = \Gamma(p) \cdot \Gamma(a_{i,\overline{n}_{e,i}}(p)+1). \end{aligned} \tag{6.2.9}$$

Hence  $\Gamma$  commutes with the crystal operators  $e_i, f_i$  on their domain of definition, and so we conclude that  $\Gamma$  is an isomorphism  $\mathcal{M}_c(\Phi) \rightarrow \mathcal{M}(\Phi)$  of crystals.

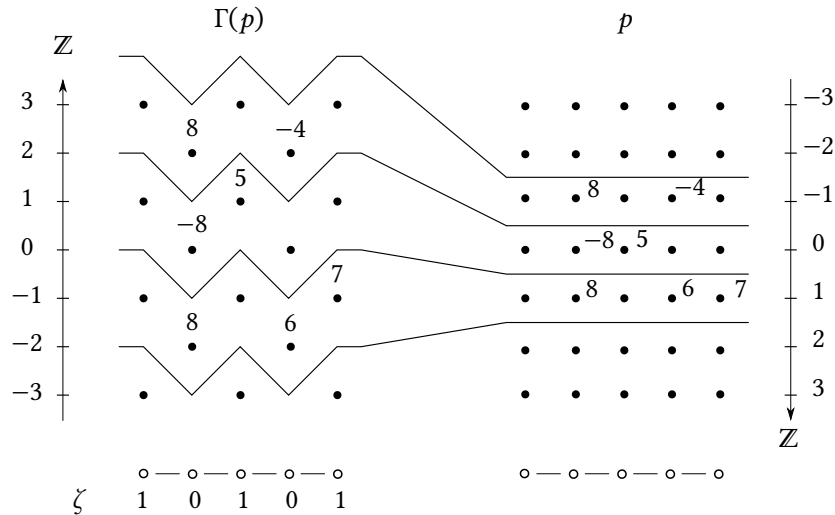
The integers  $c_{ij}$  defining the crystal  $\mathcal{M}_c(\Phi)$  in the statement of Lemma 6.2.5 are dependent on the function  $\zeta : I \rightarrow$

$\mathbb{Z}$  chosen. However, we can combine Lemma 6.2.5 and Kashiwara’s action of  $\mathbb{Z}^I$  on the space of parameters  $(c_{ij})_{i \sim j}$  to get the following result, which does not appear in the literature as far as we can tell.

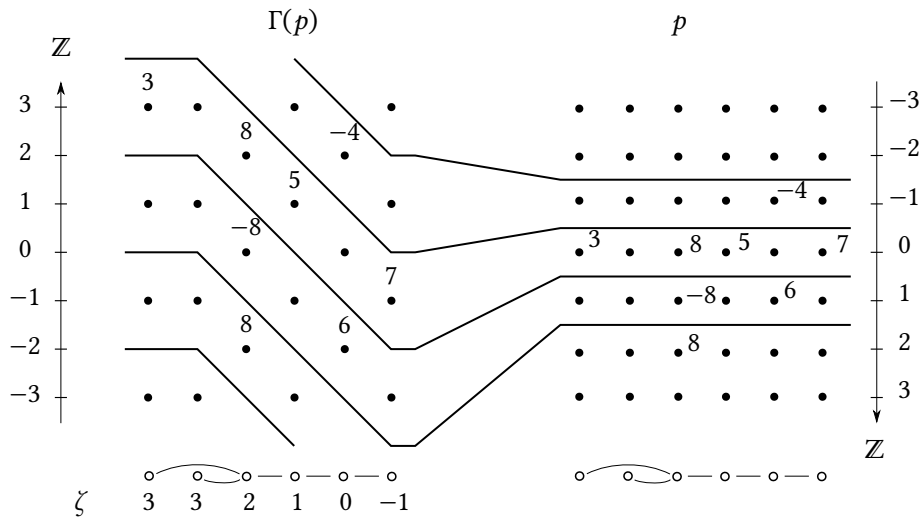
**6.2.10 Corollary**

Let  $\Phi$  be a root datum of type  $(I, \cdot)$ , where  $I$  is both bipartite and acyclic. For any two-colouring  $\zeta : I \rightarrow \mathbb{Z}/2\mathbb{Z}$  and any choice of parameters  $(c_{ij})_{i \sim j}$  satisfying  $c_{ij} + c_{ji} = 1$ , the crystal  $\mathcal{M}(\Phi)$  defined by  $\zeta$  and the crystal  $\mathcal{M}_c(\Phi)$  defined by  $c$  are isomorphic, via an isomorphism  $\mathcal{M}_c(\Phi) \rightarrow \mathcal{M}(\Phi)$  of the form  $y_{i,k} \mapsto y_{i,-2k+n_i}$  for some integers  $(n_i)_{i \in I}$ .

Here is an example of the crystal isomorphism  $\Gamma : \mathcal{M}_c(\Phi) \rightarrow \mathcal{M}(\Phi)$  in type  $(I, \cdot) = A_5$ , with the standard choice of  $\zeta(i)$  being 1 if  $i$  is odd and 0 if  $i$  is even. Note that we have drawn the  $\mathbb{Z}$ -coordinates of the monomials in  $\mathcal{M}_c(\Phi)$  increasing *down* the page rather than up the page.



Another example is shown below, this time in type  $D_6$  with a  $\zeta : I \rightarrow \mathbb{Z}$  which has more general values than just  $\{0, 1\}$ .



We conclude this section with some remarks about various monomial crystals, how they are related, and how each is related to this thesis. The first monomial crystal  $\mathcal{M}(\Phi)$  we introduced first appeared in [Nak02] in simply-laced types A and D, and was later generalised to arbitrary bipartite types in Section 3 of [Kas02b]. The definition we use comes from [HN06], which is a slight generalisation to permit arbitrary root data  $\Phi$  rather than only

simply-connected root data. The product monomial crystal (the main object of study) is defined in [Kam+19a] as a subcrystal of  $\mathcal{M}(\Phi)$ , and so this monomial crystal is the most important one for most of our work.

The varied monomial crystal  $\mathcal{M}_c(\Phi)$  first appeared in Section 4 of [Kas02b], although one can see from the discussion in Section 8.5 that this crystal more-or-less already appeared in symmetric type in Section 8 of [Nak01b]. The varied monomial crystal is the one more commonly seen in the literature, thus Lemma 6.2.5 makes the work appearing in the rest of this thesis applicable to a much wider class of crystals. The authors of [KS14] use monomial multiplication in  $\mathcal{M}_c(\Phi)$  in finite type as a model of the tensor product of crystals, where (similarly to our product monomial crystal) as long as the subcrystals are taken far enough apart vertically, the monomial-wise product is isomorphic to the tensor product of crystals. The authors of [AN18] focus on the crystal  $\mathcal{M}_c(\Phi)$  in type  $A_n$ , and show that the monomial-wise product of subcrystals is indeed a subcrystal, and give an explicit decomposition theorem. Both of these results are implied (in symmetric type) by our work, and Lemma 6.2.5.

There is a third crystal appearing in the literature under the name *modified Nakajima monomials* in [KKS07] which is commonly denoted by  $\widehat{\mathcal{M}}_c(\Phi)$ . This crystal is certainly different in nature to  $\mathcal{M}(\Phi)$  and  $\mathcal{M}_c(\Phi)$ , as the subcrystal generated by a primitive element of weight  $\lambda$  is isomorphic to the crystal  $T_\lambda \otimes \mathcal{B}(\infty)$ , which is the crystal of the Verma module  $M(\lambda)$  rather than being a crystal of an  $\mathcal{O}_q^{\text{int}}(\Phi)$ -module. The techniques in our work could probably be adapted to analyse this crystal, but there is nothing we can say about the relation between our work and this crystal directly, and so we will not mention it further.

## 6.3 The product monomial crystal

In light of the isomorphism given in Lemma 6.2.5, we could define the product monomial crystal inside either  $\mathcal{M}(\Phi)$  or the variation  $\mathcal{M}_c(\Phi)$ . We choose to go with the first crystal  $\mathcal{M}(\Phi)$ , following its original definition in [Kam+19a]. The product monomial crystal will be a monomial-wise product of various subcrystals of  $\mathcal{M}(\Phi)$ , with each subcrystal generated by a certain *dominant* monomial. We introduce some necessary notation for this.

### 6.3.1 Definition (Dominant monomials and fundamental subcrystals)

For each multiset  $\mathbf{R}$  based in  $I \times \mathbb{Z}$ , let  $y_{\mathbf{R}} \in A(\Phi)$  be the monomial  $y_{\mathbf{R}} = \prod_{(i,c) \in \mathbf{R}} y_{i,c}$ . A *dominant pair* is a pair  $(\lambda, \mathbf{R})$  of a weight  $\lambda \in X(\Phi)$  and a multiset  $\mathbf{R}$  based in  $I \times \mathbb{Z}$  such that  $e^\lambda \cdot y_{\mathbf{R}} \in \mathcal{M}(\Phi)$ . This condition is equivalent to  $\langle \lambda, \alpha_i^\vee \rangle = \sum_{l \in \mathbb{Z}} \mathbf{R}[i, l]$  for all  $i \in I$ .

If  $(\lambda, \mathbf{R})$  is a dominant pair then the monomial  $e^\lambda \cdot y_{\mathbf{R}}$  is primitive of highest weight  $\lambda$ , and hence generates a subcrystal of  $\mathcal{M}(\Phi)$  isomorphic to  $\mathcal{B}(\lambda)$  by finite-normality of  $\mathcal{M}(\Phi)$  and Theorem 4.4.2. Let  $\mathcal{M}(e^\lambda, y_{\mathbf{R}})$  denote this subcrystal.

Suppose that  $\mathbf{R}$  is concentrated in a single entry, so that  $\mathbf{R} = \{(i, c)^n\}$  for some  $n > 0$ . In this case we call a dominant pair  $(\lambda, \mathbf{R})$  a *fundamental pair*, and call the crystal generated by  $e^\lambda \cdot y_{\mathbf{R}} = e^\lambda \cdot y_{i,c}^n$  a *fundamental subcrystal*  $\mathcal{M}(e^\lambda, \mathbf{R})$ .

If  $\Phi$  is simply-connected, then a fundamental pair is always of the form  $\lambda = \omega_i^n$  and  $\mathbf{R} = \{(i, c)^n\}$ . We now define the product monomial crystal as a monomial-wise product over fundamental subcrystals.

### 6.3.2 Definition (Product monomial crystal)

For a dominant pair  $(\lambda, \mathbf{R})$ , fix a decomposition  $\lambda = \sum_{(i,c) \in \mathbf{R}} \lambda_{i,c}$  such that  $\langle \alpha_i^\vee, \lambda_{i,c} \rangle = \mathbf{R}[i, c]$  for all  $(i, c) \in \mathbf{R}$ . The *product monomial crystal* is the set defined as the monomial-wise product of the fundamental subcrystals:

$$\mathcal{M}(\lambda, \mathbf{R}) = \prod_{(i,c) \in \text{Supp } \mathbf{R}} \mathcal{M}(e^{\lambda_{i,c}} \cdot y_{i,c}^{\mathbf{R}[i,c]}). \quad (6.3.3)$$

The product monomial crystal does not depend on the decomposition of  $\lambda$ .

We remark that here we have chosen a definition where the product Eq. (6.3.3) has as few terms as possible. We will later see that product of fundamental subcrystals concentrated over the same vertex  $(i, c)$  is again a

fundamental subcrystal:

$$\mathcal{M}(e^\lambda \cdot y_{i,c}^n) \cdot \mathcal{M}(e^\mu \cdot y_{i,c}^m) = \mathcal{M}(e^{\lambda+\mu} \cdot y_{i,c}^{n+m}), \quad (6.3.4)$$

reconciling our definition of the product monomial crystal with the definition of [Kam+19a] in the simply-connected case. This fact will be implied by our character formula, and also by the proof of Theorem 6.3.5.

As defined, the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is only a subset of  $\mathcal{M}(\Phi)$ , and it is unclear whether it is closed under the crystal operators  $e_i$  and  $f_i$ . The following theorem justifies the name *crystal*.

### 6.3.5 Theorem

When  $\Phi$  is a root datum whose type  $(I, \cdot)$  is *symmetric, bipartite, and without cycles*, then  $\mathcal{M}(\lambda, \mathbf{R})$  is a subcrystal of  $\mathcal{M}(\Phi)$ .

The proof of this theorem is sketched in Section 7 of [Kam+19a], and the purpose of Chapter 8 will be to explain this proof. The restriction on  $I$  being bipartite comes from the fact that the monomial crystal  $\mathcal{M}(\Phi)$  (Definition 6.1.1) is only defined in bipartite type. The restriction on  $I$  being symmetric comes from the method of proof, using Nakajima quiver varieties. Various computer experiments suggest that the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is still a crystal even when  $I$  is of arbitrary finite type.

### 6.3.6 Remark

Our notation for the product monomial crystal differs from its original definition in [Kam+19a] in three ways. Firstly, we use the symbol  $\mathcal{M}(\lambda, \mathbf{R})$  for the crystal rather than  $\mathcal{B}(\mathbf{R})$ . Secondly, they work only in the simply-connected case making the weight term  $e^\lambda$  in the monomials unnecessary. Thirdly, they use a collection of multisets  $(\mathbf{R}_i)_{i \in I}$ , where  $\mathbf{R}_i$  is a multiset based in  $2\mathbb{Z} + \zeta(i)$ : to go between the two notations set  $\mathbf{R}[i, k] = \mathbf{R}_i[k]$ .

It was noted that in the simply-connected case, there exist embeddings of crystals (Theorem 2.2 of [Kam+19a])

$$\mathcal{B}(\lambda) \hookrightarrow \mathcal{M}(\lambda, \mathbf{R}) \hookrightarrow \bigotimes_{(i,c) \in \mathbf{R}} \mathcal{B}(\omega_i), \quad (6.3.7)$$

and that furthermore by varying  $\mathbf{R}$  while keeping  $\lambda$  fixed, both extremes  $\mathcal{M}(\lambda, \mathbf{R}) \cong \mathcal{B}(\lambda)$  and  $\mathcal{M}(\lambda, \mathbf{R}) \cong \bigotimes_{(i,c) \in \mathbf{R}} \mathcal{B}(\omega_i)$  can be achieved. The first embedding  $\mathcal{B}(\lambda) \hookrightarrow \mathcal{M}(\lambda, \mathbf{R})$  is clear to see: the monomial  $e^\lambda \cdot \mathbf{R} \in \mathcal{M}(\lambda, \mathbf{R})$  is highest weight and hence generates a subcrystal isomorphic to  $\mathcal{B}(\lambda)$ . The second embedding is much more subtle, and we will explain it in Section 7.3.

### 6.3.8 Remark

There is a case where the isomorphism  $\mathcal{M}(\lambda, \mathbf{R}) \cong \bigotimes_{(i,c) \in \mathbf{R}} \mathcal{B}(\omega_i)$  is easy to see, which is when  $(I, \cdot)$  is finite type, and the parameter  $\mathbf{R}$  is sufficiently ‘generic’. By this we mean that the elements of  $\mathbf{R}$  are spread out enough in the  $\mathbb{Z}$ -direction so that the fundamental subcrystals  $\mathcal{M}(e^{\omega_i}, (i, c))$  for  $(i, c) \in \mathbf{R}$  do not interact. For example, suppose that  $\mathbf{R} = \{(i_1, c_1), \dots, (i_N, c_N)\}$  with  $c_1 > \dots > c_N$ , and suppose further that a vertical line can be drawn on the monomial diagram, separating any element of  $\mathcal{M}(e^{\omega_{i_r}}, (i_r, c_r))$  from  $\mathcal{M}(e^{\omega_{i_{r+1}}}, (i_{r+1}, c_{r+1}))$ , for all  $r \in 1, \dots, N-1$ . (Since  $(I, \cdot)$  is finite type, each fundamental subcrystal  $\mathcal{M}(e^{\omega_i}, (i, c))$  is finite, and hence this can always be done by taking the  $c_r$  to be far enough apart). It follows that the map of sets

$$\mathcal{M}(e^{i_1}, (c_1, r_1)) \otimes \dots \otimes \mathcal{M}(e^{i_N}, (c_N, r_N)) \xrightarrow{\text{mult}} \mathcal{M}(\lambda, \mathbf{R}) \quad (6.3.9)$$

from the Cartesian product to the product monomial crystal is a bijection (as each subcrystal is vertically separated from the others, we can uniquely factorise the product), and it is straightforward to see that the crystal structure inherited from  $\mathcal{M}(\Phi)$  actually makes mult into a crystal isomorphism.



## 6.4 Labelling elements of the crystal

Let  $\mathbf{R}$  and  $\mathbf{S}$  be finite multisets based in  $I \times \mathbb{Z}$ , and define the auxiliary monomials

$$y_{\mathbf{R}} := \prod_{(i,c) \in \mathbf{R}} y_{i,c}, \quad z_{\mathbf{S}} := \prod_{(i,k) \in \mathbf{S}} z_{i,k} = \prod_{(i,k) \in \mathbf{S}} e^{\alpha_i} \cdot y_{i,k} \cdot y_{i,k+2} \cdot \prod_{j \neq i} y_{j,k+1}^{a_{ji}}, \quad z_{\mathbf{S}}^{-1} = (z_{\mathbf{S}})^{-1}. \quad (6.4.1)$$

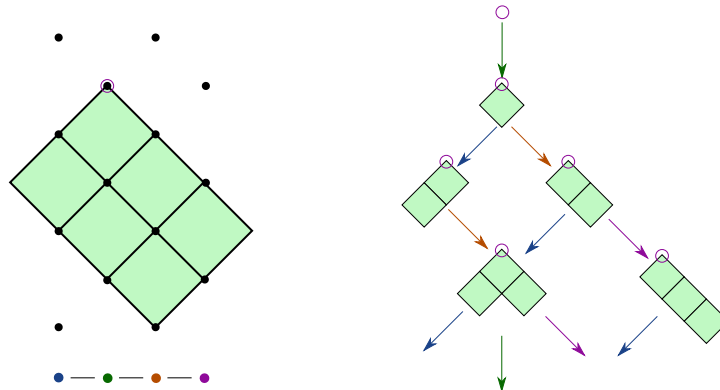
The fundamental subcrystal  $\mathcal{M}(e^\lambda \cdot y_{i,c})$  is generated by the highest-weight element  $e^\lambda \cdot y_{i,c}$ , and so every element of the crystal is of the form  $e^\lambda \cdot y_{i,c} \cdot z_{\mathbf{S}}^{-1}$  for some finite multiset  $\mathbf{S}$  based in  $I \times \mathbb{Z}$ . It follows by definition that every element of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is of the form  $e^\lambda \cdot y_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$  for some finite multiset  $\mathbf{S}$ . Since the set  $\{z_{i,k} \mid (i,k) \in I \times \mathbb{Z}\}$  of auxiliary monomials is linearly independent in  $\mathcal{M}(\Phi)$  (which can be seen by using a triangularity argument from the  $y_{i,k}$ ), a monomial  $p \in \mathcal{M}(\lambda, \mathbf{R})$  is *uniquely* determined by the  $\mathbf{S}$ -multiset appearing in the expression  $p = e^\lambda \cdot y_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$ , and we will call this the **S-labelling** of an element. Under this labelling, the exponent  $p[i,k]$  of a monomial  $p \in \mathcal{M}(\lambda, \mathbf{R})$  may be expressed as

$$p[i,k] = \mathbf{R}[i,k] - \mathbf{S}[i,k-2] - \mathbf{S}[i,k] - \sum_{j \neq i} a_{ji} \mathbf{S}[j,k-1]. \quad (6.4.2)$$

In the type A case, this  $\mathbf{S}$  labelling has a direct interpretation in terms of Young diagrams.

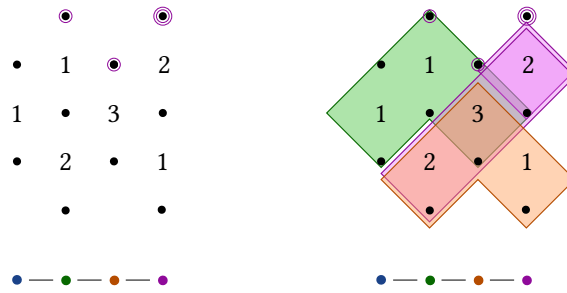
### 6.4.3 Remark

Consider the simply-connected root datum  $\Phi = \text{SL}_5$  of Cartan type  $A_4$ . The fundamental subcrystal generated by the monomial  $\mathcal{M}(e^{\omega_2} \cdot y_{2,0}) \cong \mathcal{B}(\omega_2)$  has 10 elements, indexed by the Young diagrams fitting within a  $2 \times 3$  rectangle. The empty partition corresponds to the highest-weight element  $y_{2,0}$ , and in general a monomial  $y_{2,0} z_{\mathbf{S}}^{-1}$  corresponds to a Young diagram drawn in ‘Russian’ style, where the box with bottom corner  $(j,k)$  is present if and only if  $(j,k) \in \mathbf{S}$ . Below the partition corresponding to the lowest-weight element of  $\mathcal{M}(e^{\omega_2} \cdot y_{2,0})$  is shown on the left, with some of the crystal  $\mathcal{M}(e^{\omega_2} \cdot y_{2,0})$  shown on the right. In each picture, the point  $(2,0)$  is indicated with a circle.



The crystal operators on the right are easy to remember: the operator  $f_i$  adds a box in column  $i$ , if there is an addable box in that position.

Now let  $\lambda = \omega_2 + \omega_3 + 2\omega_4$ ,  $\mathbf{R} = \{(2,0), (3,-1), (4,0)^2\}$ , and consider the problem of deciding whether a monomial  $e^\lambda \cdot y_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$  is an element of  $\mathcal{M}(\lambda, \mathbf{R})$ . We may picture the multiset  $\mathbf{R}$  by circling points on the monomial diagram, and the multiset  $\mathbf{S}$  by placing multiplicities in their corresponding box positions. Below is shown a potential  $\mathbf{S}$  on the left, with the figure on the right showing a valid ‘covering’ of  $\mathbf{S}$  by partitions fitting within boxes. The partitions used are  $(2,1)$  above the vertex 2, the partition  $(2,1)$  above the vertex 3, and the two partitions  $(1)$  and  $(1,1,1)$  above the vertex 4.



The monomial  $e^\lambda \cdot y_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$  is an element of  $\mathcal{M}(\lambda, \mathbf{R})$  if and only if there exists a valid covering of  $\mathbf{S}$  by partitions hung from the pegs described by  $\mathbf{R}$ . The diagram on the right is not the only way to resolve the multiset  $\mathbf{S}$  into overlapping partitions hung from the pegs  $\mathbf{R}$ . Each resolution corresponds to some factorisation of a monomial back into a product of monomials coming from the fundamental crystals making up  $\mathcal{M}(\lambda, \mathbf{R})$ .

This interpretation of the product monomial crystal in type A was originally described in Section 6 of [Kam+19a], and Section 2.5.3 of [WWY17]. Although we will not use it further in the paper, we found this observation invaluable in making the initial connection to generalised Schur modules, which eventually led us to our general Demazure character formula for  $\mathcal{M}(\lambda, \mathbf{R})$ .

### 6.5 A partial order

Define a partial order  $\leq$  on the set  $I \times \mathbb{Z}$  as the transitive closure of

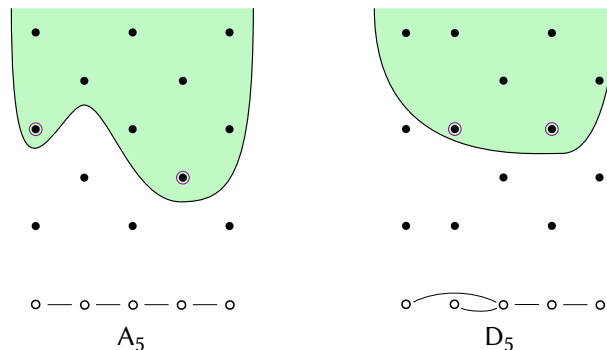
$$(i, k) \leq (i, k + 2) \quad \text{and} \quad (i, k) \leq (j, k + 1) \text{ for all } j \sim i. \tag{6.5.1}$$

(Recall that  $j \sim i$  means that the vertices  $i$  and  $j$  are connected in the Dynkin diagram, or equivalently that  $a_{ij} \neq 0$ ). A subset  $J \subseteq I \times \mathbb{Z}$  is called *upward-closed* if whenever  $x \in J$  and  $y \in I \times \mathbb{Z}$  satisfy  $x \leq y$ , then  $y \in J$ . (This condition is sometimes called being an *upper set*). A *minimal element* in an upward closed set  $J$  is an element  $x \in J$  such that for all  $y \in J$ , either  $x \leq y$ , or  $x$  and  $y$  are incomparable.

When  $I$  is connected, we will say that an upward-closed set  $J \subseteq I \times \mathbb{Z}$  is *proper* if it is a proper nonempty subset. For general  $I$ , we say an upward-closed set is proper if its restriction to each connected component is proper. For any subset  $J \subseteq I \times \mathbb{Z}$ , let  $\text{up}(J) = \{y \in I \times \mathbb{Z} \mid x \leq y \text{ for some } x \in J\}$  be the *upward-closed set generated by  $J$* . Every proper upward-closed set is a union of the upward-closed sets generated by its finitely many minimal elements. We define downward-closed sets and  $\text{down}(J)$  similarly.

#### 6.5.2 Example

The following diagram shows an example of a proper upward-closed set in type  $A_5$ , and another in type  $D_5$  (or we could also view both diagrams as a *single* proper upward-closed set in type  $A_5 \times D_5$ ). The minimal elements have been marked with a circle.



Define the *boundary* of the upward-closed set  $J$  to be

$$\partial J = \{(i, k) \in J \mid (i, k - 2) \notin J\}. \tag{6.5.3}$$

An upward-closed set  $J$  is proper if and only if  $|\partial J| = |I|$ . The figure in [Example 6.5.2](#) contains 10 boundary points in total, but only 4 minimal points: a minimal point is always a boundary point, but not conversely.

The reason we have introduced this partial order on  $I \times \mathbb{Z}$  is that the fundamental crystals  $\mathcal{M}(e^\lambda \cdot y_{i,c}^n)$  ‘grow downwards’ with respect to the order.

**6.5.4 Lemma** (*Fundamental subcrystals grow downwards*)

Let  $(\lambda, y_{i,c}^n)$  be a fundamental pair. Then if  $e^\lambda \cdot y_{i,c}^n \cdot z_S^{-1} \in \mathcal{M}(e^\lambda \cdot y_{i,c}^n)$ , then  $x \leq (i, c - 2)$  for all  $x \in \text{Supp } S$ .

**6.5.5 Proof**

The claim is vacuous for the highest-weight element  $e^\lambda \cdot y_{i,c}^n$ , since its associated  $S$ -multiset is empty. As the fundamental subcrystal  $\mathcal{M}(e^\lambda \cdot y_{i,c}^n)$  is connected, it suffices to show that the crystal  $f_i$  operators preserve the above property.

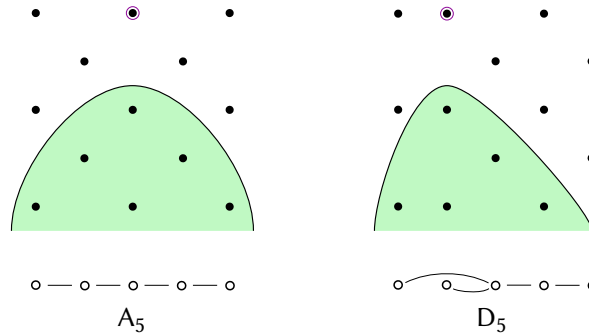
Suppose that  $p = e^\lambda \cdot y_{i,c}^n \cdot z_S^{-1} \in \mathcal{M}(e^\lambda \cdot y_{i,c}^n)$  satisfies  $\text{Supp } S \leq (i, c - 2)$ , meaning that  $x \leq (i, c - 2)$  for all  $x \in \text{Supp } S$ . Fix a vertex  $j \in I$ . If  $f_j(p) = \perp$  then there is nothing to prove, so assume instead that  $f_j(p) = p \cdot z_{j,k-2}^{-1}$  (we seek to prove that  $(j, k - 2) \leq (i, c - 2)$ ). In particular, this means that the largest upper column sum  $\varphi_j(p) > 0$  was maximised at  $(i, k)$ , and hence  $p[j, k] > 0$ . Applying this inequality to [Eq. \(6.4.2\)](#) gives

$$\mathbf{R}[j, k] + \sum_{k \neq j} |a_{jk}| S[j, k - 1] > S[j, k] + S[j, k - 2] \geq 0 \tag{6.5.6}$$

where  $\mathbf{R}$  is the multiset  $\{(i, c)^n\}$ .

If  $\mathbf{R}[j, k] = n$ , then  $(j, k) = (i, c)$  and so  $(j, k - 2) = (i, c - 2)$ . Otherwise, we must have  $\mathbf{R}[j, k] = 0$  since  $\mathbf{R}$  is concentrated in a single element. This means that  $\sum_{k \neq j} |a_{jk}| S[j, k - 1]$  is strictly positive, and hence there exists an  $l \sim j$  such that  $(l, k - 1) \in S$ , so we have found an upward neighbour of  $(j, k - 2)$  already contained in  $\text{Supp } S$ . So the claim follows by the inductive assumption and the transitivity of  $\leq$ .

We give two illustrations of [Lemma 6.5.4](#) in types  $A_5$  and  $D_5$ . In each picture, the point  $(i, c)$  has been circled, and the set  $\{(j, k) \mid (j, k) \leq (i, c - 2)\}$  has been shaded.



## 6.6 Supports of monomials

In light of [Lemma 6.5.4](#) we will define the *based support* of a monomial, a certain ‘shadow’ it makes on the underlying set  $I \times \mathbb{Z}$ . For a monomial  $p = e^\lambda \cdot y_{\mathbf{R}} \cdot z_S^{-1} \in \mathcal{M}(\lambda, \mathbf{R})$ , define its *based support* to be  $\text{Supp}_{\mathbf{R}} p = \text{Supp } \mathbf{R} \cup \text{Supp } S$ . Note that the based support of a monomial  $p$  relies in a fundamental way on  $\mathbf{R}$ , since the multiset  $\mathbf{R}$  determines the factorisation  $p = e^\lambda \cdot y_{\mathbf{R}} \cdot z_S^{-1}$  and hence determines  $S$ . The based support of a monomial can be quite large compared to its support in terms of the  $y_{i,c}$ .

**6.6.1 Example**

Let  $\Phi = \mathrm{SL}_4$ ,  $\lambda = \omega_1 + \omega_3$ ,  $\mathbf{R} = \{(1, 1), (3, 5)\}$ ,  $\mathbf{S} = \{(1, 1), (2, 2), (3, 3)\}$ , and  $p = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$ . Then as a monomial,  $p = 1$ , and hence  $\mathrm{Supp}_{\mathbf{R}}(1) = \{(1, 1), (2, 2), (3, 3), (3, 5)\}$ .

The based support is additive, in the sense that if we have two different monomials from two different product monomial crystals  $p \in \mathcal{M}(\lambda, \mathbf{R})$  and  $q \in \mathcal{M}(\mu, \mathbf{Q})$ , then we have  $\mathrm{Supp}_{\mathbf{R}+\mathbf{Q}}(p \cdot q) = \mathrm{Supp}_{\mathbf{R}}(p) + \mathrm{Supp}_{\mathbf{Q}}(q)$ , which follows from the identity of monomials

$$(e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}) \cdot (e^\mu \cdot \gamma_{\mathbf{Q}} \cdot z_{\mathbf{T}}^{-1}) = e^{\lambda+\mu} \cdot \gamma_{\mathbf{R}+\mathbf{Q}} \cdot z_{\mathbf{S}+\mathbf{T}}^{-1}. \quad (6.6.2)$$

Applying this additivity property to the definition of the product monomial crystal in terms of fundamental subcrystals, together with Lemma 6.5.4, we find that for all  $p \in \mathcal{M}(\lambda, \mathbf{R})$  the based support  $\mathrm{Supp}_{\mathbf{R}}(p)$  is contained in the downward-closed set generated by  $\mathrm{Supp}_{\mathbf{R}}$ . Examining Eq. (6.4.1) shows that if  $p[i, k] \neq 0$ , then  $(i, k) \geq x$  for some point  $x \in \mathrm{Supp}_{\mathbf{R}}(p)$ .

The next lemma shows that a monomial whose based support extends below  $\mathrm{up}(\mathbf{R})$  cannot be highest weight.

**6.6.3 Lemma (Raising)**

Let  $p = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1} \in \mathcal{M}(\lambda, \mathbf{R})$  and suppose that  $(i, k)$  is a minimal point of  $\mathrm{Supp}_{\mathbf{R}}(p)$ . Let  $q = e_i^n(p)$  be the element at the top of  $p$ 's  $i$ -string, so  $n = \varepsilon_i(p)$ . If  $(i, k) \notin \mathbf{R}$  then  $n > 0$  and  $\mathrm{Supp}_{\mathbf{R}}(q) \subseteq \mathrm{Supp}_{\mathbf{R}}(p) \setminus (i, k)$ .

**6.6.4 Proof**

Let  $p = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$  and  $q = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{T}}^{-1}$ . We have  $\mathbf{T} \subseteq \mathbf{S}$  by definition of the crystal raising operator  $e_i$ , and hence  $\mathrm{Supp}_{\mathbf{R}}(q) \subseteq \mathrm{Supp}_{\mathbf{R}}(p)$ . By the minimality of  $(i, k)$  we have  $p[i, r] = q[i, r] = 0$  for  $r < k$ , and by the assumption that  $\mathbf{R}[i, k] = 0$  we have for the lower negated column sums  $\varepsilon_i^k(p) = -\mathbf{S}[i, k]$  and  $\varepsilon_i^k(q) = -\mathbf{T}[i, k]$ . However,  $n = \varepsilon_i(p) \geq \varepsilon_i^k(p) = \mathbf{S}[i, k] > 0$  shows that  $n > 0$ , and  $0 = \varepsilon_i(q) \geq \varepsilon_i^k(q) = \mathbf{T}[i, k]$  shows that  $(i, k) \notin \mathrm{Supp}_{\mathbf{R}} q$ .

Hence we get a useful necessary condition for highest-weight monomials in the product monomial crystal.

**6.6.5 Corollary**

If  $p \in \mathcal{M}(\lambda, \mathbf{R})$  is highest-weight, then  $\mathrm{Supp}_{\mathbf{R}}(p) \subseteq \mathrm{up}(\mathbf{R})$ .

## 7 Truncations and the character formula

The product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is bounded between two extremes: on one hand it can be isomorphic to the irreducible highest-weight crystal  $\mathcal{B}(\lambda)$ , and on the other hand it can be isomorphic to a large tensor product  $\mathcal{B}(\lambda_1) \otimes \cdots \otimes \mathcal{B}(\lambda_n)$  where  $\lambda = \lambda_1 + \cdots + \lambda_n$ . We can see the product monomial crystal as *interpolating* between these two extremes (we make this precise in type A in [Chapter 10](#), where we show that the product monomial crystal is the crystal of a generalised Schur module). The question remains: what can we say about the other cases?

In this chapter we define *truncations*, a family of subsets of the product monomial crystal parametrised by upward-closed sets. These truncations are somewhat like Demazure crystals as they are finite, closed under the crystal raising operators  $e_i$ , have containment compatible with containment of upward-closed sets, and in their limit we recover the whole product monomial crystal. In fact it turns out that these truncations *are* disjoint unions of Demazure crystals, but this is not obvious from their definition.

By relating ‘nearby’ truncations to each other in purely crystal-theoretic terms, we give an inductive character formula ([Theorem 7.2.3](#)) for any of the truncations, specialising in the finite type case to a character formula for the whole product monomial crystal ([Corollary 7.3.9](#)). Since  $\mathcal{M}(\lambda, \mathbf{R})$  is the crystal associated to the categorical  $\mathfrak{g}(\Phi)$ -representation  $\mathcal{V}(\lambda, \mathbf{R})$  introduced in [Chapter 1](#), our formula has implications for the study of the *truncated shifted Yangian* algebras  $Y_\mu^\lambda(\mathbf{R})$ .

### 7.1 Truncations defined by upward sets

We saw in [Corollary 6.6.5](#) at the end of the last section that a highest-weight monomial  $p \in \mathcal{M}(\lambda, \mathbf{R})$  satisfies  $\text{Supp}_{\mathbf{R}}(p) \subseteq \text{up}(\mathbf{R})$ . Consider the subset  $\mathcal{M}(\lambda, \mathbf{R}, \text{up}(\mathbf{R}))$  of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  consisting of those monomials whose based support is contained within the upward-closed set  $\text{up}(\mathbf{R})$ . This subset is closed under the crystal raising operators, because each such operator will only ever remove elements from the based support of a monomial. Furthermore, this subset contains every highest-weight element of  $\mathcal{M}(\mathbf{R})$ . It is the prototypical example of one of our truncations.

#### 7.1.1 Definition

Let  $J \subseteq I \times \mathbb{Z}$  be an upward-closed set containing  $1\mathbf{R}$ . The *truncation* of  $\mathcal{M}(\lambda, \mathbf{R})$  by  $J$  is the subset

$$\mathcal{M}(\lambda, \mathbf{R}, J) = \{p \in \mathcal{M}(\lambda, \mathbf{R}) \mid \text{Supp}_{\mathbf{R}}(p) \subseteq J\}. \quad (7.1.2)$$

The subset  $\mathcal{M}(\lambda, \mathbf{R}, J)$  is closed under the crystal raising operators, and hence we may equip it with its unique upper-seminormal crystal structure coming from the restriction of the  $(e_i)_{i \in I}$  and  $\text{wt}$  functions on the original crystal. (This agrees with the crystal operators defined on  $\mathcal{M}(\lambda, \mathbf{R}, J)$  coming from the monomial crystal  $\mathcal{M}(\Phi)$ ).

Each of our truncations  $\mathcal{M}(\lambda, \mathbf{R}, J)$  satisfies  $\text{up}(\mathbf{R}) \subseteq J$  by definition, and therefore by [Corollary 6.6.5](#) contains every highest-weight element of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$ . As the product monomial crystal decomposes into highest-weight crystals, this means that knowing its highest-weight elements determines it up to isomorphism. Therefore in order to determine the isomorphism class of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$ , it is enough to determine the highest-weight elements of *any* truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$ .

<sup>1</sup>We use the terminology “ $J$  contains  $\mathbf{R}$ ” to mean that  $\text{Supp } \mathbf{R} \subseteq J$ . In particular, elements of  $\mathbf{R}$  may still have multiplicity greater than one.

However, only knowing the character  $\text{ch } \mathcal{M}(\lambda, \mathbf{R}, J)$  of a truncation is not enough information to recover its highest-weight elements, unless the character satisfies some special properties. We will eventually end up showing that each truncation is a disjoint union of Demazure crystals, hence by the linear independence of Demazure characters (Lemma 5.2.9) the character of any truncation will determine the isomorphism class of the product monomial crystal.

An observation we will use repeatedly is that if  $p \in \mathcal{M}(\lambda, \mathbf{R}, J)$  then  $p[i, c] = 0$  for all  $(i, c) \notin J$ . For instance, it plays a part in the next lemma which shows that using a single crystal lowering operator on a monomial repeatedly can only push it outside the truncation by a single element.

**7.1.3 Lemma (Lowering)**

Let  $p \in \mathcal{M}(\lambda, \mathbf{R}, J)$  be an element of a truncation,  $i \in I$ , and  $(i, k) \in \partial J$ . Then  $\text{Supp}_{\mathbf{R}}(f_i^n(p)) \subseteq J \cup \{(i, k-2)\}$  for all  $0 < n \leq \varphi_i(p)$ .

**7.1.4 Proof**

By definition of the monomial crystal we have that  $n_{f_i}(f_i(q)) \geq n_{f_i}(q)$  for all  $q \in \mathcal{M}(\Phi)$ . Since the monomial  $p$  lies in the truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  we have  $p[i, r] = 0$  for all  $r \leq k-2$ , and hence  $n_{f_i}(p) \geq k$ . This means that when applying  $f_i$  repeatedly, the monomial  $p$  will be multiplied by  $z_{i,r}$  for  $r \geq k-2$ . Therefore  $\text{Supp}_{\mathbf{R}}(f_i^n(p)) \subseteq J \cup \{(i, k-2)\}$  for all  $0 < n \leq \varphi_i(p)$ .

The next lemma shows that when  $J$  and  $J'$  are upward-closed sets which differ by a single minimal element, then the associated truncations are related in purely crystal-theoretic terms: the larger truncation will be an extension of strings of the smaller truncation.

**7.1.5 Lemma (Minimal points and extensions of strings)**

Suppose that  $J$  and  $J'$  are two upward-closed sets containing  $\mathbf{R}$ , which differ in a single element  $J' = J + \{(i, k)\}$ . Then  $\mathcal{M}(\lambda, \mathbf{R}, J') = \mathfrak{D}_i \mathcal{M}(\lambda, \mathbf{R}, J)$ , where  $\mathfrak{D}_i$  is the extension of  $i$ -strings (Definition 5.3.1) operator.

**7.1.6 Proof**

Since both  $J$  and  $J'$  are upward-closed and contain  $\mathbf{R}$ , the point  $(i, k)$  is minimal in  $J'$  and  $(i, k) \notin \mathbf{R}$ . (If it were not minimal, its removal from  $J'$  would result in a non-upward-closed set). Hence if a monomial  $q \in \mathcal{M}(\lambda, \mathbf{R}, J')$  satisfies  $(i, k) \in \text{Supp}_{\mathbf{R}}(q)$ , the conditions of Lemma 6.6.3 are met and so there exists a  $p \in \mathcal{M}(\lambda, \mathbf{R}, J)$  and  $n > 0$  with  $q = f_i^n(p)$ . If  $(i, k) \notin \text{Supp}_{\mathbf{R}}(q)$  then  $q \in \mathcal{M}(\lambda, \mathbf{R}, J)$  already. Hence  $\mathcal{M}(\lambda, \mathbf{R}, J') \subseteq \mathfrak{D}_i(\lambda, \mathbf{R}, J)$ .

Conversely, applying Lemma 7.1.3 to this special case gives  $\mathcal{M}(\lambda, \mathbf{R}, J') \supseteq \mathfrak{D}_i(\lambda, \mathbf{R}, J)$ .

Lemma 7.1.5 is one of the two key pieces we need for the character formula, which will give us the ‘Demazure’ part of our ‘Demazure character formula’. It allows us to relate any two truncations defined by nested upward-closed sets  $J \subseteq J'$ , for a fixed choice of  $(\lambda, \mathbf{R})$ . The second piece of our character formula will be a fact which allows us to relate truncations defined by the same upward-closed set, where  $(\lambda, \mathbf{R})$  is varied along the boundary of that set.

**7.1.7 Lemma (Factorisation along a boundary)**

Suppose  $(\lambda, \mathbf{R})$  and  $(\mu, \mathbf{Q})$  are two dominant pairs, and that  $J$  is an upward-closed set containing both  $\mathbf{R}$  and  $\mathbf{Q}$ , and further that  $\mathbf{Q}$  is supported only along the boundary of  $J$ , i.e.  $\text{Supp } \mathbf{Q} \subseteq \partial J$ . Then  $\mathcal{M}(\lambda + \mu, \mathbf{R} + \mathbf{Q}, J) = e^\mu \cdot \gamma_{\mathbf{Q}} \cdot \mathcal{M}(\lambda, \mathbf{R}, J)$ , where  $\cdot$  denotes a product of monomials.

**7.1.8 Proof**

By the definition of the product monomial crystal and the additivity of based support, we have that

$$\mathcal{M}(\lambda, \mathbf{R}, J) \cdot \mathcal{M}(\mu, \mathbf{Q}, J) = \mathcal{M}(\lambda + \mu, \mathbf{R} + \mathbf{Q}, J). \quad (7.1.9)$$

Since  $\mathbf{Q}$  is concentrated along the boundary  $\partial J$ , Lemma 6.5.4 gives that  $\mathcal{M}(\mu, \mathbf{Q}, J) = \{e^\mu \cdot \gamma_{\mathbf{Q}}\}$ .

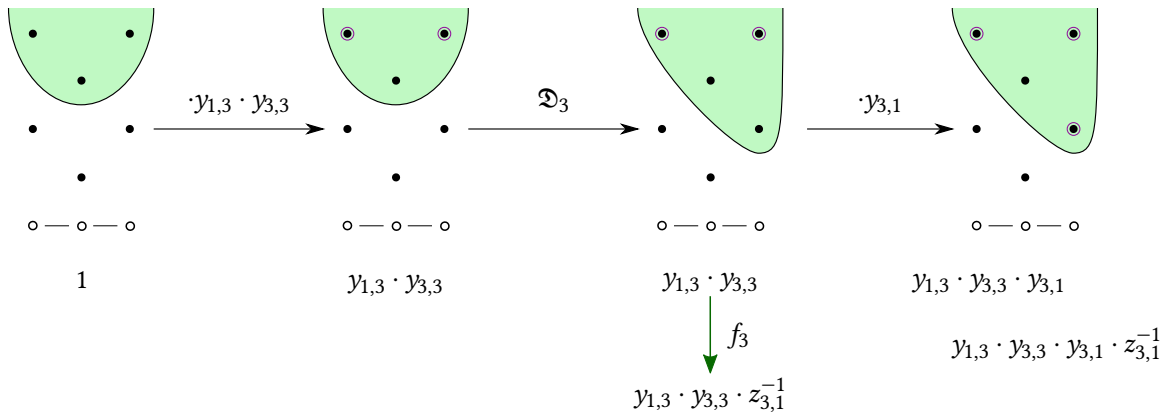
Using only Lemma 7.1.5 and Lemma 7.1.7, we may already compute decompositions of small product monomial crystals quite quickly, as we will show in the following example.

7.1.10 Example

Consider the  $\Phi = \text{SL}_4$  crystal  $\mathcal{M}(\lambda, \mathbf{R})$  defined by  $\lambda = \omega_1 + 2\omega_3$  and  $\mathbf{R} = \{(1, 3), (3, 1), (3, 3)\}$ . Since  $\Phi$  is simply-connected, we may omit the  $e^\lambda$  terms from the monomial crystal, instead declaring that  $\text{wt}(y_{i,c}) = \omega_i$ , the  $i$ th fundamental weight. We will determine the isomorphism class of  $\mathcal{M}(\lambda, \mathbf{R})$  by calculating a suitable truncation  $\mathcal{M}(\lambda, \mathbf{R}, J_3)$ .

1. Begin with  $J_0 = \text{up}((2, 2))$  and  $\mathbf{R}_0 = \emptyset$ . By definition,  $\mathcal{M}(\mathbf{R}_0, J_0) = \{1\}$ .
2. Let  $J_1 = J_0$  and  $\mathbf{R}_1 = \{(1, 3), (3, 3)\}$ . Since  $\mathbf{R}_1 - \mathbf{R}_0$  is concentrated along  $\partial J_1$ , Lemma 7.1.7 applies and we get  $\mathcal{M}(\mathbf{R}_1, J_1) = y_{1,3} \cdot y_{3,3} \cdot \mathcal{M}(\mathbf{R}_0, J_0) = \{y_{1,3} \cdot y_{3,3}\}$ .
3. Let  $\mathbf{R}_2 = \mathbf{R}_1$  and  $J_2 = \text{up}((3, 1))$ . Since  $J_2$  differs from  $J_1$  by adding the minimal element  $(3, 1)$ , Lemma 7.1.5 applies and we get  $\mathcal{M}(\mathbf{R}_2, J_2) = \mathfrak{D}_3 \mathcal{M}(\mathbf{R}_1, J_1)$ . Computing this extension of 3-strings is easy since  $y_{1,3} \cdot y_{3,3}$  is the highest-weight element of a crystal isomorphic to  $\mathcal{B}(\omega_1 + \omega_3)$ , so we get a 3-string with two elements:  $y_{1,3} \cdot y_{3,3} \xrightarrow{f_3} y_{1,3} \cdot y_{3,3} \cdot z_{3,1}^{-1}$ . We find that  $\mathcal{M}(\mathbf{R}_2, J_2) = \{y_{1,3} \cdot y_{3,3}, y_{1,3} \cdot y_{3,3} \cdot z_{3,1}^{-1}\}$ .
4. Let  $\mathbf{R}_3 = \mathbf{R}_2 \cup \{(3, 1)\}$  and  $J_3 = J_2$ . Since  $(3, 1) \in \partial J_3$ , we apply Lemma 7.1.7 again to find that  $\mathcal{M}(\mathbf{R}_3, J_3) = y_{3,1} \cdot \mathcal{M}(\mathbf{R}_2, J_2) = \{y_{1,3} \cdot y_{3,3} \cdot y_{3,1}, y_{1,3} \cdot y_{3,3} \cdot y_{3,1} \cdot z_{3,1}^{-1}\}$ .

We can represent this process graphically by drawing a monomial diagram, with each truncating set  $J_i$  overlaid, with the elements of the multiset  $\mathbf{R}_i$  shown by circling points. Beneath each diagram, we draw the elements of the subset  $\mathcal{M}(\mathbf{R}_i, J_i)$ .



The two elements of  $\mathcal{M}(\mathbf{R}_3, J_3) = \mathcal{M}(\mathbf{R}, J_3)$  are both highest-weight, of weights  $\omega_1 + 2\omega_3$  and  $\omega_1 + \omega_2$  respectively. Since the truncation  $\mathcal{M}(\mathbf{R}, J_3)$  contains all highest-weight elements of  $\mathcal{M}(\mathbf{R}, J)$ , we have

$$\mathcal{M}(\lambda, \mathbf{R}) \cong \mathcal{B}(\omega_1 + 2\omega_3) \oplus \mathcal{B}(\omega_1 + \omega_2). \tag{7.1.11}$$

7.2 A Demazure character formula

At this point the reader should re-familiarise themselves with the content of Section 5.3, specifically the string property (Definition 5.3.1) and what it means for the equivariance of the extension-of-strings operator  $\mathfrak{D}_i$  with the Demazure operator  $\pi_i$  (Lemma 5.3.11). As should be clear from Example 7.1.10 above, we can build characters of truncations by starting from the subset  $Z = \{1\}$  containing the trivial monomial, and repeatedly applying steps of the form

1.  $Z \mapsto p \cdot Z$ , for some dominant monomial  $p = e^\lambda \cdot y_{\mathbf{R}}$ , or
2.  $Z \mapsto \mathfrak{D}_i(Z)$ , where  $\mathfrak{D}_i$  is the extension-of-strings operator.

The first of these operations is straightforward on the level of characters, we have  $\text{ch}(e^\lambda \cdot \gamma_{\mathbf{R}} \cdot Z) = e^\lambda \text{ch} Z$ . For the second operation we would like to have the equivariance  $\text{ch} \mathfrak{D}_i(Z) = \pi_i(\text{ch} Z)$  where  $\pi_i$  is the Demazure operator, but in order for this to be true the subset  $Z$  must satisfy the string property. As we have remarked previously,  $Z$  satisfying the string property does *not* imply that  $\mathfrak{D}_i(Z)$  does. Furthermore, even if  $Z$  satisfies the string property, the product  $p \cdot Z$  may not. (For a counterexample, let  $\mathcal{M}(y_{1,1}) \cong \mathcal{B}(\omega_1)$  be the  $\text{SL}_2$ -crystal generated by the monomial  $y_{1,1}$ . Then  $y_{1,1} \cdot \mathcal{M}(y_{1,1})$  is a string with two elements inside  $\mathcal{M}(y_{1,1}^2) \cong \mathcal{B}(2\omega_1)$ , a violation of the string property).

We get around this problem by using the fact that all of our subsets  $Z$  have a specific form: they are truncations  $\mathcal{M}(\lambda, \mathbf{R}, J)$ . (The surprising fact here is Lemma 7.1.5, showing that the abstract extension-of-strings operation takes a truncation to another truncation, rather than some arbitrary subset of the product monomial crystal). We can show *directly* that each of our truncations has the string property.

### 7.2.1 Lemma (Truncations have the string property)

If  $J$  is an upward-closed set containing  $\mathbf{R}$ , then  $\mathcal{M}(\lambda, \mathbf{R}, J)$  has the string property.

### 7.2.2 Proof

Since  $\mathcal{M}(\mathbf{R}, J)$  is closed under the crystal raising operators  $e_i$ , it suffices to show that for any  $p \in \mathcal{M}(\mathbf{R}, J)$  such that  $f_i(p) \notin \mathcal{M}(\mathbf{R}, J) \sqcup \{\perp\}$  that  $e_i(p) = \perp$ . Suppose we have such a  $p = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1}$  with  $f_i(p) = e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S} + \{(i, k-2)\}}^{-1}$ , then by Lemma 7.1.5 we must have  $(i, k) \in \partial J$ . By the definition of  $\varphi_i$  we know that  $k$  is largest such that  $\varphi_i^k(p) = \varphi_i(p)$  and hence  $\varphi_i^{l+2}(p) < \varphi_i^k(p)$  for all  $l \geq k$ . But since  $p[i, r] = 0$  for all  $r < k$  we have that  $\varepsilon_i^l(p) = \varphi_i^{l+2}(p) - \varphi_i^k(p) < 0$  for all  $l \geq k$ , and hence  $\varepsilon_i(p) = 0$  and so  $e_i(p) = \perp$ .

We arrive at our first main result, an inductive character formula for any truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  by interpreting Lemmas 7.1.5 and 7.1.7 in terms of characters.

### 7.2.3 Theorem (A character formula for truncations)

The following rules give an inductive character formula for any truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$ :

1. If  $(\lambda, \emptyset)$  is a dominant pair, then  $\text{ch} \mathcal{M}(\lambda, \emptyset, J) = e^\lambda$  for any upward-closed set  $J$ .
2. If  $(\lambda, \mathbf{R})$  and  $(\mu, \mathbf{Q})$  are dominant pairs,  $J$  an upward-closed set containing  $\mathbf{R}$ , and  $\mathbf{Q}$  is contained in the boundary  $\partial J$ , then  $\text{ch} \mathcal{M}(\lambda + \mu, \mathbf{R} + \mathbf{Q}, J) = e^\mu \text{ch} \mathcal{M}(\lambda, \mathbf{R}, J)$ .
3. If  $(\lambda, \mathbf{R})$  is a dominant pair,  $J$  an upward-closed set containing  $\mathbf{R}$ , and  $(i, k) \notin J$  an element such that  $J \cup \{(i, k)\}$  is upward-closed, then  $\text{ch} \mathcal{M}(\lambda, \mathbf{R}, J \cup \{(i, k)\}) = \pi_i \text{ch} \mathcal{M}(\lambda, \mathbf{R}, J)$ .

An algorithm for applying the above rules to the data  $(\lambda, \mathbf{R}, J)$  defining a truncation is the following:

1. If  $\mathbf{R} = \emptyset$  then terminate with the result  $e^\lambda$ .
2. Otherwise, if there is an element of  $\mathbf{R}$  along the boundary  $\partial J$  then define  $\mathbf{Q}$  to be the multiset supported along  $\partial J$  with multiplicities  $\mathbf{Q}[i, c] = \mathbf{R}[i, c]$  for  $(i, c) \in \partial J$ , and choose a weight  $\lambda$  such that  $(\lambda, \mathbf{Q})$  is a dominant pair. Recursively compute  $e^\mu \text{ch} \mathcal{M}(\lambda - \mu, \mathbf{R} - \mathbf{Q}, J)$ .
3. Otherwise, choose a minimal element  $(i, k) \in J - \text{up}(\mathbf{R})$  satisfying  $(i, k) \in \text{down}(\mathbf{R})$  and recursively compute  $\pi_i \text{ch} \mathcal{M}(\lambda, \mathbf{R}, J - \{(i, k)\})$ .

If the Dynkin diagram  $I$  is connected, the third step of the algorithm can be simplified a little: any minimal point of  $J - \text{up}(\mathbf{R})$  will do.

We need to prove that each inductive rule is valid, and that the algorithm given terminates after finitely many steps without ‘getting stuck’ somewhere.

### 7.2.4 Proof

Each inductive step is true:

1. If  $(\lambda, \emptyset)$  is a dominant pair, then  $\mathcal{M}(\lambda, \emptyset) = \{e^\lambda\}$  which has character  $e^\lambda$ . Since  $\text{Supp}_\emptyset e^\lambda = \emptyset$ , every upward-closed set  $J$  is a valid truncation, with  $\text{ch} \mathcal{M}(\lambda, \emptyset, J) = \text{ch} \{e^\lambda\} = e^\lambda$ .
2. From Lemma 7.1.7 we have that  $\text{ch} \mathcal{M}(\lambda + \mu, \mathbf{R} + \mathbf{Q}, J) = \text{ch}(e^\mu \cdot z_{\mathbf{Q}}^{-1} \cdot \mathcal{M}(\lambda, \mathbf{R}, J)) = e^\mu \text{ch} \mathcal{M}(\lambda, \mathbf{R}, J)$ .



3. From Lemma 7.1.5 we have that  $\text{ch } \mathcal{M}(\lambda, \mathbf{R}, J \cup \{(i, k)\}) = \text{ch } \mathfrak{D}_i(\mathcal{M}(\lambda, \mathbf{R}, J))$ , which by the fact that each truncation satisfies the string property (Lemma 7.2.1) and the equivariance property of  $\mathfrak{D}_i$  and  $\pi_i$  (Lemma 5.3.11) gives that  $\text{ch } \mathcal{M}(\lambda, \mathbf{R}, J \cup \{(i, k)\}) = \pi_i(\mathcal{M}(\lambda, \mathbf{R}, J))$ .

It remains to be seen that the given algorithm terminates, which amounts to proving that step 3 can always progress to a point where step 2 can be applied, since we decrease the size of the finite multiset  $\mathbf{R}$  on each application of step 2.

Fix the parameters  $(\lambda, \mathbf{R}, J)$ , and suppose we have reached step 3 of the algorithm. Since we did not stop at step 1, the multiset  $\mathbf{R}$  is nonempty, and since we did nothing at step 2, we have  $\text{Supp } \mathbf{R} \cap \partial J = \emptyset$ . The upward-closed set  $J$  is proper, therefore there exists a minimal point  $(i, c) \in J$  such that  $(i, c) \in \text{down}(\mathbf{R})$ . This minimal point must be a boundary point, and hence lies outside of  $\text{Supp } \mathbf{R}$  so it may be removed. This decreases the size of the finite set  $\text{down}(\mathbf{R}) \cap J$ , completing the proof.

To make the theorem more concrete, we will return to a previous example and use the theorem to compute its character.

### 7.2.5 Example

We will use Theorem 7.2.3 to determine the character of the truncated crystal we computed previously in Example 7.1.10. The crystal we ended up with was obtained through applying multiplications and extension-of-strings operators:

$$\mathcal{M}(\mathbf{R}_3, J_3) = y_{3,1} \cdot \mathfrak{D}_3(y_{1,3} \cdot y_{3,3} \cdot 1). \quad (7.2.6)$$

Now that we know that each intermediate result satisfies the string property, we can take characters by replacing all the monomials by their weights, and all of the string extension operators by Demazure operators:

$$\text{ch } \mathcal{M}(\mathbf{R}_3, J_3) = e^{\omega_3} \pi_3(e^{\omega_1 + \omega_3}). \quad (7.2.7)$$

As explained in Example 3.4.16, the weight lattice  $X(\text{SL}_3)$  is isomorphic to  $\mathbb{Z}^3 / (1, 1, 1)$ , and hence we have an isomorphism of algebras  $\mathbb{Z}[X(\text{SL}_3)] \cong \mathbb{Z}[x_1, x_2, x_3] / (x_1 x_2 x_3 - 1)$  taking the weight  $\lambda = (a, b, c)$  to the element  $x_1^a x_2^b x_3^c$ . Together with  $\omega_1 = (1, 0, 0)$  and  $\omega_2 = (1, 1, 0)$ , the character formula becomes

$$\begin{aligned} \text{ch } \mathcal{M}(\mathbf{R}_3, J_3) &= x_1 x_2 x_3 \pi_3(x_1^2 x_2 x_3) \\ &= x_1^3 x_2^2 x_3 \pi_3(x_3) \\ &= x_1^3 x_2^2 x_3 (x_3 + x_4) \\ &= x_1^3 x_2^2 x_3^2 + x_1^2 x_2. \end{aligned} \quad (7.2.8)$$

We cannot yet say for sure what the isomorphism class of the truncation  $\mathcal{M}(\mathbf{R}_3, J_3)$  is from this character, since truncations are not necessarily a highest-weight crystal (and indeed this one is not). If we did know that the truncation was a Demazure crystal however, a fact we will later prove, we could decompose the character above in the basis of Demazure characters and find that it is  $\text{ch } D(\omega_1 + 2\omega_3) + \text{ch } D(\omega_1 + \omega_2)$ , as expected from the crystal computed in Example 7.1.10.

## 7.3 Truncations are Demazure crystals

As we have now pointed out several times, knowing the character of an abstract  $\Phi$ -crystal  $B$  is not very useful. If we happened to know that  $B$  was a crystal coming from the category  $\mathcal{O}_q^{\text{int}}(\Phi)$ , then its isomorphism class would be determined by the expression of  $\text{ch } B$  into the basis  $\{\text{ch } \mathcal{B}(\lambda) \mid \lambda \in X(\Phi)_+\}$ , as discussed in Section 5.1. If we knew that  $B$  was a disjoint union of Demazure subcrystals coming from crystals of  $\mathcal{O}_q^{\text{int}}(\Phi)$ , then its isomorphism class would be determined by decomposing its character into the basis of Demazure characters.

In this section we will show that every truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  is a disjoint union of Demazure crystals. The property of being a Demazure crystal is preserved by the extension-of-strings operations  $\mathfrak{D}_i$ , more or less by definition of a Demazure crystal (Definition 5.3.4). In light of our character formula (or rather the implicit computation of

the truncation which lies behind it), it suffices to show that the ‘multiplication along a boundary’ operation of Lemma 7.1.7 preserves the property of being a Demazure crystal. We begin by reformulating this ‘boundary multiplication’ in purely crystal-theoretic terms.

### 7.3.1 Lemma

Let  $\mathcal{M}(\lambda, \mathbf{R}, J)$  be a truncation, and  $(\mu, \mathbf{Q})$  a dominant pair such that  $\text{Supp } \mathbf{Q} \subseteq \partial J$ . There is a bijective, weight-preserving map

$$\Phi : \mathcal{M}(\lambda + \mu, \mathbf{R} + \mathbf{Q}, J) \rightarrow \mathcal{M}(\mathbf{R}, J) \otimes \mathcal{B}(\mu), \quad e^{\lambda + \mu} \cdot \gamma_{\mathbf{R} + \mathbf{Q}} \cdot z_{\mathbf{S}}^{-1} \mapsto e^{\lambda + \mu} \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1} \otimes b_{\mu}, \quad (7.3.2)$$

which is equivariant under the crystal raising operators  $e_i$  for all  $i \in I$ . Hence  $\mathcal{M}(\mathbf{R} + \mathbf{Q}, J) \cong \mathcal{M}(\mathbf{R}, J) \otimes \mathcal{B}(\mu)$  as upper-seminormal crystals.

### 7.3.3 Proof

The map is defined as a consequence of Lemma 7.1.7 and is bijective and weight-preserving, hence all that remains to be seen is the  $e_i$ -equivariance. Let  $q = e^{\mu} \cdot \gamma_{\mathbf{Q}}$ , fix an  $i \in I$  and a  $p = e^{\lambda} \cdot \gamma_{\mathbf{R}} \cdot z_{\mathbf{S}}^{-1} \in \mathcal{M}(\lambda, \mathbf{R}, J)$  so that  $\Phi(pq) = p \otimes b_{\mu}$ . The tensor product rule for applying  $e_i$  to a tensor product of two crystal elements gives

$$e_i(pq) = \begin{cases} e_i(p) \otimes b_{\mu} & \text{if } \varepsilon_i(p) > \langle \mu, \alpha_i^{\vee} \rangle, \\ \perp & \text{if } \varepsilon_i(p) \leq \langle \mu, \alpha_i^{\vee} \rangle. \end{cases} \quad (7.3.4)$$

Fix an  $i \in I$  and let  $(i, k) \in \partial J$ . We have  $(pq)[i, l] = p[i, l] + q[i, l]$  for all  $l$ , but  $q$  is concentrated at  $k$  and hence  $q[i, l] = \delta_{ik} \langle \mu, \alpha_i^{\vee} \rangle$ . Since  $(pq)[i, l] = 0$  for all  $l < k$  we then have

$$\varepsilon_i^l(pq) = \begin{cases} 0 & \text{for } l < k, \\ \varepsilon_i^l(p) - \langle \mu, \alpha_i^{\vee} \rangle & \text{for } l \geq k. \end{cases} \quad (7.3.5)$$

If  $e_i(pq) = \perp$  then  $\varepsilon_i(pq) = 0$  and hence  $\varepsilon_i^l(p) \leq \langle \mu, \alpha_i^{\vee} \rangle$  for all  $l \geq k$ . Therefore we are in the second case of Eq. (7.3.4) and hence  $e_i(p \otimes b_{\mu}) = \perp$  also.

If  $e_i(pq) = pqz_{i,l}$  then we must have  $l \geq k$  and also  $0 < \varepsilon_i(pq) = \varepsilon_i^l(pq)$ . Applying Eq. (7.3.5) gives that  $\langle \mu, \alpha_i^{\vee} \rangle < \varepsilon_i^l(p)$  and so  $e_i(p \otimes b_{\mu}) = e_i(p) \otimes b_{\mu}$ , so all we have remaining to check is that  $e_i(p) = pz_{i,l}$ . This is clear however, since

$$\varepsilon_i^l(p) = \begin{cases} 0 & \text{for } l < k, \\ \varepsilon_i^l(p) & \text{for } l \geq k, \end{cases} \quad (7.3.6)$$

and so the point  $l$  where  $\varepsilon_i^l(pq)$  first attains the positive value  $\varepsilon_i(pq)$  is the same as the point  $l$  where  $\varepsilon_i^l(p)$  first attains the positive value  $\varepsilon_i(p) = \varepsilon_i(pq) + \langle \mu, \alpha_i^{\vee} \rangle$ .

Repeated application of the raising operators followed by Lemma 7.3.1 leads to an embedding of  $\mathcal{M}(\lambda, \mathbf{R})$  into the tensor product  $\bigotimes_{(i,c) \in \mathbf{R}} \mathcal{B}(\omega_i)$ , as promised in Section 6.3.

We now appeal to the general fact that the tensor product of a Demazure module with a highest-weight vector is again a Demazure module. In crystal terms, this is the main result of [Jos03]: the tensor product  $X \otimes b_{\mu}$  of a Demazure crystal  $X$  with the highest-weight element  $b_{\mu} \in \mathcal{B}(\mu)$  is again a Demazure crystal.

### 7.3.7 Theorem (Truncations are Demazure crystals)

Let  $(\lambda, \mathbf{R})$  be a dominant pair, and  $J$  an upward-closed set containing  $\mathbf{R}$ . The truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  is a Demazure crystal.

### 7.3.8 Proof

For each dominant pair  $(\lambda, \mathbf{R})$  the set  $\{e^{\lambda} \cdot \gamma_{\mathbf{R}}\}$  is a Demazure crystal, isomorphic to  $\mathcal{B}_{\varepsilon}(\lambda)$ . By Proposition 3.2.3 of [Kas93], the property of being Demazure is preserved under extension of  $i$ -strings. By the main theorem of [Jos03] and Lemma 7.3.1, multiplication at the boundary  $\partial J$  of a truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  by a monomial of the form  $e^{\mu} \cdot \mathbf{Q}$  for  $\text{Supp } \mathbf{Q} \subseteq \partial J$  also preserves the property of being Demazure. Since every

| truncation can be built out of these operations ([Theorem 7.2.3](#)), every truncation is Demazure.

The upshot of this result is twofold. The fact that the truncations  $\mathcal{M}(\lambda, \mathbf{R}, J)$  are Demazure is quite interesting, since the truncations are defined straightforwardly, and globally in terms of the monomials (rather than in terms of any crystal-theoretic operations). The second is that our character formula is now useful for truncations, as it uniquely determines the isomorphism class of the truncation as a Demazure crystal. As a consequence of this, we obtain in finite type a character formula for the full crystal  $\mathcal{M}(\lambda, \mathbf{R})$  in terms of the character of any truncation of it.

**7.3.9 Corollary** (*A character formula in finite type*)

Let  $\Phi$  be a root datum of finite type  $(I, \cdot)$ ,  $w_I \in W_I$  the longest element of the Weyl group, and  $J \supseteq \text{Supp } \mathbf{R}$  any upward-closed set containing  $\mathbf{R}$ . Then a formula for the character of  $\mathcal{M}(\lambda, \mathbf{R})$  is

$$\text{ch } \mathcal{M}(\lambda, \mathbf{R}) = \pi_{w_I} \text{ch } \mathcal{M}(\lambda, \mathbf{R}, J). \quad (7.3.10)$$

**7.3.11 Proof**

As it is a disjoint union of highest-weight crystals, the character of the product monomial crystal is

$$\text{ch } \mathcal{M}(\lambda, \mathbf{R}) = \sum_{h \in H} \text{ch } \mathcal{B}(\text{wt } h), \quad (7.3.12)$$

where  $H \subseteq \mathcal{M}(\lambda, \mathbf{R})$  is the set of highest-weight elements. Since the truncation  $\mathcal{M}(\lambda, \mathbf{R}, J)$  has the same set of primitive elements  $H$  and is a Demazure crystal, there exists some function  $w : H \rightarrow W_I$  into the Weyl group such that

$$\mathcal{M}(\lambda, \mathbf{R}, J) \cong \bigoplus_{h \in H} \mathcal{B}_{w(h)}(\text{wt } h), \quad (7.3.13)$$

which on the level of characters gives the sum of Demazure characters

$$\text{ch } \mathcal{M}(\lambda, \mathbf{R}, J) = \sum_{h \in H} \pi_{w(h)}(e^{\text{wt } h}). \quad (7.3.14)$$

Since the Demazure operators give a zero-Hecke action ([Remark 5.2.10](#)) on the character ring  $\mathbb{Z}[X(\Phi)]$ , we have

$$\pi_{w_I} \text{ch } \mathcal{M}(\lambda, \mathbf{R}, J) = \sum_{h \in H} \pi_{w_I}(e^{\text{wt } h}), \quad (7.3.15)$$

which is precisely the character of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$ , since by the Demazure character formula [Theorem 5.2.5](#) we have  $\pi_{w_I}(e^{\text{wt } h}) = \text{ch } \mathcal{B}(\text{wt } h)$ .

We now have two of our main results: the general inductive formula [Theorem 7.2.3](#) for the character of a truncation, and the finite-type specialisation [Corollary 7.3.9](#) to the character of the whole crystal finite type. Where should we go from here? We still have not proven that the product monomial crystal is actually a crystal ([Theorem 6.3.5](#)), and that will be the purpose of the next chapter. After that we will move on to putting this character formula to good use, showing in type A that the product monomial crystal is the crystal of a particular module called a *generalised Schur module*.



## 8 Nakajima quiver varieties

Nakajima has defined varieties  $\mathfrak{M}(\lambda, \mu)$ , now called *Nakajima quiver varieties*, which *geometrise* the  $\mu$ -weight space of the integrable highest-weight representation  $L(\lambda)$ . By this we mean that there is symplectic structure on  $\mathfrak{M}(\lambda, \mu)$  and a Lagrangian subvariety  $\mathfrak{L}(\lambda, \mu) \subseteq \mathfrak{M}(\lambda, \mu)$  such that the top homology of the Lagrangian subvariety has the same dimension as  $L(\lambda)_\mu$ . The top homology (with complex coefficients) of the variety  $\mathfrak{L}(\lambda) := \bigcup_{\mu \leq \lambda} \mathfrak{L}(\lambda, \mu)$  possesses an action of the Kac-Moody algebra  $\mathfrak{g}(\Phi)$ , making the homology into a module isomorphic to  $L(\lambda)$ . Furthermore, the set  $\text{Irr } \mathfrak{L}(\lambda)$  of irreducible components can be given a crystal structure, isomorphic to the crystal  $\mathcal{B}(\lambda)$ .

The goal of this chapter is to give a relatively self-contained explanation of why the product monomial crystal, which was defined as a subset of the monomial crystal  $\mathcal{M}(\Phi)$ , is actually a subcrystal – this was [Theorem 6.3.5](#) whose proof we had deferred. The proof will use a subvariety of a quiver variety called a *graded* quiver variety, and a construction (again by Nakajima) of a  $\Phi$ -crystal on the set of connected components of the graded quiver variety. We will show that Nakajima’s crystal structure agrees with the analogue of the product monomial crystal inside the variation  $\mathcal{M}_c(\Phi)$  of the monomial crystal. The method of proof we use is not our own, instead following Section 7 of [[Kam+19a](#)].

One caveat to this proof is that Nakajima quiver varieties only work when the Cartan datum  $(I, \cdot)$  is *symmetric* and bipartite, meaning that out of the finite and affine type Cartan matrices this proof will only be valid for finite types A, D, and E, as well as their untwisted affinisations (excluding those  $A_n^{(1)}$  which are an odd cycle). Various computer experiments suggest that the product monomial crystal is *always* a genuine subcrystal of  $\mathcal{M}(\Phi)$ , and all of the results in this thesis still hold true in this generality, assuming one could somehow prove [Theorem 6.3.5](#) for any bipartite type  $(I, \cdot)$ .

### 8.1 Representations of quivers

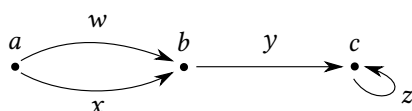
A *quiver* is a generalisation of a directed graph, which is allowed to have arbitrarily many doubled edges and self-loops. The quivers we deal with will be finite in both the number of vertices and the number of edges, however we give the general definition below.

#### 8.1.1 Definition (Quiver)

A *quiver* is a quadruple  $Q = (I, E, \text{tail}, \text{head})$ , where

1.  $I$  is a set, called the *vertex set*,
2.  $E$  is a set, called the *edge set*,
3.  $\text{tail} : E \rightarrow I$  is a function, giving the *tail* vertex of each edge, and
4.  $\text{head} : E \rightarrow I$  is a function, giving the *head* vertex of each edge.

Quivers are drawn in the same way as directed graphs, where vertices are represented by dots, and the edge  $e \in I$  is represented as an arrow from the tail of  $e$  to the head of  $e$ , i.e.  $\text{tail}(e) \xrightarrow{e} \text{head}(e)$ . For example, below is a quiver on the vertex set  $I = \{a, b, c\}$  and the edge set  $E = \{w, x, y, z\}$ .



A *representation* of a quiver is an assignment of a vector space to each vertex, and a linear map to each arrow from the tail vector space to the head vector space.

### 8.1.2 Definition (Quiver representation)

A *representation*  $(V, \varphi)$  of the quiver  $Q = (I, E, \text{tail}, \text{head})$  over the field  $\mathbb{k}$  is the data of:

1. For each vertex  $i \in I$  a  $\mathbb{k}$ -vector space  $V_i$ , and
2. for each edge  $e = (i \xrightarrow{e} j) \in E$  a  $\mathbb{k}$ -linear map  $\varphi_e : V_i \rightarrow V_j$ .

Let  $(V, \varphi)$  and  $(W, \psi)$  be two  $\mathbb{k}$ -representations of the same quiver  $Q$ . A *morphism* of quiver representations  $T : (V, \varphi) \rightarrow (W, \psi)$  is a collection  $T = (T_i)_{i \in I}$  of  $\mathbb{k}$ -linear maps  $T_i : V_i \rightarrow W_i$  such that for each edge  $e = (i \xrightarrow{e} j) \in E$  in the quiver, the obvious square  $T_j \varphi_e = \psi_e T_i$  commutes.

Let  $Q$  be a quiver and  $\mathbb{k}$  a field. Define  $\text{Rep}_{\mathbb{k}} Q$  to be the category whose objects are representations of the quiver  $Q$  over the field  $\mathbb{k}$ , and whose morphisms are morphisms of quiver representations.

It is a pleasant exercise with the axioms to show that the category  $\text{Rep}_{\mathbb{k}} Q$  is an abelian category. An alternative way of seeing this fact is to find an associative algebra  $\mathbb{k}Q$  (commonly called the *path algebra*) whose representation category is equivalent to  $\text{Rep}_{\mathbb{k}} Q$ .

## 8.2 Moduli spaces of quiver representations

Fix a quiver  $Q = (I, E)$  and an  $I$ -graded vector space  $V = \bigoplus_{i \in I} V_i$ . Let  $\text{Rep}(Q, V)$  denote the set of representations of  $Q$  with underlying vector spaces  $V$ . This is simply a direct product of morphism spaces:

$$\text{Rep}(Q, V) = \{(\varphi_e)_{e \in E} \mid e = (i \xrightarrow{e} j) \in E \text{ and } \varphi_e \in \text{Hom}_{\mathbb{k}}(V_i, V_j)\}. \quad (8.2.1)$$

Hence  $\text{Rep}(Q, V)$  is a  $\mathbb{k}$ -vector space. (The vector space structure of  $\text{Rep}(Q, V)$  is not so easy to see from the  $\mathbb{k}Q$ -module perspective). Furthermore, it carries a  $G_V = \prod_{i \in I} \text{GL}(V_i)$ -action by base-change, where the action on the component  $\varphi_e$  for  $(i \xrightarrow{e} j)$  is given by  $g \cdot \varphi_e = g_j \varphi_e g_i^{-1}$ . The subgroup  $\mathbb{k}^\times \hookrightarrow G_V$  of diagonally embedded scalars acts trivially on  $\text{Rep}(Q, V)$ .

### 8.2.2 Example

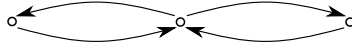
Consider the quiver  $Q$  consisting of two vertices  $I = \{1, 2\}$  and a single edge  $1 \rightarrow 2$ . Suppose we choose the  $I$ -graded vector space  $V$  with  $V_1 = \mathbb{C}$  and  $V_2 = \mathbb{C}^2$ . Then the space  $\text{Rep}(Q, V) \cong \mathbb{A}_{\mathbb{C}}^2$  since we may identify a map  $\varphi_{1 \rightarrow 2} : \mathbb{C} \rightarrow \mathbb{C}^2$  with the column vector  $\varphi_{1 \rightarrow 2}(1) = \begin{pmatrix} x \\ y \end{pmatrix}$ . The group  $G_V = \text{GL}_1 \times \text{GL}_2$  acts by

$$(g_1, g_2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = g_2 \begin{pmatrix} x \\ y \end{pmatrix} g_1^{-1}. \quad (8.2.3)$$

### 8.3 Nakajima quiver varieties

In this section we will go over the basics of Nakajima’s construction of the quiver variety  $\mathfrak{M}(\lambda)$  for any dominant weight  $\lambda$ .

Let  $(I, \cdot)$  be a symmetric Cartan datum – for our purposes, we will also assume that it is bipartite and hence admits a 2-colouring  $\zeta : I \rightarrow \{0, 1\}$ . Define a quiver by taking  $I$  as the vertex set, and adding  $|i \cdot j|$  oriented edges between  $i$  and  $j$ , pointing from the odd vertex to the even vertex: call the collection of all these edges  $\Omega$ . Similarly, add another  $|i \cdot j|$  edges between  $i$  and  $j$ , this time pointing from the even vertex to the odd vertex: call the collection of all these edges  $\bar{\Omega}$ . Then  $Q = (I, \Omega \sqcup \bar{\Omega})$  is a quiver. For example, if  $(I, \cdot)$  is  $A_3$ , the resulting quiver  $Q$  looks like

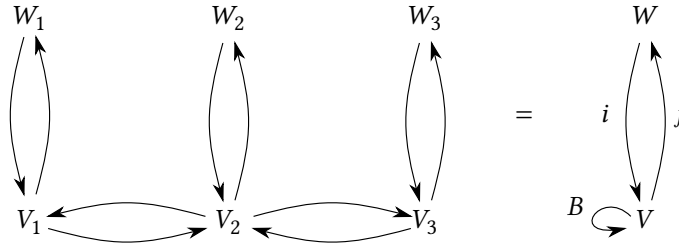


Fix a root datum  $\Phi$  of type  $(I, \cdot)$ , and a pair of weights  $\lambda, \mu$  such that  $\lambda$  is dominant and  $\mu \leq \lambda$ . In order to construct the quiver variety  $\mathfrak{M}(\lambda, \mu)$  we choose  $I$ -graded vector spaces  $W$  and  $V$  which will ‘represent’ the weights  $\lambda$  and  $\mu$ . The *framing space*  $W$  is any  $I$ -graded vector space such that  $\langle \lambda, \alpha_i^\vee \rangle = \dim W_i$  for all  $i \in I$ , while the other vector space  $V$  is any  $I$ -graded vector space such that  $\lambda - \mu = \sum_{i \in I} (\dim V_i) \alpha_i$ .

Now that we have our fixed choice of  $W$  and  $V$ , define the large vector space

$$\mathbf{M}(V, W) = \text{Rep}_{\mathbb{C}}(Q, V) \oplus \text{Hom}_{\mathbb{C}I}(W, V) \oplus \text{Hom}_{\mathbb{C}I}(V, W). \tag{8.3.1}$$

This space can be thought of as a kind of  $\text{Rep}_{\mathbb{C}}(Q^\heartsuit, V, W)$  for a *framed quiver*  $Q^\heartsuit$ . The notation used in [Nak01b] is  $(B, i, j) \in \mathbf{M}(V, W)$  for a typical element, so  $B : V \rightarrow V$  are the horizontal maps,  $i : W \rightarrow V$  goes downwards, and  $j : V \rightarrow W$  goes upwards. In the  $A_3$  example the framed quiver looks like this, with a ‘summary schematic’ on the right.



Since we really want to keep to Nakajima’s notation here, in this chapter only we will use  $i, j$  to mean these maps of quivers, and switch to using the letters  $k, l \in I$  to index vertices of the Dynkin diagram.

The reductive group  $G_V = \prod_{k \in I} \text{GL}(V_k)$  acts on the left of  $\mathfrak{M}(V, W)$  by base change automorphisms:  $g \cdot (B, i, j) = (gBg^{-1}, gi, jg^{-1})$ . Choosing any function  $\epsilon : \Omega \rightarrow \mathbb{C}^\times$  satisfying  $\epsilon(e) + \epsilon(\bar{e}) = 0$  defines a symplectic form on the vector space  $\mathbf{M}(V, W)$  by

$$\omega((B, i, j), (B', i', j')) = \text{tr}((\epsilon B)B') + \text{tr}(ij' - i'j), \tag{8.3.2}$$

where  $\epsilon B$  means to scale each map  $B_e : V_{\text{tail}(e)} \rightarrow V_{\text{head}(e)}$  by the scalar  $\epsilon(e)$ . The action of  $G_V$  preserves the symplectic form:

$$\begin{aligned} \omega(g \cdot (B, i, j), g \cdot (B', i', j')) &= \omega((gBg^{-1}, gi, jg^{-1}), (gB'g^{-1}, gi', j'g^{-1})) \\ &= \text{tr}(g(\epsilon B)B'g^{-1}) + \text{tr}(gij'g^{-1} - gi'jg^{-1}) \\ &= \omega((B, i, j), (B', i', j')). \end{aligned} \tag{8.3.3}$$

The moment map (which we stress is *nonlinear*) associated to the action of  $G_V$  is defined up to an additive constant. The moment map  $\mu$  which vanishes at the origin is given by

$$\mu : \mathbf{M}(V, W) \rightarrow (\text{Lie } G_V)^*, \quad \mu(B, i, j) = (\epsilon B)B + ij, \tag{8.3.4}$$

where  $\text{Lie } G_V$  has been identified with its dual via the trace pairing, so that for  $C \in \text{Lie } G_V = \prod_{k \in I} \text{End}_{\mathbb{C}}(V_k)$  we have

$$\mu(B, i, j)(C) = \text{tr}(C((\epsilon B)B + ij)). \quad (8.3.5)$$

The preimage  $\mu^{-1}(0)$  of the moment map is an affine algebraic variety, not necessarily reduced.

The first kind of quiver variety is the *categorical quotient*

$$\mathfrak{M}_0(V, W) = \mu^{-1}(0) // G_V = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^{G_V}. \quad (8.3.6)$$

This is an affine algebraic variety, whose underlying set is identified with the set of closed  $G_V$ -orbits in  $\mu^{-1}(0)$ .

A point  $(B, i, j) \in \mu^{-1}(0)$  is *stable* if the only  $B$ -invariant  $I$ -graded subspace  $S \subseteq V$  contained in  $\ker j$  is 0. We write  $\mu^{-1}(0)^s$  for the set of stable points, which is a  $G_V$ -invariant set. Define the second kind of quiver variety as the GIT (geometric invariant theory) quotient

$$\mathfrak{M}(V, W) = \mu^{-1}(0)^s / G_V. \quad (8.3.7)$$

The underlying set of points of the GIT quotient is the quotient of the  $G_V$ -set  $\mu^{-1}(0)^s$  of stable points by the group  $G_V$ . The  $G_V$  action is free, and  $\mathfrak{M}(V, W)$  is a smooth projective variety inheriting a symplectic form from  $\mathbf{M}(V, W)$ . There is a projective morphism

$$\pi : \mathfrak{M}(V, W) \rightarrow \mathfrak{M}_0(V, W), \quad (8.3.8)$$

sending the equivalence class  $[B, i, j]$  to that unique closed orbit contained in the orbit closure  $\overline{G_V \cdot (B, i, j)}$ .

### 8.3.9 Example (Nakajima quiver varieties in type $A_1$ )

In type  $A_1$ , the Coxeter graph consists of a single vertex, and hence the framed quiver  $Q^\heartsuit$  has only two vertices. Let  $W$  be the framing vector space and  $V$  the one corresponding to the unique vertex of the original quiver  $Q$ . The linear space simplifies since  $Q$  contains no edges:

$$\mathbf{M}(V, W) = \text{Hom}_{\mathbb{C}}(W, V) \oplus \text{Hom}_{\mathbb{C}}(V, W). \quad (8.3.10)$$

The  $G_V = \text{GL}(V)$ -action on the point  $(i, j) \in \mathfrak{M}(V, W)$  is by  $g \cdot (i, j) = (gi, jg^{-1})$ , with the moment map  $\mu(i, j) = ij \in \text{End}_{\mathbb{C}}(V)$ . The condition  $(i, j) \in \mu^{-1}(0)$  is precisely  $ij = 0 \in \text{End}_{\mathbb{C}}(V)$ , or in other words  $\text{im } j \subseteq \ker i$ . The stability condition is equivalent to requiring that  $j$  is injective, and hence we have

$$\mu^{-1}(0) = \{(i : W \rightarrow V, j : V \rightarrow W) \mid j \text{ is injective, and } ij = 0\}. \quad (8.3.11)$$

Let  $\text{Gr}(V, W) = \{U \subseteq W \mid U \cong V\}$  be the Grassmannian of  $V$ -planes in  $W$ . Alternatively, we can think of  $\text{Gr}(V, W)$  as the quotient of the space  $\text{Hom}_{\mathbb{C}}(V, W)^\circ$  of injective linear maps by the right action of  $G_V$ , identifying all injective maps with the same image. Fix a splitting  $W = V \oplus V'$ , then any point  $U \in \text{Gr}(V, W)$  may be written as the image of the map  $\begin{pmatrix} j_V \\ \sigma_U \end{pmatrix} : V \rightarrow V \oplus V' = W$ , where  $j_V : V \hookrightarrow W$  is the inclusion and  $\sigma_U : V \rightarrow V'$ . This defines an affine open neighbourhood of the point  $V \in \text{Gr}(V, W)$  isomorphic to  $\text{Hom}(V, V')$ . Correspondingly, the cotangent space at the point  $V$  is the affine space  $\text{Hom}(V', V) \cong \{j \in \text{Hom}(W, V) \mid j(V) = 0\}$ . The cotangent bundle of the Grassmannian is precisely the quotient of  $\mu^{-1}(0)$  by the action of  $G_V$ , hence we have

$$\mu^{-1}(0)/G_V = \mathfrak{M}(V, W) \cong T^* \text{Gr}(V, W). \quad (8.3.12)$$

We find that  $\mathfrak{M}(V, W)$  is nonempty if and only if  $\dim V \leq \dim W$ , and that if  $\dim V = 0$  or  $\dim V = \dim W$  then  $\mathfrak{M}(V, W)$  is a point.

Define the *Nakajima quiver variety* associated to  $W$  to be  $\mathfrak{M}(W) = \bigsqcup_V \mathfrak{M}(W, V)$ , where we let  $V$  vary over a fixed set of vector spaces, one for each possible  $I$ -graded dimension. As we have already seen in [Example 8.3.9](#), there will typically be many  $V$  for which  $\mathfrak{M}(W, V)$  is empty. It is important to remember that whenever a  $V$  appears



in the discussion of the variety  $\mathfrak{M}(W)$ , we are implicitly working inside the piece  $\mathfrak{M}(W, V)$ . This is obvious once pointed out, but might trip up the casual reader otherwise.

Lastly, recall that the  $I$ -graded dimensions of  $W$  and  $V$  were determined by a dominant weight  $\lambda$  and a weight  $\mu \leq \lambda$ . We define  $\mathfrak{M}(\lambda, \mu) = \mathfrak{M}(W, V)$  and  $\mathfrak{M}(\lambda) = \mathfrak{M}(W)$  when we prefer to speak of weights rather than graded vector spaces.

## 8.4 Vector bundles on quiver varieties

For a given pair  $(W, V)$ , both  $W$  and  $V$  can be considered as left  $G_V$  representations, where  $V$  is the defining representation of  $G_V$ , and  $W$  is a trivial representation. As the action of  $G_V$  on the set  $\mu^{-1}(0)^s$  of stable points is free, the projection map  $\mu^{-1}(0)^s \rightarrow \mathfrak{M}(W, V)$  is a left principal  $G_V$ -bundle. We may perform the associated bundle construction (Chapter 4.5 of [Hus94]) yielding two  $I$ -graded vector bundles  $\mathcal{V}$  and  $\mathcal{W}$  over  $\mathfrak{M}(W, V)$ .

We have vector bundles  $\text{Rep}(Q, \mathcal{V})$ ,  $\text{Hom}_{\text{CI}}(\mathcal{W}, \mathcal{V})$  and  $\text{Hom}_{\text{CI}}(\mathcal{V}, \mathcal{W})$  analogously as before.

## 8.5 Graded quiver varieties

In [Nak01b], Nakajima constructs representations of tensor products  $L(\lambda_1) \otimes \cdots \otimes L(\lambda_n)$  by considering a suitable subvariety inside  $\mathfrak{M}(\lambda_1 + \cdots + \lambda_n)$ , defined by picking a grading on the framing space  $W$  which separates it into pieces of ‘sizes’  $\lambda_1, \dots, \lambda_n$ . Later in the same paper (Section 8), a more general grading is considered, defining a subvariety which we will call a *graded Nakajima quiver variety*. A crystal structure is defined on the connected components of this graded quiver variety, which we will show is isomorphic to the product monomial crystal.

We will now consider  $W$  to be not just an  $I$ -graded vector space, but an  $(I, \mathbb{Z})$ -bigraded vector space, meaning that we put a  $\mathbb{Z}$ -grading on each  $I$ -homogeneous piece  $W_k$ . We will continue to use lower indices for the  $I$ -grading, and start using upper indices for the  $\mathbb{Z}$ -grading, so that  $W_k^p$  is the subspace of the  $I$ -graded piece  $W_k$  which has  $\mathbb{Z}$ -grading  $p$ . Up to conjugacy in  $\text{GL}_I(W)$ , the extra data of such a  $\mathbb{Z}$ -grading is determined by a finite multiset  $\mathbf{Q}$  based in  $I \times \mathbb{Z}$ , where  $\dim W_i^{p-1} = \mathbf{Q}[i, p]$  (the indexing shift  $p-1$  we use will make the morphism to the varied monomial crystal  $\mathcal{M}_c(\Phi)$  more pleasant later on). Let  $\rho_{\mathbf{Q}} : \mathbb{C}^\times \rightarrow \text{GL}_I(W)$  be a morphism of algebraic groups giving the grading  $\mathbf{Q}$  (again,  $\rho_{\mathbf{Q}}$  is determined by  $\mathbf{Q}$  only up to conjugacy, so we fix any such morphism for the remainder of this section).

From this point we are closely following Section 8 of [Nak01b], albeit choosing notation we find more clear. Using this extra  $\mathbb{Z}$ -grading on  $W$ , we define a  $\mathbb{C}^\times$ -action on  $\mathbf{M}(W, V)$  by the formulas

$$t \diamond_{\mathbf{Q}} B_e = \begin{cases} B_e & \text{if } e \in \Omega, \\ tB_e & \text{if } e \in \bar{\Omega}, \end{cases} \quad t \diamond_{\mathbf{Q}} i = i\rho_{\mathbf{Q}}(t)^{-1}, \quad t \diamond_{\mathbf{Q}} j = t\rho_{\mathbf{Q}}(t)j, \quad (8.5.1)$$

which together define an action  $t \diamond_{\mathbf{Q}}(B, i, j)$ . The  $\diamond_{\mathbf{Q}}$  action preserves the set  $\mu^{-1}(0)^s$  of stable points and commutes with the  $G_V$ -action, and hence descends to an action on the quiver variety  $\mathfrak{M}(W)$ . The *graded quiver variety* is the set of fixed points  $\mathfrak{M}(W)^{\mathbf{Q}} \subseteq \mathfrak{M}(W)$  under the  $\diamond_{\mathbf{Q}}$  action, and we will soon see how to equip its set of connected components with a crystal structure.

Consider a fixed point  $[B, i, j] \in \mathfrak{M}(W, V)^{\mathbf{Q}}$ . For each representative  $(B, i, j) \in \mathbf{M}(W, V)$  of the fixed point  $[B, i, j]$ , there exists a unique map  $\rho : \mathbb{C}^\times \rightarrow G_V$  of algebraic groups satisfying

$$t \diamond_{\mathbf{Q}}(B, i, j) = \rho^{-1}(t) \cdot (B, i, j). \quad (8.5.2)$$

The conjugacy class of  $\rho$  is independent of the choice of representative  $(B, i, j)$ , and hence we obtain a map

$$F_{\mathbf{Q}} : \mathfrak{M}(W, V)^{\mathbf{Q}} \rightarrow \text{Hom}_{\text{AlgGrp}}(\mathbb{C}^\times, G_V)/G_V, \quad (8.5.3)$$

taking a fixed point  $[B, i, j]$  to the conjugacy class  $[\rho]$ . It turns out that the fibres of this map are precisely the connected components of the graded quiver variety  $\mathfrak{M}(W, V)^{\mathcal{Q}}$ , and so we have a map

$$\mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{C}^\times, G_V)/G_V \rightarrow \pi_0(\mathfrak{M}(W, V)^{\mathcal{Q}}), \quad [\rho] \mapsto F_{\mathcal{Q}}^{-1}([\rho]). \quad (8.5.4)$$

Technically, in order to construct a crystal structure on the set of connected components, one should consider a larger Lagrangian subvariety containing the graded quiver variety, of points whose  $t \rightarrow \infty$   $\diamond_{\mathcal{Q}}$ -limit lands inside the graded quiver variety. This gives ‘enough information’ to determine where the crystal operators should go. However, it turns out that all of this information is implied by the conjugacy class  $[\rho]$  indexing a connected component, and so we will not mention these attracting sets further.

The crystal structure on the set of connected components is spelled out in Proposition 8.3 and Lemma 8.4 of [Nak01b]. We will write it down here in compatible notation. Our main change is replacing a conjugacy class  $[\rho] \in \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{C}^\times, G_V)/G_V$  by a finite multiset  $\mathbf{T}_\rho : I \times \mathbb{Z} \rightarrow \mathbb{N}$  based in  $I \times \mathbb{Z}$ , noting that the conjugacy class of  $\rho$  in  $G_V$  is entirely determined by the dimensions  $\mathbf{T}_\rho[i, p] = \dim V_i^p$ , where the  $\mathbb{Z}$ -grading on  $V$  is coming from  $\rho$ . Therefore we have that each connected component of the graded quiver variety  $\mathfrak{M}(W, V)^{\mathcal{Q}}$  is indexed by a finite multiset  $\mathbf{T}$ .

Over the connected component  $F_{\mathcal{Q}}^{-1}(\mathbf{T})$  indexed by the multiset  $\mathbf{T}$ , there is a complex of  $(I, \mathbb{Z})$ -graded vector bundles, whose  $(k, p)$ -graded component is

$$C_k^p(\mathbf{T}) : V_k^p \rightarrow W_k^{p-1} \bigoplus_{(k \xrightarrow{e} l) \in \Omega} V_l^{p-1} \bigoplus_{(k \xrightarrow{e} l) \in \bar{\Omega}} V_l^p \rightarrow V_k^{p-1}, \quad (8.5.5)$$

where the middle term is in homological degree zero (recall that the dimensions of the  $V_k^p$  appearing in the complex above are determined by  $\mathbf{T}$ , via  $\mathbf{T}[i, p] = \dim V_i^p$ ). The *rank* of this complex (alternating sum of dimensions) features in the definition of the crystal structure, so we compute it explicitly here. Recall that our partition  $\Omega \sqcup \bar{\Omega}$  was determined by a two-colouring  $\zeta : I \rightarrow \{0, 1\}$ , where we declared that  $\Omega$  contains edges going from odd to even vertices, while  $\bar{\Omega}$  goes from even to odd vertices, where ‘odd’ or ‘even’ is given by the two-colouring  $\zeta$ . This means that for each vertex, one of the two summations in Eq. (8.5.5) is zero, depending on the parity of the vertex. For even vertices, only the  $V_l^p$  term contributes, while for odd vertices only the  $V_l^{p-1}$  term contributes. Together with our contrived choice of indexing shift  $\dim W_k^p = \mathcal{Q}[k, p + 1]$  we get

$$\mathrm{rank} C_k^p(\mathbf{T}) = \mathcal{Q}[k, p] - \mathbf{T}[k, p] - \mathbf{T}[k, p - 1] - \sum_{l \neq k} a_{lk} \mathbf{T}[l, p - \zeta(l)]. \quad (8.5.6)$$

We can now give the crystal structure on the set  $\pi_0(\mathfrak{M}(W)^{\mathcal{Q}})$  of connected components of the graded Nakajima quiver variety.

### 8.5.7 Definition (Crystal structure on the graded quiver variety)

This is Lemma 8.4 of [Nak01b]<sup>1</sup>. Let  $\Phi$  be a root datum of symmetric bipartite type  $(I, \cdot)$ ,  $\lambda \in X(\Phi)_+$  a dominant weight,  $W$  an  $I$ -graded vector space with  $\dim W_i = \langle \lambda, \alpha_i^\vee \rangle$ , and  $\mathcal{Q}$  a multiset giving the  $(I, \mathbb{Z})$ -bigrading on  $W$  via  $\dim W_i^{p-1} = \mathcal{Q}[i, p]$ . Over the connected component  $F_{\mathcal{Q}}^{-1}(\mathbf{T}) \subseteq \mathfrak{M}(W, V)^{\mathcal{Q}} \subseteq \mathfrak{M}(W)^{\mathcal{Q}}$ , define

$$\begin{aligned} \varepsilon_k^p(\mathbf{T}) &= - \sum_{q > p} \mathrm{rank} C_k^q(\mathbf{T}), & \varphi_k^p(\mathbf{T}) &= \sum_{q \leq p} \mathrm{rank} C_k^q(\mathbf{T}), \\ \varepsilon_k(\mathbf{T}) &= \max_{p \in \mathbb{Z}} \varepsilon_k^p(\mathbf{T}), & \varphi_k(\mathbf{T}) &= \max_{p \in \mathbb{Z}} \varphi_k^p(\mathbf{T}), \end{aligned} \quad (8.5.8)$$

and let

$$n_{e,k}(\mathbf{T}) = \max \{q \mid \varepsilon_k^q(\mathbf{T}) = \varepsilon_k(\mathbf{T})\}, \quad n_{f,k}(\mathbf{T}) = \min \{q \mid \varphi_k^q(\mathbf{T}) = \varphi_k(\mathbf{T})\}. \quad (8.5.9)$$

Define the weight on the connected component indexed by  $\mathbf{T}$  to be

$$\begin{aligned} \text{wt}(\mathbf{T}) &= \lambda - \sum_{k \in I} (\dim V_k) \alpha_k \\ &= \lambda - \sum_{k \in I} \sum_{p \in \mathbb{Z}} \mathbf{T}[k, p] \alpha_k. \end{aligned} \quad (8.5.10)$$

Define the crystal operators by

$$e_k(\mathbf{T}) = \begin{cases} 0 & \text{if } \varepsilon_k(\mathbf{T}) = 0, \\ \mathbf{T} - \{(k, n_{e,k}(\mathbf{T}))\} & \text{if } \varepsilon_k(\mathbf{T}) > 0, \end{cases} \quad f_k(\mathbf{T}) = \begin{cases} 0 & \text{if } \varphi(\mathbf{T}) = 0, \\ \mathbf{T} + \{(k, n_{f,k}(\mathbf{T}))\} & \text{if } \varphi_k(\mathbf{T}) > 0. \end{cases} \quad (8.5.11)$$

Then  $(\text{wt}, \varepsilon_k, \varphi_k, e_k, f_k)$  gives the set of connected components  $\pi_0(\mathfrak{M}(\lambda)^{\mathcal{Q}})$  the structure of a seminormal crystal.

We can give an embedding of the crystal  $\pi_0(\mathfrak{M}(\lambda)^{\mathcal{Q}})$  into the varied monomial crystal  $\mathcal{M}_c(\Phi)$  of Definition 6.2.1 with parameters  $c_{kl} = \zeta(k)$ . Recall the auxiliary monomial from Definition 6.2.1:

$$a_{k,p} = e^{\alpha_k} \cdot y_{k,p} \cdot y_{k,p+1} \cdot \prod_{l \neq k} y_{l,p+c_{lk}}^{a_{lk}}. \quad (8.5.12)$$

Extracting the exponent of  $y_{k,p}$  from the monomial  $e^\lambda \cdot y_{\mathcal{Q}} \cdot a_{\mathbf{T}}^{-1}$  we get

$$(e^\lambda \cdot y_{\mathcal{Q}} \cdot a_{\mathbf{T}}^{-1})[k, p] = \mathcal{Q}[k, p] - \mathbf{T}[k, p] - \mathbf{T}[k, p-1] - \sum_{l \neq k} a_{lk} \mathbf{T}[p - c_{lk}], \quad (8.5.13)$$

precisely the same expression as  $\text{rank } C_k^p(\mathbf{T})$  in Eq. (8.5.6) since  $c_{lk} = \zeta(l)$ . Define the map of sets

$$\phi : \pi_0(\mathfrak{M}(W)^{\mathcal{Q}}) \rightarrow \mathcal{M}_c(\Phi), \quad F_{\mathcal{Q}}^{-1}(\mathbf{T}) \mapsto e^\lambda \cdot y_{\mathcal{R}} \cdot z_{\mathbf{T}}^{-1}. \quad (8.5.14)$$

It is straightforward to verify that  $\phi$  is a strict inclusion of crystals, by directly comparing Definitions 6.2.1 and 8.5.7 using Eq. (8.5.13) to convert between monomials and multisets.

We now wish to show that the image of  $\phi$  is the product monomial crystal  $\mathcal{M}_c(\lambda, \mathcal{Q})$ , by which we mean the analogue of the product monomial crystal  $\mathcal{M}(\lambda, \mathcal{Q})$ , but defined inside the varied monomial crystal  $\mathcal{M}_c(\Phi)$  rather than the monomial crystal  $\mathcal{M}(\Phi)$ .

### 8.5.15 Theorem

Let  $\Phi$  be a root datum of symmetric, bipartite, acyclic Cartan type  $(I, \cdot)$ . Then the image of the map  $\phi : \pi_0(\mathfrak{M}(W)^{\mathcal{Q}}) \rightarrow \mathcal{M}_c(\Phi)$  is the varied product monomial crystal  $\mathcal{M}_c(\lambda, \mathcal{Q})$ , where

$$\mathcal{M}_c(\lambda, \mathcal{Q}) = e^{\lambda_0} \cdot \prod_{(i,c) \in \text{Supp } \mathcal{Q}} \mathcal{M}(e^{\lambda_{i,c}} \cdot y_{i,c}^{\mathbf{R}[i,c]}), \quad (8.5.16)$$

and  $\lambda = \lambda_0 + \sum_{(i,c) \in \mathbf{R}} \lambda_{i,c}$  is any decomposition of  $\lambda$  such that  $\langle \alpha_i^\vee, \lambda_{i,c} \rangle = \mathbf{R}[i, c]$  for all  $(i, c) \in \mathbf{R}$ .

The proof follows the approach outlined in Proposition 7.7 of [Kam+19a].

### 8.5.17 Proof

Firstly, we reason that this works for a monomial concentrated in a single point. When the multiset  $\mathcal{Q}$  is concentrated, i.e. it is of the form  $\mathcal{Q} = \{(k, p)^n\}$ , then the vector space  $W$  is concentrated in the term  $W_k^p$ , so it is concentrated over a single vertex with a  $\mathbb{C}^\times$ -action by a single weight. In this case a result<sup>2</sup> of Nakajima's (references in Section 8 of [Nak01b]) gives that the graded quiver variety (or rather, the set attracting to it)

<sup>1</sup>This is *almost* exactly what appears in [Nak01b], but we have switched the min and max appearing in the definitions of  $n_{e,k}$  and  $n_{f,k}$ . We believe that this is a typo in Nakajima's original paper, since the  $e_i$  and  $f_i$  crystal operators are not obviously partially inverse without this change.

is the usual Lagrangian variety  $\mathfrak{L}(\lambda) \subseteq \mathfrak{M}(\lambda)$ , and hence the crystal afforded by the quiver variety is the highest weight crystal  $\mathcal{B}(\lambda)$ . Therefore in this case we have  $\pi_0(\mathfrak{M}(W)^{\mathcal{Q}}) \cong \mathcal{B}(\lambda) \cong \mathcal{M}(e^\lambda \cdot \gamma_{\mathcal{Q}})$ , and the claim follows since  $\phi$  is a nonzero crystal morphism between two connected seminormal crystals, and hence must be an isomorphism.

Next, we consider the general case. Fix a factorisation of  $(\lambda, \mathbf{R})$  into dominant pairs

$$(\lambda_1, (k_1, p_1)^{n_1}), \dots, (\lambda_N, (k_N, p_N)^{n_N}). \quad (8.5.18)$$

An arbitrary monomial  $p$  of the product monomial crystal  $\mathcal{M}_c(\lambda, \mathbf{Q})$  has an according factorisation  $p = p_1 \cdots p_N$  into monomials coming from the fundamental subcrystals.

Let  $\mathbf{Q}_r = \{(k_r, p_r)^{n_r}\}$ , so that each monomial  $p_r$  is an element of the fundamental subcrystal  $\mathcal{M}_c(\lambda_r, \mathbf{Q}_r)$ , and let  $W = W[1] \oplus \cdots \oplus W[r]$  be a factorisation of  $W$  such that  $W[r]$  is concentrated in the  $(I, \mathbb{Z})$ -degree  $(k_r, p_r)$  with dimension  $n_r$ . Since the claim holds on fundamental subcrystals, for each  $r$  there is an isomorphism  $\phi[r] : \pi_0(\mathfrak{M}(W[r])^{\mathbf{Q}_r}) \xrightarrow{\sim} \mathcal{M}_c(\lambda_r, \mathbf{Q}_r)$  of crystals, and hence for each monomial  $p_r$  there exists some  $I$ -graded vector space  $V[r]$  and fixed point  $(B[r], i[r], j[r]) \in \mathcal{M}(W[r], V[r])^{\mathbf{Q}_r}$  mapping to  $p_r$  under  $\phi[r]$ . It is then straightforward to check that

$$(B[1] \oplus \cdots \oplus B[r], i[1] \oplus \cdots \oplus i[r], j[1] \oplus \cdots \oplus j[r]) \in \mathcal{M}(W, V[1] \oplus \cdots \oplus V[r]) \quad (8.5.19)$$

is a  $\diamond_{\mathbf{Q}}$ -fixed point mapping to  $p$  under  $\phi$ . This shows that  $\mathcal{M}_c(\lambda, \mathbf{Q}) \subseteq \text{im } \phi$ .

To get the opposite inclusion, consider a  $\mathbf{T}$  such that  $F_{\mathbf{Q}}^{-1}(\mathbf{T})$  is nonempty (and hence is a connected component of  $\mathfrak{M}(W)^{\mathbf{Q}}$ ). By Proposition 4.1.2 of [Nak01a] the variety  $F_{\mathbf{Q}}^{-1}(\mathbf{T})$  is homotopic to its projective subvariety  $F_{\mathbf{Q}}^{-1}(\mathbf{T}) \cap \mathfrak{L}(W)$ . The maximal torus  $T_W \subseteq \text{GL}_I(W)$  acts on this subvariety, and hence there exists a fixed point by Borel's theorem. Lemma 3.2 of [Nak01b] implies that such a fixed point can be decomposed as a sum of  $(B[r], i[r], j[r])$  as above, where each  $(B[r], i[r], j[r])$  is an element of  $\pi_0(\mathfrak{M}(W[r])^{\mathbf{Q}_r})$ , and hence  $\phi(F_{\mathbf{Q}}^{-1}(\mathbf{T})) \in \mathcal{M}_c(\lambda, \mathbf{Q})$ , showing that  $\text{im } \phi \subseteq \mathcal{M}_c(\lambda, \mathbf{Q})$ .

Having shown that  $\phi$  is a crystal isomorphism, it is immediate that the subset  $\mathcal{M}_c(\lambda, \mathbf{Q})$  is a seminormal abstract crystal of an  $\mathcal{O}_q^{\text{int}}(\Phi)$ -module, and together with the isomorphism  $\mathcal{M}_c(\Phi) \xrightarrow{\sim} \mathcal{M}(\Phi)$  given in Lemma 6.2.5 implies that the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is indeed a crystal: Theorem 6.3.5 is proven.

Finally, we also note that Theorem 8.5.15 together with the fact that  $\pi_0(\mathfrak{M}(W)^{\mathbf{Q}})$  is connected when  $\mathbf{Q}$  is concentrated in a single point imply that the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  is well-defined, no matter how the multiset  $\mathbf{R}$  is broken up into fundamental pairs, since the identity Eq. (6.3.4) is proven.

<sup>2</sup>The acyclic assumption on  $(I, \cdot)$  is necessary for this result to hold.

## 9 Generalised Schur modules

The irreducible polynomial representations of  $\mathrm{GL}_n(\mathbb{C})$  are parametrised by partitions  $\lambda$  with at most  $n$  rows, each simple module  $L(\lambda)$  having an explicit construction by applying a certain *Schur functor*  $\mathcal{S}_\lambda : \mathrm{Vect}_{\mathbb{C}} \rightarrow \mathrm{Vect}_{\mathbb{C}}$  to the defining representation of  $\mathrm{GL}_n$ . For this reason, the representation  $L(\lambda)$  is sometimes called a *Schur module*: the image of the defining representation under a Schur functor. There is a natural generalisation of the functor  $\mathcal{S}_\lambda$  to any *diagram*  $D$  of boxes in the plane, recovering  $\mathcal{S}_D = \mathcal{S}_\lambda$  in the case that  $D$  is a Young diagram of the partition  $\lambda$ . The image of the defining representation under such a functor  $\mathcal{S}_D$  is accordingly called a *generalised Schur module*.

Our eventual aim for these final two chapters is to show a correspondence between diagrams  $D$  and multisets  $\mathbf{R}$  such that the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\mathrm{GL}_n)$  is the crystal of a generalised Schur module  $\mathcal{S}_D(\mathbb{C}^n)$ . In order to do this we put two character formulae to work, the first being our own formula [Theorem 7.2.3](#) on the product monomial crystal side, and the second formula due Magyar, Reiner, and Shimozono on the generalised Schur module side. Unfortunately, this only gets us as far as the statement ‘they match for large enough  $n$ ’, and so we spend [Chapter 10](#) showing some stability properties of the monomial crystal (which are interesting in their own right) to deduce the result in general.

In this chapter we will first define generalised Specht modules, the symmetric group analogues of the generalised Schur modules, which can be used to give a quick definition of the Schur modules from which their stability properties are evident. We then introduce the generalised Schur functors  $\mathcal{S}_D$  which give a second definition of a generalised Schur module as the image of the defining representation under  $\mathcal{S}_D$ , and show that (in characteristic zero) these two different definitions of generalised Schur modules coincide. The Schur functors are special cases of *flagged* Schur functors, in a similar way to how the highest-weight modules  $L(\lambda)$  are special cases of Demazure modules  $L_w(\lambda)$ . The flagged Schur functors are more amenable to inductive analysis, and we give results due to Magyar, Reiner, and Shimozono about the characters of flagged Schur modules. Finally, we give a direct correspondence between diagrams  $D$  and multisets  $\mathbf{R}$  which we use to show that the characters of the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R})$  and the generalised Schur module  $\mathcal{S}_D(\mathbb{C}^n)$  coincide for *column-convex* diagrams  $D$ , provided that  $n$  is large enough.

### 9.1 Generalised Specht modules

Perhaps the quickest way of defining generalised Schur modules is via their analogue for representations of symmetric groups, the generalised *Specht* modules. We first recall some necessary combinatorics.

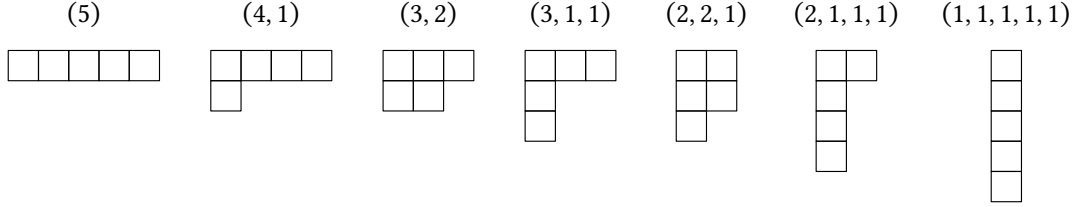
#### 9.1.1 Definition (Partitions)

A *partition* is a weakly decreasing finitely supported sequence  $\lambda : \mathbb{P} \rightarrow \mathbb{N}$ . A partition  $\lambda$  can be regarded as a finite list  $\lambda = (\lambda_1, \dots, \lambda_l)$  where  $\lambda_1 \geq \dots \geq \lambda_l > 0$  and  $\lambda_r = 0$  for  $r > l$ . Each element  $\lambda_i \geq 1$  is called a *part* of  $\lambda$ , the sum of the parts  $|\lambda| = \sum_{i \geq 1} \lambda_i$  is called the *size* of  $\lambda$ , and the size of the support  $\ell(\lambda) = l$  is called the *length* of  $\lambda$ . When  $\lambda$  has size  $n$  we say that  $\lambda$  is a *partition of  $n$* . The *empty partition* is the unique partition of 0 and is denoted by  $\emptyset$ . We define the sets

1. Part of all partitions,
2.  $\mathrm{Part}(\leq l)$  of all partitions of length at most  $l$ ,
3.  $\mathrm{Part}_n$  of all partitions of size  $n$ ,
4.  $\mathrm{Part}_n(\leq l) = \mathrm{Part}_n \cap \mathrm{Part}(\leq l)$  of all partitions of  $n$  with length at most  $l$ .

The first two sets are always infinite (besides the special case  $\text{Part}(\leq 0) = \{\emptyset\}$ ), while the last two sets are always finite.

It is common to represent a partition  $\lambda$  pictorially via its *Young diagram*  $D(\lambda) \subseteq \mathbb{P} \times \mathbb{P}$ , where  $D(\lambda)$  is the subset consisting of those  $(i, j)$  such that  $j \leq \lambda_i$ . The coordinates  $(i, j)$  are read similarly to matrices, with  $i$  increasing down the page and  $j$  increasing to the right. The Young diagrams of the partitions  $\lambda \in \text{Part}_5$  are pictured as follows:



The partitions  $\text{Part}_d$  of size  $d$  index the conjugacy classes in the symmetric group  $\mathfrak{S}_d$ , classifying a permutation by its sorted list of cycle lengths. By the general theory of representations of finite groups, the partitions of size  $d$  also index the set  $\{\Sigma_\lambda \mid \lambda \in \text{Part}_d\}$  of pairwise nonisomorphic irreducible representations of  $\mathbb{C}[\mathfrak{S}_d]$ . The constructions we will give of these modules  $\Sigma_\lambda$  are called *Specht modules*, and the reader can find a full account of this from the traditional perspective (which we will briefly go over here) in Chapter 7 of [Ful96]. We also mention that there is an alternative beautiful derivation of these modules, including a canonical decomposition into lines, due to Okounkov and Vershik [VO05] and explained well in Chapter I.2 of [Kle05].

In order to construct the Specht module  $\Sigma_\lambda$ , we first fix a *bijection tableau*  $T$  of shape  $\lambda$ , which is a bijective map  $T : D(\lambda) \rightarrow [d]$ . The symmetric group  $\mathfrak{S}_d$  has a free transitive action on the set of bijection tableaux by postcomposition,  $\sigma \cdot T = \sigma \circ T$ . The tableau  $T$  determines a *row stabilising* subgroup  $R_T \subseteq \mathfrak{S}_d$  of elements permuting the entries of  $T$  within their rows, and a *column stabilising* subgroup  $C_T$  of elements permuting the entries of  $T$  within their columns. For example, shown below is a bijection tableau  $T$  of  $\lambda = (3, 2)$  together with its row and column stabilising subgroups:

$$\begin{array}{c} T \\ \begin{array}{|c|c|c|} \hline 2 & 5 & 4 \\ \hline 1 & 3 & \\ \hline \end{array} \end{array} \quad R_T = \mathfrak{S}(\{2, 4, 5\}) \times \mathfrak{S}(\{1, 3\}) \quad C_T = \mathfrak{S}(\{1, 2\}) \times \mathfrak{S}(\{3, 5\}) \times \mathfrak{S}(\{4\})$$

Define the following elements of the group algebra  $\mathbb{C}[\mathfrak{S}_d]$ :

$$c_T = \sum_{\pi \in C_T} (-1)^\pi \pi, \quad r_T = \sum_{\sigma \in R_T} \sigma, \quad y_T = c_T r_T. \tag{9.1.2}$$

The element  $y_T$  is called the *Young symmetriser* associated to  $T$  and is a *pseudo-idempotent* of  $\mathbb{C}[\mathfrak{S}_d]$ , meaning that  $y_T^2 = zy_T$  for some nonzero scalar  $z \in \mathbb{C}^\times$ . The left submodule  $\Sigma_T = \mathbb{C}[\mathfrak{S}_d]y_T$  is called a *Specht module* associated to  $\lambda$ . Since  $R_{\sigma T} = \sigma R_T \sigma^{-1}$  and  $C_{\sigma T} = \sigma C_T \sigma^{-1}$ , we have  $y_{\sigma T} = \sigma y_T \sigma^{-1}$  and hence a different choice of bijection tableau  $\sigma T$  will yield an isomorphic Specht module, with the map

$$\Sigma_T \rightarrow \Sigma_{\sigma T}, \quad x \mapsto x\sigma^{-1} \tag{9.1.3}$$

giving an isomorphism of representations. Hence we can define the Specht module  $\Sigma_\lambda$  *up to isomorphism* to be any one of the  $\Sigma_T$  where  $T$  is a bijection tableau of shape  $\lambda$ .

**9.1.4 Theorem**

Fix a  $d \geq 1$ . The Specht modules  $\{\Sigma_\lambda \mid \lambda \in \text{Part}_d\}$  give a complete list of pairwise nonisomorphic irreducible representations of the symmetric group  $\mathfrak{S}_d$  over the complex numbers  $\mathbb{C}$ .

Everything we have done so far is completely classic. We will generalise this construction in one of the most naïve ways possible, by replacing the Young diagram  $D(\lambda) \subseteq \mathbb{P} \times \mathbb{P}$  with an arbitrary subset  $D \subseteq \mathbb{P} \times \mathbb{P}$  of size  $d$ ,

called a *diagram*. We can define bijective tableaux  $T : D \rightarrow [d]$  and the row and column stabilising subgroups  $R_T$  and  $C_T$  in the same way as before, and we end up with a *generalised Young symmetriser*  $y_T$ . This symmetriser is still a pseudo-idempotent of the group algebra, but the associated *generalised Specht module*  $\Sigma_D \cong \Sigma_T = \mathbb{C}[\mathfrak{S}_d]y_T$  is no longer irreducible in general. As the representation theory of  $\mathbb{C}[\mathfrak{S}_d]$  is semisimple, the generalised Specht module  $\Sigma_D$  decomposes as a direct sum of Specht modules.

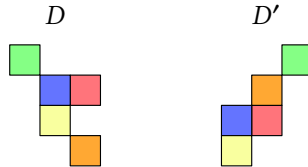
**9.1.5 Definition (Generalised Littlewood-Richardson coefficients)**

The multiplicity  $c_D^\lambda := [\Sigma_\lambda : \Sigma_D]$  of the irreducible module  $\Sigma_\lambda$  inside the generalised Specht module  $\Sigma_D$  is called a *generalised Littlewood-Richardson coefficient*. As the isomorphism class of  $\Sigma_D$  is invariant under row or column permutations of  $D$ , so are the coefficients  $c_D^\lambda$ .

We will justify the name *generalised Littlewood-Richardson coefficients*. A diagram is *skew* if it is equal to the difference  $D(\nu) \setminus D(\lambda)$  of two Young diagrams. By a theorem in Section 3 of [JP79], the multiplicity of the Specht module  $\Sigma_\mu$  in the generalised Specht module  $\Sigma_{D(\nu) \setminus D(\lambda)}$  is equal to the inner product  $\langle s_{\nu/\lambda}, s_\mu \rangle$  of symmetric functions, which is the Littlewood-Richardson coefficient  $c_{\lambda\mu}^\nu$  counting the number of Littlewood-Richardson tableaux of shape  $D(\nu) \setminus D(\lambda)$  and weight  $\mu$  (Section I.9 of [Mac95]). Hence the generalised Littlewood-Richardson coefficient  $c_{D(\nu) \setminus D(\lambda)}^\mu$  is equal to the Littlewood-Richardson coefficient  $c_{\lambda\mu}^\nu$ .

**9.1.6 Example**

Let  $D$  be the 5-box diagram  $D = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 3)\} \subseteq \mathbb{P} \times \mathbb{P}$ . By applying the row permutation (234) and the column permutation (132),  $D$  may be rearranged into a skew diagram  $D' = D(3, 2, 2, 1) \setminus D(2, 1)$ . We show this rearrangement pictorially, using colours to mark the original and final positions of boxes:



We can then state the multiplicities of Specht modules inside the generalised (skew) Specht module by

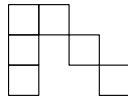
$$c_{D(3,2,2,1) \setminus D(2,1)}^\lambda = [\Sigma_\lambda : \Sigma_{D(3,2,2,1) \setminus D(2,1)}] = \langle s_{(3,2,2,1)/(2,1)}, s_\lambda \rangle, \tag{9.1.7}$$

which are easily calculated by any software package<sup>1</sup> dealing with symmetric functions or Littlewood-Richardson coefficients. In this case, we get

$$\Sigma_D \cong \Sigma_{D'} \cong \Sigma_{(2,1,1,1)} \oplus \Sigma_{(3,1,1)} \oplus \Sigma_{(2,2,1)}^{\oplus 2} \oplus \Sigma_{(3,2)}. \tag{9.1.8}$$

Hence the generalised Littlewood-Richardson coefficient  $c_D^{(2,2,1)} = 2$ .

The generalised Littlewood-Richardson coefficients are really a *strict* generalisation of Littlewood-Richardson coefficients, since not all diagrams can be made via row and column permutations into a skew shape. For example the following diagram



cannot be rearranged by column and row permutations to be skew. In addition to [JP79], the modules  $\Sigma_D$  have been studied in [Liu10], [Liu15], and various works of Reiner and Shimozono which we will cite in what follows.

<sup>1</sup>Or by hand, if it's a rainy day.

## 9.2 Generalised Schur modules

Let  $V$  be a finite-dimensional complex vector space, and  $\mathrm{GL}(V)$  the associated general linear group. The tensor power  $V^{\otimes d}$  of the defining representation is naturally a  $(\mathrm{GL}(V), \mathfrak{S}_d)$ -bimodule, where  $\mathrm{GL}(V)$  acts along the diagonal  $g \cdot (v_1 \otimes \cdots \otimes v_d) = gv_1 \otimes \cdots \otimes gv_d$  and  $\mathfrak{S}_d$  acts on the right by permuting tensor factors. For each diagram  $D$  we get a left  $\mathrm{GL}(V)$ -module  $\mathcal{S}_D(V) = V^{\otimes d} \otimes_{\mathbb{C}[\mathfrak{S}_d]} \Sigma_D$  called a *generalised Schur module*, where  $d = |D|$  is the number of boxes in the diagram  $D$ . When  $D = D(\lambda)$  is the Young diagram of a partition, the generalised Schur module  $\mathcal{S}_\lambda(V)$  is irreducible, and we call it a *Schur module*.

As a consequence of Schur-Weyl duality, the generalised Schur module  $\mathcal{S}_D(V)$  decomposes in terms of the irreducibles  $\mathcal{S}_\lambda(V)$  with the same decomposition multiplicities  $c_D^\lambda$  given by the generalised Littlewood-Richardson coefficients, where the sum is taken over the restricted set  $\mathrm{Part}_d(\leq \dim V)$  of partitions with length at most  $(\dim V)$ :

$$\mathcal{S}_D(V) \cong \bigoplus_{\ell(\lambda) \leq \dim V} c_D^\lambda \mathcal{S}_\lambda(V). \quad (9.2.1)$$

Note that if  $\ell(\lambda) > \dim V$  then  $\mathcal{S}_\lambda(V) = 0$ , so the above sum would still be valid without this restriction, but must be read more carefully. Eq. (9.2.1) has an important consequence for what we will call the *stability* of the generalised Littlewood-Richardson coefficients: although the decomposition of the generalised Schur module  $\mathcal{S}_D(V)$  depends on  $\dim V$ , the coefficients  $c_D^\lambda$  do not. Via an alternative definition of the generalised Schur module (Definition 9.2.5 and Lemma 9.2.7) and the Pieri rule, we can see that  $c_D^\lambda = 0$  whenever the length of  $\lambda$  is larger than the number of rows of  $D$ .

Eq. (9.2.1) gives the following interpretation of generalised Littlewood-Richardson coefficients, which should be read in two parts: firstly a *stability* part about the ideal coefficients  $c_D^\lambda$  being determined for  $\dim V$  large enough, and secondly a *restriction* part about how to apply those coefficients to unstable  $V$ .

### 9.2.2 Corollary (Generalised Littlewood-Richardson coefficients via Schur modules)

Let  $D$  be a diagram of  $d$  boxes, and  $V$  a vector space such that  $\dim V$  is at least the number of rows of  $D$ . Then the coefficients  $c_D^\lambda$  appearing in the decomposition  $\mathcal{S}_D(V) \cong \bigoplus_\lambda c_D^\lambda \mathcal{S}_\lambda(V)$  are the generalised Littlewood-Richardson coefficients of Definition 9.1.5. Furthermore, if  $U$  is any vector space, then the decomposition of  $\mathcal{S}_D(U)$  into irreducible modules is given by  $\mathcal{S}_D(U) \cong \bigoplus_{\ell(\lambda) \leq \dim U} c_D^\lambda \mathcal{S}_\lambda(U)$ .

Although the definition we have given above of generalised Schur modules is concise and naturally gives the stability result Corollary 9.2.2, it is quite difficult to use for more ‘hands-on’ work and does not lend itself to a straightforward filtration by smaller modules. We turn instead to the notion of a *Schur functor*.

The symmetric algebra  $S^*(V)$  and exterior algebra  $\Lambda^*(V)$  are graded-commutative Hopf algebras associated to  $V$ . Fixing a degree  $d \in \mathbb{N}$ , iterated comultiplication followed by taking the  $(1, \dots, 1)$ -graded piece gives maps into the tensor power  $V^{\otimes d}$ :

$$\begin{aligned} S^d(V) &\xrightarrow{\Delta} V^{\otimes d}, & v_1 \cdots v_d &\mapsto \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \\ \Lambda^d(V) &\xrightarrow{\Delta} V^{\otimes d}, & v_1 \cdots v_d &\mapsto \sum_{\sigma \in \mathfrak{S}_d} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}. \end{aligned} \quad (9.2.3)$$

Similarly, we have maps from the tensor power  $V^{\otimes d}$  to the degree- $d$  part of the symmetric and exterior algebras by taking iterated multiplication:

$$\begin{aligned} V^{\otimes d} &\xrightarrow{m} S^d(V), & v_1 \otimes \cdots \otimes v_d &\mapsto v_1 \cdots v_d, \\ V^{\otimes d} &\xrightarrow{m} \Lambda^d(V), & v_1 \otimes \cdots \otimes v_d &\mapsto v_1 \cdots v_d. \end{aligned} \quad (9.2.4)$$

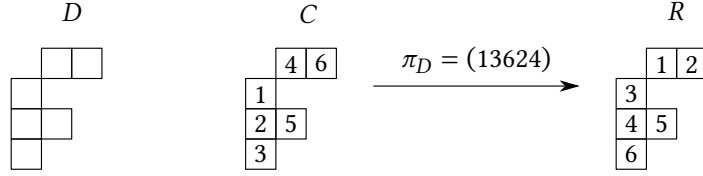
For a diagram  $D$ , let  $\mathrm{cols}(D) : \mathbb{P} \rightarrow \mathbb{N}$  and  $\mathrm{rows}(D) : \mathbb{P} \rightarrow \mathbb{N}$  be the finitely supported functions counting the number of boxes in each column or row. For a finitely supported function  $\alpha : \mathbb{P} \rightarrow \mathbb{N}$ , let  $S^\alpha(V)$  be the tensor product of symmetric powers

$$S^\alpha(V) = S^{\alpha(1)}(V) \otimes S^{\alpha(2)}(V) \otimes S^{\alpha(3)}(V) \otimes \cdots,$$



noting that  $S^0(V) \cong \mathbb{C}$  and hence this tensor product is isomorphic to a finite tensor product. We similarly define  $\Lambda^\alpha(V)$ , hence we get the spaces  $\Lambda^{\text{cols}(D)}(V)$  and  $S^{\text{rows}(D)}(V)$ .

There are two distinguished bijective tableaux associated to  $D$ , the *column ordered* tableau  $C$  corresponding to ordering columns from left-to-right, and from top-to-bottom within a column, and the *row ordered* tableau  $R$  corresponding to ordering rows from top-to-bottom, and left-to-right within a row. There is a unique permutation  $\pi_D$  such that  $\pi_D \circ C = R$ .



We may now define the Schur functor associated to  $D$ .

**9.2.5 Definition (Generalised Schur functor)**

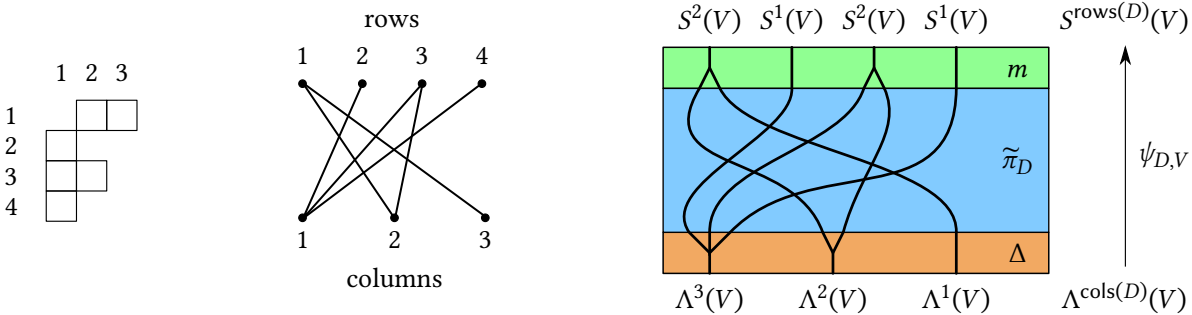
Let  $D \subseteq \mathbb{P} \times \mathbb{P}$  be a diagram of  $d$  boxes in the plane, with  $\text{cols}(D)$ ,  $\text{rows}(D)$ , and  $\pi_D$  as defined above. Given a vector space  $V$ , define the map  $\psi_{D,V}$  to be the composition

$$\psi_{D,V} : \Lambda^{\text{cols}(D)}(V) \xrightarrow{\Delta} V^{\otimes d} \xrightarrow{\pi_D} V^{\otimes d} \xrightarrow{m} S^{\text{rows}(D)}(V). \tag{9.2.6}$$

Define the *Schur functor* on a vector space by  $\mathcal{S}_D(V) = \text{im } \psi_{D,V}$ . For a map  $f : V \rightarrow W$ , let  $\mathcal{S}_D(f) : \mathcal{S}_D(V) \rightarrow \mathcal{S}_D(W)$  be the map obtained by restricting the natural map  $f^{\text{rows}(D)} : S^{\text{rows}(D)}(V) \rightarrow S^{\text{rows}(D)}(W)$  to  $\text{im } \psi_{D,V}$ .

It is straightforward to check that the image of  $\mathcal{S}_D(f)$  does indeed land in the subspaces  $\text{im } \psi_{D,V}$ , and hence  $\mathcal{S}_D$  defines an endofunctor in the category of complex vector spaces. Endofunctors of vector spaces naturally produce new representations from old representations: if  $\rho : G \rightarrow \text{End}_{\mathbb{C}}(V)$  is a representation of a group  $G$  on the vector space  $V$ , then we obtain a new representation of  $G$  on  $\mathcal{S}_D(V)$  by letting the group element  $g \in G$  act by  $\mathcal{S}_D(\rho(g))$ . Our second definition of a generalised Schur module of  $\text{GL}(V)$  is the image  $\mathcal{S}_D(V)$  of the defining representation under the Schur module. We note that column permutations of  $D$  do not affect the functor  $\mathcal{S}_D$  at all, while row permutations of  $D$  give functors isomorphic to  $\mathcal{S}_D$ .

We can also interpret Definition 9.2.5 in terms of a ‘wiring diagram’ defined by  $D$ . The data  $D$  is equivalent to a bipartite graph on  $\mathbb{P} \times \mathbb{P}$  (and the isomorphism class of  $\mathcal{S}_D$  depends only on the isomorphism class of this bipartite graph). Drawing all of the edges in this graph in an ordered fashion, we can then group edges together and treat the whole thing as a string diagram.



Above we have shown a diagram  $D$ , its associated bipartite graph embedded into the plane, and a string diagram (read from bottom-to-top). The junctions in the bottom orange part of the string diagram should be understood as iterated comultiplication in the exterior algebra followed by projection onto the  $(1, \dots, 1)$ -graded piece, as in Eq. (9.2.3). The junctions in the top green part of the diagram are the iterated multiplications  $V^{\otimes k} \rightarrow S^k(V)$  into the symmetric power, as in Eq. (9.2.4). The middle blue part of the diagram is a map  $\tilde{\pi}_D : V^{\otimes d} \rightarrow V^{\otimes d}$  permuting tensor factors according to the permutation  $\pi_D = (13624)$  defined before.

Now that we have given two different definitions of generalised Schur modules, we should show that they are equivalent. In this proof we will have to make use of the characteristic-zero assumption on the base field, a reflection of how semisimplicity of both categories is partly responsible for this equivalence.

**9.2.7 Lemma** (*Equivalence of definitions of generalised Schur modules*)

Let  $D$  be a diagram of size  $d$ , with  $C$  its column ordered tableau,  $R$  its row ordered tableau, and  $\pi_D \in \mathfrak{S}_d$  the unique permutation such that  $\pi_D \circ C = R$ . Recall that to the column-stabilising subgroup of  $C$  we associate the alternating sum over its elements  $c_C = \sum_{\sigma} (-1)^\sigma \sigma$ , and to the row-stabilising subgroup of  $R$  we associate the sum over its elements  $r_R = \sum_{\sigma} \sigma$ . Let  $\Gamma^k(V) \subseteq V^{\otimes d}$  denote the subspace of tensors which are symmetric under the action of  $\mathfrak{S}_k$ . Consider the following diagram:

$$\begin{array}{ccccc}
 \Lambda^{\text{cols}(D)}(V) & \xrightarrow{\Delta} & V^{\otimes d} & \xrightarrow{\pi_D} & V^{\otimes d} & \xrightarrow{m} & \mathcal{S}^{\text{rows}(D)}(V) \\
 & \nearrow & & & \searrow^{r_R} & & \downarrow \Delta \\
 & & V^{\otimes d}_{c_C} & & & & \Gamma^{\text{rows}(D)}(V)
 \end{array} \tag{9.2.8}$$

It is easy to check that the triangle on the right commutes, just by checking that the operation  $\Delta \circ m$  is equal to the action of  $r_R$ . The two maps on the left have the same image inside  $V^{\otimes d}$ , and since the vertical map  $\Delta$  is an isomorphism (a strictly characteristic-zero phenomenon) we have that the image  $\text{im } \psi_{D,V}$  of the top horizontal map is isomorphic to the image of the bottom composition  $V^{\otimes d}_{c_C} \pi_D r_R$ .

As subgroups of  $\mathfrak{S}_d$  we have  $\pi_D \text{RowStab}_R \pi_D^{-1} = \text{RowStab}_{\pi_D^{-1} \circ R} = \text{RowStab}_C$ , hence  $\pi_D r_R \pi_D^{-1} = r_C$ , showing that

$$V^{\otimes d}_{c_C} \pi_D r_R = V^{\otimes d}_{c_C r_C} \pi_D \cong V^{\otimes d}_{c_C r_C}, \tag{9.2.9}$$

where the last isomorphism is because right multiplication by a permutation is an isomorphism  $V^{\otimes d} \rightarrow V^{\otimes d}$ . Therefore we have that the image of  $\psi_{D,V}$  is isomorphic to  $V^{\otimes d}_{c_C r_C} = V^{\otimes d} \otimes_{[\mathfrak{S}_d]} \Sigma_C$  where  $\Sigma_C$  is the Specht module associated to the tableau  $C$  of the diagram  $D$ .

We briefly remark on how this setup can be extended to the positive characteristic case.

**9.2.10 Remark**

There is a dual notion of *Weyl functor*  $\mathcal{W}_D$ , defined as the image of the composition

$$\varphi_{D,V} : \Gamma^{\text{rows}(D)}(V) \xrightarrow{\Delta} V^{|D|} \xrightarrow{\pi_D^{-1}} V^{|D|} \xrightarrow{m} \Lambda^{\text{cols}(D)}(V), \tag{9.2.11}$$

where  $\Gamma^d(V)$  denotes the  $d$ th divided power algebra (see Appendix 2.4 of [Eis95] for a definition of the divided power algebra). On a morphism  $f : V \rightarrow W$ , the Weyl functor is the restriction of the natural map  $f^{\text{cols}(D)} : \Lambda^{\text{cols}(D)}(V) \rightarrow \Lambda^{\text{cols}(D)}(W)$  to the image of  $\varphi_{D,V}$ . The Schur and Weyl functors  $\mathcal{S}_D$  and  $\mathcal{W}_D$  make sense for modules over any commutative unital ring  $k$ , and behave well under base change (in fact, when  $D$  is a Young diagram or skew diagram, both functors are universally free [ABW82]).

For a partition  $\lambda$ , the Schur module  $\mathcal{S}_\lambda(V)$  is what we would call the *induced module* of  $\text{GL}(V)$  corresponding to the weight  $\lambda$ , isomorphic to the space of sections  $\Gamma(G/B, \mathcal{L}_\lambda)$ , while the Weyl module  $\mathcal{W}_\lambda(V)$  is its dual. When we say *dual*, we mean the composition of the normal dual with the anti-involution of  $G$  switching positive and negative roots – in the case of a module  $M$  over  $\text{GL}_n$ , this is the vector space  $\text{Hom}_k(M, k)$  with the action  $(g \cdot f)(v) = f(g^T v)$  where  $g^T$  is the transpose matrix.

In the characteristic-zero case the functors  $\mathcal{S}_D$  and  $\mathcal{W}_D$  are isomorphic, and furthermore both are isomorphic to the functor  $(-) \otimes_{[\mathfrak{S}_D]} \Sigma_D$  of tensoring with a generalised Specht module, however this is no longer true in the case of positive characteristic. A modern perspective for the study of the functors  $\mathcal{S}_D$  and  $\mathcal{W}_D$  is notion of *polynomial functors*.

### 9.3 Flagged Schur modules

The generalised Schur functors  $\mathcal{S}_D$  are not straightforward to study. The results of [JP79; ABW82], which give a basis of  $\mathcal{S}_D(V)$  when  $D$  is a Young diagram or skew shape, depend in an essential way on a partial ordering on semistandard tableaux. Such a partial ordering simply does not exist when  $D$  is an arbitrary diagram, so a different approach is required. The approach we use here will follow that of Magyar, Reiner, Shimozono and others, studying the Schur module  $\mathcal{S}_D(V)$  by certain  $B$ -stable quotients defined by the rows of the diagram  $D$ .

We now need to be clear about what we mean by a Borel subgroup  $B$ . For each  $n \geq 0$ , let  $\mathbb{C}^n$  be the full flag of quotient spaces  $\mathbb{C}^n = (\mathbb{C}^n \rightarrow \mathbb{C}^{n-1} \rightarrow \dots \rightarrow \mathbb{C})$ , where the map  $\mathbb{C}^i \rightarrow \mathbb{C}^{i-1}$  sends the coordinate vector  $e_i$  to zero. We say that a group element  $g \in \text{GL}(V)$  *preserves* a flag  $F$ , if  $g(F_i) \subseteq F_i$  for all  $i$ . Let  $B(\mathbb{C}^n) \subseteq \text{GL}(\mathbb{C}^n)$  be the subgroup preserving the coordinate flag  $\mathbb{C}^n$ , which is precisely the subgroup of upper-triangular matrices. Defining  $T(\mathbb{C}^n) \subseteq \text{GL}(\mathbb{C}^n)$  to be the subgroup of diagonal matrices, we get a pinning  $(T(\mathbb{C}^n) \subseteq B(\mathbb{C}^n) \subseteq \text{GL}(\mathbb{C}^n))$  of the reductive algebraic group  $\text{GL}(V)$ , a realisation of the root datum  $\text{GL}_n$  of type  $A_{n-1}$ .

#### 9.3.1 Definition (Flagged Schur module)

Let  $D$  be a diagram with  $d$  boxes fitting within the first  $r$  rows, meaning that  $D \subseteq [r] \times \mathbb{P}$ , and fix a full flag  $\mathbb{C}^r$  of quotient spaces. When row  $i$  of the diagram has  $\text{row}_i(D)$ -many boxes, we can apply the symmetric power functor  $S^{\text{row}_i(D)}$  to the surjection  $\mathbb{C}^r \rightarrow \mathbb{C}^i$  determined by the flag  $\mathbb{C}^r$  to get a surjection  $S^{\text{row}_i(D)}(\mathbb{C}^r) \rightarrow S^{\text{row}_i(D)}(\mathbb{C}^i)$ . Taking the tensor product of these and precomposing with the map  $\psi_{D, \mathbb{C}^r}$  defining the Schur module gives a map

$$\Lambda^{\text{cols}(D)}(\mathbb{C}^r) \xrightarrow{\psi_{D, \mathbb{C}^r}} S^{\text{rows}(D)}(\mathbb{C}^r) \xrightarrow{\phi_D} \bigotimes_{i \geq 1} S^{\text{row}_i(D)}(\mathbb{C}^i). \tag{9.3.2}$$

The image of this map is the *flagged Schur module*  $\mathcal{F}_D(\mathbb{C}^r)$ . It is naturally a  $B(\mathbb{C}^r)$ -module, but is no longer in general a  $\text{GL}(\mathbb{C}^r)$ -module.

Note that we have  $\mathcal{F}_D(\mathbb{C}^n) = \mathcal{F}_D(\mathbb{C}^r)$  for all  $n \geq r$ , and furthermore that the definition does not make sense for  $n < r$ . Therefore we can use the notation  $\mathcal{F}_D(\mathbb{C}^r) = \mathcal{F}(D)$  and leave it implied we are working inside some flag of length at least  $r$ .

This quotient is quite straightforward to reason about in terms of a basis of the target space  $S^{\text{rows}(D)}(\mathbb{C}^r)$  of the Schur functor map  $\psi_{D, \mathbb{C}^r}$ . As a tensor product of symmetric powers,  $S^{\text{rows}(D)}(\mathbb{C}^r)$  has a basis indexed by tableaux  $T : D \rightarrow [r]$  which are *row-semistandard*, meaning that the entries along each row weakly increase. In terms of this basis, the quotient  $S^{\text{rows}(D)}(\mathbb{C}^r) \rightarrow \bigotimes_{i \geq 1} S^{\text{row}_i(D)}(\mathbb{C}^i)$  simply kills any tableaux having an entry in row  $i$  which is larger than  $i$ .

#### 9.3.3 Example

Below are shown two row-semistandard tableaux  $T_1, T_2$  for a diagram  $D$ , and another row-semistandard tableau  $T'_3$  for a different diagram  $D'$ .

$T_1$	$T_2$	$T'_3$
$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \\ \hline 2 & 3 \\ \hline \end{array}$	$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{ c c } \hline 1 & 1 \\ \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{ c c } \hline & \\ \hline 1 & 1 \\ \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$
$e_{11} \otimes e_2 \otimes e_{23}$	$e_{11} \otimes e_3 \otimes e_{12}$	$e_{11} \otimes e_3 \otimes e_{12}$

The tableau  $T_1$  survives the quotient  $\phi_D$  because every entry of row  $i$  is at most  $i$ . The tableau  $T_2$  gets killed in the quotient  $\phi_D$  because there is a 3 in row 2. The third tableau  $T'_3$  survives the quotient  $\phi_{D'}$ , because by shifting  $T_2$  down a row we have removed the only problem. However,  $T'_3$  is a tableau for a different diagram to  $T_2$ , and they do not represent elements in the same flagged Schur module.

The example above shows that the isomorphism class of the flagged Schur module  $\mathcal{F}_D(\mathbb{C}^r)$  is *not* invariant under row permutations of the diagram  $D$ , in contrast to the Schur module  $\mathcal{S}_D(\mathbb{C}^r)$ . However, it is clear from the defini-

tions that a permutation of the columns of  $D$  leaves the flagged Schur module unchanged. For further properties of these modules (and their dual equivalents, the *flagged Weyl modules*), the reader can consult Sections 2 and 5 of [RS99].

Many of the results known about generalised Schur modules are due to geometric constructions of this module as sections of a line bundle over a (generally singular) variety, first explored in [Mag98a; Mag98b]. In this setting, the flagged Schur module (or the dual construction, the flagged Weyl module) naturally arise, and in [RS95; RS98] a Demazure-type character formula is given for the characters of the flagged Schur modules of *percentage-avoiding* diagrams  $D$ . The only diagrams we will encounter are *northwest* (or can be made northwest via a column permutation) which are automatically percentage-avoiding, and hence these results apply. (A diagram  $D$  is *northwest* if whenever  $(j, k), (i, l) \in D$  with  $(i < j)$  and  $(k < l)$ , then  $(i, k) \in D$ ).

From now on, we will restrict ourselves to diagrams which are *column-convex*, meaning that the columns have no gaps. A column-convex diagram satisfies the northwest property after a column permutation has been applied, and hence the results in the above paper will apply to our situation.

In order to use the results of Reiner and Shimozono to write down a character formula for the flagged Schur module  $\mathcal{F}_D$  for a column-convex diagram  $D$ , it will help to have a convenient way of encoding the data of a column-convex diagram  $D$ . We will encode a column-convex diagram  $D$  fitting within  $r$  rows as a sequence of  $r$  partitions.

**9.3.4 Definition** (*Partition sequences and diagrams*)

A *partition sequence* of length  $r \geq 0$  is a sequence  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$  of partitions, such that  $\ell(\lambda^{(i)}) \leq i$  (the  $i$ th partition has at most  $i$  rows). For each  $0 \leq i \leq r$ , let  $\underline{\lambda}^i = (\lambda^{(1)}, \dots, \lambda^{(i)})$  denote the *prefix* of  $\underline{\lambda}$  of length  $i$ .

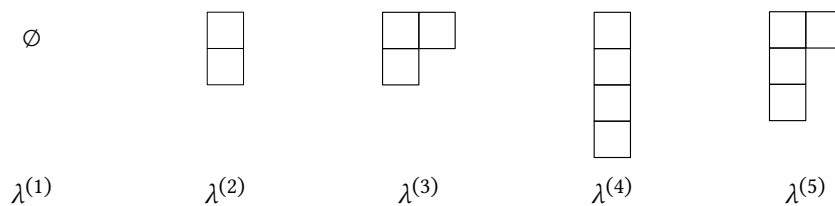
When  $\underline{\lambda}$  is a partition sequence of length  $r$ , we define the *associated diagram*  $D(\underline{\lambda})$  inductively by:

1. For  $i = 0$ ,  $D(\underline{\lambda}^0) = \emptyset$ , the empty diagram,
2. For  $i > 0$ ,  $D(\underline{\lambda}^i)$  is obtained by shifting the contents of the previous diagram  $D(\underline{\lambda}^{i-1})$  down one row, and placing the Young diagram of the partition  $\lambda^{(i)}$  to the right of the previous diagram, with the first row of  $\lambda^{(i)}$  in row 1.

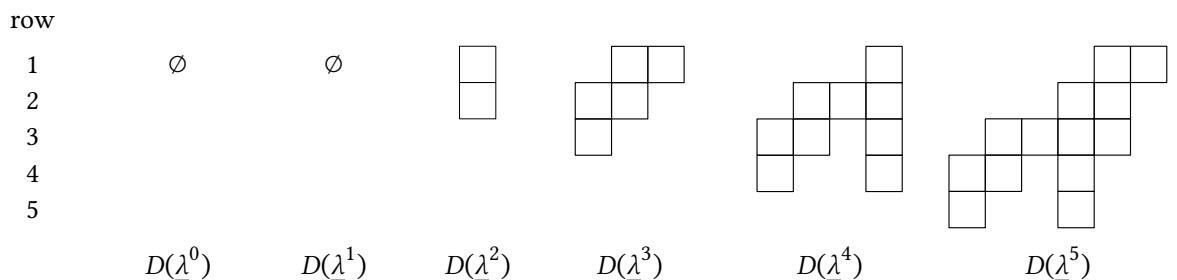
The diagram  $D(\underline{\lambda}^i)$  is always column-convex and contained within the rows  $\{1, \dots, i\}$ .

**9.3.5 Example**

Consider the partition sequence  $\underline{\lambda} = (\emptyset, (1, 1), (2, 1), (1, 1, 1, 1), (2, 1, 1))$ , which we could picture as the following sequence of Young diagrams:



The associated sequence of diagrams  $D(\underline{\lambda}^i)$  is shown below.



The above example makes it clear that the diagram  $D(\underline{\lambda})$  is always column-convex. Conversely, given any column-convex diagram  $D$  its columns may be sorted in such a way that there exists a partition sequence  $\underline{\lambda}$  with  $D(\underline{\lambda}) = D$ . This sorting can be done by placing columns with entries appearing higher to the right, and breaking ties by putting longer columns on the left.

In order to read the next lemma, we remind the reader that any integer vector  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$  can be treated as a weight of  $GL_n$ , the function  $e^\alpha : T(\mathbb{C}^n) \rightarrow \mathbb{C}^\times$  taking the torus element  $\text{diag}(t_1, \dots, t_n)$  to  $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ . (This is explained further in Section 9.4). Each integer partition  $\lambda = (\lambda_1, \dots)$  of length at most  $n$  defines a *dominant weight*  $e^\lambda$  of  $GL_n$ .

### 9.3.6 Lemma (Character of a flagged Schur module)

Let  $\underline{\lambda}$  be a partition sequence of length  $r$ . The characters of the flagged Schur modules  $\mathcal{F}(D(\underline{\lambda}^i))$  satisfy the following recurrence:

1. For  $i = 0$ ,  $\text{ch } \mathcal{F}(D(\underline{\lambda}^0)) = \text{ch } \mathcal{F}(\emptyset) = 1$ ,
2. For  $i > 0$ ,  $\text{ch } \mathcal{F}(D(\underline{\lambda}^i)) = e^{\lambda^{(i)}} \cdot \pi_1 \cdots \pi_{i-1} (\text{ch } \mathcal{F}(D(\underline{\lambda}^{i-1})))$ .

### 9.3.7 Proof

The case for  $i = 0$  is clear. The inductive case follows from Theorem 23 of [RS98], noting that moving the diagram  $D(\underline{\lambda}^{i-1})$  down one row can be done by applying the successive row permutations  $(i-1, i), \dots, (1, 2)$ , which corresponds to the application of Demazure operators  $\pi_{i-1}, \dots, \pi_1$ , and adding a Young diagram  $\lambda$  in the top row corresponds to multiplication by  $e^\lambda$ .

### 9.3.8 Example

Consider Example 9.3.5 above. By the recursion rule given in Lemma 9.3.6, the character of the flagged Demazure module  $\mathcal{F}(D(\underline{\lambda}))$  is

$$\text{ch } \mathcal{F}(D(\underline{\lambda})) = e^{(2,1,1)} \pi_1 \pi_2 \pi_3 \pi_4 (e^{(1,1,1,1)} \pi_1 \pi_2 \pi_3 (e^{(2,1)} \pi_1 \pi_2 (e^0 \pi_1 e^0))). \quad (9.3.9)$$

## 9.4 Polynomial characters of $GL_n$

In this section we will explain how the characters of flagged Schur modules fit into the more well-known framework of characters of Schur modules. We begin by fixing a choice of root datum of  $GL_n$ .

### 9.4.1 Definition

Fix an  $n \geq 1$ . The root datum  $\Phi = GL_n$  is the type  $(I, \cdot) = A_{n-1}$  root datum with the following presentation:

$$\begin{aligned} X(GL_n) &= \mathbb{Z} \{ \epsilon_1, \dots, \epsilon_n \}, & \alpha_i &= \epsilon_i - \epsilon_{i+1} \text{ for } 1 \leq i \leq n-1, \\ X^\vee(GL_n) &= \mathbb{Z} \{ \epsilon_1^\vee, \dots, \epsilon_n^\vee \}, & \alpha_i^\vee &= \epsilon_i^\vee - \epsilon_{i+1}^\vee \text{ for } 1 \leq i \leq n-1, \\ \langle \epsilon_i, \epsilon_j^\vee \rangle &= \delta_{ij}. \end{aligned} \quad (9.4.2)$$

A weight  $\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n$  is dominant iff  $\lambda_1 \geq \cdots \geq \lambda_n$ , and *polynomial* if  $\lambda_i \geq 0$  for all  $1 \leq i \leq n$ . There is an alternative basis of weights which we call *fundamental weights* by abuse of notation (as  $GL_n$  is not semisimple, it does not really have fundamental weights). These fundamental weights are  $\omega_i = \epsilon_1 + \cdots + \epsilon_i$  for  $1 \leq i \leq n$ , and we have  $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$  for  $1 \leq i, j \leq n-1$ , with  $\omega_n = \epsilon_1 + \cdots + \epsilon_n$  generating the null space  $X(GL_n)_0 = \bigcap_{i \in I} \ker \langle -, \alpha_i^\vee \rangle$ . A weight is *dominant polynomial* if and only if it expands positively in the basis of the  $\omega_i$ .

The polynomial weights of  $GL_n$  are indexed by *compositions* of  $n$ , meaning finite sequences  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ . A polynomial weight is dominant if and only if it is a partition. The category of representations with polynomial weights is closed under taking direct sums and tensor products, and furthermore to reach any non-polynomial representation one only needs to tensor with an appropriate negative power of the one-dimensional determinant representation  $L(\omega_1)$ . From here on, we restrict our attention only to polynomial weights, which we will denote by  $X(GL_n)^p$ .

There is an isomorphism of algebras  $\mathbb{Z}[X(\mathrm{GL}_n)^p] \cong \mathbb{Z}[x_1, \dots, x_n]$  between the monoid algebra of the polynomial weights and the polynomial ring in  $n$  variables, taking  $e^\lambda$  to  $x^\lambda := x_1^{\lambda_1} \dots x_n^{\lambda_n}$ . Furthermore, the Weyl group action on  $X(\mathrm{GL}_n)$  preserves the polynomial weights, and hence descends to the usual action of the symmetric group  $\mathfrak{S}_n$  on the polynomial ring  $\mathbb{Z}[x_1, \dots, x_n]$  by permuting coordinates. Since we have realised the polynomial ring and the ring of symmetric polynomials  $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$  as the character ring and Weyl-invariant character ring of a root datum, we will get interesting new bases of these rings by considering characters of Demazure modules and highest-weight modules respectively.

When  $\lambda$  is a partition (a dominant polynomial weight), the Schur module  $\mathcal{S}_\lambda(\mathbb{C}^n)$  gives an explicit construction of the highest-weight module  $L(\lambda)$ , and when the resulting characters are viewed inside  $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$  they are called *Schur polynomials* and usually denoted  $\mathrm{ch} L(\lambda) = s_\lambda(x_1, \dots, x_n)$ . The Schur polynomials form a basis for the ring of invariants  $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ , by a slight modification of the argument in Lemma 5.1.2 for polynomial weights. The Schur polynomials are extremely well-studied, and we could point the reader to several references [Ful96; Mac95; Sta97].

When  $\alpha$  is a composition (an arbitrary polynomial weight), the character of the Demazure module  $D(\alpha)$  with Demazure lowest weight  $\alpha$  is called a *key polynomial*, and usually denoted  $\mathrm{ch} D(\alpha) = \kappa_\alpha$ . Computing these key polynomials is quite straightforward. If  $\alpha$  is a partition then  $D(\alpha) = L_e(\alpha) = L(\alpha)_\alpha$  is the one-dimensional weight space of weight  $\alpha$  and hence  $\kappa_\alpha = x^\alpha$ . In terms of Demazure operators, we have that  $\kappa_\alpha = \pi_w(x^\lambda)$ , where  $\lambda$  is dominant and  $w \in W$  is the shortest permutation such that  $w\lambda = \alpha$ . This together with the zero-Hecke property (Remark 5.2.10) gives that

$$\pi_i(\kappa_\alpha) = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1}, \\ \kappa_\alpha & \text{otherwise.} \end{cases} \tag{9.4.3}$$

Finally, we can rewrite the Demazure operator  $\pi_i : \mathbb{Z}[x_1, \dots, x_n] \rightarrow \mathbb{Z}[x_1, \dots, x_n]$  using the fact that  $e^{\alpha_i} = \frac{x_i}{x_{i+1}}$ :

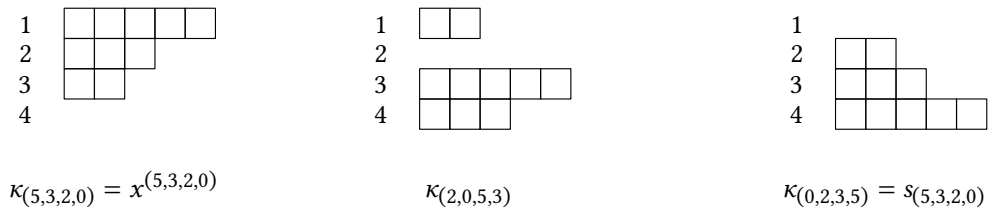
$$\pi_i(x^\lambda) = \frac{x_i x^\lambda - x_{i+1} x^{s_i\lambda}}{x_i - x_{i+1}}. \tag{9.4.4}$$

Some example of key polynomial computations are below:

$$\begin{aligned} \kappa_{(1,1,0)} &= x^{(1,1,0)} \\ \kappa_{(1,0,1)} &= \pi_2(\kappa_{(1,1,0)}) = x_1 \cdot (x_2 + x_3) = x^{(1,1,0)} + x^{(1,0,1)} \\ \kappa_{(0,1,1)} &= \pi_1(\kappa_{(1,0,1)}) = x^{(1,1,0)} + x^{(1,0,1)} + x^{(0,1,1)}. \end{aligned} \tag{9.4.5}$$

The last character  $\kappa_{(0,1,1)}$  is symmetric, and equal to the Schur polynomial  $s_{(1,1,0)}$ . This is a general fact: the Schur polynomial  $s_{(\lambda_1, \dots, \lambda_n)}$  in  $n$  variables is equal to the key polynomial  $\kappa_{(\lambda_n, \dots, \lambda_1)}$ .

It was shown in [RS95] that, similarly to how the highest-weight module  $L(\lambda)$  can be constructed as a Schur module  $\mathcal{S}_\lambda(\mathbb{C}^n)$  using the Young diagram associated to  $\lambda$ , the flagged Schur module  $\mathcal{F}(\alpha)$  has character  $\kappa_\alpha$ , where  $\alpha$  is treated as a left-justified diagram having  $\alpha_i$  boxes in row  $i$ . The next image shows three diagrams  $D$ , with the character of the flagged Schur module  $\mathcal{F}(D)$  written below.



We again see how Demazure modules interpolate between a single highest-weight space and the full space of a representation, and here they have a very concrete interpretation in terms of boxes in the plane. To the author's knowledge, it is still unknown whether  $\mathcal{F}(\alpha)$  is isomorphic to the Demazure module  $D(\alpha)$ , despite them both having the same character  $\kappa_\alpha$  (the character of a  $B$ -module does not determine its isomorphism class). There are

combinatorial interpretations of these diagrams as well, for example a Demazure crystal for  $D(\alpha)$  can be built using *key tableau*, which are certain fillings of the diagrams above by the letters  $1, \dots, n$ . For a good survey, one can read [AG19].

## 9.5 Type A truncations

Throughout this section, we work in type  $(I, \cdot) = A_n$  for  $n$  large enough. When considering a partition sequence of length  $r$ , we will need to be working in  $A_n$  for  $n \geq r$ . Since we will only be working with product monomial crystals whose data is supported over the first  $r$  nodes of the diagram, the particular choice of  $n$  does not matter for any of the statements we make, provided  $n \geq r$ .

### 9.5.1 Definition (Multisets and truncations associated to partition sequences)

Let  $\underline{\lambda}$  be a partition sequence of length  $r$ . Define the *associated multisets*  $\mathbf{R}(\underline{\lambda}^i)$  and the *associated upward-closed set*  $J_i$  inductively as follows:

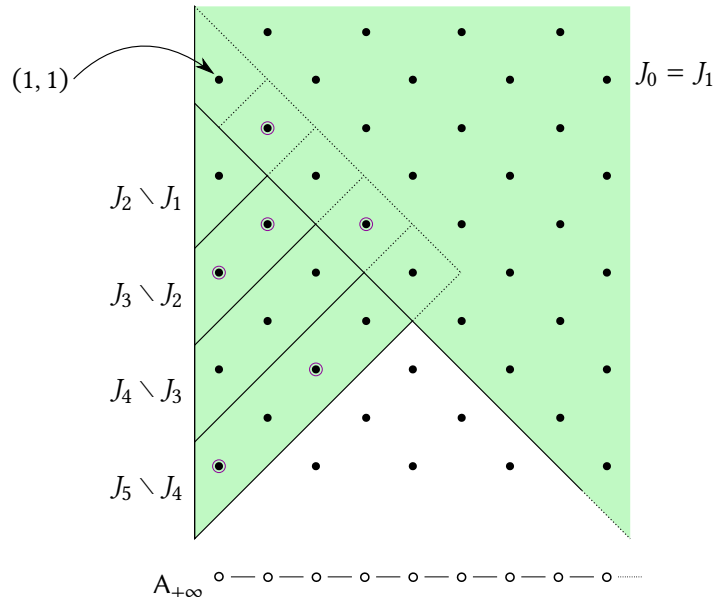
1. For  $i = 0$ , the multiset  $\mathbf{R}(\underline{\lambda}^0) = \emptyset$  is empty, and  $J_0$  is the complement of the downward-closed set generated by  $(1, -1)$ .
2. For  $i > 0$ , let  $J_i$  be the union of  $J_{i-1}$  with the upward-closed set generated by  $(1, 3 - 2i)$ , and let the difference  $\mathbf{R}(\underline{\lambda}^i) - \mathbf{R}(\underline{\lambda}^{i-1})$  be supported on the elements along the north-east diagonal beginning at  $(1, 3 - 2i)$  and have weight  $\lambda^{(i)}$ .

When  $\underline{\lambda}$  has length  $r$ , we set  $\mathbf{R}(\underline{\lambda}) = \mathbf{R}(\underline{\lambda}^r)$  and  $J(\underline{\lambda}) = J_r$ .

The definition above is more easily interpreted as a picture in terms of monomial diagrams. Rather than saying ‘ $A_n$  for  $n$  large enough’, we will draw the half-infinite path  $A_{+\infty}$  and work in this setting until we need to compute the character of a whole crystal  $\mathcal{M}(\lambda, \mathbf{R})$ . While we are working with the truncations  $\mathcal{M}(\lambda, \mathbf{R}, J)$  it will not matter which  $A_n$  we are working in, so long as all monomial in question are supported over  $A_n \times \mathbb{Z}$ .

### 9.5.2 Example

Taking the same partition sequence  $\underline{\lambda} = (\emptyset, (1, 1), (2, 1), (1, 1, 1, 1), (2, 1, 1))$  as in Example 9.3.5, a schematic picture of what the data  $\mathbf{R}(\underline{\lambda}^i)$  and  $J_i$  looks like for every  $0 \leq i \leq 5$  is shown below.



We start with the upward-closed set  $J_0$  (note that  $J_0$  is not finitely-generated, since  $A_{+\infty}$  is infinite). The set  $J_1$  adds a single point  $(1, 1)$ , the set  $J_2$  adds two more points, then  $J_3$  three more, and so on. We end up with

$J_5$ , which is the entire green shaded region.

In terms of the ‘fundamental weights’ of  $A_{+\infty}$ , our partition sequence  $\underline{\lambda}$  is the sequence of weights  $(0, \omega_2, \omega_1 + \omega_2, \omega_4, \omega_1 + \omega_3)$ . The construction of the  $\mathbf{R}(\underline{\lambda}^i)$  multiset from the previous one is simply to include elements from  $J_i \setminus J_{i-1}$  with multiplicities according to the weights given by  $\lambda^{(i)}$ . For example, here the weight of  $\lambda^{(3)}$  is  $\omega_1 + \omega_2$ , and therefore we include into  $\mathbf{R}$  the two points  $(1, -3)$  and  $(2, -2)$  of  $J_3 \setminus J_2$  which will give that weight.

Our construction of the  $\mathbf{R}(\underline{\lambda}^i)$  and  $J_i$  is designed to be compatible with our inductive character formula [Theorem 7.2.3](#). From the picture above it is clear that we can go from the set  $J_{i-1}$  to  $J_i$  by adding  $(i - 1)$  points to the truncating set, and each one added can be taken to be minimal by adding the rightmost one first. After this is done, we place the new elements of  $\mathbf{R}(\underline{\lambda}^i)$  over the new points  $J_i$ , noting that  $(J_i \setminus J_{i-1}) \subseteq \partial J_i$  and so every new element of  $\mathbf{R}(\underline{\lambda}^i)$  is added along the boundary. We get a recurrence of characters for the truncations  $\mathcal{M}(\text{wt } \underline{\lambda}^i, \mathbf{R}(\underline{\lambda}^i), J_i)$  which matches the recurrence for the flagged Schur modules [Lemma 9.3.6](#).

### 9.5.3 Lemma

Let  $\underline{\lambda}$  be a partition sequence of length  $r$ . The characters of the truncations  $\mathcal{M}(\text{wt } \underline{\lambda}^i, \mathbf{R}(\underline{\lambda}^i), J_i)$  satisfy the recurrence

1. For  $i = 0$ ,  $\text{ch } \mathcal{M}(0, \emptyset, J_1) = 1$ ,
2. For  $i > 0$ ,

$$\begin{aligned} \text{ch } \mathcal{M}(\text{wt } \underline{\lambda}^i, \mathbf{R}(\underline{\lambda}^i), J_i) &= e^{\lambda^{(i)}} \cdot \text{ch } \mathcal{M}(\text{wt } \underline{\lambda}^{i-1}, \mathbf{R}(\underline{\lambda}^{i-1}), J_i) \\ &= e^{\lambda^{(i)}} \cdot \pi_1 \cdots \pi_{i-1} \text{ch } \mathcal{M}(\text{wt } \underline{\lambda}^{i-1}, \mathbf{R}(\underline{\lambda}^{i-1}), J_{i-1}). \end{aligned} \tag{9.5.4}$$

Therefore, the character of the flagged Schur module  $\mathcal{F}(D(\underline{\lambda}))$  matches the character of the truncated crystal  $\mathcal{M}(\text{wt } \underline{\lambda}, \mathbf{R}(\underline{\lambda}), J(\underline{\lambda}))$ . We can apply two different stabilisation results to see that the characters of the corresponding product monomial crystal and Schur module match.

### 9.5.5 Theorem

Let  $\underline{\lambda}$  be a partition sequence of length  $r$ , and work in type  $I = A_n$  for  $n \geq r$ . Then the characters of the Schur module  $\mathcal{S}_{D(\underline{\lambda})}(\mathbb{C}^r)$  and the product monomial crystal  $\mathcal{M}(\text{wt } \underline{\lambda}, \mathbf{R}(\underline{\lambda}))$  are equal.

### 9.5.6 Proof

By [Corollary 7.3.9](#), the character of the whole product monomial crystal can be obtained from any of its truncations by applying the Demazure operator  $\pi_{w_I}$  corresponding to the longest element of the Weyl group  $W_I \cong \mathfrak{S}_{n+1}$ . Hence

$$\text{ch } \mathcal{M}(\text{wt } \underline{\lambda}, \mathbf{R}(\underline{\lambda})) = \pi_{w_I} \text{ch } \mathcal{M}(\text{wt } \underline{\lambda}^r, \mathbf{R}(\underline{\lambda}^r), J_r). \tag{9.5.7}$$

By Theorem 21 of [RS98], the character of the Schur module  $\mathcal{S}_D(\mathbb{C}^n)$  can be obtained from the character of the flagged Schur module  $\mathcal{F}_D(\mathbb{C}^n)$  by applying the Demazure operator  $\pi_{w_I}$ , hence

$$\text{ch } \mathcal{S}_D(\mathbb{C}^n) = \pi_{w_I} \text{ch } \mathcal{F}_D(\mathbb{C}^n). \tag{9.5.8}$$

Since both the characters appearing on the right-hand sides of the above expressions are equal, the result follows.

It is worth pointing out that we really did need to use both results [Corollary 7.3.9](#) and (Theorem 21, [RS98]) in the statement above. Despite the fact that we know that the character of the flagged Schur module matches the character of the truncated product monomial crystal, and that both are positive sums of Demazure characters even, we cannot apply the stabilising operator  $\pi_{w_I}$  and expect to get the character of the containing representation (resp. crystal) unless we know for sure that the representation  $\mathcal{F}_D(\mathbb{C}^n)$  has a filtration by Demazure modules (resp. the crystal is a sub-Demazure crystal). In this case due to our result that the truncations of the product monomial crystals are Demazure ([Theorem 7.3.7](#)) we know that each truncation is a sub-Demazure crystal. Interestingly it is still unknown whether the flagged Schur modules admit a filtration by Demazure modules (Reiner, personal communication). Theorem 21 of [RS98] is proven in an entirely different manner, by a construction due to Magyar [[Mag98b](#)] realising  $\mathcal{S}_D(\mathbb{C}^n)$  as the space of sections of a line bundle over a ‘configuration variety’ depending on



the diagram  $D$ .

The result [Theorem 9.5.5](#) is good, but is not as strong as we want since it only holds for  $n$  large enough compared to the height of the diagram  $D$ . If the partition sequence  $\underline{\lambda}$  is of length  $r$ , but each partition has length at most 5 (for example), then it is easy to see from [Example 9.5.2](#) that the corresponding multiset  $\mathbf{R}(\underline{\lambda})$  is defined in type  $A_5$ . However our above result [Theorem 9.5.5](#) is not applicable in type  $A_5$  if  $r > 5$ . In the following chapter, we will develop some stability properties of the product monomial crystals defined when  $\mathbf{R}$  is held fixed and the type  $A_n$  varies, so that we can strengthen the above result to all  $n$ .



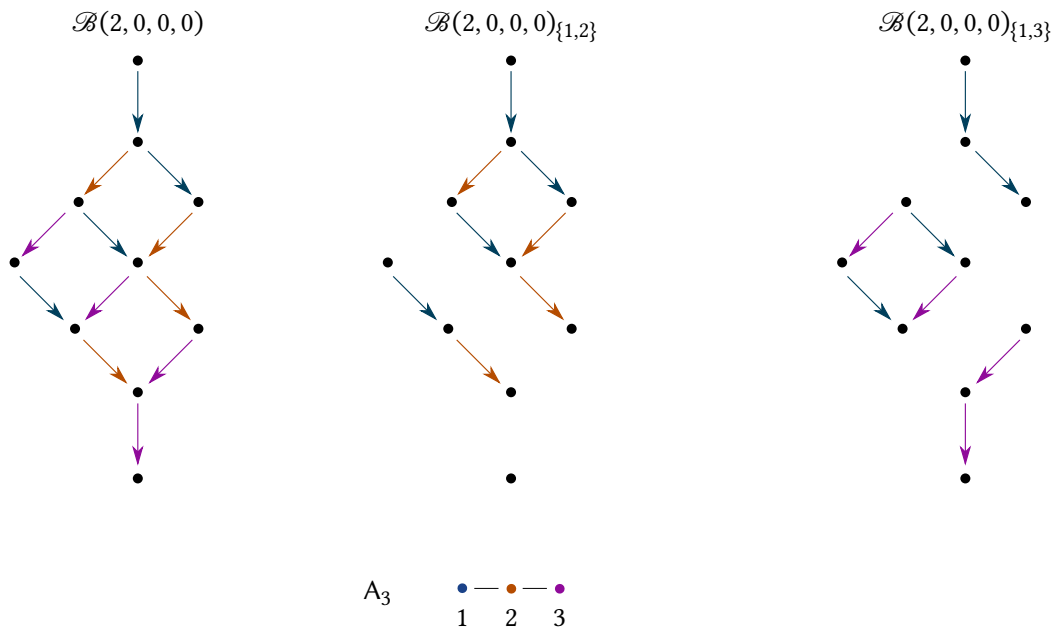
# 10 Stability of decomposition

In Chapter 9 we discussed the *generalised Littlewood-Richardson coefficients*  $c_D^\lambda$  associated to a diagram  $D$ , and remarked on their *stability properties* in terms of Schur functors (Corollary 9.2.2). Namely, for any vector space whose dimension is large enough, every coefficient  $c_D^\lambda$  can be read off from the decomposition of the generalised Schur module  $\mathcal{S}_D(V)$ , and furthermore these *stable coefficients* can be used to deduce the decomposition of  $\mathcal{S}_D(U)$  for any  $\dim U \leq \dim V$ .

In this chapter we will see that the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\mathrm{GL}_n)$  has similar behaviour: there exist stable coefficients  $c_{\mathbf{R}}^\lambda$  giving the decomposition of  $\mathcal{M}(\lambda, \mathbf{R})$  into irreducible crystals whenever  $n$  is large enough, and furthermore these stable coefficients can be used to deduce the decomposition of  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\mathrm{GL}_m)$  whenever  $m \leq n$ . Once we have this result, we can apply Theorem 9.5.5 to show that whenever  $D$  and  $\mathbf{R}$  correspond, we have  $c_D^\lambda = c_{\mathbf{R}}^\lambda$ , and hence the product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\mathrm{GL}_n)$  is the crystal of the generalised Schur module  $\mathcal{S}_D(\mathbb{C}^n)$  for all  $n$ .

## 10.1 Stability of restriction

Let  $\Phi$  be a root datum of Cartan type  $(I, \cdot)$ . Recall from Definitions 3.1.1 and 3.4.1 that each subset  $J \subseteq I$  defines a Cartan datum  $(J, \cdot)$  by restriction, and a restricted root datum  $\Phi_J$  of type  $(J, \cdot)$  by keeping the weight and coweight lattices the same, but only remembering the simple roots and coroots indexed by  $J$ . For each  $\Phi$ -crystal  $B$ , we have a  $\Phi_J$ -crystal  $B_J$  by restriction: keeping the set  $B$  the same, but remembering only the crystal operators indexed by  $J$ . As an example, we show the  $\mathrm{GL}_4$ -crystal  $\mathcal{B}(2, 0, 0, 0)$ , and the restrictions  $\mathcal{B}(2, 0, 0, 0)_{\{1,2\}}$  and  $\mathcal{B}(2, 0, 0, 0)_{\{1,3\}}$ .



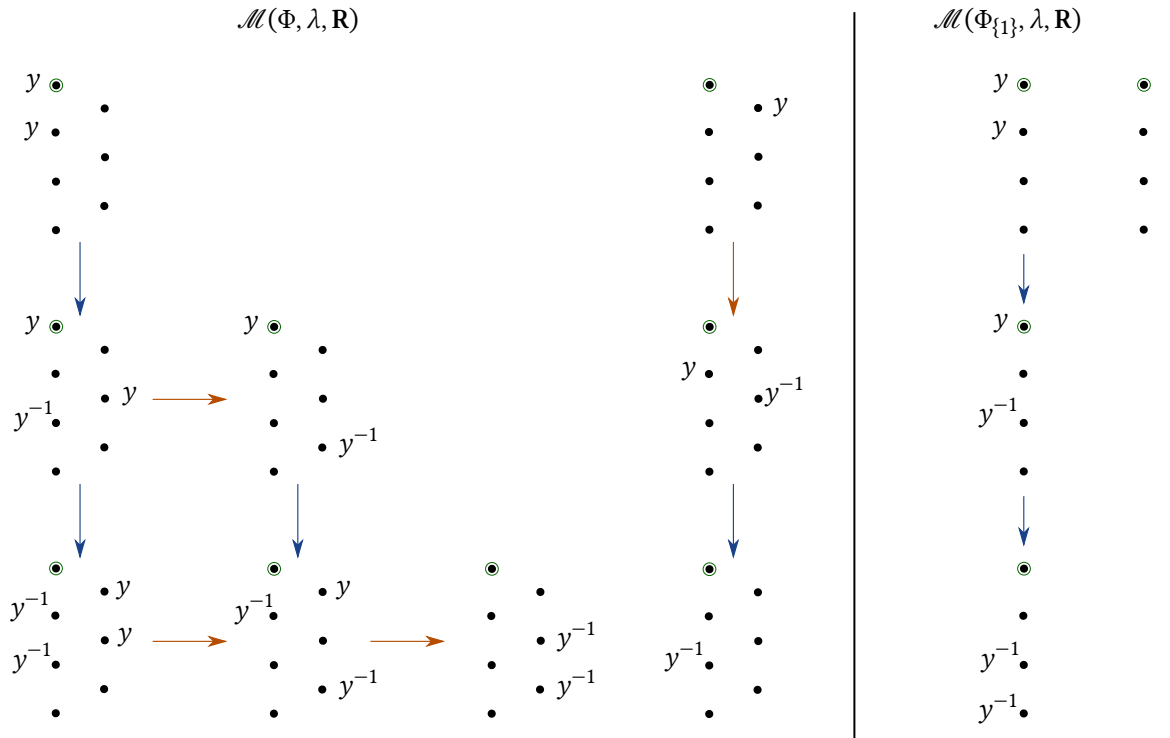
As the example shows, we expect a restricted crystal  $B_J$  to often have many connected components, for example with  $J = \{1, 2\}$  in the example we see that  $\mathcal{B}(2, 0, 0, 0)_J$  is a disjoint union of three connected crystals. When

$J$  is a single element  $J = \{i\}$ , rather than writing  $\Phi_{\{i\}}$  and  $B_{\{i\}}$ , we will simply write  $\Phi_i$  and  $B_i$ . The connected components of the restricted crystal  $B_i$  are exactly the  $i$ -strings inside  $B$ .

Suppose that  $(\lambda, \mathbf{R})$  is a dominant pair defining a product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\Phi)$ . Whenever  $J \subseteq I$  is a subset such that the multiset  $\mathbf{R}$  is supported over  $J \times \mathbb{Z}$ , this data also defines a product monomial crystal  $\mathcal{M}(\lambda, \mathbf{R}) \subseteq \mathcal{M}(\Phi_J)$ , a  $\Phi_J$ -crystal rather than a  $\Phi$ -crystal. In this case we will say that the multiset  $\mathbf{R}$  *lives over*  $J \subseteq I$ . We write  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$  for the  $\Phi$ -crystal, and  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  for the  $\Phi_J$ -crystal to keep notation as clear as possible. We get a *different*  $\Phi_J$ -crystal by the restriction  $\mathcal{M}(\Phi, \lambda, \mathbf{R})_J$ , and the arguments following will have to do with the interplay between the  $\Phi$ -crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$ , its restricted  $\Phi_J$ -subcrystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})_J$ , and the  $\Phi_J$ -crystal  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$ .

**10.1.1 Example**

Let  $\Phi = \text{GL}_3$ , a root datum of Cartan type  $(I, \cdot) = A_2$ , with vertices  $I = \{1, 2\}$ . The data  $\lambda = 2\epsilon_1$ ,  $\mathbf{R} = \{(1, 3), (1, 1)\}$  defines a  $\Phi$ -product monomial crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$ , and also a  $\Phi_{\{1\}}$ -product monomial crystal  $\mathcal{M}(\Phi_{\{1\}}, \lambda, \mathbf{R})$ . These two crystals are pictured below using monomial diagrams, where the point  $(1, 3)$  is circled in green, and we write a  $y^k$  next to the point  $(i, c)$  to indicate that the multiplicity of  $y_{i,c}$  in that monomial is  $k$ .



Both components of the  $\Phi_{\{1\}}$ -crystal  $\mathcal{M}(\Phi_{\{1\}}, \lambda, \mathbf{R})$  appear inside the restricted  $\Phi_{\{1\}}$ -crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})_{\{1\}}$ . This is a general fact, which we will make precise in what follows.

Let  $\mathcal{Z}(\Phi) \subseteq \mathcal{M}(\Phi)$  be the subgroup generated by the  $z_{i,k}$  for all  $(i, k) \in I \times \mathbb{Z}$ , so that the product monomial crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$  is contained inside the coset  $e^\lambda \cdot \gamma_{\mathbf{R}} \cdot \mathcal{Z}(\Phi)$ . It is clear from the definition of the monomial crystal (Definition 6.1.1) that each coset of  $\mathcal{Z}(\Phi)$  is a subcrystal of  $\mathcal{M}(\Phi)$ . For any subset  $J \subseteq I$ , let  $\gamma_J : \mathcal{Z}(\Phi_J) \hookrightarrow \mathcal{Z}(\Phi)$  be the linear inclusion map taking  $z_{i,k} \in \mathcal{Z}(\Phi_J)$  to  $z_{i,k} \in \mathcal{Z}(\Phi)$  (note that the definition of  $z_{i,k}$  depends on the Cartan matrix, so this map is less obvious than it seems). If  $(\lambda, \mathbf{R})$  lives over  $J$  then the product monomial crystal  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  makes sense. We define an inclusion of cosets

$$\psi_{\lambda, \mathbf{R}, J} : e^\lambda \gamma_{\mathbf{R}} \mathcal{Z}(\Phi_J) \hookrightarrow e^\lambda \gamma_{\mathbf{R}} \mathcal{Z}(\Phi), \quad e^\lambda \gamma_{\mathbf{R}} z \mapsto e^\lambda \gamma_{\mathbf{R}} \gamma_J(z), \quad (10.1.2)$$

which is in fact a map of  $\Phi_J$ -crystals, where the crystal on the right is equipped with the  $\Phi_J$ -crystal structure through restriction. The image of  $\psi_{\lambda, \mathbf{R}, J}$  consists of all monomials whose  $z_{i,k}$ -exponent is zero for  $i \notin J$ . Note that the map  $\psi_{\lambda, \mathbf{R}, J}$  is an *affine*, rather than linear, map of abelian groups.

**10.1.3 Lemma**

The inclusion  $\psi_{\lambda, \mathbf{R}, J}$  maps the product monomial crystal  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  into the set  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$ . Furthermore, the image of this map consists of all monomials  $p \in \mathcal{M}(\Phi, \lambda, \mathbf{R})$  such that  $\text{Supp}_{\mathbf{R}}(p) \subseteq J \times \mathbb{Z}$ .

**10.1.4 Proof**

First consider the case where  $(\lambda, \mathbf{R})$  is a fundamental pair concentrated at some  $(i, c)$ . The restricted crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})_J$  decomposes into a disjoint union of  $\Phi_J$ -crystals. Let us write  $\mathcal{M}(\Phi, \lambda, \mathbf{R})_J = B \sqcup C$  where  $B$  is the connected component containing the highest-weight element  $e^\lambda \cdot \gamma_{\mathbf{R}}$  and  $C$  are the other connected components. Since the inclusion  $\psi_{\lambda, \mathbf{R}, J}$  maps the  $J$ -highest-weight element  $e^\lambda \cdot \gamma_{\mathbf{R}} \in \mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  to the  $I$ -highest-weight element  $e^\lambda \cdot \gamma_{\mathbf{R}}$  of  $B$ , the  $\Phi_J$ -crystal map  $\psi_{\lambda, \mathbf{R}, J}$  gives an isomorphism  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R}) \rightarrow B$ .

It remains to be seen that  $\text{Supp}_{\mathbf{R}}(p) \not\subseteq J \times \mathbb{Z}$  for all  $p \in C$ . As  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$  is connected as a  $\Phi$ -crystal, every monomial  $p \in \mathcal{M}(\Phi, \lambda, \mathbf{R})$  may be written  $p = f_{i_1}^{r_1} \cdots f_{i_l}^{r_l}(e^\lambda \cdot \gamma_{\mathbf{R}})$  for some  $l \geq 0, r_i \geq 1$  and  $i_k \in I$ . The monomial  $p$  belongs to the top component  $B$  if and only if  $\{i_1, \dots, i_l\} \subseteq J$  because  $B$  is a highest-weight  $\Phi_J$ -crystal with highest-weight element  $e^\lambda \cdot \gamma_{\mathbf{R}}$ . Hence every monomial  $p \in C$  has  $\text{Supp}_{\mathbf{R}}(p) \not\subseteq J \times \mathbb{Z}$ , completing the proof of the claim in the case where  $(\lambda, \mathbf{R})$  is a fundamental pair.

For the general case where  $(\lambda, \mathbf{R})$  is a dominant pair, we use a factorisation into fundamental pairs  $(\lambda_1, \mathbf{R}_1), \dots, (\lambda_r, \mathbf{R}_r)$ , and note that for each factorised monomial  $p = e_0^\lambda p_1 \cdots p_r \in \mathcal{M}(\Phi, \lambda, \mathbf{R})$  we have

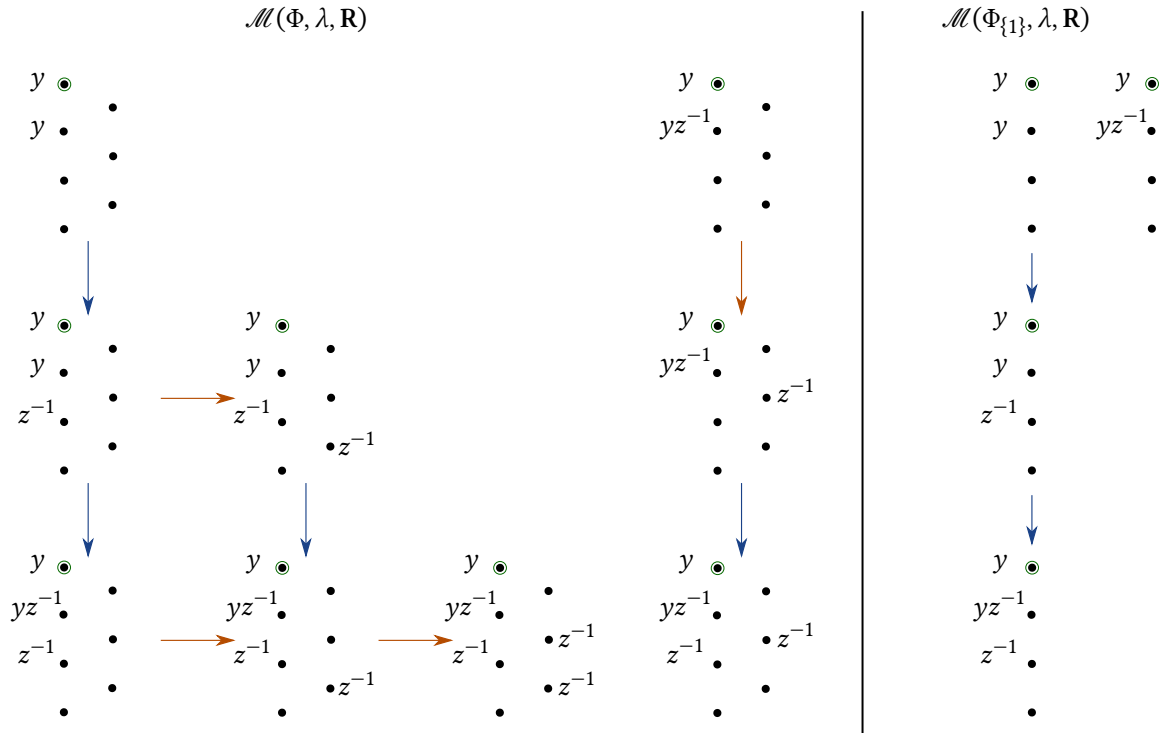
$$\psi_{J, \lambda, \mathbf{R}}(p_1 \cdots p_r) = \psi_{J, \lambda_1, \mathbf{R}_1}(p_1) \cdots \psi_{J, \lambda_r, \mathbf{R}_r}(p_r). \tag{10.1.5}$$

The result then follows from the case of a fundamental pair.

In order to really understand what the affine map  $\psi_{\lambda, \mathbf{R}, J}$  is doing in Lemma 10.1.3, we return to our previous example, this time written out in terms of the  $e^\lambda \cdot \gamma_{\mathbf{R}} \cdot z_S^{-1}$  labelling.

**10.1.6 Example**

The crystals  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$  and  $\mathcal{M}(\Phi_{\{1\}}, \lambda, \mathbf{R})$  of Example 10.1.1 are shown as monomial diagrams below, this time always using the factorisation  $\gamma_{\mathbf{R}} \cdot z_S^{-1}$ .



The claims of Lemma 10.1.3 should be clear in the picture above.

A highest-weight element  $p \in \mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  of the smaller  $\Phi_J$ -crystal gets mapped to a highest-weight element  $\psi_{\lambda, \mathbf{R}, J}(p)$  of the larger  $\Phi$ -crystal: we can see that  $e_j(\psi_{\lambda, \mathbf{R}, J}(p)) = \perp$  for all  $j \in J$  because  $\psi$  is a morphism of  $\Phi_J$ -crystals, and if  $i \in I \setminus J$  then all exponents  $\psi_{\lambda, \mathbf{R}, J}(p)[i, k]$  are zero or negative by the definition of the auxiliary monomial  $z_{i, k}$ . Therefore we get the following:

**10.1.7 Lemma**

The map  $\psi_{\lambda, \mathbf{R}, J}$  restricted to highest-weight elements is an injective map of sets

$$\Psi_{\lambda, \mathbf{R}, J} : \mathcal{M}(\Phi_J, \lambda, \mathbf{R})^{\text{h.w.}} \hookrightarrow \mathcal{M}(\Phi, \lambda, \mathbf{R})^{\text{h.w.}}. \tag{10.1.8}$$

Furthermore, the image of this map consists of all highest-weight monomials  $p \in \mathcal{M}(\Phi, \lambda, \mathbf{R})^{\text{h.w.}}$  such that  $\text{Supp}_{\mathbf{R}}(p) \subseteq J \times \mathbb{Z}$ .

**10.1.9 Proof**

The map is injective since  $\psi_{\lambda, \mathbf{R}, J}$  is, and its image consists of highest-weight elements as reasoned in the previous discussion. By Lemma 10.1.3, all we need to prove about the claim describing the image is one inclusion: that a highest-weight monomial  $p \in \mathcal{M}(\Phi, \lambda, \mathbf{R})^{\text{h.w.}}$  satisfying  $\text{Supp}_{\mathbf{R}}(p) \subseteq J \times \mathbb{Z}$  is in the image of  $\Psi_{\lambda, \mathbf{R}, J}$ . By Lemma 10.1.3 it has a preimage  $q \in \mathcal{M}(\Phi, \lambda, \mathbf{R})$ , so we just need to show that  $q$  is highest-weight, but this is clear because  $\psi_{\lambda, \mathbf{R}, J}$  is a morphism of  $\Phi_J$ -crystals.

The inclusions  $\Psi_{\lambda, \mathbf{R}, J}$  can be viewed as a directed system consisting of those subsets  $J \subseteq I$  such that  $\mathbf{R}$  lives over  $J$ , in the sense that the inclusions  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R}) \hookrightarrow \mathcal{M}(\Phi_K, \lambda, \mathbf{R}) \hookrightarrow \mathcal{M}(\Phi, \lambda, \mathbf{R})$  are compatible for  $J \subseteq K \subseteq I$ . One can ask: is there a smaller subset  $J \subseteq I$  such that the product monomial crystal  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$  captures all necessary information to determine the isomorphism class of  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$ ?

**10.1.10 Lemma**

Let  $(\lambda, \mathbf{R})$  be a dominant pair for  $\Phi$ . Consider the set  $X \subseteq I \times \mathbb{Z}$  defined by

$$X = \text{up}(\mathbf{R}) \cap (\text{down}(\{(i, c - 2) \mid (i, c) \in \mathbf{R}\}) \cup \text{Supp } \mathbf{R}). \tag{10.1.11}$$

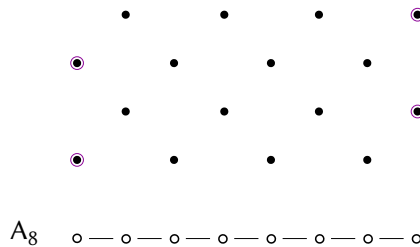
Then if  $X$  is contained in  $J \times \mathbb{Z}$  for some  $J \subseteq I$ , the map  $\Psi_{\lambda, \mathbf{R}, J} : \mathcal{M}(\Phi_J, \lambda, \mathbf{R})^{\text{h.w.}} \hookrightarrow \mathcal{M}(\Phi, \lambda, \mathbf{R})^{\text{h.w.}}$  is a bijection.

**10.1.12 Proof**

By Lemma 6.5.4 and Corollary 6.6.5, every highest-weight element  $q \in \mathcal{M}(\Phi, \lambda, \mathbf{R})^{\text{h.w.}}$  satisfies  $\text{Supp}_{\mathbf{R}}(q) \subseteq X \subseteq J \times \mathbb{Z}$ , and hence by the description of the image in Lemma 10.1.7 the claim follows.

## 10.2 Application: decomposing a product

The results of the previous section can be applied to decompose the product monomial crystal into a product of crystals, in some cases. Suppose we are working with the root datum  $\Phi = \text{SL}_9$  of Cartan type  $(I, \cdot) = A_8$ , and we have  $\lambda = 2\omega_1 + 2\omega_8$  with the multiset  $\mathbf{R} = \{(1, 1), (1, 3), (8, 2), (8, 4)\}$  as pictured here:



In this special case, the set  $X$  appearing in Lemma 10.1.10 is precisely  $X = \text{Supp } \mathbf{R}$ , which lives over  $J = \{1, 8\}$ . Therefore to determine the highest-weight elements of the crystal  $\mathcal{M}(\Phi, \lambda, \mathbf{R})$  it suffices to determine the highest-weight elements of the (much smaller) crystal  $\mathcal{M}(\Phi_J, \lambda, \mathbf{R})$ . Since the Cartan datum  $(J, \cdot)$  is disconnected, we have



and whenever  $\mathbf{R}$  lives over  $[m - 1] \times \mathbb{Z}$  for some  $m \geq 1$  we have

$$\mathcal{M}(\Phi_{[m-1]}, \lambda, \mathbf{R}) \cong \bigoplus_{\ell(\mu) \leq m} \mathcal{B}(\mathrm{GL}_m, \mu)^{\oplus c_{\lambda, \mathbf{R}}^\mu} \quad \text{as } \mathrm{GL}_m\text{-crystals.} \quad (10.3.3)$$

**Lemma 10.3.1** should be compared with **Corollary 9.2.2**, as they are similar in two ways: firstly they assert the existence of certain stable coefficients, and secondly they give a restriction rule, and the restriction rules match.

**10.3.4 Proof**

Since  $\mathbf{R}$  is finite, the set  $X \subseteq \mathbb{A}_{+\infty} \times \mathbb{Z}$  appearing in **Lemma 10.1.10** is also finite, so there exists some  $n \geq 1$  such that  $X \subseteq [n - 1] \times \mathbb{Z}$  which we use to define the coefficients  $c_{\lambda, \mathbf{R}}^\mu$  according to **Eq. (10.3.2)**. **Lemma 10.1.10** gives that **Eq. (10.3.3)** holds for all  $m \geq n$ . Now suppose that  $m < n$ , and consider the map  $\Psi : \mathcal{M}(\Phi_{[m-1]}, \lambda, \mathbf{R})^{\mathrm{h.w.}} \hookrightarrow \mathcal{M}(\Phi_{[n-1]}, \lambda, \mathbf{R})^{\mathrm{h.w.}}$  appearing in **Lemma 10.1.7** which has image consisting of those highest-weight monomials whose  $\mathbf{R}$ -support is contained in  $[m - 1] \times \mathbb{Z}$ . A highest-weight monomial  $p$  of  $\mathcal{M}(\Phi_{[n-1]}, \lambda, \mathbf{R})$  has  $\mathbf{R}$ -support contained in  $[m - 1] \times \mathbb{Z}$  if and only if  $\mathrm{wt}(p)$  is a partition with at most  $m$  parts, giving that **Eq. (10.3.3)** holds when  $m < n$ .

We remark that the statement of **Lemma 10.3.1** would be a lot more ugly if we were to use  $\mathrm{SL}_n$  rather than  $\mathrm{GL}_n$ , since the property of a monomial having  $\mathbf{R}$ -support contained in  $[m - 1] \times \mathbb{Z}$  cannot be checked purely from its weight. In order to compute decompositions in the  $\mathrm{SL}_n$  case, one should do computations for  $\mathrm{GL}_n$ , and restrict to  $\mathrm{SL}_n$  as a last step.

We now get our main result: for  $\mathrm{GL}_m$ , the product monomial crystal is the crystal of a generalised Schur module, which follows directly from comparing **Lemma 10.3.1** with **Corollary 9.2.2**, and checking that they both agree for some large  $n$  as a result of **Theorem 9.5.5**.

**10.3.5 Corollary**

Let  $\underline{\lambda}$  be a partition sequence of length  $r$ , defining both a diagram  $D(\underline{\lambda})$  by **Definition 9.3.4** and a multiset  $\mathbf{R}(\underline{\lambda})$  by **Definition 9.5.1**. Let  $m \geq 1$  be an integer such that  $\mathbf{R}(\underline{\lambda})$  lives over  $[m - 1] \times \mathbb{Z}$ . Then the product monomial crystal  $\mathcal{M}(\mathrm{GL}_m, \mathrm{wt} \underline{\lambda}, \mathbf{R}(\underline{\lambda}))$  is the crystal of the generalised Schur module  $\mathcal{S}_{D(\underline{\lambda})}(\mathbb{C}^m)$  associated to the column-convex diagram  $D(\underline{\lambda})$ .

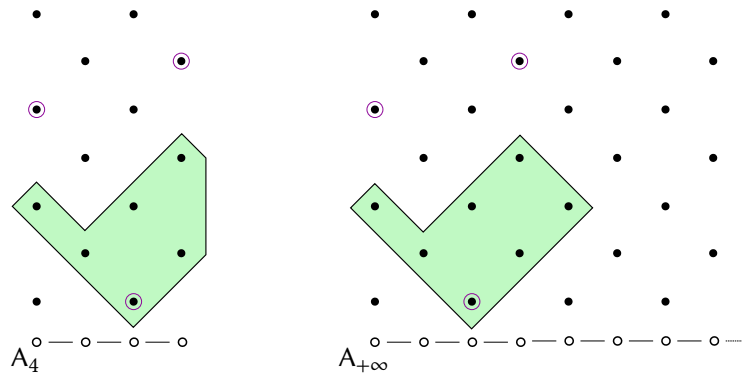
**10.3.6 Remark**

**Corollary 10.3.5** applies to *all* product monomial crystals of some  $\mathrm{GL}_m$ , since after applying a vertical shift  $\mathbf{R} \mapsto \mathbf{R}'$  of the form  $\mathbf{R}'[i, c] = \mathbf{R}[i, c + 2k]$  for some  $k \in \mathbb{Z}$ ,  $\mathbf{R}'$  can be brought to the form where  $\mathbf{R}' = \mathbf{R}(\underline{\lambda})$  for some partition sequence  $\underline{\lambda}$ . Similarly, **Corollary 10.3.5** applies to all *column-convex* diagrams, which are all of the form  $D(\underline{\lambda})$  after applying some column permutation.

To finish this section, we give a worked example of using **Lemma 10.3.1** to find stable coefficients and restrict them to a smaller  $\mathrm{GL}_m$ .

**10.3.7 Example**

Let  $\mathbf{R} = \{(1, 5), (3, 1), (4, 6)\}$ , with associated dominant weight  $\lambda = \omega_1 + \omega_3 + \omega_4$ , which define a product monomial crystal  $\mathcal{M}(\mathrm{GL}_5, \lambda, \mathbf{R})$  in type  $A_4$ . The figure below shows the set  $\mathrm{Supp} \mathbf{R}$  as the circled points, and the set  $\mathrm{down}((i, c - 2) \mid (i, c) \in \mathbf{R}) \cup \mathrm{up}(\mathbf{R})$  in green: their union is the set  $X$  appearing in **Lemma 10.1.10**.





The figure shows that  $\mathbf{R}$  is not stable for  $GL_5$ , but it is stable for  $GL_6$  and upwards, so we can determine the stable coefficients  $c_{\lambda, \mathbf{R}}^\mu$  by computing the decomposition of the product monomial crystal  $\mathcal{M}(GL_n, \lambda, \mathbf{R})$  for any  $n \geq 6$ . Using a computer, we determine the decomposition of  $\mathcal{M}(GL_6, \lambda, \mathbf{R})$  to be

$$\mathcal{M}(\Phi_6, \lambda, \mathbf{R}) \cong \mathcal{B}(2\omega_4) \oplus \mathcal{B}(\omega_3 + \omega_5) \oplus \mathcal{B}(\omega_2 + \omega_6) \oplus \mathcal{B}(\omega_1 + \omega_3 + \omega_4) \oplus \mathcal{B}(\omega_1 + \omega_2 + \omega_5) \oplus \mathcal{B}(2\omega_1 + \omega_6). \quad (10.3.8)$$

We can then apply the restriction rule in [Lemma 10.3.1](#) to deduce the decomposition of  $\mathcal{M}(GL_5, \lambda, \mathbf{R})$ : we must discard all partitions with length greater than 5. As  $GL_5$ -crystals, we get

$$\mathcal{M}(GL_5, \lambda, \mathbf{R}) \cong \mathcal{B}(2\omega_4) \oplus \mathcal{B}(\omega_3 + \omega_5) \oplus \mathcal{B}(\omega_1 + \omega_3 + \omega_4) \oplus \mathcal{B}(\omega_1 + \omega_2 + \omega_5). \quad (10.3.9)$$

We could even further restrict to  $SL_5$ -crystals along the morphism  $SL_5 \rightarrow GL_5$  of  $A_4$  root data, which has the effect of quotienting the weight lattice by  $\mathbb{Z}\omega_5$ :

$$\mathcal{M}(SL_5, \lambda, \mathbf{R}) \cong \mathcal{B}(2\omega_4) \oplus \mathcal{B}(\omega_3) \oplus \mathcal{B}(\omega_1 + \omega_3 + \omega_4) \oplus \mathcal{B}(\omega_1 + \omega_2). \quad (10.3.10)$$



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