MATH 595 Thursday 26 April

Ruled surfaces; divisors on elliptic curves

(1) **Exercise V.2.3 (a)** If $\mathscr{E}$ is a locally free sheaf of rank r on a non-singular curve C, prove that there is a sequence

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_r = \mathscr{E}$$

such that $\mathscr{E}_i/\mathscr{E}_{i+1}$ is locally free of rank 1 for each r. (We say that $\mathscr{E}$ is a *successive extension* of invertible sheaves.)

Hint: use Exercise II.8.2 and induction. Note that this may fail for base varieties X of dimension ≥ 2!)

(2) Let C be an elliptic curve.
   (a) Prove that $K \sim 0$.
   (b) Let $\mathscr{L}$ be any invertible sheaf. Prove that there is a unique point $P \in C$ such that $\mathscr{L} \cong \mathscr{L}(P)$. This gives a one-to-one correspondence between degree one divisors and points of C.
   (c) Let P, Q be two distinct points of C, and consider the linear system $|P + Q|$. Prove that it is base-point free and of dimension 1, hence determining a map $X \to \mathbb{P}^1$. Prove that this map is ramified at exactly four points, each of ramification index 2. Letting R be one of the ramification points, conclude that $P + Q \sim 2R$.