Math. Scientist, 1984, **9**, 79–94

Lewis Carroll as a Probabilist and Mathematician

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(Received: 25 June 1984)

Abstract

This article focuses on the thirteen problems, some improperly posed, of a probabilistic nature, published with answers and solutions by Lewis Carroll (C. L. Dodgson) among his 72 *Pillow Problems* of 1893. These are analysed in Sections 3 and 4 in the context of the state of English probability at the time, especially the writings of De Morgan, Venn, and Whitworth pertaining to Bayes' theorem, described in Section 2. Section 5 briefly reassesses, from the standpoint of linear algebra, Dodgson's current standing as an undistinguished mathematician.

1. Introduction

Lewis Carroll, the Reverend Charles Lutwidge Dodgson (1832–1898), was a don at Christ Church, Oxford, and a mathematical lecturer in the University. Little has been written, however, about the mathematical aspect of Dodgson's creativity; most mathematicians know him mainly as the celebrated author of *Alice* and other literary works for children (see Carroll (1982)). Even less attention has been devoted to Dodgson's interest in probability; the exploration of this facet of his work is the main aim of this article.

The centenary of Dodgson's birth witnessed a number of tributes to his creativity, including several to his activity as a logician and mathematician (Braithwaite (1932); Russell (1932); Eperson (1933)). Of these authors, Eperson makes a serious attempt at appraisal of his mathematics, with emphasis on Dodgson's preoccupation with Euclidean geometry, but with several pages devoted to the 72 mathematical problems entitled *Pillow Problems* (see Carroll (1958)). It seems that it is exclusively in these that Dodgson's probabilistic interests manifest themselves. All 72 mathematical problems are claimed to have been formulated and worked out mentally in bed at night. The questions are stated together in Chapter 1, with a date of solution, and page references to the answer and a solution. Chapter 2 contains the answers, while Chapter 3 gives the mental solution procedure by which the answers were obtained. Twelve probability problems, Nos. 5, 10, 16, 19, 23, 27, 38, 41, 45, 50, 58, 66 occur in the 'Subjects Classified' at the beginning of the book under the subheading *Chances* of the heading *Algebra*. A thirteenth problem (No. 72) occurs under the rather mysterious heading *Transcendental Probabilities*. The last 12 of these 13 problems are reproduced in our Section 3 (No. 5 is subsumed by No. 16). All these problems are dated between March 1876 (No. 38) and August 1890 (No. 10).

The only one of the 13 problems to have attracted more than fleeting attention is No. 72. Obviously the problem cannot be solved, and Dodgson's reasoning (leading to the conclusion that there is one black counter and one white counter) is incorrect. Eperson (1933), pp. 98–99, sets out Dodgson's solution, but leaves the reader to discover the fallacy in it, adding that he will content himself

> 'with remarking that if one applies a similar argument to the case of a bag containing 3 unknown counters, black or white, one reaches the still more paradoxical conclusion that there cannot be 3 counters in the bag!'

Eperson does not, however, exclude the possibility that Dodgson was indulging in a little leg-pulling, in spite of the serious tone of Dodgson's May 1893 Introduction, which reads in part:

> 'If any of my readers should feel inclined to reproach me with having worked too uniformly in the region of common-place and with never having ventured to wander out of the beaten tracks, I can proudly point to my one problem in Transcendental Probabilities—a subject in which, I believe, very little has yet been done by even the most enterprising of mathematical explorers. To the casual reader it may seem abnormal, and even paradoxical; but I would have such a reader ask himself, candidly, the question "Is Life not itself a Paradox?"'

Warren Weaver (1956), p. 119, says much more bluntly on the topic of No. 72,

> 'In his attack on the problem (which as stated cannot be solved) he makes two dreadful mistakes. First he assumes, incorrectly, that the statement implies the probabilities of *BB*, *BW*, and *WW* ... are $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$ respectively.'

and later (p. 120) adds '... in the probability problem cited he failed to grasp the principle of insufficient reason'. We shall show in passing that these comments, relating to the first of the 'dreadful mistakes' are, in the context of Dodgson's milieu, misleading. It is a pity that Weaver (1956) confines himself only to No. 72 amongst the probabilistic problems within his article dealing with the mathematical aspect of Dodgson's work, since he himself is well known as a popularizer of probability theory. The publisher's notes preceding Weaver's *Lady Luck* (1964) describe him as an *aficionado* of Lewis Carroll, and this is emphasized by an earlier article of Weaver (1954). However, of the three places where Lewis Carroll is mentioned in his book, the first is concerned with Carroll's logical arguments, and the other two are fleeting references.

There are, as we shall see, others amongst Carroll's probabilistic problems which are not properly formulated (in particular the two problems of 1884, Nos. 45 and 58), whose solution is therefore generally incorrect. Yet by studying the structure of Dodgson's problems as a whole, and their solution procedures, we obtain an insight into the standing and understanding of probability theory within the English mathematical community of the time. This study thus seeks to go beyond any curiosity value that the problems and their solutions might have, simply as a product of Dodgson's imagination and mathematics; it necessitates a brief look at the probabilistic background against which *Pillow Problems* was created, to which we shall pass in Section 2. Sections 3 and 4 briefly analyse the problems and their solutions.

Section 5, in line with the modest revival of interest in Lewis Carroll, of which Gattégno's (1977) recent work is representative, addresses itself briefly to Dodgson's significant contribution to determinants and linear equation systems, to which generally little weight has been accorded in the past. This will perhaps offset Weaver's (1956) rather negative assessment of Dodgson's mathematics in general: 'In all of Dodgson's mathematical writings', Weaver writes, 'it is evident that he was not an important mathematician.' This evaluation has spread among various non-mathematical biographers, even though the article of Eperson by which it was partly influenced takes a more moderate view.

We conclude this introduction by listing those probabilistic problems which have been mentioned in the literature, in addition to the above citations. Eperson (1933) also mentions No. 5. Fisher (1975), pp. 221–227, as well as echoing Eperson and Weaver on No. 72, mentions problems Nos. 5 and 10 (which were solved correctly by Dodgson). Russell (1932), in passing and without comment, mentions No. 45. Other allusions to these problems may exist in literature that we have not seen.

2. English probability in the nineteenth century

Dodgson gives no indication as to where he might have acquired his knowledge of probability. In France, books on probability by Condorcet, Laplace, Poisson, Cournot and Bertrand had followed the earlier European treatises of Huygens, Jacob Bernoulli and Montmort. In contrast, until the appearance of *Pillow Problems* in 1893, this branch of mathematics appears to have played a relatively minor role in England. (We shall not be concerned in this paper with the development of statistics.)

The first important work on probability to appear in English in book form had been De Moivre's celebrated *Doctrine of Chances* (1718; 1738; 1756). The next work with a major impetus was De Morgan's (1838) *Essay on Probabilities* (followed by an entry in the *Encyclopaedia Metropolitana*, Volume 2, in 1845). Augustus De Morgan (1806–1871), Professor of Mathematics at University College, London, had strong connections with Cambridge. Partly in consequence of his influence, Cambridge University seems to have become the centre of mathematical work on probability in England, through the appearance of monographs in 1865, 1866 and 1867 respectively, by Isaac Todhunter (1820–1884), William Whitworth (1840–1905) and John Venn (1834–1923). Also associated with Cambridge were Robert Ellis (1817–1859) and James Glaisher (1848–1928). Of the eminent English figures in probability outside Cambridge at that time, we mention Morgan Crofton (1826–1915), of the Royal Military Academy at Woolwich, who wrote a remarkable entry on probability in Volume 2 of the ninth edition of the *Encyclopaedia Britannica* (1885) and the Savilian Professor of Astronomy at the University of Oxford, W. F. Donkin, who wrote on probability in the 1850s.

The prevailing probabilistic climate is expressed in the first sentence of Venn's (1867) *Logic of Chance*:

> 'Any work on Probability by a Cambridge man will be so likely to have its scope and its general treatment of the subject prejudged, that it may be well to state at the outset that the following Essay is in no sense mathematical.'

Venn alludes in this same preface to De Morgan's (1847) *Formal Logic* and George Boole's† (1854) *The Laws of Thought*, both of which, while approaching probability from the direction of logic, have substantial mathematical content. (See Kolmogorov and Yushkevich (1978) for further details on the English logical school of probability.) It is tempting to suppose that Dodgson, with his interest in logic, came to an interest in probability through an acquaintance with such writings.

Be that as it may, with the emphasis on 'inverse probabilities' in his problems of drawing counters from bags, and his approach to the formulation of ignorance in a prior distribution, it is not unlikely that Dodgson eventually turned to De Morgan's (1838) *Essay*, and also probably to Whitworth's *Choice and Chance*, whose first edition had appeared in 1867, to gain his probabilistic knowledge.

De Morgan's *Essay* seems, in its time, to have played the same role as Feller's Volume I assumed in more recent years as a standard modern introduction to probability in English. It has a brief historical sketch in its Preface; the first six chapters and one appendix are devoted to the principles of probability with emphasis on gaming, and drawing balls from bags. The remaining seven chapters and five appendices are devoted to what was then the most common application of this theory, the consideration of life contingencies. De Morgan's historical sketch indicates that he was well acquainted with continental work on probability, even including the 1837 treatise of Poisson. After speaking of De Moivre, he states:

> 'De Moivre, nevertheless, did not discover the inverse method. This was first used by the Rev. T. Bayes, in *Phil. Trans. liii* 370; and the author, though now almost forgotten, deserves the most honourable remembrance from all who treat the history of this science.'

There is, accordingly, a chapter entitled 'On Inverse Probabilities' (following one entitled 'On Direct Probabilities'). We note (in reference to Dodgson's problem No. 10) that the notion of mathematical expectation is very clearly presented by De Morgan on p. 97, as one would expect in a treatise dealing with actuarial matters. It is these chapters of De Morgan's *Essay* together with Examples 134–139 in the 'Accession of Knowledge' section, and possibly several further sections involving 'inverse probabilities' such as the section entitled 'Credibility of Testimony', of Whitworth's book, that seem to have motivated Dodgson's probability problems.

For convenience, we state here versions of what are commonly known today as the theorem of total probability and Bayes' theorem.‡ If A_i, $i=1,\ldots,n$, is a set of mutually exclusive and exhaustive events such that $P(A_i)>0$, $i=1,\ldots,n$ and B is any event, then

$$P(B)=\sum_{i=1}^{n} P(B|A_i)P(A_i), \tag{2.1}$$

(theorem of total probability). Further, if $P(B)>0$, then

$$P(A_j|B)=\frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \tag{2.2}$$

† Boole was at Queen's College, Cork, but was in contact with De Morgan.
‡ We do not consider here the question of the discovery of this result (Stigler (1983)).

$j=1,\ldots,n$ (Bayes' theorem). We may regard $P(A_i)$, $i=1,\ldots,n$ as prior (or antecedent) probabilities of the partition $\{A_i\}$, $i=1,\ldots,n$ and $P(A_i|B)$, $i=1,\ldots,n$ as the posterior probabilities of the same partition. In the simple case where $P(A_1)=P(A_2)=\cdots=P(A_n)$, (2.2) becomes

$$P(A_j|B)=\frac{P(B|A_j)}{\sum_{i=1}^{n} P(B|A_i)}. \tag{2.3}$$

This is the case to which most emphasis was given by the early writers; in it, $P(A_j|B)$ can be calculated for a fixed j once the ratios $P(B|A_1)$: $P(B|A_2)$: $\cdots$:$P(B|A_n)$ are known.

De Morgan (1838), pp. 53–55, proceeds through two examples in his chapter on inverse probabilities concerned with drawing balls from urns. These lead to the formulation of (2.3) as Principle V of probability, followed by three further examples on it, the first one on 'credibility of evidence'. The third of these (to give the flavour) reads as follows:

> 'There are two urns, having certainly 3 and 2 white balls; and in one or the other, but which is not known, is a black ball. A ball is drawn and replaced; and this process is repeated, but whether out of the same urn as before is not known. Both drawings give a white ball: what is the probability of the several cases from which this result might have happened?'

His solution proceeds as follows:

> 'Since the black ball is as likely to be in one or in the other, the antecedent state of things (so far as a single drawing is concerned) is the same as if there were four urns, as follows:
> I. (3 white) II. (3 white, 1 black) III. (2 white) IV. (2 white, 1 black).
> There are 16 possible cases, ...'.

That is, he takes the 4^2 ordered pairings of Roman numerals as the equally likely antecedent states (the A_i) for the two drawings. It is, however, not difficult to see that a pairing such as (I, III) is not appropriate, and there are only eight possible ordered pairs, if the black ball is fixed once and for all.

The next example is aimed at showing how to handle a partition A_i, $i=1,\ldots,n$ with unequal prior probabilities via (2.2), then (bottom of p. 59) he begins to discuss an example to illustrate the situation:

> '... having only a first event by which to judge of the preceding state of things, we ask what is the probability of a second event yet to come.'

This amounts to the calculation at the first stage of the posterior probabilities $P(A_j|B)$, $j=1,\ldots,n$ on the basis of observation of 'a first event' B, and then using these to calculate the probability of a second event yet to come, say C, by the theorem of total probability in the form

$$P(C|B)=\sum_{i=1}^{n} P(C|B,A_i)P(A_i|B). \tag{2.4}$$

This is enunciated verbally as Principle VI of probability (on p. 61; Principles I to IV are elementary rules for manipulating probabilities, and are enunciated in the chapter on direct probabilities).

This principle is followed by an example which begins as follows:

> 'PROBLEM. There is a lottery of 10 balls, each one white or black, but which is not known: drawings are made, *after each of which the ball is replaced.* The first five drawings are white; what chance is there that the next two drawings shall be white.'

His subsequent argument implies that the 11 initial states A_i: (i white, $10-i$ black balls), $i=0, \ldots, 10$ are equiprobable, thus expressing ignorance by a uniform prior, $P(A_1)=P(A_2)=\cdots=P(A_n)$. He develops this example into the well-known 'Rule of Succession' for N (large) binomial trials, where one event A has occurred m times and the complementary event $\bar{A}$ has occurred $n=N-m$ times, giving the probability of the next trial resulting in an A as

$$(m+1)/(m+n+2).$$

De Morgan's specification of prior probabilities $P(A_i)$, $i=1, \ldots, n$, is vague in those instances when he in fact wishes to express ignorance, but in practice he does so through a uniform prior. The objection of Venn (Chapter VII of the 3rd edition, Venn (1888); not to mention Venn's remarks on the 'Rule of Succession' in Chapter VIII) to De Morgan's and others' inverse probability calculations is precisely to the ambiguity in allocating prior probabilities, especially in the presence of ignorance. In particular Venn (1888), p. 183, alludes to a question of Whitworth's ((1901), question 136) which reads:

> 'A purse contains ten coins, each of which is either a sovereign or a shilling: a coin is drawn and found to be a sovereign, what is the chance that this is the only sovereign?'

Whitworth in the solution of this problem assumes that initially each coin in the bag has probability $\frac{1}{2}$ of being a sovereign, so that the prior distribution of the number of sovereigns is binomial. Question 137 of Whitworth (1901) is, however, worded as follows:

> 'A purse contains ten coins, which are either sovereigns or shillings, and all possible numbers of each are equally likely: a coin is drawn and found to be a sovereign, what is the chance it is the only sovereign?'

This question implies a uniform prior over the number of sovereigns, in the manner of De Morgan. Whitworth (1901) carries the following footnote to his Answer to Question 136:

> 'In the statement of this question the words "each of which" implies that the purse has been filled in such a way that each coin separately is equally likely to be a sovereign or a shilling. For instance each coin may have been taken from either of two bags at random, one containing sovereigns and the other shillings. This case is carefully marked off from that of Qn. 137. Mr Venn in

> his strictures on this solution (*Logic of Chance*, Second Edition, pp. 166, 167) appears to overlook the significance of the words "each of which", and implies that the solution of Qn. 137 would have been applicable to Qn. 136.'

When Whitworth approaches the subject entitled 'Inverse probabilities' later in the text, he writes (1901), p. 183:

> 'The term inverse probability is used by many writers to denote those cases in which the *a priori* probability of a cause is modified by the observation of some effect due to the cause.
>
> But there seems no reason to regard these cases as belonging to a special category. All probability is based on the limitation of our knowledge, and every accession of knowledge in regard to a contingent event alters the probability (to us) of the event.'

Distaste for the notion of inverse probability had by 1900 in some quarters reached the level expressed by George Chrystal (1906), Honorary Fellow of Corpus Christi College, Cambridge, in the chapter on probability in his respected textbook:

> 'All matter of debatable character or of doubtful utility has been excluded. Under this head fall, in our opinion, the theory of *a priori* or inverse probability, and the applications to the theory of evidence. The very meaning of some of the propositions usually stated in some of these theories seems to us to be doubtful. Notwithstanding the weighty support of Laplace, Poisson, De Morgan, and others, we think that many of the criticisms of Mr. Venn on this part of the doctrine of chances are unanswerable. The mildest judgement we could pronounce would be the following words of De Morgan himself, who seems, after all, to have "doubted":— "My own impression ... is that mathematical results have outrun their interpretation".'

In the solution of his problems using Bayes' theorem, Dodgson did not concern himself with the more dubious aspects of its use, such as that embodied in the 'Rule of Succession', apart perhaps from problem No. 66. His statements of problems do initially appear to suffer from some imprecision in formulating the prior distribution $P(A_i)$, $i = 1, \ldots, n$. However on examining his solutions to them, and then their formulation, we see that in general he uses the binomial as the prior distribution in the manner of Whitworth, when he alludes to a bag containing counters each of which may be of one kind or another. This explains, for example, why in No. 72 he gives the prior probabilities for *BB*, *BW* and *WW* as $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, which is hardly a dreadful mistake, or a failure to grasp the principle of insufficient reason.

3. Dodgson's bags and counters probability problems

These problems are Nos. 5, 16, 19, 23, 27, 38, 41, 50, 66, 72. The problem is in each case stated as in *Pillow Problems*, and a brief analysis given. The reader may verify for himself that to solve such difficult problems mentally, and on the whole correctly even making use of recipes such as De Morgan's Principles V and VI, requires, as Eperson (1933), p. 99, remarks, 'unusual ability'.

Nos. 5 and 16

> There are two bags, one containing a counter, known to be either white or black; the other containing 1 white and 2 black. A white is put into the first, the bag shaken, and a counter drawn out, which proves to be white. Which course will now give the best chance of drawing a white—to draw from one of the two bags without knowing which it is, or to empty one bag into the other and then draw?

As stated this is actually Dodgson's No. 16. No. 5 requires only the calculation of the posterior distribution of the contents of the first bag after the white is drawn out; the prior distribution is uniform, so (2.3) can be used. The last part of the question then requires use of the theorem of total probability (for each of the alternatives). The problem as a whole serves as an instance for applying De Morgan's Principle VI, just as its first part needs only the direct application of Principle V. The answers obtained by Dodgson, $\frac{1}{2}$ and $\frac{5}{12}$ respectively, are correct; the problems are dated 8/9/1887 and 10/1887 (that is 8 September 1887 and October 1887).

No. 19

> There are 3 bags; one containing a white counter and a black one, another 2 white and a black, and the third 3 white and a black. It is not known in what order the bags are placed. A white counter is drawn from one of them, and a black from another. What is the chance of drawing a white counter from the remaining bag?

This problem requires first the determination of the posterior distribution over the possible pairs of bags in view of the drawings, then the application of the theorem of total probability in form (2.4) to determine the probability of white from the remaining bag. Note that the prior distribution of the six ordered pairs of bags is uniform, thus again entailing only the use of (2.3). This is an illustration of the use of De Morgan's Principles V and VI. Dodgson's solution here, $\frac{11}{17}$, is correct; the problem is undated.

No. 23

> A bag contains 2 counters, each of which is known to be black or white. 2 white and a black are put in, and 2 white and a black drawn out. Then a white is put in, and a white drawn out. What is the chance that it now contains 2 white?

The prior distribution for the states *BB*, *BW* and *WW* (*B* for black and *W* for white) is taken by Dodgson to be binomial $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, presumably in line with Whitworth and the specification '... each of which is known to be black or white.' The problem involves a modification of this prior to a posterior distribution $(\frac{1}{7}, \frac{4}{7}, \frac{1}{7})$ in view of the first deterministic input and subsequent random withdrawal (without replacement) of the two white and a black (in that or any other order). This posterior distribution is then used as prior distribution in relation to the deterministic input and random withdrawal of the white to give the second-stage posterior distribution $(\frac{1}{15}, \frac{8}{15}, \frac{6}{15})$. The problem is thus a two-stage application of (2.2). Dodgson's answer, $\frac{6}{15}=\frac{2}{5}$, is correct here; the problem is dated 25/9/1887.

No. 27

> There are 3 bags, each containing 6 counters; one contains 5 white and one black; another, 4 white and 2 black; the third, 3 white and 3 black. From two of the bags (it is not known which) 2 counters are drawn, and prove to be black and white. What is the chance of drawing a white counter from the remaining bag?

Dodgson considers the three possible pairs of bags from which the choices are made without regard to order. The uniform distribution over the pairs $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, is modified in virtue of the choice of one black and one white, to a posterior distribution $(\frac{7}{25}, \frac{9}{25}, \frac{9}{25})$ using (2.3). The theorem of total probability is then used to calculate the ultimate probability required, $\frac{17}{25}$, correctly. Thus, this is an application of De Morgan's Principle VI. However, Dodgson reasons in the process that if the bags from which the two choices are made are the second and third mentioned above, the probability of a black and a white is $\frac{1}{2}(\frac{2}{6}\,.\,\frac{3}{6}+\frac{4}{6}\,.\,\frac{3}{6})$ depending on the order of choice of bag. The spurious $\frac{1}{2}$ so introduced cancels in the application of (2.3). The problem is dated 4/3/1880.

No. 38

> There are 3 bags, '*A*', '*B*', and '*C*'. '*A*' contains 3 red counters, '*B*' 2 red and one white, '*C*' one red and 2 white. Two bags are taken at random, and a counter drawn from each: both prove to be red. The counters are replaced, and the experiment is repeated with the same two bags: one proves to be red. What is the chance of the other being red?

The appropriate approach would seem to be take the uniform prior distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ over the three possible choices of pairs from which the initial choices of counter are made, and to modify it in virtue of the choices of the two reds to the posterior distribution $(\frac{6}{11}, \frac{2}{11}, \frac{3}{11})$ using (2.3); in this it would be similar to No. 27. The wording of the question then suggests that, using the same two bags, a counter is drawn from each, with at least one being red; the question then seems to ask: what is the probability that both are red? Thus the next stage would seem to require the use of the theorem of total probability to calculate the probabilities of the (unordered) outcome pairs, *RR*, *WR*, *WW* (*W* for white and *R* for red), and then the conditional probability of *RR*, given *RR* or *WR*. This gives the answer $\frac{49}{95}$.

Dodgson's answer is $\frac{49}{72}$, obtained by beginning with the following reasoning, which ingeniously aims to facilitate the mental calculation. By noting that the bag from which the unspecified counter is drawn had a red drawn from it in the first stage, he takes the six possible ordered arrangements of *A*, *B*, *C*, the order corresponding to the bag from which the unknown counter is drawn, the bag from which the red one is twice drawn, and the remaining bag. He rates these prior 'states' to be equiprobable, uses conditional probabilities given each arrangement (for the arrangement *ABC*, for example, this is $1\,.\,(\frac{2}{3})^2$) to calculate posterior probabilities of arrangements using (2.3), and then uses the theorem of total probability. This procedure seems to be incorrect, as the 'state' ABC, for example, specifies not only bags from which drawings were made, but also the outcomes of the drawing from the start. The problem is dated March 1876, and is the earliest of the probability problems.

No. 41

> My friend brings me a bag containing four counters, each of which is either black or white. He bids me draw two, both of which prove to be white. He then says
>
> 'I meant to tell you, before you began, that there was at least *one* white counter in the bag. However, you know it now, without my telling you. Draw again.'
>
> (1) What is now my chance of drawing white?
> (2) What would it have been, if he had not spoken?

The phrase 'each of which is either black or white' is again taken, following Whitworth, to imply that the prior probability that the number of white counters is x is given by the binomial expression $\binom{4}{x}(\frac{1}{2})^4$, $x = 0, 1, 2, 3, 4$; that is $2^{-4}(1, 4, 6, 4, 1)$. Part (2) of the problem is straightforward, requiring an application of Bayes' theorem in form (2.2) to obtain the posterior distribution of the number of white counters in the bag. The theorem of total probability then gives the final answer, $\frac{1}{2}$. Dodgson's approach to part (1) is to understand the friend's information to mean that one white ball is definitely put into the bag (at the beginning, say), and the remaining three are each independently white or black with probability $\frac{1}{2}$, so the prior probability distribution for the number of white balls is now $2^{-3}(0, 1, 3, 3, 1)$. The answer which follows is $\frac{7}{12}$. This problem is dated September 1887.

We might now interpret the information given by the friend as: there is at least one white ball initially present. This conditioning would alter the binomial distribution given under (2) to the prior $(\frac{1}{15})$ $(0, 4, 6, 4, 1)$ for (1). Problems No. 38 and No. 41 both suggest that Dodgson may have had some difficulty in handling direct conditional probabilities.

No. 50

> There are 2 bags, H and K, each containing 2 counters: and it is known that each counter is either black or white. A white counter is added to bag H, the bag is shaken up, and one counter transferred (without looking at it) to bag K, where the process is repeated, a counter being transferred to bag H. What is now the chance of drawing a white counter from bag H?

This is perhaps the most complex problem of the set, to which Dodgson devotes over a page of calculation in setting out the correct solution, $\frac{17}{27}$. Again, he takes the interpretation of 'each counter is either black or white' as equivalent to a binomial prior $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ for the number of white counters. Mental solution would seem to require a phenomenal memory. One needs to proceed by specifying the four possible arrangements of counter colours according to the first transfer and second transfer. Both Bayes' theorem and the theorem of total probability are needed several times. The problem is undated.

No. 66

> Given that there are 2 counters in a bag, as to which all that was originally known was that each was either white or black. Also given that the experiment has been tried, a certain number of times, of drawing a counter, looking at it, and replacing it; that it has been white every time; and that, as a result, the chance of drawing white, next time, is $\alpha/(\alpha+\beta)$. Also given that the same experiment is repeated m times more, and that it still continues to be white every time. What would then be the chance of drawing white?

This problem shows the clear influence of the 'Rule of Succession', mentioned briefly in our Section 2. The prior distribution of the number of whites in the bag according to Dodgson's conventional interpretation of such wording should be $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. Be that as it may, if x denotes the posterior probability of *WW* and therefore $(1-x)$ of *WB* after the 'certain number of times', the probability of drawing a white next time is, by the theorem of total probability $x \cdot 1 + (1-x) \cdot \frac{1}{2} = \alpha/(\alpha+\beta)$, whence $x = (\alpha-\beta)/(\alpha+\beta)$. Thus the posterior distribution, of (*WW*, *WB*), for the first stage of the experiment is specified in terms of α and β, and may be used as a prior distribution to determine a new posterior for (*WW*, *WB*), by Bayes' theorem. The theorem of total probability can then be used again, to obtain Dodgson's answer, $\{2^m(\alpha-\beta)+\beta\}/\{2^m(\alpha-\beta)+2\beta\}$. This can be done in a few lines. Dodgson's detailed solution to this problem begins in the above manner, but mysteriously runs to $2\frac{1}{2}$ long-winded printed pages. It does not seem like a problem which could be solved mentally. The problem is dated September 1889.

No. 72

> A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colours without taking them out of the bag.

This badly formulated problem has been alluded to several times already, and the reason for the prior distribution $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ for *BB*, *BW* and *WW* explained. The problem is dated 8/9/87.

Dodgson's solution proceeds by noting that if a black were added to the bag, the chance of drawing a black, by the theorem of total probability, would be $\frac{2}{3}$. He also notes that the only composition of a bag with three counters which gives black a chance of $\frac{2}{3}$ of being drawn is two blacks and a white. He therefore deduces, incorrectly, that before the black was added, the composition had to be one B and one W. The error consists in this: since a probability calculated from the theorem of total probability is equal to the probability of an element of the partition $\{A_i\}$, $i = 1, 2, 3$ which this theorem uses, then this element of the partition must necessarily obtain. This is hardly a 'dreadful mistake' insofar as logic is concerned. The solution does not involve Bayes' theorem, but has an element of 'inverse' reasoning.

We have, in this light, an explanation of Eperson's (1933) comment that in the case of a bag with three unknown counters, by this reasoning one reaches the conclusion that there cannot be three counters in the bag. The prior distribution, with the added black counter shown in parentheses, is

$BBB(B)$	$BBW(B)$	$BWW(B)$	$WWW(B)$
$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The probability of drawing a black is thus

$$(\tfrac{1}{8}).1+(\tfrac{3}{8}).(\tfrac{3}{4})+(\tfrac{3}{8}).(\tfrac{2}{4})+(\tfrac{1}{8}).(\tfrac{1}{4})=\tfrac{5}{8}.$$

As the number $\frac{5}{8}$ does not coincide with the probability of drawing a black from any one member of the partition, Dodgson's line of deductive reasoning breaks down. A similar situation obtains if white is added, instead of black.

4. Dodgson's other probability problems

No. 10

> A triangular billiard-table has 3 pockets, one in each corner, one of which will hold only one ball, while each of the others will hold two. There are 3 balls on the table, each containing a single coin. The table is tilted up, so that the balls run into one corner, it is not known which. The 'expectation', as to the contents of the pocket, is two shillings and sixpence. What are the coins?

This problem, dated August 1890, involves the concept of expectation. In fact Dodgson's correct solution, that the coins must be two florins and a sixpence, or else a half-crown and two shillings, involves the prior calculation of conditional expectations of coin value in the pocket, conditional on the pocket containing two balls and one ball, respectively. Ultimately the concept of total expectation is used.

No. 45

> If an infinite number of rods be broken: find the chance that one at least is broken in the middle.

Dodgson's answer to this improperly formulated problem is '0·6321207 etc.' and his solution begins:

> 'Divide each rod into $(n+1)$ parts, where n is assumed to be odd, and the n points of division are assumed to be the only points where the rod will break, and be equally frangible.'

Thus the probability that a single rod breaks at a point other than the middle one is $(n-1)/n$; and the probability that n rods do so is $(1-n^{-1})^n$, so the probability that at least one of the n breaks in the middle is $1-(1-n^{-1})^n \to 1-e^{-1}=0{\cdot}6321207$ etc. as $n\to\infty$. Even accepting Dodgson's formulation as quoted, a second fallacy is that the number n of rods is taken to be the same as the number of break-points before passing to the limit. The problem is dated May 1884.

No. 58

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle.

This problem is dated 20/1/1884 and manifests Dodgson's preoccupation, noted by Eperson (1933), with Euclidean geometry. It is not properly posed since the idea of a uniform distribution of a point over the plane, which would correspond to the idea of 'random' embodied in the description, cannot be realized. The problem is interesting in that it is Dodgson's only excursion into 'continuous' probability.

Dodgson's approach to the problem, according to his solution, is to consider the longest side of an arbitrary triangle AB, which will therefore subtend the largest angle, and to consider the three-sided figure bounded by AB, and circular arcs with centres A, B respectively, and radii AB meeting at D. The third vertex, C, of the triangle is then taken to have a uniform distribution over the area of the three-sided figure ABD. If the point is located within the area of the semicircle centred at the midpoint of AB, with radius $AB/2$, the angle ACB will be obtuse, otherwise acute. The solution for the probability is thus the ratio of the area of the semicircle to the area of the three-sided figure, which is easily calculated to be $(\pi/8)/[(\pi/3)-(\sqrt{3}/4)]$, the answer given by Dodgson.

5. Dodgson and linear algebra

Gattégno (1977), p. 146, mentions that '... Bourbaki is of a somewhat different opinion...' to Weaver (1956) on Dodgson's standing as a mathematician. The reference is to Bourbaki (1974), p. 87, which (in free English translation) speaks of

> '... Kronecker's definitive form of theorems on linear equation systems with real (or complex) coefficients, which are also elucidated, in an obscure treatise, with the attention to detail characteristic of him, by the celebrated author of *Alice in Wonderland*....'

Bourbaki gives no specific references, either in the passage or in his Bibliography.

In fact the allusion is to Dodgson's main contribution to mathematics, his *Elementary Treatise on Determinants* (1867). On this, Eperson (1933), p. 94, had already concluded that

> 'It exhibits those same qualities of originality and logical consistency as his former books. It is a treatise for the specialist....'

More recently, the authors of Chapter 2 of Kolmogorov and Yushkevich (1978), p. 69, comment (in free English translation):

> 'The concept of rank and the theorem of Kronecker–Capelli were discovered independently by several investigators. The first printed proof of this theorem is due to C. L. Dodgson (1832–1898), author of the splendid stories *Alice in Wonderland* and *Alice through the Looking Glass*. The

theorem was published in his *An Elementary Treatise on Determinants*, London, 1867 in the following formulation: "For a system of n inhomogeneous equations with m unknown to be consistent, it is necessary and sufficient that the order of the largest minor different from zero be the same in the augmented and non-augmented array of the system".'

Perhaps the best authority on the standing in its day of Dodgson's work on determinants and linear equation systems is Muir (1920). In his authoritative historical survey, two works of Dodgson are mentioned: on pp. 17–18, a paper of Dodgson (1866), which gives a little-known rule for the evaluation of determinants by condensation; and the book by Dodgson (1867) which is mentioned twice (on pp. 24–32, in Muir's section entitled 'Determinants in General'; and on pp. 86–90, in the section entitled 'Linear Equations'). On p. 24, Muir begins: 'This is a textbook quite unlike all its predecessors. Professedly its main aim is logical exactitude.' On p. 86 Muir comments:

> 'Chapters iii and iv establish conditions under which equations of a set are consistent, inconsistent, interdependent, and so forth: also, consequences flowing from consistency: and, finally, necessary and sufficient tests for consistency. They may thus be expected to contain extensions and improvements of previous incidental work on the subject.'

We shall confine ourselves to just one of Dodgson's results (Muir (1920), p. 89) the main one of Chapter iv, the chapter which deals with tests for consistency. This is in the form of a necessary and sufficient condition, as is the result cited from Kolmogorov and Yushkevich earlier. The arrays of Dodgson are generally rectangular. He considers unaugmented arrays and augmented arrays of a linear equation system. If the equation system is written $Ax = b$, the unaugmented array is A and the augmented array is (A, b). A central concept is the 'evanescence' of an array: according to Muir (p. 87) this means that all the primary minors of the array vanish, or that for an $(m \times n)$ array B, where $r = \min(m, n)$, the determinant of every $(r \times r)$ submatrix of B should be zero. The condition reads:

> 'The necessary and sufficient test for a set of linear equations, which are not all homogeneous, being consistent is that either (1) there is one of them such that when it is taken along with every one of the remaining equations separately, each pair of equations so formed has its augmented array evanescent: or (2) there are m of them, $m > 1$, which contain at least m unknowns and have their unaugmented array not evanescent and are such that, when they are taken along with one of the remaining equations, each so formed set of equations has its augmented array evanescent'.

Muir's assessment of Dodgson's contributions is in general very positive. Entries on the work of other authors repeatedly contain comments on their work as it relates to Dodgson's (see e.g. Muir (1920), pp. 37, 50, 76, 90–92). It is also clear that Muir's self-assurance about his standards was high; illustrative of this is a comment (p. 16) on two papers by Renshaw, which in conclusion states that '. . . both papers are unimportant'.

We conclude by remarking that while the evidence presented in this brief survey of Dodgson's linear algebra indicates that his contributions in this field were very important, their unusual format and manner of presentation caused them to be largely ignored in the development of the subject.

6. Notes on other mathematical writings

Weaver (1954) briefly describes Dodgson's unpublished manuscripts now held at the Department of Rare Books and Special Collections of the Princeton University Library. Nothing of probabilistic interest is described in the article. Eperson (1933) had access to Dodgson's diaries, which have since been published (Green (1953)), from which several mathematical items are mentioned, none of which is probabilistic, but one of which on closer examination again indicates one of those strange lapses in the

> '... meticulous care in being precise and logical at all costs, not only when dealing with his beloved Euclid, but also in algebra and arithmetic.'

traditionally accorded to Dodgson (Eperson (1933)). The proposition is that any number whose square is the sum of two squares is itself the sum of two squares. Dodgson's solution (5 November 1890) reads:

> 'Assume x, y, z to have no common factor. If $x^2 + y^2 = z^2$, then $y^2 = (z - x)(z + x)$. Then if y has a pair of factors μ, ν, (where $\mu > \nu$ and ν may be unity),
>
> $$z + x = \mu^2,$$
> $$z - x = \nu^2,$$
>
> and $z = \frac{1}{2}(\mu^2 + \nu^2) = \frac{1}{4}[(\mu + \nu)^2 + (\mu - \nu)^2]$.
>
> Q.E.F.'

The proof rests on the supposition that if x, y, z are assumed to have no factor in common, neither will $(z - x)$ and $(z + x)$. From this, the step $z + x = \mu^2$ and $z - x = \nu^2$ follows (take x, y, z all to be positive). In fact, it is easy to see, for example if $x = 5$, $y = 12$, $z = 13$, that this is evidently not true. The argument can be extended to cover such a case by first noting that z and x can have no factor in common (for it would then be a factor of y also), and if $z + x$ and $z - x$ are both divisible by $d \geq 2$, then so is $2z$ and $2x$, whence finally, since z and x have no factors in common, $d = 2$. Then $z + x = 2\mu^2$, $z - x = 2\nu^2$ whence the conclusion follows. In the original case treated by Dodgson, where $(z - x)$ and $(z + x)$ are relatively prime, $(\mu + \nu)/2$ and $(\mu - \nu)/2$ are both integers, as both μ and ν must be odd, since $z + x$ and $z - x$ must both be odd or even, and cannot be relatively prime if both are even.

A good recent bibliography of Lewis Carroll's works is contained in Fisher (1975), and an extensive selection of his writings with a mathematical flavour is given in Carroll (1982), including an interesting piece on 'Lawn Tennis Tournaments.' Gattégno (1977), p. 148, mentions Dodgson's interest in proportional representation, and there is a recent article on this by Abeles (1981).

Acknowledgements

The author is indebted to Steve Stigler for informative discussions on the persons mentioned in Section 2. This article owes its existence to Irene Buschtedt, who provided sources and encouragement.

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