A Liouville theorem for p-harmonic functions on exterior domains

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THE LIOUVILLE THEOREM.

Theorem (Cauchy [Cau1844]).
Any bounded entire function of a single complex variable must be constant.

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Theorem (cf. [AxBouRam01, Theorem 3.1]).
Let $d \geq 2$ and let u be a real harmonic function on $\mathbb{R}^d$, bounded either from above or below. Then u must be constant.

Theorem (cf. [AxBouRam01, Corollary 3.3]).
Let $d = 2$ and let u be a harmonic function on the exterior domain $\mathbb{R}^2 \setminus \{0\}$, bounded either from above or below. Then u must be constant.

Remarks.

- We call a domain $\Omega \subseteq \mathbb{R}^d$ an *exterior domain* provided the complement $\Omega^c = \mathbb{R}^d \setminus \Omega$ is compact and nonempty.
- Liouville's theorem fails on $\mathbb{R}^d \setminus \{0\}$ for $d \geq 3$.
  Counter-example: *fundamental solution* $x \mapsto \mu_2(x) := |x|^{2-d}$

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Theorem (cf. [SerZou02, Thm II] or [HeiKilMar93, Cor. 6.11]).
Let $d \geq 2$ and let $1 < p < \infty$. Suppose u is a p-harmonic function on $\mathbb{R}^d$, bounded either from above or below. Then u must be constant.

Recall. We call a real-valued function u on an open set $\Omega \subseteq \mathbb{R}^d$ *p-harmonic* if $u \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ and a solution of

$$-\Delta_p u = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Here: $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is called *p-Laplace operator*.

Theorem (cf. [KiVé86, Corollary 2.2]).
Let $d \geq 2$ and let u be a real d-harmonic function on the exterior domain $\mathbb{R}^d \setminus \{0\}$, bounded either from above or below. Then u must be constant.

Remark. Liouville's theorem fails on $\mathbb{R}^d \setminus \{0\}$ for $p > d \geq 1$.

Counter-example: *fundamental solution* $x \mapsto \mu_p(x) := |x|^{(p-d)/(p-1)}$

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- (cf. [Ser72]) For positive solutions of

$$-\Delta u + f(u, \nabla u) = 0 \qquad \text{on } \mathbb{R}^d,$$

- or of the *stationary Strödinger* equation (e.g., [BreChi08, FraPin11]),

$$-\nabla(A(x)\nabla u(x)) + V(x)u(x) = 0 \qquad \text{on } \mathbb{R}^d,$$

- or of the *generalized Lane-Emden* equation

$$-\Delta_p u = u^{q-1} \qquad \text{on } \mathbb{R}^d$$

  (cf. [GidSpr81, BiVéPo01, SerZou02]).

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$$-\Delta_p u = u^{q-1} \qquad \text{on } \Omega \supseteq \{|x| > R_0 > 0\}$$

(cf. [GidSpr81, BiVéPo01, SerZou02]) "exterior domains".

More Liouville-type theorems.

These results are all about elliptic equations with lower-order terms!!!

Question.

What is known about Liouville-type results for solutions of

$$-\Delta_p v = 0 \qquad \text{in } \Omega,$$

when Ω is a general exterior domain?

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What is known about Liouville-type results for solutions of

$$\begin{aligned} -\Delta_p u &= 0 && \text{in } \Omega, \\ \mathcal{B}v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

when Ω is a general exterior domain?

Answer. Not much is known!!!

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Question.

Why we are interested in Liouville-type results?

Answer.

- Intimate relation between Liouville-type theorems and pointwise a priori estimates (cf. [SerZou02, p.82] and [PolQuiSou07, p.556]):

  Liouville's theorem $\Leftrightarrow$ univ. upper bounds for pos. solut.

- Convergence of domain perturbation problems of elliptic boundary value problems (cf. [DDH13])
- much more...

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Theorem. (Dancer, Daners, H. [DDH13-Lio]) *Let Ω be an exterior domain. Then:*

- *Let $1 < p < d$ and u be a positive solution of problem*

$$(1) \qquad \begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

Then u must be constant.

Example.

Consider the function

$$u(x) := \begin{cases} \log|x| & \text{if } p = d, \\ |x|^{(p-d)/(p-1)} - 1 & \text{if } p > d \end{cases}$$

for every $x \in \overline{B}_1^c := \{x \in \mathbb{R}^d \mid |x| > 1\}$. Then u is a positive solution of

$$\begin{cases} -\Delta_p u = 0 & \text{in } \overline{B}_1^c, \\ u = 0 & \text{on } \partial\overline{B}_1. \end{cases}$$

Remark.

Similarly, one can easily construct an example of a positive non-trivial p-harmonic function on $\overline{B}_1^c$ satisfying zero Robin boundary conditions.

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$$(1) \qquad \begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

  Then v must be constant.

- *Let $p \geq d$. Then every positive solution u of (1) is either constant or $u \sim \mu_p$ as $|x| \to \infty$ with*

$$\mu_p(x) := \begin{cases} |x|^{(p-d)/(p-1)} & \text{if } p \neq d, \\ \log|x| & \text{if } p = d. \end{cases}$$

1. Step.

- Determine the asymptotic behavior near infinity,

2. Step.

- Use integration techniques with suitable test functions to establish Liouville's theorem.

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- Suppose u is a positive harmonic function on $\overline{B}_1^c$, $d > 2$;
- Let $K[u](x) := |x|^{2-d}u(x/|x|^2)$ be the Kelvin transform of u;

$\Rightarrow$ $K[u]$ is a positive harmonic on $B_1 \setminus \{0\}$ (cf. [AxBouRam01]);

$\Rightarrow$ $\exists$ harmonic w on B_1, $\exists\, b \geq 0$ such that

$$K[u](x) = w(x) + b\,|x|^{2-d} \quad \text{or} \quad K[u-b](x) = w(x)$$

(Bôcher's theorem (cf. [AxBouRam01]);

$\Rightarrow$ Again Kelvin's transform, $u(x) - b = |x|^{2-d}\,w(x/|x|^2)$ on $\overline{B}_1^c$

$\Rightarrow$ Since $w(x/|x|^2) \to w(0)$ as $|x| \to \infty$, $\exists\, r_0 \geq 1$, $C \geq 0$ s.t.

$$|u(x) - b| \leq C\,|x|^{2-d} \qquad \text{for } |x| \geq r_0.$$

Lemma.

- Let $\varphi \in C_c^\infty(R^d)$ s.t. $0 \leq \varphi \leq 1$ on $\mathbb{R}^d$, $\varphi \equiv 1$ on $\overline{B}_1$, $\varphi \equiv 0$ on $\overline{B}_2^c$.
- Set $\varphi_r(x) = \varphi(x/r)$ for every $x \in \mathbb{R}^d$ and all $r > 0$.
- Suppose $\exists\ C_0, C_1 \geq 0$, $r_0 \geq 1$, and $b \in \mathbb{R}$ s.t. $u \in W^{1,p}_{loc}(\Omega)$ satisfies

$$\int_{\Omega\cap B_{2r}} |\nabla v|^2\, \varphi_r^2 \,\mathrm{d}x \leq \tfrac{C_0}{r} \left[\int_{(\Omega\cap B_{2r})\setminus B_r} |\nabla u|^2\, \varphi_r^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_{(\Omega\cap B_{2r})\setminus B_r} |u-b|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}$$

and

$$\tfrac{1}{r^2} \int_{(\Omega\cap B_{2r})\setminus B_r} |u-b|^2 \,\mathrm{d}x \leq C_1 \qquad \text{for all } r \geq r_0.$$

Then, u is constant.

Since u satisfies

$$|u(x) - b| \leq C\,|x|^{2-d} \qquad \text{for } |x| \geq r_0,$$

we achieve to

$$\frac{1}{r^2}\int_{B_{2r}\setminus B_r} (u-b)^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{B_{2r}\setminus B_r} |x|^{2(2-d)}\,\mathrm{d}x$$
$$= \frac{C_1}{r^2}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_r^{2r} s^{3-d}\,\mathrm{d}s. \qquad \square$$

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SECOND MAIN THEOREM.

Theorem. (Dancer, Daners, H. [DDH13-Lio]) *Let Ω be an exterior domain, and $1 < p < \infty$. Suppose u is a p-harmonic function on Ω, which is bounded from below or above and satisfies zero Neumann boundary conditions. Then u must be constant.*

Thank you for your attention!!!

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