INTRODUCTION TO DERIVED CATEGORIES OF COHERENT SHEAVES

ABSTRACT. Notes from a course given by Tom Bridgeland in Sydney Mathematical Research Institute, 2020. Notes by Gaston Burrull and Linyuan Liu.

0. History

(1) (**early 1960s**) Grothendieck's student Jean-Louis Verdier introduced the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ to provide a framework for Grothendieck duality, which is a massive generalization of Serre duality to include relative settings and singular schemes. More precisely, given a proper map $f : X \to Y$ between finite type schemes, then the functor $Rf_* : D\operatorname{Coh}(X) \to D\operatorname{Coh}(Y)$ has a right adjoint $f^!$.

(2) (**1979**) Beĭlinson proved in [1] that there exists an equivalence of triangulated categories $D^b\operatorname{Coh}(\mathbb{P}^n) \cong D^b\operatorname{Mod}(\Lambda)$, where Λ is a finite dimensional non-commutative algebra.

Remark 0.1. Naively, one would think that the right hand side was easier than the left hand side, but it is not really the case in general.

This has been generalized to the theory of exeptional collections during Rudakov Seminar in Moscow, 1980s ([2]).

(3) (**1981**) Mukai proved that if $A, \widehat{A}$ are dual abelian varieties, then $D^b\operatorname{Coh}(A) \cong D^b\operatorname{Coh}(\widehat{A})$ as triangulated categories ([3]). This provides the first example of derived equivalence between two non-isomorphic varieties. In the case where $A \cong \widehat{A}$, we get an automorphism of $D^b\operatorname{Coh}(A)$. He applied this result to the study of stable bundles on A. It has been generalized in the 1990s to K3 surfaces.

(4) (**mid 1990s**) By a more systematic study of $D^b\operatorname{Coh}(X)$, Bondal and Orlov obtained a semiorthogonal decomposition for the derived category of coherent sheaves on the intersection of two even dimensional quadrics and investigated the behavior of derived categories with respect to birational maps ([4]). In the same article, they made the following

Conjecture 0.2. *Let X_1 and X_2 be two birationally isomorphic 3-dimensional Calabi–Yau's. Then*

$$D^b\operatorname{Coh}(X_1) \cong D^b\operatorname{Coh}(X_2).$$

(5) (**1994**, Kontsevich's Homological Mirror Symmetry) In his ICM talk, Kontsevich conjectured that if X and $\check{X}$ are mirror dual Calabi-Yau 3-folds, then we have

$$D^b\operatorname{Coh}(X) \cong D^b\operatorname{Fuk}(\check{X}),$$

$$D^b\operatorname{Coh}(\check{X}) \cong D^b\operatorname{Fuk}(X),$$

where $\operatorname{Fuk}(X)$ is the Fukaya category of X ([5]).

1. The derived category of an abelian category

1.1. **Basics on $D(\mathcal{A})$.** Let $\mathcal{A}$ be an abelian category, e.g. $\mathrm{Mod}(R)$ of $\mathrm{Coh}(X)$. Let $C(\mathcal{A})$ denote the category of cochain complexes in $\mathcal{A}$. A morphism $f^\bullet : M^\bullet \to N^\bullet$ in this category looks as follows

$$\begin{tikzcd} \cdots \arrow[r] & M^{i-1} \arrow[r, "d^i"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "d^i"] \arrow[d, "f^i"] & M^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r] & N^{i-1} \arrow[r, "d^{i-1}"] & N^i \arrow[r, "d^i"] & N^{i+1} \arrow[r, "d^{i+1}"] & \cdots . \end{tikzcd}$$

Such a morphism $f^\bullet : M^\bullet \to N^\bullet$ is called a quasi-isomorphism if the induced map on cohomology objects $H^i(f^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ is an isomorphism for all i.

Example 1.1. If

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is a resolution of a module M, then

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism.

Then there exists a category $D(\mathcal{A})$, which will be called the derived category of $\mathcal{A}$, and a localisation functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ which is universal with the property that it takes quasi-isomorphisms to isomorphisms. The objects of $D(\mathcal{A})$ can be taken to be the same as the objects of $C(\mathcal{A})$

From the universal property, there exist well-defined functors $H^i : D(\mathcal{A}) \to \mathcal{A}$ sending a complex to its cohomology objects. The bounded derived category is defined to be the full subcategory

$$D^b(\mathcal{A}) = \{E \in D(\mathcal{A}) : H^i(E) = 0 \text{ for } |i| \gg 0\}.$$

Facts 1.2.
(1) $D(\mathcal{A})$ is an additive category[1] and the localisation functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ is additive.
(2) The natural functor

$$\begin{aligned} I : &\mathcal{A} \longrightarrow D(\mathcal{A}) \\ &M \longmapsto \left(\substack{\cdots \to 0 \to M \to 0 \to \cdots \\ M \text{ at position } 0} \right) \end{aligned}$$

is fully faithful, with essential image

$$\{N^\bullet \in D^b(\mathcal{A}) \mid H^i(N^\bullet) = 0 \text{ for all } i \neq 0\}$$

(objects concentrated in degree 0).

Remark 1.3. Let E, F be two objects in $D(\mathcal{A})$. In general, we have

$$\Big(H^i(E) \cong H^i(F) \quad \forall i \in \mathbb{Z}\Big) \not\Rightarrow E \cong F.$$

We will give an explicit example of this fact later.

Well-behaved functors $F : \mathcal{A} \to \mathcal{B}$ between abelian categories induce derived functors $\underline{F} : D(\mathcal{A}) \to D(\mathcal{B})$. The classical derived functors of F are

$$\begin{aligned} F^i : &\mathcal{A} \xrightarrow{I} D(\mathcal{A}) \xrightarrow{\underline{F}} D(\mathcal{B}) \xrightarrow{H^i} \mathcal{B} \\ &A \quad \longmapsto \quad H^i(\underline{F}(A)). \end{aligned}$$

[1] Comment by note takers: In general, $D(\mathcal{A})$ is not abelian. In fact, $D(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.

1.2. **An example: duality for modules.** When $R = \mathbb{C}$, the duality functor

$$\mathbb{D}(-) = \mathrm{Hom}_R(-, R) : \mathrm{Mod}_{\mathrm{fg}}(R) \to \mathrm{Mod}_{\mathrm{fg}}(R)^{\mathrm{op}}$$

is an equivalence satisfying $\mathbb{D}^2 = \mathrm{Id}$.

What happens when we take $R = \mathbb{C}[x, y]$? For finitely generated projective modules (i.e. free modules in this case since R is a polynomial ring), we still have an equivalence of additive categories

$$\mathbb{D} : \mathrm{Proj}_{\mathrm{fg}}(R) \xrightarrow{\sim} \mathrm{Proj}_{\mathrm{fg}}(R)^{\mathrm{op}}$$

satisfying $\mathbb{D}^2 = \mathrm{Id}$. But if we take the torsion module $M = R/(x)$, we will have $\mathbb{D}(M) = \mathrm{Hom}_R(R/(x), R) = 0$. Let's try the classical derived functors

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M, R) \quad \text{for} \quad i \geq 0.$$

To compute these, take the free (=projective) resolution

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

to apply the functor $\mathbb{D}(-) = \mathrm{Hom}_R(-, R)$ to get

$$0 \leftarrow R \xleftarrow{x} R \leftarrow 0.$$

Hence we have $\mathbb{D}^1(M) \cong R/(x) = M$, and $\mathbb{D}^1 \circ \mathbb{D}^1(M) \cong M$.

Similarly, let $M = R/(x, y)$, using the free resolution

$$0 \to R \xrightarrow{(y,-x)} R^{\oplus 2} \xrightarrow{\binom{x}{y}} R \to M \to 0,$$

we get

$$\mathbb{D}^i(M) = \begin{cases} M & i = 2 \\ 0 & i \neq 2 \end{cases},$$

so $\mathbb{D}^2 \circ \mathbb{D}^2(M) \cong M$.

Now consider modules M fitting into a short exact sequence

$$(*) \qquad 0 \to R/(x, y) \to M \to R/(x) \to 0.$$

By the long exact sequence in $\mathrm{Ext}^\bullet_R$, we get

$$\mathbb{D}^i(M) = \begin{cases} R/(x) & i = 1 \\ R/(x, y) & i = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Note that extensions $(*)$ are classified by

$$\mathrm{Ext}^1_R(R/(x), R/(x, y)) = \mathbb{C} \neq 0,$$

so there exist non-split extensions M, but $\mathbb{D}^i(M)$ is the same as that of a split extension for all i. Hence M is not determined by $\mathbb{D}^i(M)$'s.

The solution is (of course) to define an equivalence

$$\underline{\mathbb{D}} : D^b \mathrm{Mod}_{\mathrm{fg}}(R) \to D^b \mathrm{Mod}_{\mathrm{fg}}(R)^{\mathrm{op}}$$

by replacing an object by a quasi-isomorphic free complex[2]. Then we have $\underline{\mathbb{D}}^2 = \mathrm{Id}$, because the duality functor $\mathbb{D}$ works well for finitely generated free modules. Hence we have

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M, R) = H^i(\underline{\mathbb{D}}(M))$$

[2]Comment by note takers: we have an equivalence of triangulated categories $K^b \mathrm{Proj}_{\mathrm{fg}}(R) \cong D^b \mathrm{Mod}_{\mathrm{fg}}(R)$ via the obvious functor, where $K^b(\mathcal{A})$ denotes the bounded homotopy category of $\mathcal{A}$. In fact, in general, if $\mathcal{A}$ is an abelian category with enough projectives, then we have an equivalence of triangulated categories $D(\mathcal{A}) \cong K(\mathcal{A})$ where $K(\mathcal{A})$ is the homotopy category. When $\mathcal{A} = \mathrm{Mod}(R)$, we deduce the bounded version equivalence from Hilbert Syzygy Theorem.

for all $M \in \mathrm{Mod}_{\mathrm{fg}}(R)$, but these don't determine $\mathbb{D}(M)$ or M[3].

1.3. **Sturcture of $D(\mathcal{A})$.**

1.3.1. *Triangulated structure.* The additive category $D(\mathcal{A})$ has shift functors

$$[n] : D(\mathcal{A}) \to D(\mathcal{A}),$$

$$M^\bullet[n]^i = M^{n+i}, \quad d^i_{M^\bullet[n]} = (-1)^n d^{i+n}_{M^\bullet}$$

and a collections of (distinguished) triangles

$$M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{g} P^\bullet \xrightarrow{h} M^\bullet[1],$$

which will be usually denoted as

$$\begin{tikzcd} M^\bullet \arrow[rr, "f"] & & N^\bullet \arrow[dl, "g"] \\ & P^\bullet \arrow[ul, dashed, "h"] & \end{tikzcd}$$

obtained from the mapping cone construction. Any such a triangle is a sequence of maps

$$\cdots \to P^\bullet[-1] \xrightarrow{h[-1]} M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{g} P^\bullet \xrightarrow{h} M^\bullet[1] \to \cdots .$$

We have the following properties:

- Triangles of a triangulated category are analogues of short exact sequences of an abelian category.
- All derived functors are triangulated: they take triangles to triangles.
- Define
  $$\mathrm{Hom}^i_{D(\mathcal{A})}(M^\bullet, N^\bullet) := \mathrm{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet[i]).$$
  If $M, N \in \mathcal{A}$, then
  $$\mathrm{Hom}^i_{D(\mathcal{A})}(M, N) \cong \mathrm{Ext}^i_{\mathcal{A}}(M, N).$$
  Fix $X \in D(\mathcal{A})$ and let
  $$\begin{tikzcd} M^\bullet \arrow[rr, "f"] & & N^\bullet \arrow[dl, "g"] \\ & P^\bullet \arrow[ul, dashed, "h"] & \end{tikzcd}$$
  be a triangle, then we have a long exact sequence:
  $$\begin{tikzcd} & & \cdots \arrow[dll, bend left] \\ \mathrm{Hom}^i_{D(\mathcal{A})}(X, M^\bullet) \arrow[r] & \mathrm{Hom}^i_{D(\mathcal{A})}(X, N^\bullet) \arrow[r] & \mathrm{Hom}^i_{D(\mathcal{A})}(X, P^\bullet) \arrow[dll, bend left] \\ \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(X, M^\bullet) \arrow[r] & \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(X, N^\bullet) \arrow[r] & \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(X, P^\bullet) \arrow[dll, bend left] \\ \cdots . & & \end{tikzcd}$$

[3]Comment by note takers: We have already seen why these $\mathbb{D}^i(M)$'s do not determine M. They don't determine $\mathbb{D}(M)$ either since $\mathbb{D}(\mathbb{D}(M)) \cong M$. This provides an example for remark 1.3.

- Every short exact sequence
$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$
in $\mathcal{A}$ gives rise to a triangle in $D(\mathcal{A})$:
$$\begin{tikzcd} M \arrow[rr, "f"] & & N \arrow[dl, "g"] \\ & P \arrow[ul, dashed, "h"] & \end{tikzcd}$$
where h is the element in $\mathrm{Hom}_{D(\mathcal{A})}(P, M[1]) = \mathrm{Ext}^1_{\mathcal{A}}(P, M)$ corresponding to the extension $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$.

2. Lecture 2

2.1. **Structure of $D(\mathcal{A})$ (Cont.)** Let $\mathcal{A}$ be an abelian category and $D(\mathcal{A})$ its derived category. $D(\mathcal{A})$ is an additive category with two types of structures:

- **Triangulated category.** Preserved by derived functors.
- **A standard t-structure.** Usually not preserved by derived functors.

2.1.1. *Triangulated categories.* Now we are to go over some of the axioms of a triangulated category. Let $\mathcal{C}$ be an additive category with an additive auto-equivalence $T\colon \mathcal{C} \to \mathcal{C}$. We call T the translation functor on $\mathcal{C}$. If X is an object of $\mathcal{C}$, we denote $T^n(X) = X[n]$ for any $n \in \mathbb{Z}$. A *triangle* in $\mathcal{C}$ is a sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1].$$

We are going to represent a triangle diagrammatically as

$$\begin{tikzcd} M \arrow[rr, "f"] & & N \arrow[dl, "g"] \\ & P \arrow[ul, dashed, "h"] & \end{tikzcd} .$$

The category $\mathcal{C}$ is a *triangulated category* if it is equipped with a family of triangles called *distinguished triangles* (the analogues of short exact sequences in $\mathcal{C}$), satisfying several axioms. The first one being: “any triangle isomorphic to a distinguished triangle is a distinguished triangle”. The second one is that for every object $M \in \mathrm{Ob}(\mathcal{C})$ the following diagram

$$\begin{tikzcd} M \arrow[rr, "\mathrm{id}_M"] & & M \arrow[dl] \\ & 0 \arrow[ul, dashed] & \end{tikzcd} .$$

is a distinguished triangle. We will only give intuition on two more of that axioms and omit the actual definition.

Octahedral axiom. (Looks scary, but not.) In an abelian category, given short exact sequences

$$0 \to A \to B \to C \to 0$$
$$0 \to B \to D \to E \to 0,$$

we can make the following diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\ & & D \arrow[d] & & \\ & & E \arrow[d] & & \\ & & 0 & & \end{tikzcd} .$$

Since composing two monomorphisms gives us another monomorphism (and the same for epimorphisms), we may complete it to a commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, equal] \arrow[d] & E & \\
 & & 0 & &
\end{tikzcd},$$

where $F \cong D/A$. Finally, we can find a map $C \to F$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, equal] \arrow[d] & E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}.$$

The existence of the last map is equivalent to the statement $F/C \cong E$, which is the same thing as

$$(D/A)/(B/A) \cong D/B.$$

This is the third isomorphism theorem for abelian categories.

The octahedral axiom for a triangulated category is the same thing if we replace the short exact sequences above by distinguished triangles.

Henceforth, the terms “triangle" and “distinguished triangle" will be used interchangeably.

Turning the triangle axiom. This axiom tells us that the diagram

$$\begin{tikzcd}
M \arrow[rr, "f"] & & N \arrow[dl, "g"] \\
 & P \arrow[ul, dashed, "h"] &
\end{tikzcd}$$

is a triangle if and only if the diagram

$$\begin{tikzcd}
N \arrow[rr, "g"] & & P \arrow[dl, "h"] \\
 & M[1] \arrow[ul, dashed, "-f[1]"] &
\end{tikzcd}$$

is a triangle.

Remark 2.1. The mysterious sign appearing in front of $f[1]$ has the following meaning. If

$$\begin{tikzcd} M \arrow[rr, "f"] & & N \arrow[dl, "g"] \\ & P \arrow[ul, dashed, "h"] & \end{tikzcd}$$

is a distinguished triangle then in the long sequence

$$\cdots \to P[-1] \xrightarrow{-h[-1]} M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1] \xrightarrow{-f[1]} N[1] \xrightarrow{-g[1]} P[1] \to \cdots .$$

each four-term sequence gives a distinguished triangle.

Remark 2.2. This axiom makes the situation totally different regarding triangles in triangulated categories compared to short exact sequences in abelian categories, where $0 \to M \to N \to P \to 0$ can be interpreted as "M and P are smaller than N". In a distinguished triangle, we do not have a notion of which elements are the "smaller" ones.

Remark 2.3. The only simple objects of $\mathrm{Coh}(X)$ are the skyscraper sheaves on points of X. Since they are simple, they could be called "small" objects in the sense of the previous remark. If a functor $\mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ preserves "small" things, then one would obtain a map $X \to Y$ between the underlying topological spaces. This is of course not the case. In general, our derived functors are not going to preserve short exact sequences. In particular, they are not going to preserve "small" objects, so in the case of $\mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ they will not induce a map between the underlying topological spaces. Derived functors are just going to preserve the collection of distinguished triangles.

Exercise 2.4. Let $\mathcal{C}$ be a triangulated category and $X \in \mathcal{C}$. Use the triangulated category axioms to prove that

$$\begin{tikzcd} M \arrow[rr, "f"] & & N \arrow[dl, "g"] \\ & P \arrow[ul, dashed, "h"] & \end{tikzcd}$$

induces a long exact sequence in $\mathrm{Hom}^i_{\mathcal{C}}(X, -)$:

$$\begin{tikzcd} & & \cdots \arrow[dll, bend left] \\ \mathrm{Hom}^i_{\mathcal{C}}(X,M) \arrow[r] & \mathrm{Hom}^i_{\mathcal{C}}(X,N) \arrow[r] & \mathrm{Hom}^i_{\mathcal{C}}(X,P) \arrow[dll, bend left] \\ \mathrm{Hom}^{i+1}_{\mathcal{C}}(X,M) \arrow[r] & \mathrm{Hom}^{i+1}_{\mathcal{C}}(X,N) \arrow[r] & \mathrm{Hom}^{i+1}_{\mathcal{C}}(X,P) \arrow[dll, bend left] \\ \cdots . & & \end{tikzcd}$$

Use this to prove that $h = 0$ implies $N \cong M \oplus P$.

2.1.2. *Standard t-structure.* Let $\mathcal{A}$ be an abelian category. Let us recall the fully faithful functor

$$I \colon \mathcal{A} \to \mathcal{D}(\mathcal{A})$$
$$M \mapsto \left(\substack{\cdots \to 0 \to M \to 0 \to \cdots \\ M \text{ at position } 0} \right).$$

Let us also recall that if $M, N \in \mathcal{A}$, then $\mathrm{Hom}^i_{D(\mathcal{A})}(M,N) \cong \mathrm{Ext}^i_{\mathcal{A}}(M,N)$. Note this is 0, for $i < 0$.

Definition 2.5. Let $i \in \mathbb{Z}$, the *truncation functor* $\tau_{\leq i}\colon C(\mathcal{A}) \to C(\mathcal{A})$ on $C(\mathcal{A})$ is given by the formula

$$\tau_{\leq i}\left(\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \cdots\right) = \left(\cdots \to M^{i-1} \to \operatorname{Ker}(d^i) \to 0 \to 0 \to \cdots\right).$$

Remark 2.6. We have,

$$H^j(\tau_{\leq i}(M^\bullet)) = \begin{cases} H^j(M^\bullet) & \text{if } j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the truncation functors descend to $D(\mathcal{A})$, but they are not triangulated. If we had defined the truncation functors by the rule

$$\left(\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \cdots\right) \mapsto \left(\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \to 0 \to 0 \to \cdots\right)$$

instead, they would not descend to $D(\mathcal{A})$. However, they descend to the homotopy category $K(\mathcal{A})$. These truncations are called *"stupid" truncations.*

We now describe a "filtration" for $D(\mathcal{A})$ using the truncation functors. A triangle

$$\begin{tikzcd} M^\bullet \arrow[rr, "f"] & & N^\bullet \arrow[dl, "g"] \\ & P^\bullet \arrow[ul, dashed, "h"] & \end{tikzcd}$$

induces a long exact sequence in $\mathcal{A}$, i.e.,

$$\begin{tikzcd} & & \cdots \arrow[dll, out=-10, in=170] \\ H^i(M^\bullet) \arrow[r] & H^i(N^\bullet) \arrow[r] & H^i(P^\bullet) \arrow[dll, out=-10, in=170] \\ H^{i+1}(M^\bullet) \arrow[r] & \cdots . & \end{tikzcd}$$

For $M^\bullet \in D(\mathcal{A})$ define $C_i \in D(\mathcal{A})$ by the triangle

$$\begin{tikzcd} \tau_{\leq i-1}(M^\bullet) \arrow[rr] & & \tau_{\leq i}(M^\bullet) \arrow[dl] \\ & C_i \arrow[ul, dashed] & \end{tikzcd} .$$

This induces a long exact sequence in $H^i(-)$:

$$\begin{tikzcd} & & \cdots \arrow[dll, out=-10, in=170] \\ 0 \arrow[r] & H^i(\tau_{\leq i}(M^\bullet)) \arrow[r] & H^i(C_i) \arrow[dll, out=-10, in=170] \\ 0 \arrow[r] & \cdots . & \end{tikzcd}$$

Hence

$$H^j(C_i) = \begin{cases} H^i(M^\bullet) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

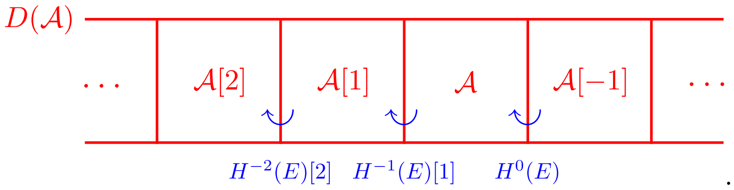
Therefore, C_i is concentrated in degree i with

$$C_i \cong H^i(M^\bullet)[-i] \in \mathcal{A}[-i].$$

Then we get a canonical "filtration" for any $E = M^\bullet \in D(\mathcal{A})$ with factors in $\mathcal{A}[d]$

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\leq i-1}(E) \arrow[rr] & & \tau_{\leq i}(E) \arrow[dl] \arrow[rr] & & \tau_{\leq i+1}(E) \arrow[dl] \arrow[r] & \cdots \\
& & H^i(E)[-i] \arrow[ul, dashed] & & H^{i+1}(E)[-(i+1)] \arrow[ul, dashed] & &
\end{tikzcd}.$$

Therefore, every object E in $D(\mathcal{A})$ has composition series in $\mathcal{A}[d]$ for $d \in \mathbb{Z}$:



The previous picture caricatures the canonical filtration of the derived category as a "film". Describing an object of E is equivalent of the fact of passing from the right to the left through every "frame" in which we collect the data of an element $H^{-i}(E)[i]$ of $\mathcal{A}[-i]$.

Remark 2.7. A derived functor does not preserve the t-structure since it does not preserve $\mathcal{A}$. For example, the derived functor $\mathrm{RHom}(-, \mathbb{C}[x])$ on $D(\mathcal{A})$, where $\mathcal{A} = \mathrm{Mod}_{\mathrm{fg}}(\mathbb{C}[x])$ for $i > 0$.

Example 2.8. (2-step complexes) Fix $A, B \in \mathcal{A}$. Consider $E \in D(\mathcal{A})$ such that

$$H^i(E) = \begin{cases} A & \text{if } i = -1, \\ B & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The composition factors of E are B and $A[1]$, i.e. we have the filtration

$$\begin{tikzcd}
0 \arrow[r, "\cong"] & \tau_{\leq -2}(E) \arrow[rr] & & \tau_{\leq -1}(E) \arrow[dl, equal] \arrow[rr] & & \tau_{\leq 0}(E) \arrow[dl] \arrow[r, equal] & E \\
& & A[1] \arrow[ul, dashed] & & B \arrow[ul, dashed] & &
\end{tikzcd}.$$

Therefore,

$$\begin{tikzcd}
A[1] \arrow[rr] & & E \arrow[dl] \\
& B \arrow[ul, dashed, "\eta"] &
\end{tikzcd}$$

is a triangle. Then E is classified up to isomorphism by $\eta \in \mathrm{Hom}_{D(\mathcal{A})}(B, A[2]) \cong \mathrm{Ext}^2_{\mathcal{A}}(B, A)$.

2.2. **Grothendieck groups.** Let D be a triangulated category.

Definition 2.9. The *Grothendieck group* $K_0(D)$ of D is given by

$$K_0(D) := \bigoplus_{M \in D/\cong} \mathbb{Z} \cdot [M] \Big/ \left\{ \begin{array}{c} [N]=[M]+[P], \text{ for each triangle} \\ \begin{tikzcd} M \arrow[rr] & & N. \arrow[dl] \\ & P \arrow[ul, dashed] & \end{tikzcd} \end{array} \right\}.$$

Proposition 2.10. *The inclusion $I \colon \mathcal{A} \to D^b(\mathcal{A})$ induces a map*

$$K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$$

which is an isomorphism of groups.

Proof. For any $M \in \mathcal{A}$, the diagram

$$\begin{tikzcd} M \arrow[rr, "\mathrm{id}_M"] & & M \arrow[dl] \\ & 0 \arrow[ul, dashed] & \end{tikzcd}$$

is a triangle. By the "turning the triangle" axiom we have

$$\begin{tikzcd} M \arrow[rr] & & 0 \arrow[dl] \\ & M[1] \arrow[ul, dashed] & \end{tikzcd}$$

is a triangle too. Then $[0] = [M] + [M[1]]$, so $[M[1]] = -[M]$. Taking the canonical filtration we get

$$\begin{aligned} [M] &= \sum_{i \in \mathbb{Z}} \left[H^i(M)[-i]\right] \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \left[H^i(M)\right]. \end{aligned}$$

Consider the map $K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$ defined by

$$M^\bullet \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \left[H^i(M^\bullet)\right].$$

This is the inverse of the obvious map induced by I. Hence is an isomorphism. □

Exercise 2.11. Suppose $\mathrm{gldim}(\mathcal{A}) \leq 1$ (global dimension of $\mathcal{A}$), i.e., for all $M, N \in \mathcal{A}$ the group

$$\mathrm{Ext}^i_{\mathcal{A}}(M, N) = 0 \text{ for all } i > 1.$$

Then any $E \in D^b(\mathcal{A})$ satisfies

$$E \cong \bigoplus_{i \in \mathbb{Z}} H^i(E)[-i].$$

2.3. **Coherent sheaves.** Let X be a variety over $\mathbb{C}$, i.e., a reduced, irreducible scheme of finite type over $\mathrm{Spec}(\mathbb{C})$. Denote its ringed space by $(X, \mathcal{O}_X)$.

Example 2.12. For the affine space $\mathbb{A}^n_{\mathbb{C}} = (\mathbb{C}^n, \mathcal{O}_X)$, and a (Zariski) open subset $U \subset \mathbb{A}^n$ we have

$$\begin{aligned} \mathcal{O}_X(U) &:= \{f(x_1, \ldots, x_n)/g(x_1, \ldots, x_n) \mid f, g \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } g \neq 0 \text{ on } U\}, \\ \mathcal{O}_{X,x} &:= \{f(x_1, \ldots, x_n)/g(x_1, \ldots, x_n) \mid f, g \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } g(x) \neq 0\}. \end{aligned}$$

Definition 2.13. The set $\mathrm{Coh}(X) \subset \mathrm{Mod}(\mathcal{O}_X)$ of *coherent sheaves* is given by objects $E \in \mathrm{Mod}(\mathcal{O}_X)$ satisfying the following condition: for every $x \in X$ there exist an open set U such that $x \in U \subset X$ and there is a surjection of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{\oplus n}|_U \to E|_U \to 0,$$

for some $n \in \mathbb{N}$.

Definition 2.14. The *fibre* $E_{(x)}$ of $E \in \mathrm{Coh}(X)$ at the (closed) point $x \in X$ is defined by

$$E_{(x)} := E_x \otimes_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} \big/ m_{X,x}\right),$$

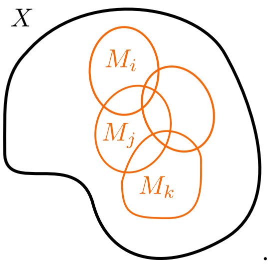
where E_x is the stalk and $m_{X,x}$ is the unique maximal ideal of the local ring $\mathcal{O}_{X,x}$.

Remark 2.15. $E_{(x)}$ is a finite dimensional vector space over the residue field

$$\mathbb{C} \cong \mathcal{O}_{X,x} \big/ m_{X,x}.$$

Remark 2.16. If X is affine ($V(I) \subset \mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]/I$) then
$$\mathrm{Coh}(X) \cong \mathrm{Mod}_{\mathrm{fg}}(\mathbb{C}[X]).$$

Remark 2.17. If X is any variety. For each $E \in \mathrm{Coh}(X)$, X can be covered by a family of affine open sets $\{U_i = \mathrm{Spec}(A_i)\}$ such that $E|_{U_i} \cong \widetilde{M_i}$ for M_i a finitely generated A_i-module



2.3.1. *An example: Vector bundles.* Let X be a variety.

Definition 2.18. A *vector bundle* $\mathcal{E}$ of rank n on X is a locally-free $\mathcal{O}_X$-module of rank n in $\mathrm{Coh}(X)$, i.e., $\mathcal{E} \in \mathrm{Coh}(X)$ such that there is a open cover $\{U_i\}$ of X together with isomorphisms of $\mathcal{O}_X$-modules
$$\mathcal{O}_X^{\oplus n}|_{U_i} \xrightarrow{\varphi_i} \mathcal{E}|_{U_i},$$
such that $\varphi_j^{-1} \circ \varphi_i \in GL_n(\mathcal{O}_X, U_i \cap U_j)$.

Definition 2.19. An *algebraic vector bundle* $\mathbb{E}$ of rank n on X is the following data:

- A scheme $\mathbb{E}$.
- A morphism $\pi \colon \mathbb{E} \to X$.
- An open covering $\{U_i\}$ of X.
- Isomorphisms $\varphi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$.

Making the following diagrams commute
$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[d, "\pi"'] & U_i \times \mathbb{C}^n \arrow[d, "\pi_i"] \\
U_i \arrow[r, equal] & U_i,
\end{tikzcd}$$
and such that the glueing maps $\varphi_j^{-1} \circ \varphi_i \in GL_n(\mathcal{O}_X, U_i \cap U_j)$.

Both descriptions are characterised by cocycles in $GL_n(\mathcal{O}_X, U_i \cap U_j)$. We will show that both descriptions are equivalent.

Definition 2.20. A *section* of an algebraic vector bundle $\mathbb{E}$ over an open set $U \subset X$ is a map $s \colon U \to \pi^{-1}(U)$ such that $\pi \circ s = \mathrm{id}_U$.

Definition 2.21. Let F, G be two $\mathcal{O}_X$-modules, consider the presheaf
$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X}(F|_U, G|_U).$$
It is a sheaf, and it is called the sheaf $\mathcal{H}om$ between F and G. It is also an $\mathcal{O}_X$-module and it is denoted by
$$\mathcal{H}om_{\mathcal{O}_X}(F, G).$$

Given an algebraic vector bundle $\mathbb{E}$, with data (U_i, φ_i) we can produce a locally free sheaf. For every open set $U \subset X$ define $E^\vee(U)$ as the set of sections of $\pi \colon \mathbb{E} \to X$ over U. This defines an $\mathcal{O}_X$-module $E^\vee$. Define $E := \mathcal{H}om_{\mathcal{O}_X}(E^\vee, \mathcal{O}_X)$. Then $E \in \mathrm{Coh}(X)$ with the data (U_i, φ_i) satisfy the conditions of a vector bundle.

For the other direction, we will need the following theorem.

Definition/Theorem 2.22. *Let X be a variety and E a coherent sheaf of $\mathcal{O}_X$-algebras, i.e., a coherent sheaf of $\mathcal{O}_X$-modules and a sheaf of rings at the same time. Then, there is a unique scheme $\underline{\mathrm{Spec}}_X(\mathcal{A})$ and a morphism $f\colon \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ such that the following conditions hold:*

- *For every $V \subset X$ open affine scheme of X*
$$f^{-1}(V) \cong \mathrm{Spec}\,\mathcal{A}(V).$$
- *The map f is compatible with restrictions $U \hookrightarrow V$ on X, i.e., the inclusion $f^{-1}(U) \hookrightarrow f^{-1}(V)$ correspond to the restriction homomorphism of rings*
$$\mathcal{A}(V) \to \mathcal{A}(U).$$

The scheme $\underline{\mathrm{Spec}}_X(\mathcal{A})$ is called the relative spectrum *of $\mathcal{A}$ over X.*

Given a vector bundle $E \in \mathrm{Coh}(X)$, by definition we have an open covering and data (U_i, φ_i). Consider the $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(E)$ which is the symmetric algebra of E with respect to the tensor product $\otimes_{\mathcal{O}_X}$. The algebraic vector bundle associated to E is given by $\pi\colon \underline{\mathrm{Spec}}_X(\mathrm{Sym}_{\mathcal{O}_X}(E)) \to X$.

Remark 2.23. Both assignments are inverse each other. Therefore both notions of vector bundles and algebraic vector bundles coincide.

Let us compute the fibre of a vector bundle $E \in \mathrm{Coh}(X)$ at the point $x \in X$,
$$\begin{aligned} E_{(x)} &\cong \left(\mathcal{O}_X^{\oplus n}\right)_{(x)} \\ &\cong \mathbb{C}^n \\ &= \frac{\{\text{sections of } \mathbb{E} \text{ near } x\}}{\{\text{sections which vanish at } x\}} \\ &= \pi^{-1}(x). \end{aligned}$$

Example 2.24. Let $Z \overset{i}{\hookrightarrow} X$ be a closed subvariety. Consider the short exact sequence of $\mathcal{O}_X$-modules
$$0 \to \mathcal{L}_Z \to \mathcal{O}_X \to i_*(\mathcal{O}_Z) \to 0,$$
where $\mathcal{L}_Z$ is the ideal sheaf of Z. Let us compute the fibres of $i_*(\mathcal{O}_Z)$
$$(i_*(\mathcal{O}_Z))_{(x)} = \begin{cases} \mathbb{C} & \text{if } x \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following theorem relating the vanishing of fibres with the vanishing of the stalks of a coherent sheaf.

Theorem 2.25 (Nakayama's Lemma)**.** *Let $E \in \mathrm{Coh}(X)$ and $x \in X$ then*
$$E_x = 0 \leftrightarrow E_{(x)} = 0.$$
i.e, the stalk is 0 if and only if the fibre is 0.

Corollary 2.26. *Let $E \in \mathrm{Coh}(X)$ then:*

- $\dim_{\mathbb{C}} E_{(x)}$ *is upper semi-continuous on X, i.e., the set*
$$S_i = \{\ x \in X \mid \dim_{\mathbb{C}} E_{(x)} \geq i\}$$
is closed.
- *$E \in \mathrm{Coh}(X)$ is locally free if and only if* $\dim_{\mathbb{C}} E_{(x)}$ *is constant.*
- *There is a stratification*
$$X = \bigsqcup_{i \in I} X_i,$$
where $X_i \subset X$ are locally closed subvarieties, such that $E|_{X_i}$ is locally-free.

- $\operatorname{supp}(E) = \{x \in X | E_{(x)} \neq 0\}$ *is a closed subset.*

In particular, E is a vector bundle with fibres which "jump up" on closed subsets.

Sketch of proof. Let us suppose $X = \operatorname{Spec}(A)$ is affine. By definition of $\operatorname{Coh}(X)$, we have for every point $x \in X$ an exact sequence:

$$\cdots \to A^{\oplus m} \xrightarrow{\varphi} A^{\oplus n} \to E \to 0.$$

If we take fibers at x we obtain,

$$\cdots \to \mathbb{C}^m \xrightarrow{\varphi \otimes_A (A/m_x)} \mathbb{C}^n \to E_{(x)} \to 0.$$

Where the map $\varphi \otimes_A (A/m_x)$ is just an $n \times m$-complex valued matrix. Finding the rank of this matrix and using Nakayama's lemma proves the corollary. □

2.4. f_* **and** f^***.** Let $f\colon X \to Y$ be a map of varieties. We have a functor,

$$f^*\colon \operatorname{Coh}(Y) \to \operatorname{Coh}(X).$$

If f is proper then we have another functor,

$$f_*\colon \operatorname{Coh}(X) \to \operatorname{Coh}(Y).$$

We have $f^* \dashv f_*$. In particular, f^* is right exact and f_* is left exact.

Example 2.27. Suppose X, Y affine, then

$$\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B,$$

$$B \xrightarrow{f^\#} A.$$

Suppose M is a finitely generated A-module and N is a finitely generated B-module, then $\widetilde{M} \in \operatorname{Coh}(\operatorname{Spec} A)$ and $\widetilde{N} \in \operatorname{Coh}(\operatorname{Spec} B)$. Furthermore,

$$f_*(\widetilde{M}) = \widetilde{M_B},$$

$$f^* \widetilde{N} = \widetilde{N \otimes_B A}.$$

Example 2.28 (Non-proper map)**.** Let $f\colon \mathbb{A}^1_{\mathbb{C}} \to *$, then $f^\#\colon \mathbb{C} \to \mathbb{C}[t]$. The sheaf $f_*\left(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}\right)$ is not coherent. Hence, f_* do not preserve coherent sheaves.

Example 2.29 (Inclusion of a point)**.** Let $\{x\} \overset{i}{\hookrightarrow} X$. Then,

$$i^*(E) = E \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x}/m_{X,x})$$

and $i_*(\mathcal{O}_{\operatorname{Spec}\mathbb{C}}) = \mathcal{O}_x$ is the skyscraper sheaf at $x \in X$. If $X = \operatorname{Spec} A$, we obtain the fibre of E at x

$$i^*(E) = E \otimes_{A_{(x)}} (A/m_{X,x}) = E_{(x)}.$$

Where $i^\#\colon A \to A/m_{X,x}$. By adjunction we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_x) &= \operatorname{Hom}_{\mathcal{O}_X}(E, i_*(\mathcal{O}_{\operatorname{Spec}\mathbb{C}})) \\ &= \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}\mathbb{C}}}(i^*E, \mathcal{O}_{\operatorname{Spec}\mathbb{C}}) \\ &= \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}\mathbb{C}}}(E_{(x)}, \mathcal{O}_{\operatorname{Spec}\mathbb{C}}) \\ &= \left(E_{(x)}\right)^{\vee}. \end{aligned}$$

Exercise 2.30. Let $f\colon X \to Y$ prove that

$$(f^*E)_x \cong E_{(f(x))}.$$

This is saying in some extent that the pullback is "easy" and the pushforward is "hard".

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