MATH 402 Review for December 7–10

Topics: The topics include Möbius transformations, cross products, and Poincaré isometries. These were covered in lecture and on Homework 12.

1. **Recall from earlier in the term:**

   - We classified all isometries: we have reflections, rotations, translations, and glide reflections. Of these, the group or *orientation-preserving isometries* consists of all rotations and translations.
   - In Euclidean geometry, we could use Cartesian coordinates (x, y) to write down nice formulas for translation, reflection across the x-axis and y-axis, and rotation about 0. This was useful to us: it made proving certain theorems much easier!
   - Recall that we introduced the extended complex plane, along with complex multiplication and addition.

2. **Things to know about Möbius transformations:**

   (a) A *Möbius transformation* is a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

   $$f(z) = \frac{az + b}{cz + d},$$

   where a, b, c, d are complex constants with the property that $ad - bc = 0$. (This condition, $ad - bc = 0$ is equivalent to the statement that f is bijective.) Möbius transformations are exactly the bijective functions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which are *conformal*, which means they preserve angles and preserve local scale.

   (b) Möbius transformations form a group (where the group operation is composition), denoted by $\mathcal{M}$.

   (c) A Möbius transformation with three or more fixed points must be the identity.

   (d) The choice of constants a, b, c, d is not unique: we can scale them all by any non-zero complex number λ, without changing the function f.

3. **Things to know about the cross-ratio:**

   (a) The *cross-ratio* of four complex numbers (a, b, c, d) is

   $$\frac{a - c}{a - d}\frac{b - d}{b - c}.$$

   (Note that it could be ∞, if $a = d$ or $b = c$, but that's fine since we're working in the extended complex plane. It is *not* well-defined if at least three of the numbers a, b, c, d are not distinct, because in that case we have $\frac{0}{0}$, which is not defined.)

   (b) Given three distinct complex numbers b, c, d, we can use the cross-ratio to define a function $f(z) = (z, b, c, d)$, which has the property that $f(b) = 1, f(c) = 0$, and $f(d) = \infty$. Likewise, given another three distinct complex numbers (b', c', d'), we can define $f'(z)$, and then the composition $(f')^{-1} \circ f(z)$ sends $b \mapsto b'$ etc.

   (c) A Möbius transformation preserves cross-ratios.

   (d) The cross-ratio of four complex numbers (a, b, c, d) is real (rather than complex) if and only if all four of those numbers lie on a circle or a line. We can use this to prove that a Möbius transformation sends a circle or a line c to a circle or a line.

(e) We can use the cross-ratio to calculate Poincaré distance between two points $z_0, z_1 \in \text{Disk}_P$: let w_0, w_1 be the omega-points of the Poincaré line through z_0 and z_1. Then

$$d_P(z_0, z_1) = |\ln(z_0, z_1, w_1, w_0)|.$$

4. **Things to know about the group $\mathcal{M}^P$**

   (a) By definition $\mathcal{M}^P$ consists of all Möbius transfomations which preserve the boundary and the interior of the unit disk. (Hence they send a Poincaré point to a Poincaré point.)

   (b) One can prove that every $f(z) \in \mathcal{M}^P$ has the form

   $$f(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1},$$

   where $\alpha, \beta \in \mathbb{C}$ with $|\alpha| < 1$, $|\beta| = 1$.

   (c) A function $f \in \mathcal{M}^P$ preserves the Poincaré distance function, and hence gives an isometry in the Poincaré model.

   (d) For example, when $\alpha = 0$ and $\beta = e^{i\theta}$ we get rotation about $(0, 0)$ by angle $\phi + 180°$. When $\beta = 0$, we get translation along the line through α and $(0, 0)$.

   (e) When α and β vary, we get many more isometries: in fact, we get *all orientation-preserving isometries.*

   (f) Recall that a translation was defined to be the composition of two reflections across lines which are parallel. In hyperbolic geometry, we can ask whether those two parallel lines are ultra-parallel or limiting parallel. When the lines are ultraparallel, there is a unique common perpendicular line l, and the resulting isometry is *hyperbolic translation along this line.* (The distance of course depends on how far the original lines are from each other.) When the lines are limiting parallel, there is actually no common perpendicular, and the isometry has no invariant lines. Instead of calling this a translation, we call it a *parallel displacement.*

   (g) What about an *orientation-reversing isometry* g of the Poincaré disk? Choose a favourite reflection r (often we take r to be reflection across the x-axis, which is given by $z \mapsto \overline{z}$). Then $g \circ r$ is orientation-preserving, so we can write $g \circ r = f$ for some $f \in \mathcal{M}^P$. So $g = f \circ r$; every orientation-reversing isometry has this form for some choice of f.

Practice Questions

Draw some pictures to convince yourself of the following facts. (Choose two reflections that can be used to define the isometry.) Recall that a reflection has: one fixed line, all perpendicular lines as invariant lines, and two fixed omega-points. Recall also that all of these isometries can be thought of as functions defined on all of $\overline{\mathbb{C}}$, and can have up to two fixed points anywhere in $\overline{\mathbb{C}}$ (but not more!), but when we view them as isometries of the Poincaré disk, we only care about fixed points inside the disk.

- A (non-identity) rotation has one fixed point, no invariant lines (unless it is a half-turn, in which case it has all lines through the centre of rotation as invariant lines), and no fixed omega-points.

- A hyperbolic translation has no fixed points, one invariant line, and two fixed omega-points.

- A parallel displacement has no fixed points, no invariant lines, and one fixed omega-point.