Factorization structures via the non-commutative Hilbert scheme of points in $\mathbb{C}^3$

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17 March, 2018

Section 1

The question

Let X be a smooth complex surface (e.g. $\mathbb{C}^2$).

The Hilbert scheme of n points of X parametrizes 0-dimensional subschemes of X of length n.

Write $\mathrm{Hilb}_X = \bigsqcup_{n \geq 0} \mathrm{Hilb}_X^n$ and

$$\mathbb{H} = H^*(\mathrm{Hilb}_X) = \bigoplus_{n \geq 0} H^*(\mathrm{Hilb}_X^n).$$

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It follows from the work of many people in geometry and in algebra that

1. $\mathbb{H}$ is an irreducible representation of the Heisenberg Lie algebra $\mathfrak{h}_X$. [Nakajima, Grojnowski]
2. $\mathbb{H}$ is isomorphic to the Heisenberg vertex algebra. [Frenkel–Lepowski–Meurmann]
3. On any smooth curve C, there is associated to Hilb_X the Heisenberg chiral algebra. [Huang–Lepowski, Frenkel–Ben-Zvi]
4. On any smooth curve C, there is a Heisenberg factorization algebra $\mathcal{H}_C$. [Beilinson–Drinfeld, Francis–Gaitsgory]

Open problem: Given a smooth curve C and a smooth surface X, find a way to construct the factorization algebra $\mathcal{H}_C$ directly from the geometry of X and C and the Hilbert scheme, without passing through all of the formal algebra.

Strategy:

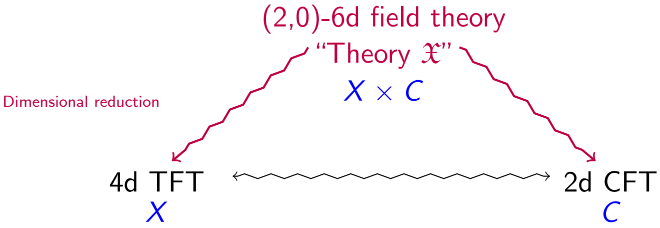
1. Construct a *factorization space* over C whose fibres are built from copies of the Hilbert scheme.
2. Linearize (e.g. taking by cohomology along the fibres) to obtain a factorization algebra with fibres copies of $\mathbb{H}$.

Section 2

The physics

The AGT correspondence

Why?



In math:	Moduli space of G-instantons on X	Vertex algebra: $\mathcal{W}$-algebra for $\mathfrak{g}^L$
$G = U(1)$:	Hilb_X	Heisenberg vertex algebra

New strategy:

1. Build a factorization space over $X \times C$.
2. Use dimensional reduction to get a space over C.
3. Linearize.

Section 3

The math

Factorization spaces

Let Z be a separated scheme.

The Ran space of Z parametrizes non-empty finite subsets $S \subset Z$.

Definition

A factorization space over Z is a space living over the Ran space,

$$\mathcal{Y} \to \operatorname{Ran} Z,$$

whose fibres $\mathcal{Y}_S$ are equipped with compatible **factorization isomorphisms***:*

- *Given some points $\{S_i\}_{i=1}^n \subset \operatorname{Ran} Z$ such that, as subsets of Z, the S_i are pairwise disjoint,we have*

$$F_{\{S_i\}} : \prod_{i=1}^{n} \mathcal{Y}_{S_i} \xrightarrow{\sim} \mathcal{Y}_{\sqcup S_i}.$$

The Hilbert scheme factorization space

In this case, we have $Z = C$, a smooth complex curve.

We define a space $\mathcal{H}ilb_{X\times C}$, whose fibre over

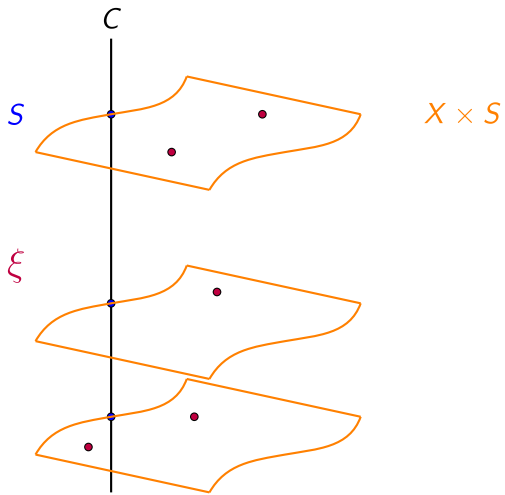
$$S = \{c_1, \ldots, c_n\} \in \operatorname{Ran} C$$

is given by

$$\mathcal{H}ilb_{X\times C,S} = \left\{\xi \in \mathrm{Hilb}_{X\times C} \;\middle|\; \operatorname{Supp}\xi \subset \bigsqcup_{i=1}^{n} (X \times \{c_i\})\right\}$$

$$\cong \prod_{i=1}^{n} \mathcal{H}ilb_{X\times C,\{c_i\}}.$$

The Hilbert scheme factorization space



The Hilbert scheme as a critical locus

e.g. when $X = \mathbb{C}^2$, $C = \mathbb{C}^3$, we can write $\mathrm{Hilb}^n_{X \times C}$ as a critical locus inside the non-commutative Hilbert scheme as follows:

$$\mathrm{Hilb}^n_{\mathbb{C}^3} \cong \left\{ (X,Y,Z,v) \;\middle|\; \begin{array}{l} X,Y,Z \in M_n(\mathbb{C}), \\ \quad [X,Y]=[Y,Z]=[X,Z]=0; \\ v \in \mathbb{C}^3 \\ \quad \text{a cyclic vector under } X,Y,Z \end{array} \right\} / GL_n(\mathbb{C}).$$

$$\mathrm{NCHilb}^n_{\mathbb{C}^3} := \left\{ (X,Y,Z,V) \;\middle|\; \begin{array}{l} X,Y,Z \in M_n(\mathbb{C}); \\ v \in \mathbb{C}^3 \\ \quad \text{a cyclic vector under } X,Y,Z \end{array} \right\} / GL_n.$$

$$\begin{aligned} W : \mathrm{NCHilb}^n_{\mathbb{C}^3} &\to \mathbb{C} \\ [X,Y,Z,v] &\mapsto \mathrm{Tr}(X,[Y,Z]). \end{aligned}$$

$$\mathrm{Hilb}^n_{\mathbb{C}^3} = \mathrm{Crit}(W).$$

Generalizing the factorization structure

For $S \in \operatorname{Ran} C$, a point $\xi = [X, Y, Z, v] \in \mathrm{Hilb}_{\mathbb{C}^3}$ lives in the fibre $\mathcal{H}ilb_{\mathbb{C}^3, S}$ whenever the eigenvalues of Z are contained in the set $S \subset \mathbb{C}$.

The factorization maps of Hilb are given by creating block diagonal matrices.

Definition

We define a space $\mathcal{NCH}ilb_{\mathbb{C}^3}$ whose fibre over $S \in \operatorname{Ran} C$ consists of those points $[X, Y, Z, v] \in \mathrm{NCHilb}_{\mathbb{C}^3}$ such that the eigenvalues of Z are contained in the set S.

Remark: In general, if we start with two points $[X_1, Y_1, Z_1, v_1]$, $[X_2, Y_2, Z_2, v_2]$, there is no reason to hope that the data

$$\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

will again be stable.

However, in the case that the eigenvalues of Z_1 and Z_2 are distinct, stability is ensured.

This gives us factorization maps

$$F^{NC}_{\{S_i\}} : \prod_{i=1}^{n} \mathcal{NCH}ilb_{\mathbb{C}^3, S_i} \to \mathcal{NCH}ilb_{\mathbb{C}^3, \sqcup S_i}.$$

Results (jt. with Itziar Ochoa)

- The maps F^{NC} are closed embeddings, not isomorphisms.
- The factorization space $\mathcal{H}ilb_{\mathbb{C}^3}$ can be realized as a critical locus in $\mathcal{NCH}ilb_{\mathbb{C}^3}$.
- Over this critical locus, F^{NC} restrict to the factorization isomorphisms.
- We have a perverse sheaf $\mathcal{PV}$ of vanishing cycles on $\mathcal{H}ilb_{\mathbb{C}^3}$, a candidate for linearizing the factorization space to get a factorization algebra on $C = \mathbb{C}$.

Work in progress: Is this sheaf compatible with the factorization structure on $\mathcal{H}ilb_{\mathbb{C}^3}$?

- After applying results of Brav–Bussi–Dupont–Joyce–Szendroi, this amounts to checking vanishing of (or adjusting $\mathcal{PV}$ to account for) certain $\mathbb{Z}/2\mathbb{Z}$-bundles $J_{F^{NC}}$ on spaces associated to $\mathcal{H}ilb_{\mathbb{C}^3}$.