Geometric phase and periodic orbits of the equal-mass, planar three-body problem with vanishing angular momentum

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Mathematics Postgraduate Seminar Series

Introduction

- Basic ideas:
  - Geometric phase
  - 3-body problem
  - Symmetry
  - Symmetry reduction
  - Regularisation
  - Periodic orbits
- A theorem about geometric phase with illustrations.
- More detailed results and observations.

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Geometric phase in cats

Cat: a non-rigid system of connected weights with an inbuilt control system.

- When dropped from an inverted position, able to land on its feet.
- How?!
- By exploiting geometric phase: rotation independent of angular momentum.

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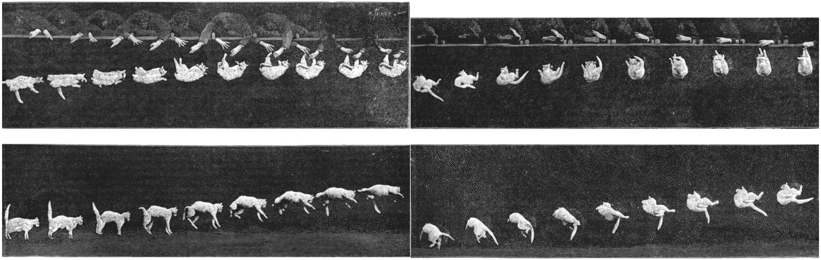

FIG. 1.—Side view of a falling cat. (The series runs from right to left.)

FIG. 2.—End view of a falling cat. (The series runs from right to left.)

And something more fun (click)...

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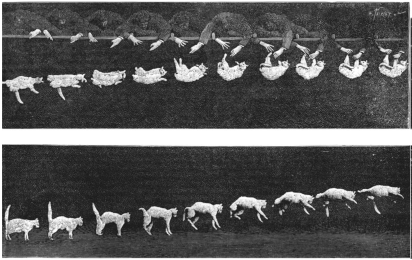

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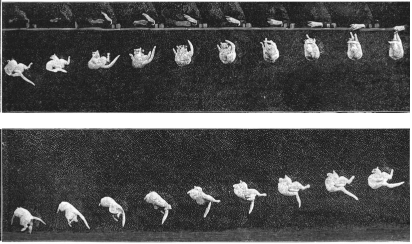

Fig. 2.—End view of a falling cat. (The series runs from right to left.)

And something more fun (click)...

Save the cats! (Or: let's do something that hasn't been done already)

- An antique problem: that of three bodies under mutual gravitation.
- The 3-body problem is intractable. Many questions remain open, such as: do periodic orbits exist with geometric phase at zero angular momentum? If so, when or why it occurs? (Or doesn't...)
- Why should we care? Aside from the fact that maths for its own sake is and has always been beautiful and valuable in unpredictable ways?
- Direct applications: understanding astronomical systems.
- Transferable solutions: insights gained here may transfer or translate to similar problems (e.g. molecular dynamics).

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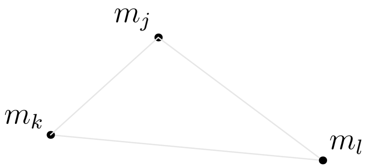

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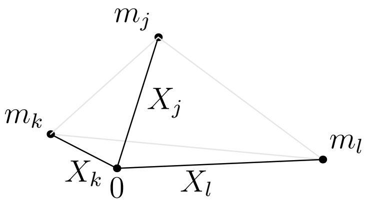

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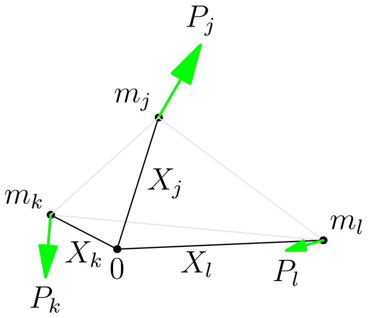

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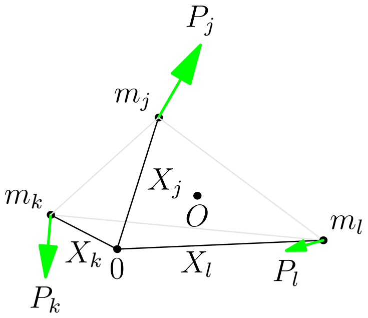

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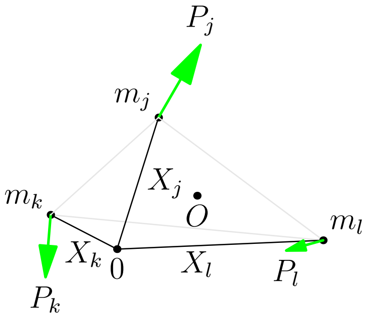

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The 3-body problem

- Three-body problem is also a non-rigid system of masses connected by gravitational force.
- "Control system" is the Hamiltonian:

$$H = \sum \frac{|P_j|^2}{2m_j} - \sum \frac{m_k m_l}{|X_l - X_k|} \qquad (1)$$

producing Hamilton's equations

$$X_j' = \frac{dX_j}{dt} = \frac{\partial H}{\partial P_j}, \; P_j' = \frac{dP_j}{dt} = -\frac{\partial H}{\partial X_j}, \qquad (2)$$

governing the motion.

(Summation convention: (j, k, l) cyclic permutations of $(1, 2, 3)$. Each case is substituted, and then all three are added.)

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The 3-body problem

Write Hamilton's equations more compactly as the vector field

$$z' = J\nabla H(z) = F(z), \text{ where} \tag{3}$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \text{ and}$$

$$z = (X_1, X_2, X_3, P_1, P_2, P_3)^T \in \Omega,$$

and Ω is the phase space (in this case $\mathbb{C}^6$).

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Symmetries

Key idea in dynamical systems: symmetry.

- What is symmetry? An excess of information. E.g. A square can be described from a $\frac{1}{8}$-th wedge if you know its symmetries.
- If $S : \Omega \longrightarrow \Omega$ such that $S \circ F(z) = F \circ S(z)$, then we say S is a symmetry of the vector field F.
- What types of symmetries can we have?
  - Continuous: e.g., translations: $S(z) = z + a$, $a = (a_1, a_2, a_3, 0, 0, 0) \in \mathbb{C}^6$, rigid rotations: $S(z) = e^{i\theta} z$, $\theta \in \mathbb{R}$.
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Why we reduce

- Noether's theorem relates continuous symmetries and conserved quantities:
  - Symmetry under spatial translation implies conservation of linear momentum.
  - Symmetry under rotation implies conservation of angular momentum.
  - Symmetry under time translation implies conservation of energy.
- So we usually choose:
  - Centre of mass $O = \frac{1}{m}\sum m_j X_j = 0$.
  - Centre of momentum $\sum P_j = 0$ (fixes centre of mass).
  - Angular momentum fixed by initial choices of X_j, P_j.
- Two paths to not have to worry about these choices: 1) the "shape sphere"; and 2) "elimination of the nodes".

Reducing by symmetries can reveal "important" structure of system.

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Reducing by symmetries can reveal "important" structure of system.

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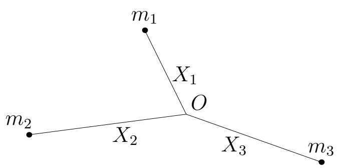
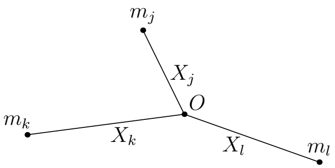
Two reductions





Original triangle.

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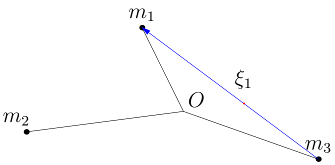
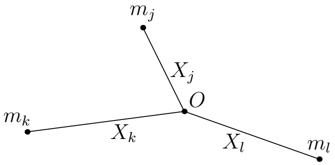
Two reductions





$\xi_1 = X_1 - X_3$.

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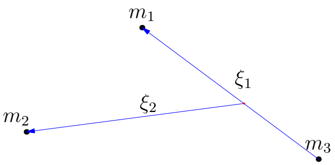
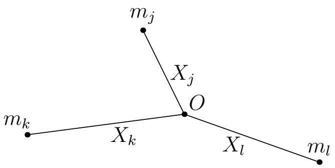
Two reductions





$\xi_2 = X_2 - \frac{m_1 X_1 + m_3 X_3}{m_1 + m_3}$.

Original triangle.

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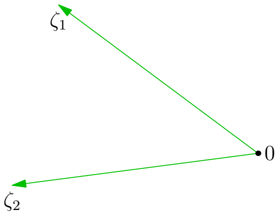
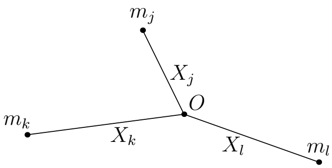




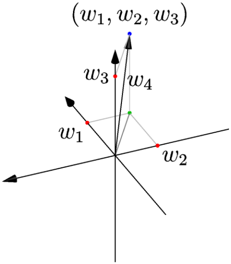
Original triangle.

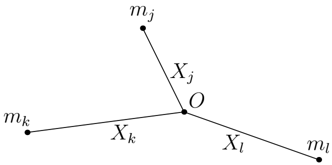
$$\frac{1}{\tilde{\mu}_1} = \frac{1}{m_1} + \frac{1}{m_3}$$

$$\frac{1}{\tilde{\mu}_2} = \frac{1}{m_2} + \frac{1}{m_1 + m_3}$$

$$\zeta_1 = \sqrt{\tilde{\mu}_1}\xi_1, \ \zeta_2 = \sqrt{\tilde{\mu}_2}\xi_2$$

Two reductions

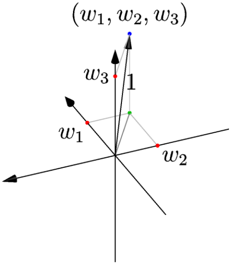




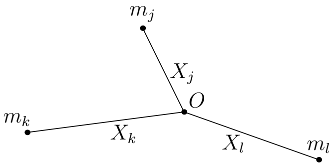
Original triangle.

$$\begin{aligned} w_1 &= |\zeta_1|^2 - |\zeta_2|^2 \\ w_2 + i w_3 &= 2\bar{\zeta}_1 \zeta_2 \\ w_4 &= |\zeta_1|^2 + |\zeta_2|^2 \\ &= \sqrt{w_1^2 + w_2^2 + w_3^2}. \end{aligned}$$

Two reductions



$w_1^2 + w_2^2 + w_3^2 = 1$.



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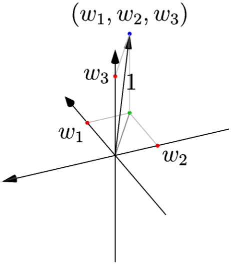

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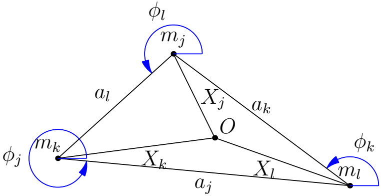
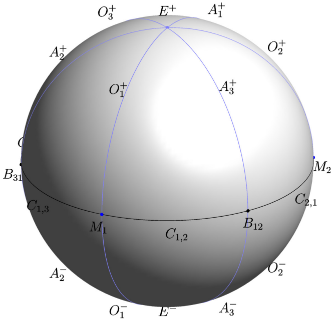
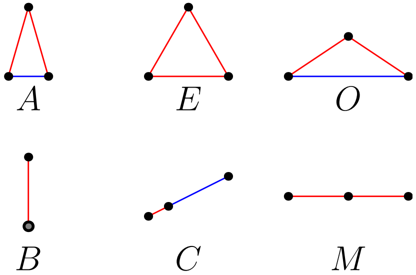


$$a_j e^{i\phi_j} = X_l - X_k$$

$$\phi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3).$$

Shape sphere





Features when $m_1 = m_2 = m_3$:

- Equilateral points (Lagrange configurations): $E^{\pm}$.
- Isosceles curves: $A_j^{\pm}$ (acute), $O_j^{\pm}$ (obtuse).
- Collinear curves: $C_{j,k}$.
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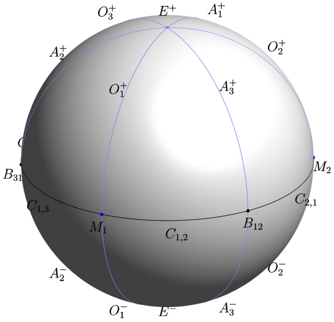

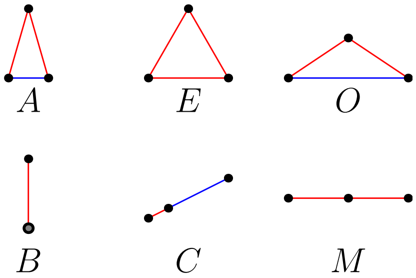

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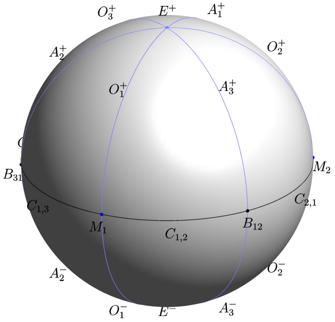

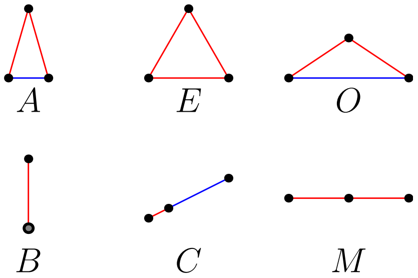

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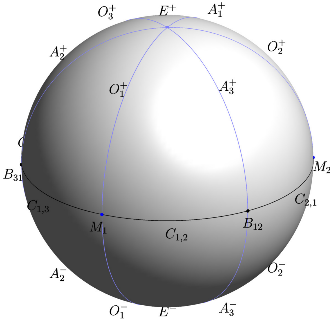

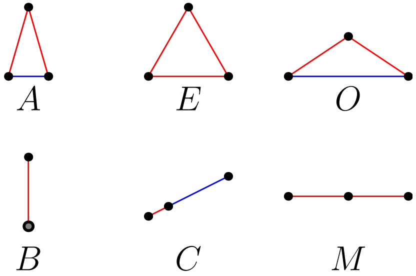

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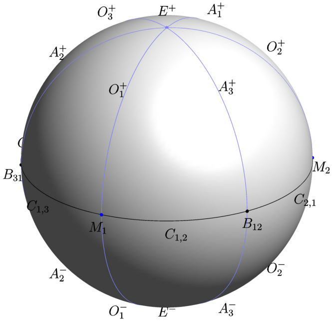

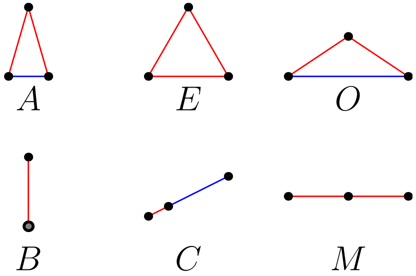

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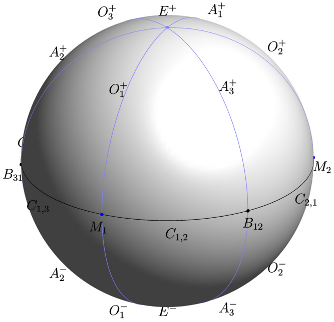

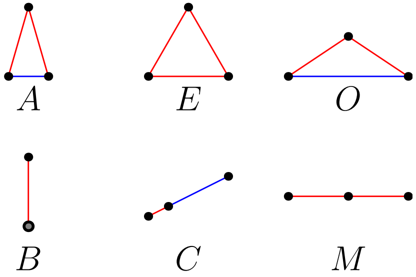

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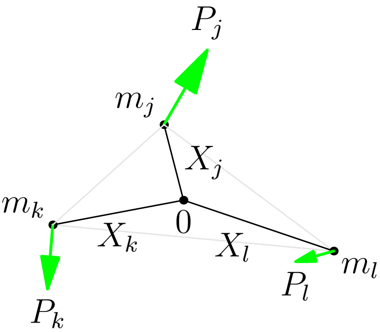

In the physical problem:

- σ_j swaps indices k, l.
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- ρ reflects whole configuration in space.
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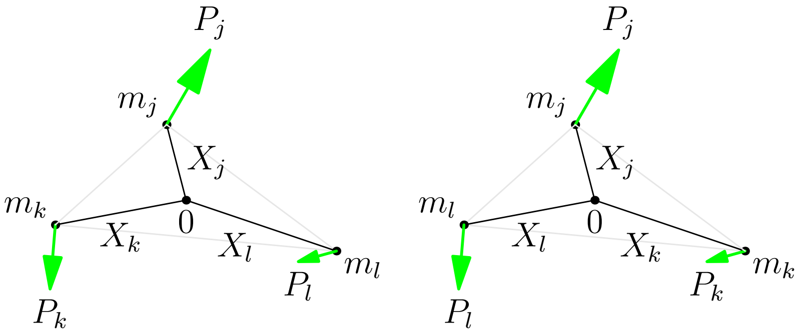

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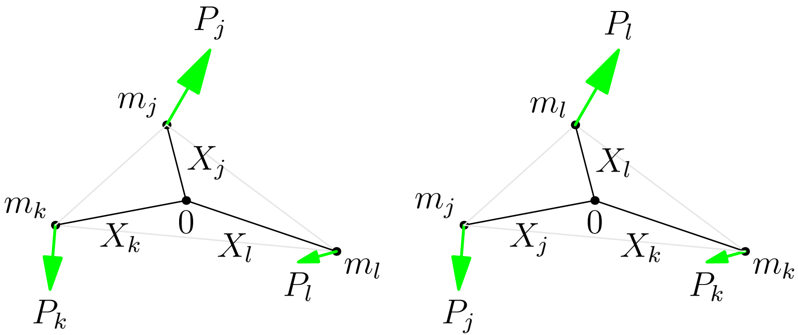

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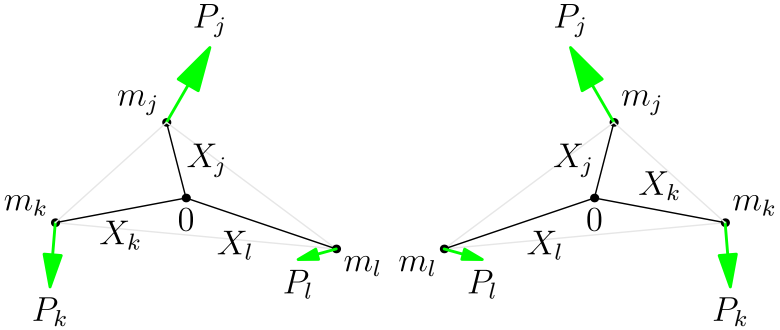

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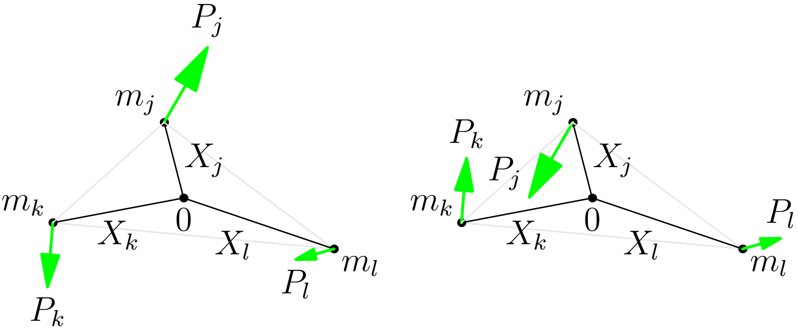

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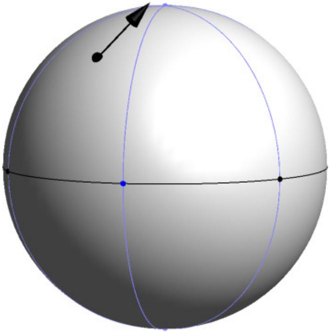

On the shape sphere:

- σ_j rotates about axis through B_{kl}-M_j by π.
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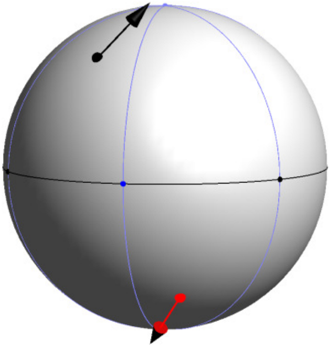

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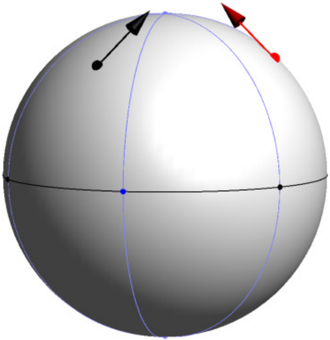

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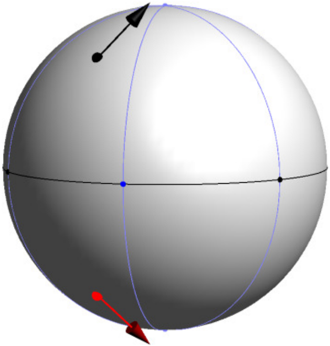

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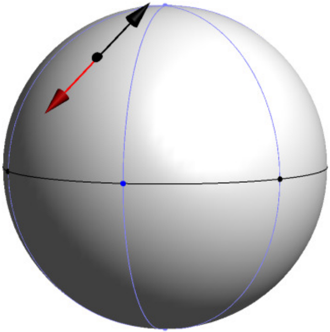

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- Recall $S \circ F(z) = F \circ S(z) \iff S \in \mathfrak{G}_S$ is a symmetry of F.
- Observe that $\tau \circ F(z) = -F \circ \tau(z)$ means τ is an *antisymmetry* of F.
- We call τ a *reversing symmetry*.
- Composition $R = \tau \circ S$ is also a reversing symmetry.
- Turns out τ commutes with every symmetry.

We now have a *reversing symmetry group* $\mathfrak{G}_R \cong S_3 \times Z_2^2$ (order 24). Note that $Z_2^2 = V_4 = \{I, \rho, \tau, \tau\rho\}$ is in the centre of $\mathfrak{G}_R$.

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  - A method of "smoothing out" singularities.
- 3-body problem is singular at binary collisions and triple collision.
- We can regularise all binary collisions simultaneously.
- Do so by making more space and more time.

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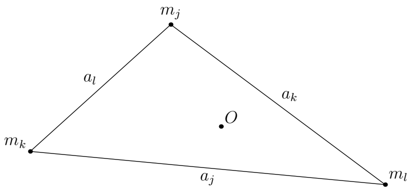

Simultaneous regularisation of all binary collisions (due to Lemaître [1]):

- New coordinates α_j such that $a_j = \alpha_k^2 + \alpha_l^2$.
  - $\alpha_j = 0$ gives collinearity with m_j in eclipse.
  - $\alpha_k = \alpha_l = 0$ gives collision between m_k and m_l.
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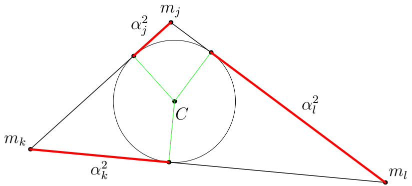

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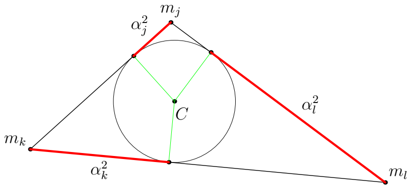

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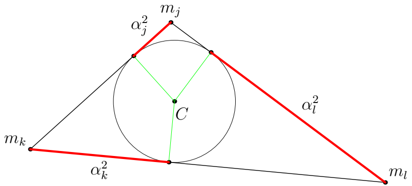

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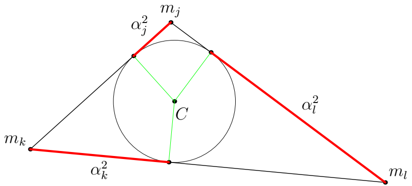

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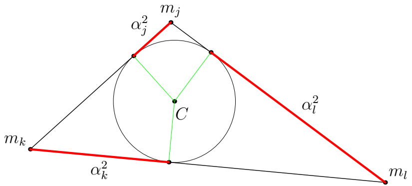

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Sub new variables into old Hamiltonian. Now one final step.

- We need to slow down time near collisions.
- Define new time variable τ such that $\frac{dt}{d\tau} = a_1 a_2 a_3$, then
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- New equations of motion by

$$\dot{z} = \frac{dz}{d\tau} = J\nabla K = F(z).$$

- Shape changes by $\dot{\alpha}_j, \dot{\pi}_j$, while $\dot{\phi} = \dot{\phi}(z)$ governs rotations. Now set $p_\phi = 0$ everywhere.
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The regularised system

- Collisions are now allowed. Act like elastic rebound.
- Regularised shape space/sphere is “bigger”:

- Can also write w_1, w_2, w_3 in terms of $\alpha_1, \alpha_2, \alpha_3$ and masses.

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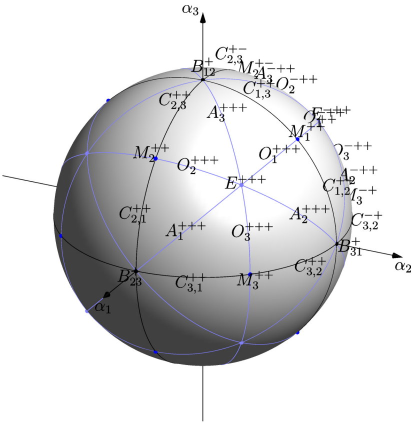

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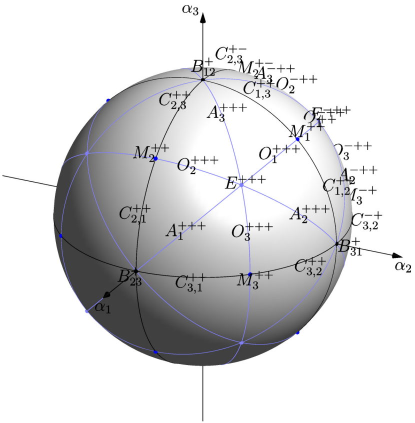

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Discrete symmetries again

Symmetries can be put in terms of regularised coordinates. Choose:

- $\sigma_1(\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3) = -(\alpha_1, \alpha_3, \alpha_2, \pi_1, \pi_3, \pi_2)$, etc.,
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- $s_1(\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3) = (\alpha_1, -\alpha_2, -\alpha_3, \pi_1, -\pi_2, -\pi_3)$, etc..
- Subgroup $\{I, s_1, s_2, s_3\} \cong V_4$. Elements interact with S_3 by semidirect product $S_3 \ltimes V_4 = S_4$ (order 24). Elements written uniquely as composition $S \circ s_j$, $S \in S_3$.

New (reversing) symmetry group $\mathfrak{G}_R \cong S_4 \times Z_2^2$ (order 96), with same centre as before.

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- Important idea: fixed sets of symmetries.
- Involutions (S such that $S = S^{-1} \iff S^2 = I$) may have interesting fixed sets.
- Non-reversing fixed sets give invariant subspaces (isosceles and collinear).
- Shape space is divided into regions by fixed sets of non-reversing involutions (that act as reflections only).
- Smallest region enclosed is the *fundamental domain* (FD). We pick just one.
- Solutions can be “reflected” back into FD at boundaries (“billiards”).
- Some solutions can reflect back the way they came...

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Reversing fixed sets

- Fixed sets of reversing symmetries are not invariant.
- Solution with points in fixed sets of reversing involutions *run in reverse possibly with some other symmetry applied after that instant in time.*

Theorem

A solution connecting two points in the fixed sets of reversing involutions R_1, R_2 is periodic.

- These generate the solution's symmetries (after restricting to an invariant subspace, collinear or isosceles, if necessary).

Proof.

Suppose at $\tau = 0$ we have $z(0) \in \operatorname{Fix} R_1$ and at $\tau = \tau_0$ we have $z(\tau_0) \in \operatorname{Fix} R_2$. Now at $\tau = 2\tau_0$ we observe that $z(2\tau_0) \in \operatorname{Fix} R_1 R_2 R_1 = \operatorname{Fix} R_1 S$, where S is non-reversing of order k (i.e. $S^k = I$). If $R_1 = R_2$ then $S = I$ and orbit is periodic with period $2\tau_0$. Else periodic with period $2k\tau_0$.

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A solution connecting two points in the fixed sets of reversing involutions R_1, R_2 is periodic.

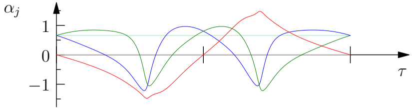
- These generate the solution's symmetries (after restricting to an invariant subspace, collinear or isosceles, if necessary).

Proof.

Suppose at $\tau = 0$ we have $z(0) \in \mathrm{Fix}\, R_1$ and at $\tau = \tau_0$ we have $z(\tau_0) \in \mathrm{Fix}\, R_2$. Now at $\tau = 2\tau_0$ we observe that $z(2\tau_0) \in \mathrm{Fix}\, R_1 R_2 R_1 = \mathrm{Fix}\, R_1 S$, where S is non-reversing of order k (i.e. $S^k = I$). If $R_1 = R_2$ then $S = I$ and orbit is periodic with period $2\tau_0$. Else periodic with period $2k\tau_0$. □

Example reversing orbit

- An orbit generated by $R_1 = R_2 = \tau\sigma_3 s_3$ (which looks like...)



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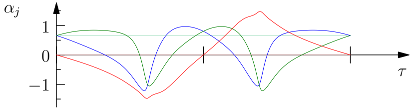
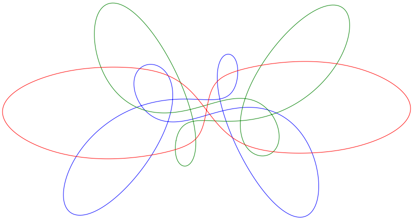
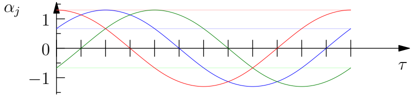
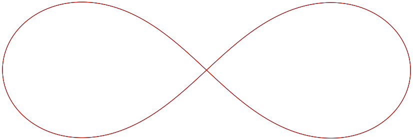




More examples

$R_1 = \tau\rho\sigma_3 s_3,\ R_2 = \tau\sigma_2 s_2$





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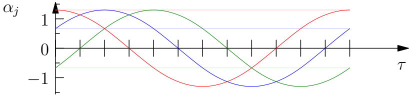

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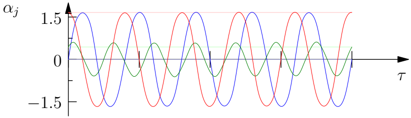

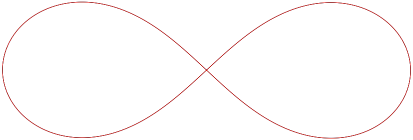

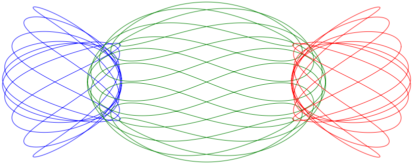

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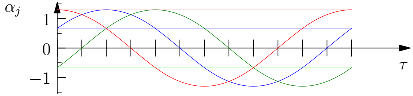

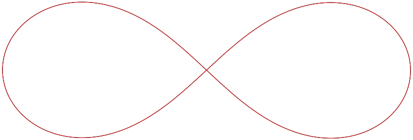

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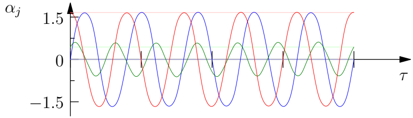

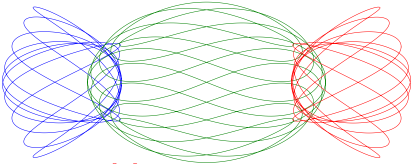

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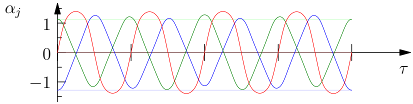

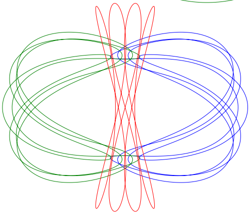

Non-reversing symmetries

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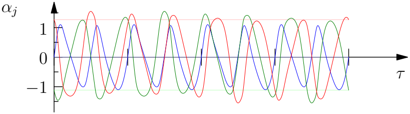
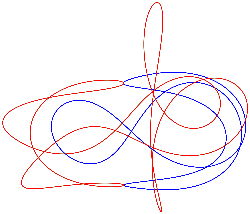
$S = \sigma_1 s_3$





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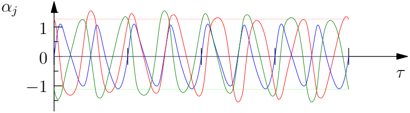

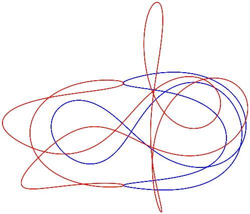

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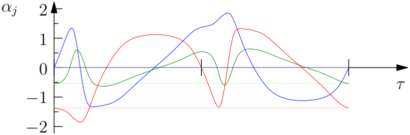

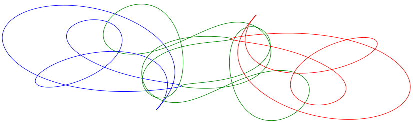

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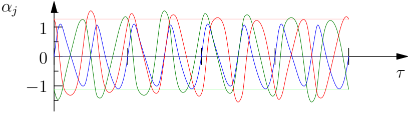

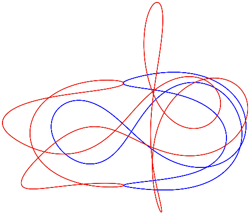

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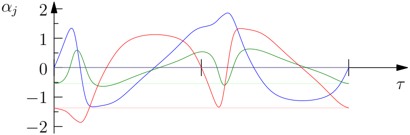

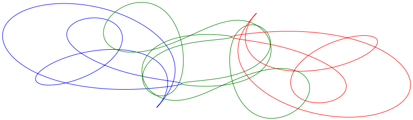

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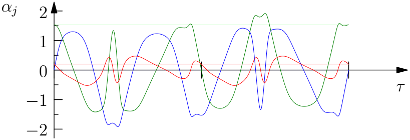

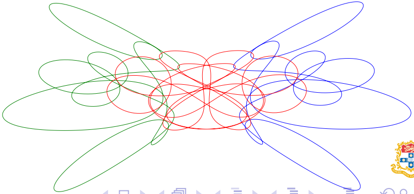

Orbits in invariant subspaces

Some orbits live in the fixed sets of non-reversing symmetries:

- ρs_j are collinear with m_j in eclipse,
- $\rho\sigma_j$ or $\rho\sigma_j s_j$ are isosceles with m_j on axis of symmetry,
- c, c^2, cs_j or $c^2 s_j$ are equilateral,
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Collinear and isosceles are interesting, but equilateral and isosceles collinear are "too small" (only blow up from and collapse to triple collision).

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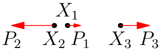

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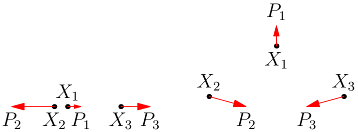

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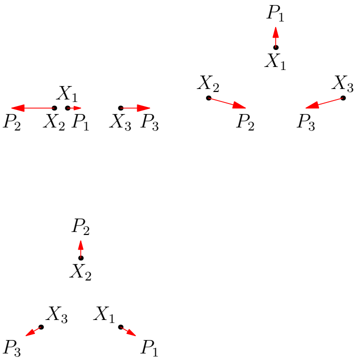

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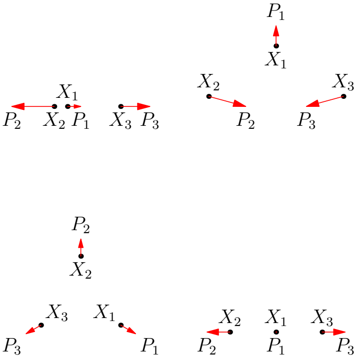

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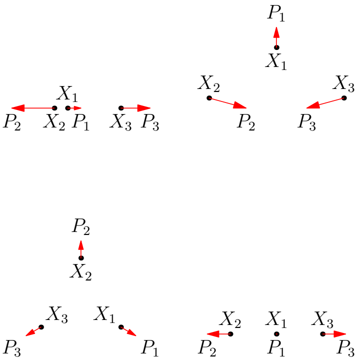

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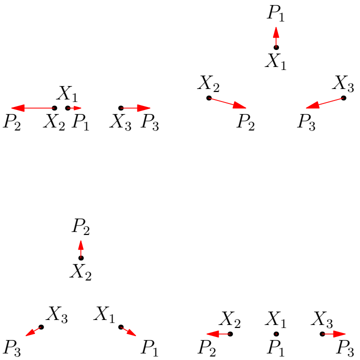

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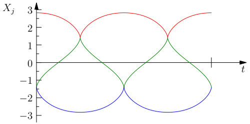

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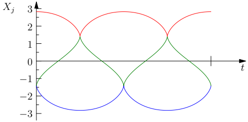

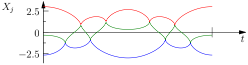

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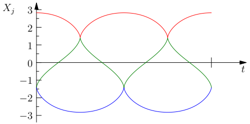

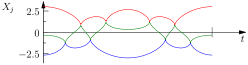

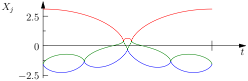

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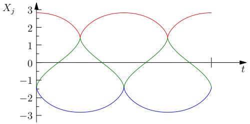

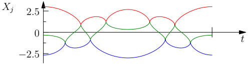

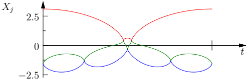

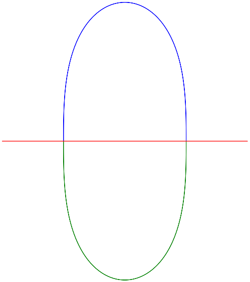

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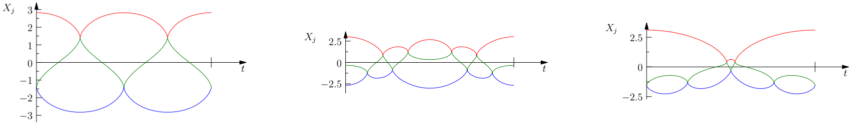

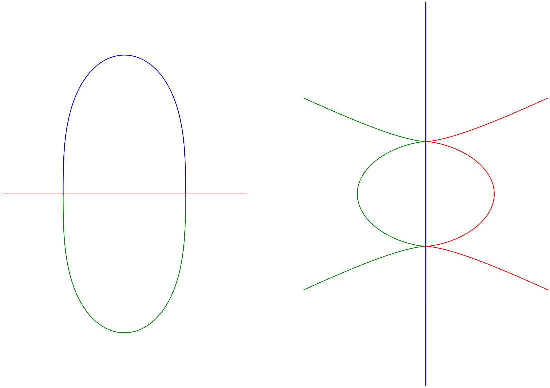

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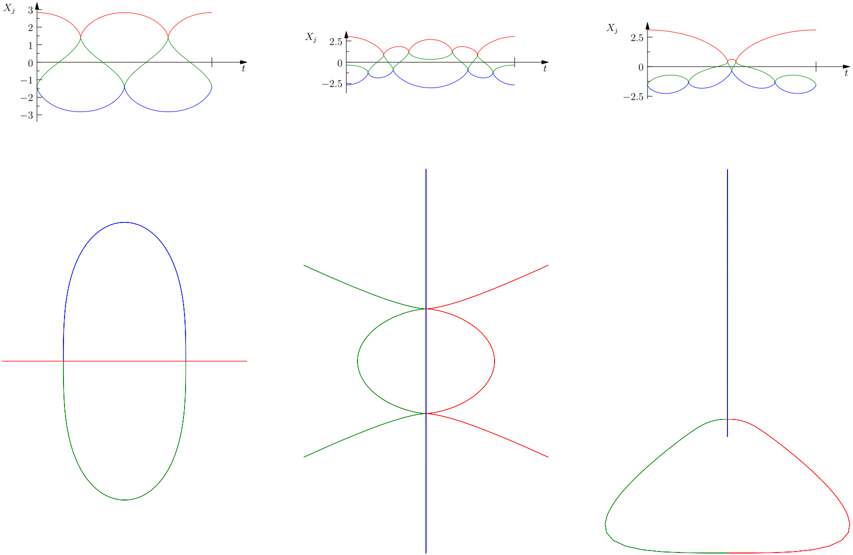

Reversing fixed sets

The fixed sets of reversing symmetries: (the interesting ones)

- $\tau\rho s_j$ are collinear reversing with m_j momentarily in eclipse,
- τs_j are collision between m_k, m_l with m_j momentarily at rest,
- $\tau\rho\sigma_j$ or $\tau\rho\sigma_j s_j$ are isosceles reversing with m_j momentarily on the axis of symmetry,
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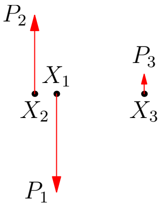

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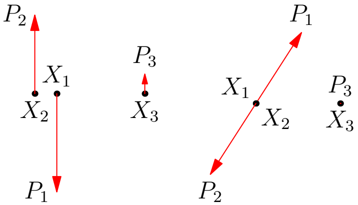

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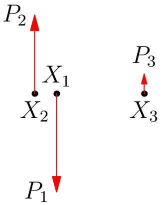

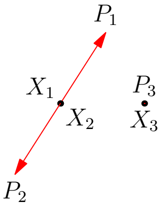

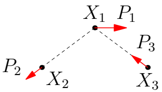

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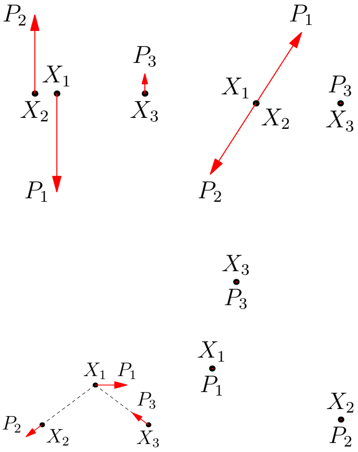

Reversing fixed sets

The fixed sets of reversing symmetries: (the interesting ones)

- $\tau\rho s_j$ are collinear reversing with m_j momentarily in eclipse,
- τs_j are collision between m_k, m_l with m_j momentarily at rest,
- $\tau\rho\sigma_j$ or $\tau\rho\sigma_j s_j$ are isosceles reversing with m_j momentarily on the axis of symmetry,
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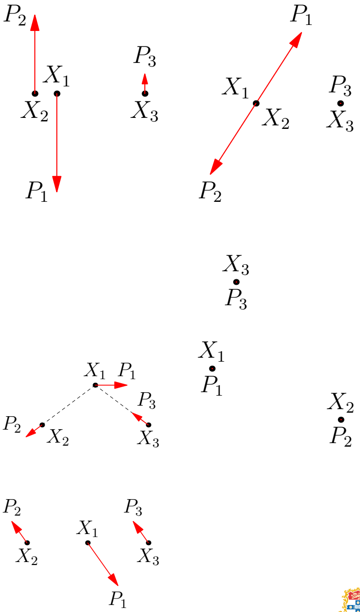

Our use for symmetries

- Symmetries are beautiful, but why care so much?
- Because it turns out that the right symmetries force geometric phase to cancel over an orbit.
- Need one more thing: isotropy subgroup, Σ_z. The subgroup of the reversing symmetry group of F that map an orbit $z(\tau)$ back to itself (possibly with a time shift).
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Isotropy subgroup structures

Working hypothesis: isotropy subgroups generated by:

- reversing involutions R_1, R_2 such that $(R_2R_1)^k = I$: dihedral D_k, order $2k$.
- non-reversing S of order k with shift $\frac{T}{k}$: cyclic Z_k, order k.
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- Montgomery [3] shows calculation of geometric phase.
- "Area enclosed by a loop on the shape sphere."

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dG &= -\frac{1}{2} w_3 d\theta, \text{ where } \theta = \arg(w_1 + i w_2) \\
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- We calculate geometric phase over an orbit of period T by

$$G(T) = \int_0^T U(z(\tau))d\tau. \tag{4}$$

- Symmetries divide an orbit's period evenly.
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Symmetries and antisymmetries of U

- Consider $S \in S_4$ any composition of elements from $\{I, s_1, s_2, s_3, \sigma_1, \sigma_2, \sigma_3, c, c^2\}$.
- Observe that $U \circ S(z) = U(z)$.
- But $U \circ (\rho \circ S)(z) = -U(z)$ and $U \circ (\tau \circ S)(z) = -U(z)$.
- Which also means that $U \circ (\tau \circ \rho \circ S)(z) = U(z)$.
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Recall $dG = -\frac{1}{2}w_3 d\theta$, $\theta = \arg(w_1 + iw_3)$, $S \in Z_4$.

- On shape sphere, s_1, s_2, s_3 do nothing, so dG invariant.
- c, c^2 rotate by $\frac{2\pi}{3}$, fixing equilateral points, so w_3, $d\theta$ invariant.
- σ_1, σ_2, σ_3 rotate paths by π about axes through B_{23}-M_1, B_{31}-M_2, B_{12}-M_3, so $w_3 \to -w_3$, $d\theta \to -d\theta$, leaving dG invariant.
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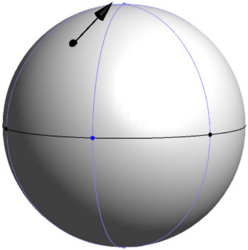

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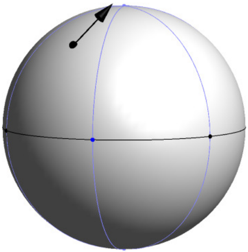

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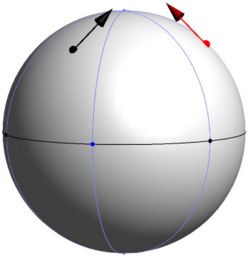

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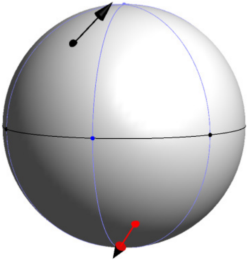

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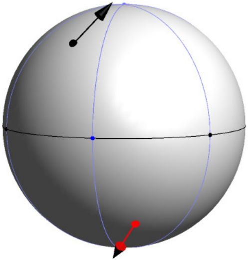

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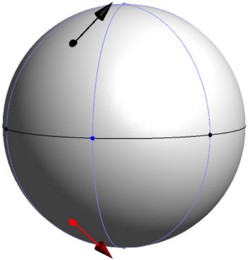

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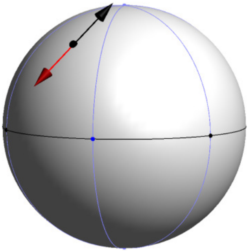

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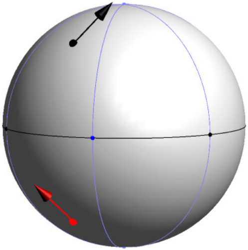

Antisymmetries of U in periodic orbits

Consider periodic solution $z(\tau) = z(\tau + T)$. Cases when isotropy subgroup Σ_z generated by antisymmetry of U, per working hypothesis:

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1. Orbit in invariant subspace: $U(z(\tau)) \equiv 0$, as either $w_3 \equiv 0$ (collinear) or $d\theta = 0$ (isosceles). Any other symmetries don't matter.
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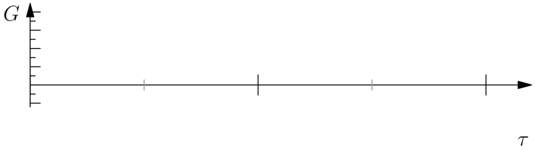
Recall $k \geq 2$ antisymmetry of U is even. Thus result follows for non-reversing case.



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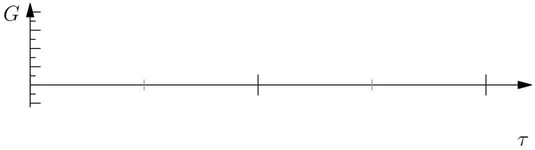

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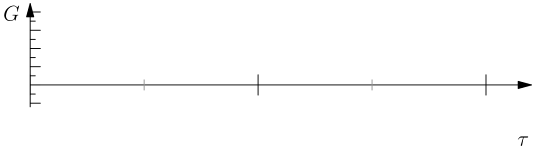

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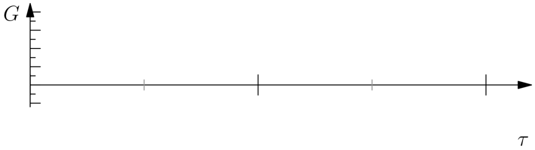

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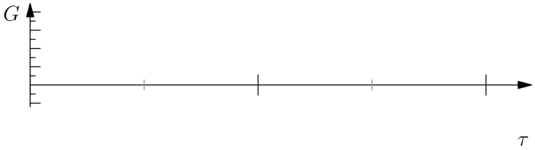

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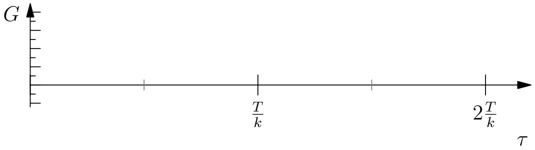

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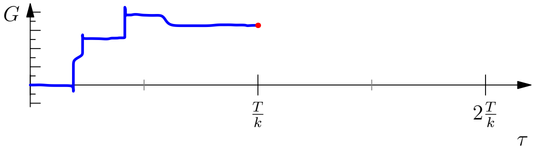

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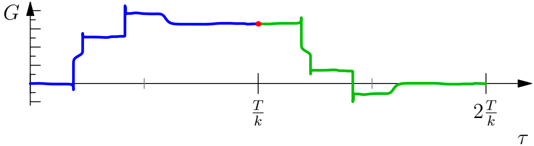

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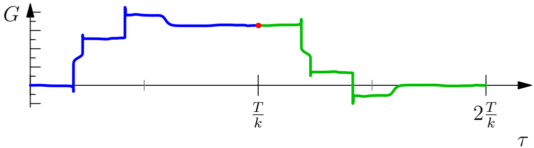

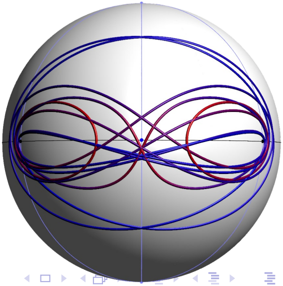

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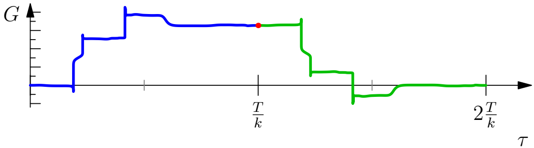

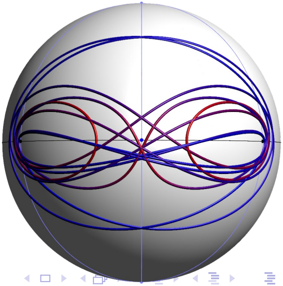

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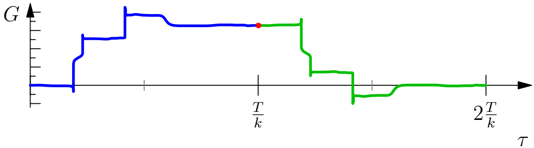

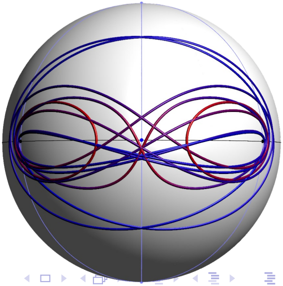

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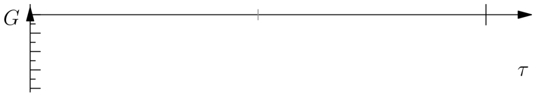

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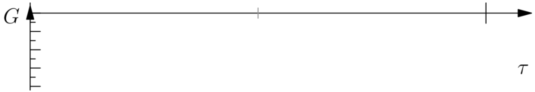

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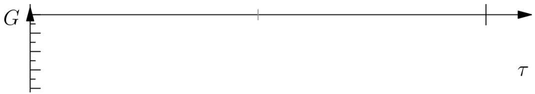

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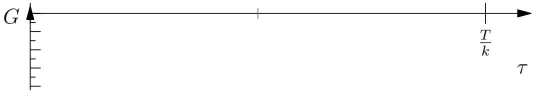

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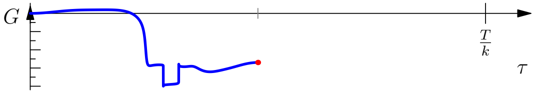

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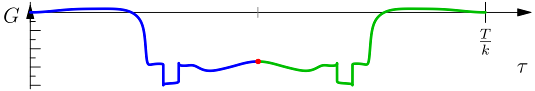

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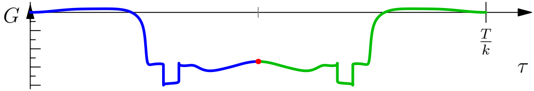

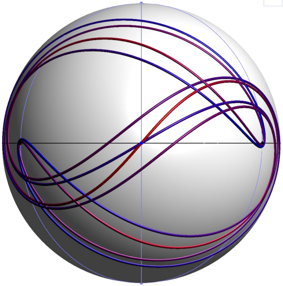

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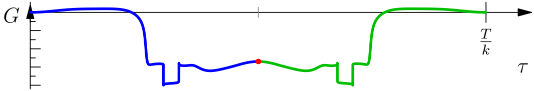

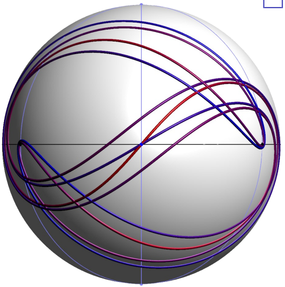

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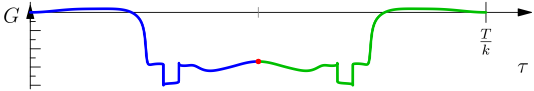

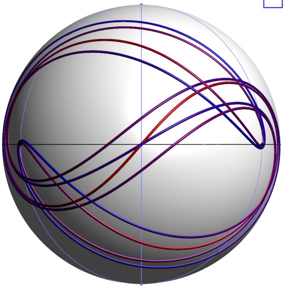

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I.e. any other case, can only vanish by "accident". Requires more knowledge of possible isotropy subgroups of orbits.

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Theorem 2 gives us lots of subgroups forcing no geometric phase, but many, many left over.

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- If isotropy subgroup contains any symmetry of form ρS or τS ($S \in S_4$ as before), geometric phase vanishes.
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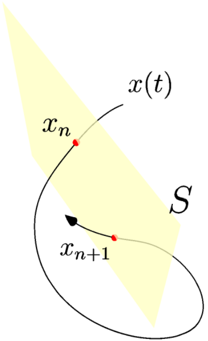
- Introduce the Poincaré map: turns a continuous dynamical system into a discrete one.
- If continuous system D has phase space Ω, Poincaré map is

$$P : S \longrightarrow S,$$

  where $S \subset \Omega$ is the *Poincaré surface of section*.
- Surface of section defined by appropriately chosen $S(x_1, \ldots, x_n) = 0$, with $(x_1, \ldots, x_n) \in \Omega$.
- Poincaré map can be used to find periodic orbits in D.
- Trajectory between section points computed with method from [4].



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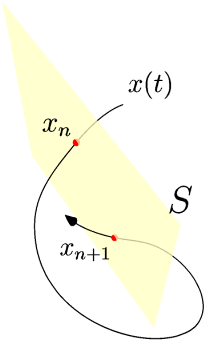

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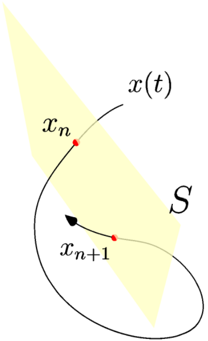

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- If continuous system D has phase space Ω, Poincaré map is

$$P : S \longrightarrow S,$$

  where $S \subset \Omega$ is the *Poincaré surface of section*.
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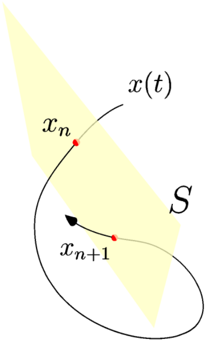

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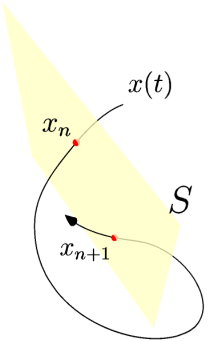

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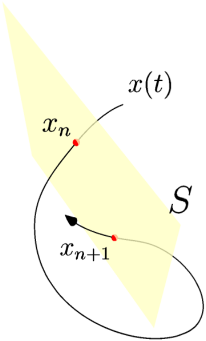

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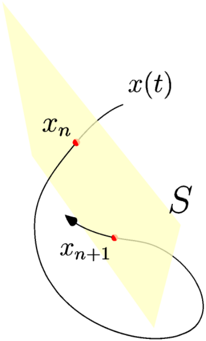

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- Now fix $m_1 = m_2 = m_3 = 1$.
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- Grid points approx 0.05 apart for each dimension. Order of ten million initial conditions.
- Reduce size of search space by integrating up to next section points after $\tau = 250$, looking for near-periodic points of any length in Poincaré map.
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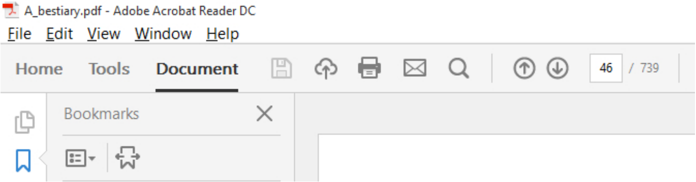

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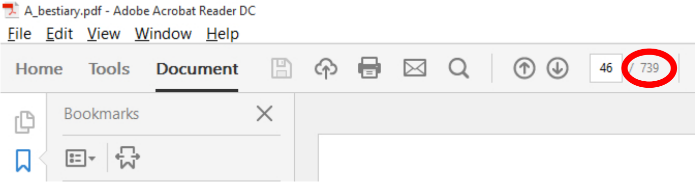

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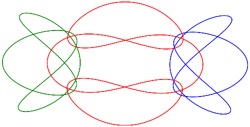

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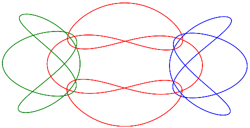

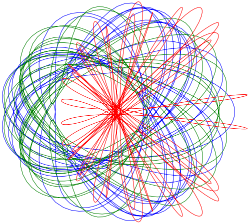

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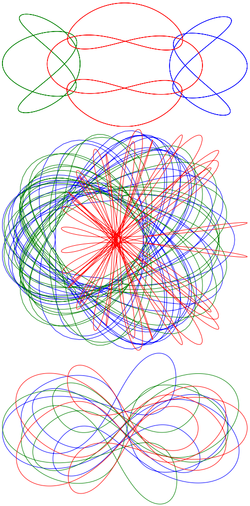

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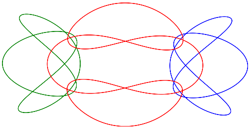
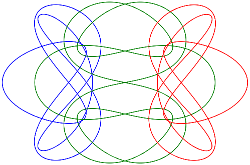
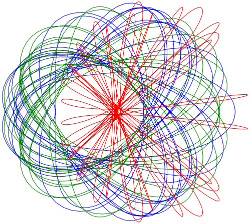
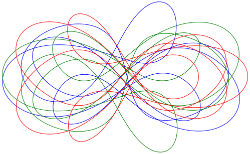


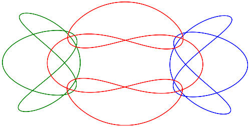


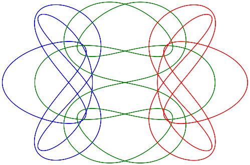


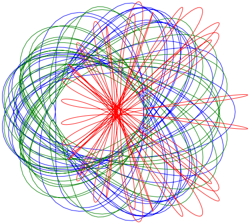


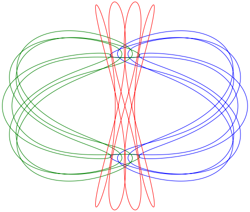
Summary of results

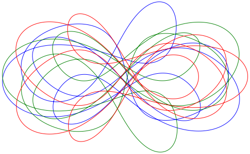
MANY with geometric phase - seems to be the norm. But a substantial number without.











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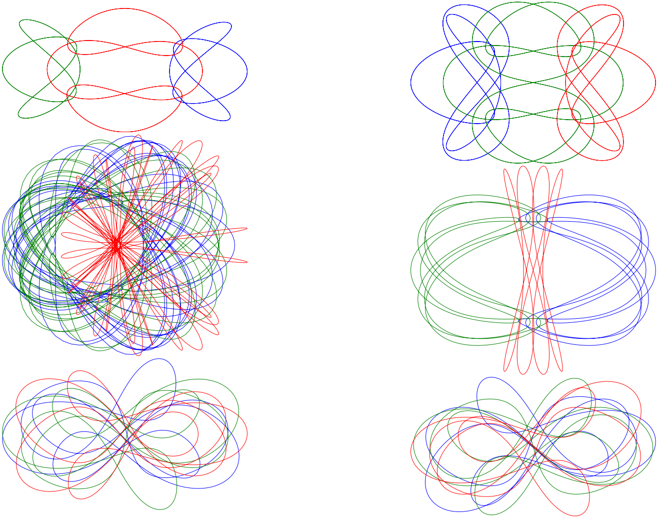
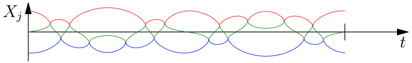


Summary of results

Many collinear orbits. A handful of isosceles orbits.



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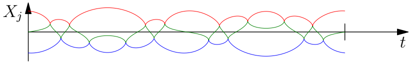

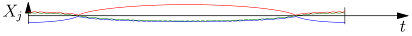

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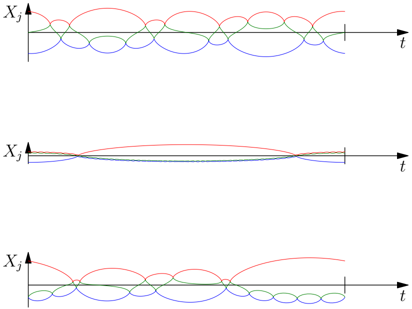

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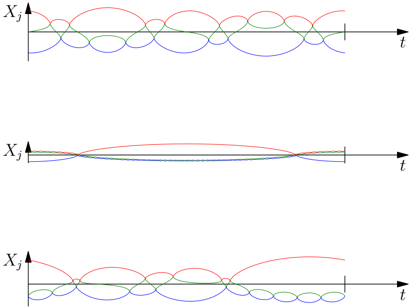

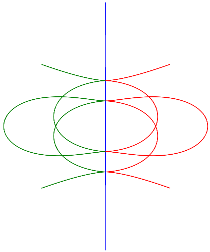

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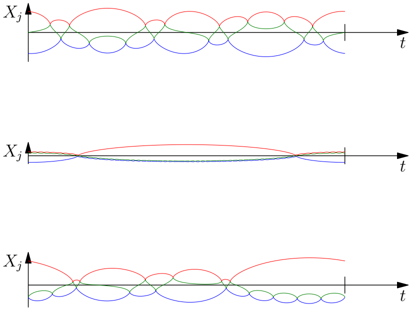

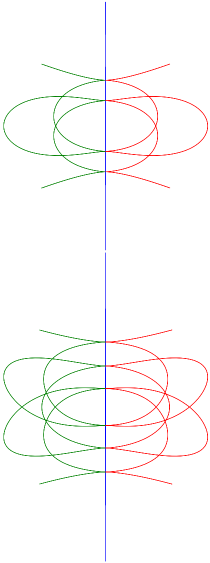

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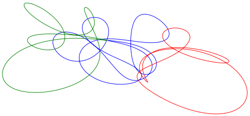
Most other orbits collisionless.



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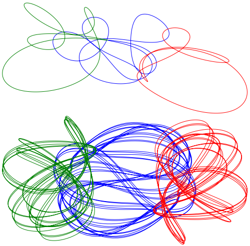

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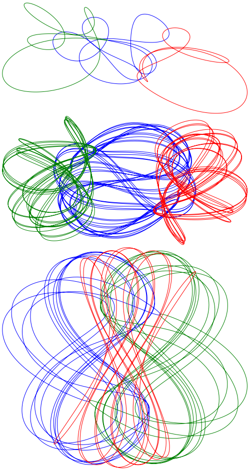

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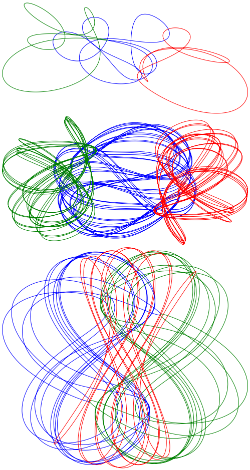

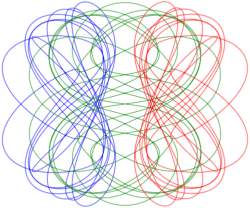

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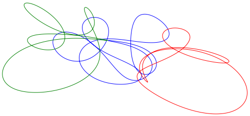

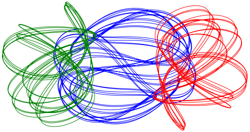

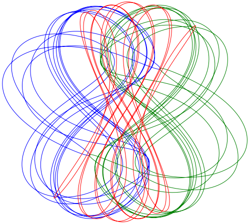

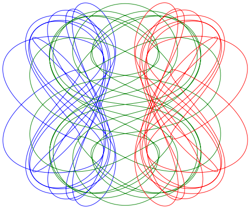

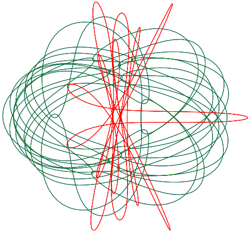

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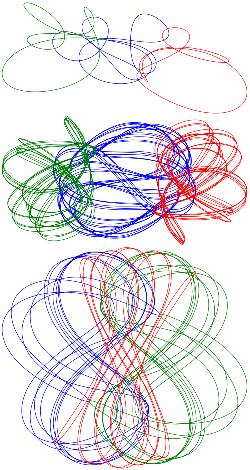

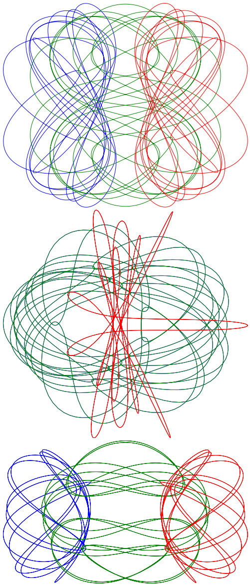

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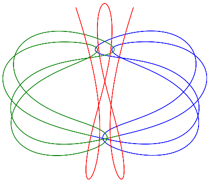

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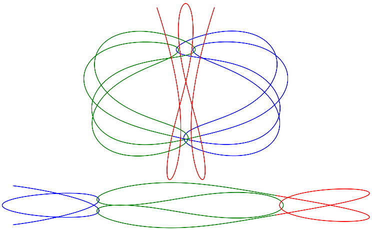

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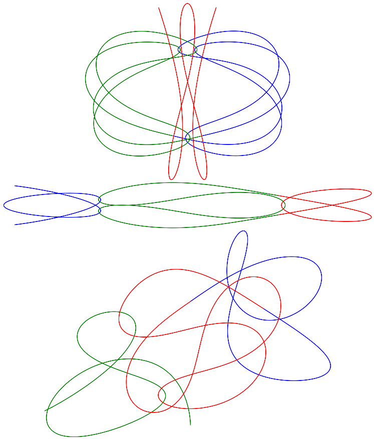

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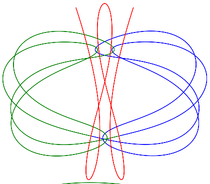

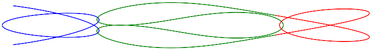

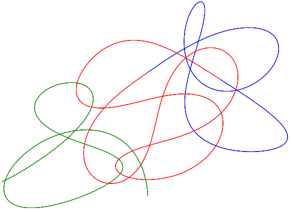

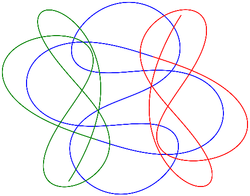

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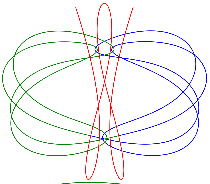

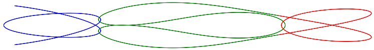

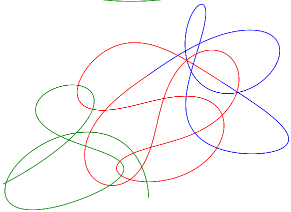

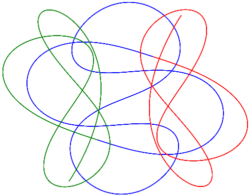

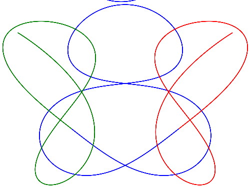

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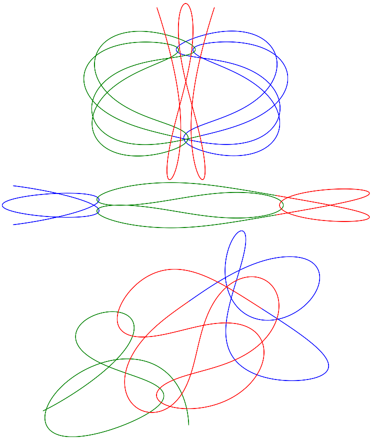

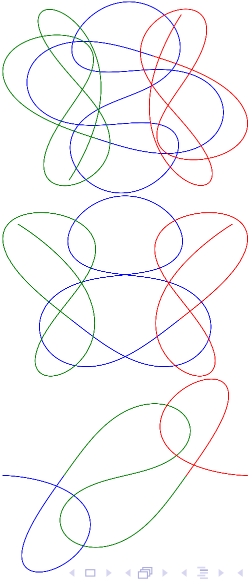

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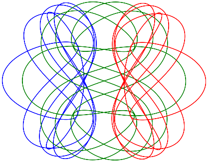

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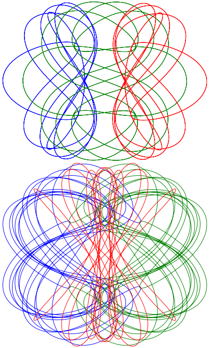

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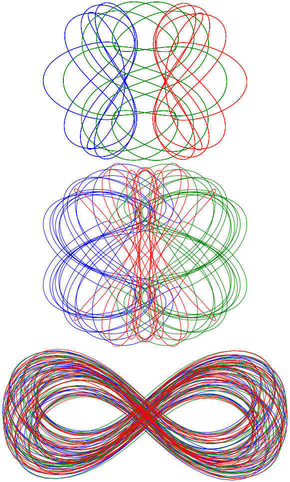

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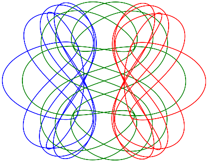

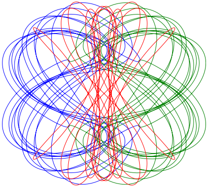

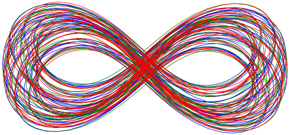

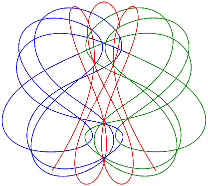

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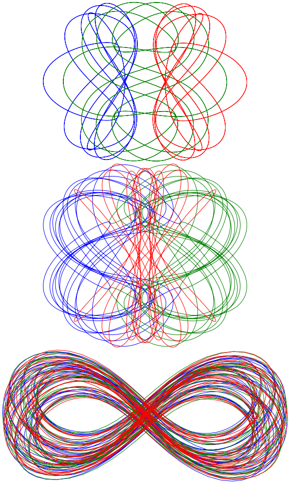

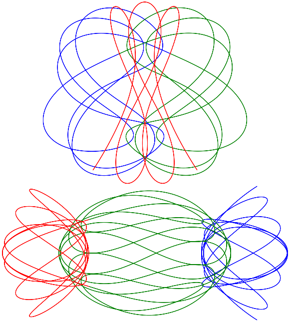

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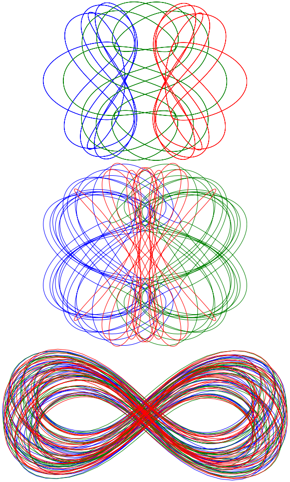
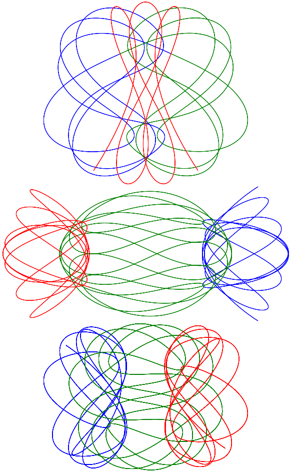
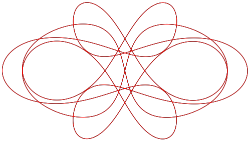




Summary of results

Some simple choreographies.

One simple relative choreography.



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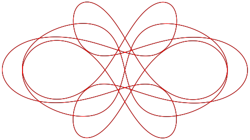

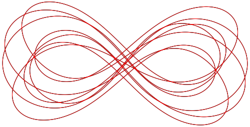

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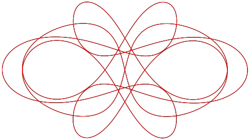
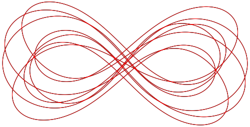
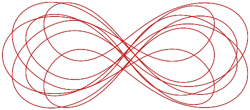
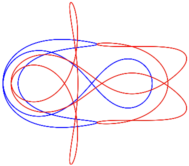




Summary of results

Relative *partial* choreographies. Reversing symmetries on A and C such that $R_1 R_2$ is at least order 2. One known case with cyclic symmetries only. And a fair few more.



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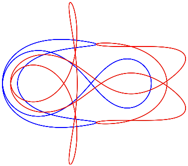

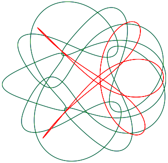

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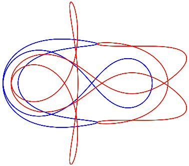

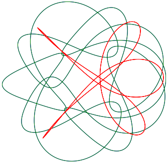

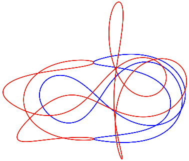

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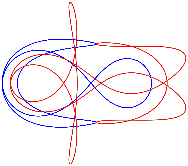

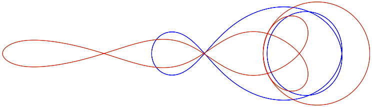

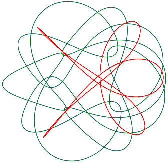

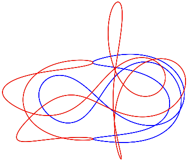

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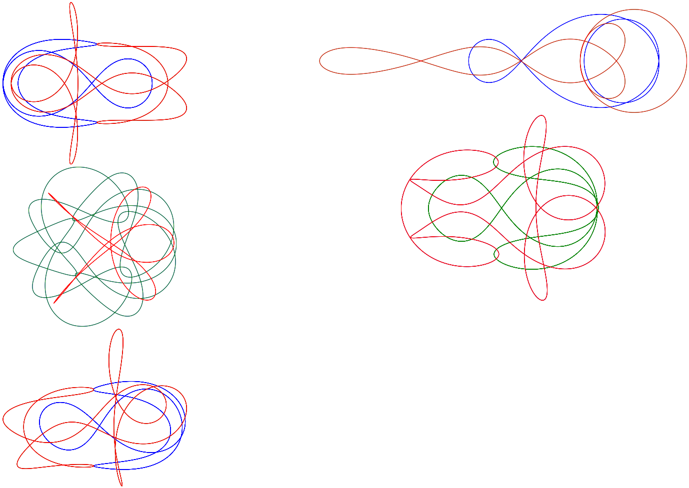

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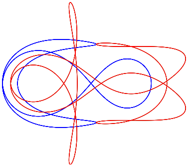

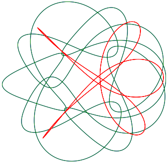

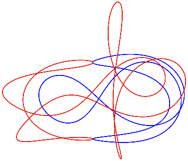

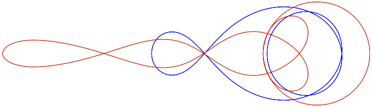

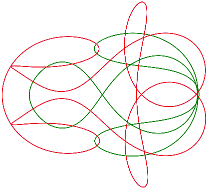

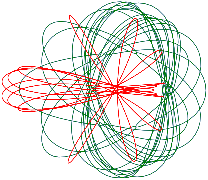

New classification

- Possibly most important observation:
- (Almost) all orbits seem to live near planes orthogonal to lines of symmetry in regularised shape space.
- Possible classification by “mode”. Each mode has characteristic appearance and permits different symmetries.

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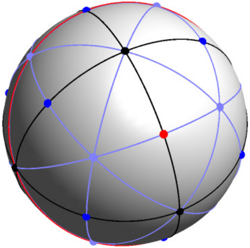

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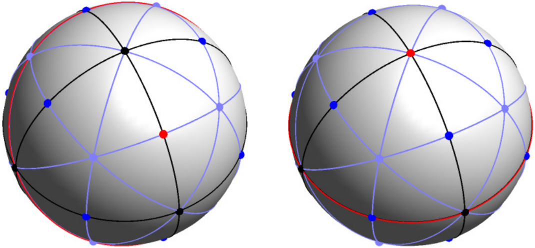

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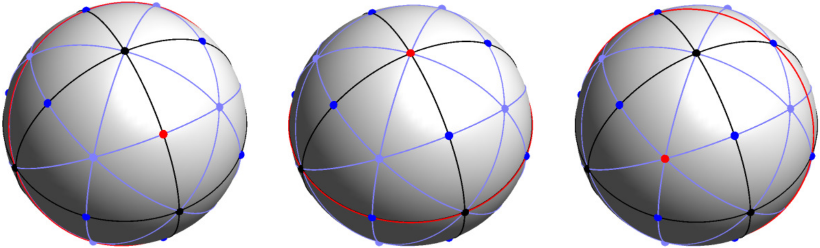

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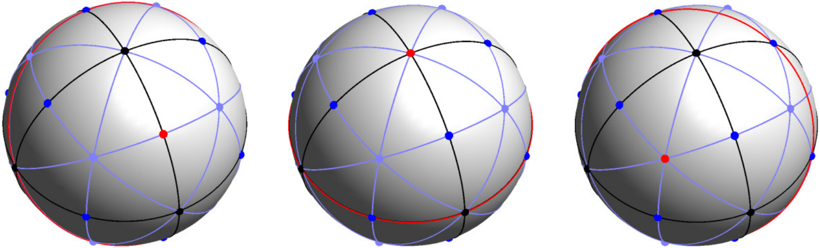

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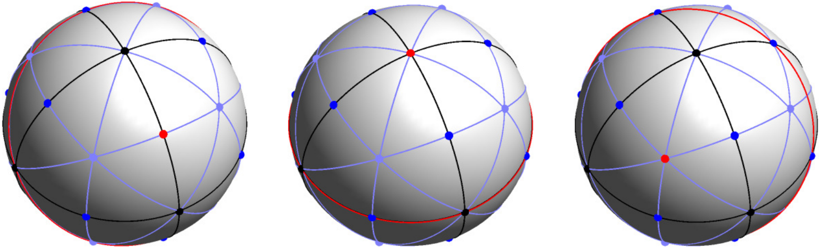

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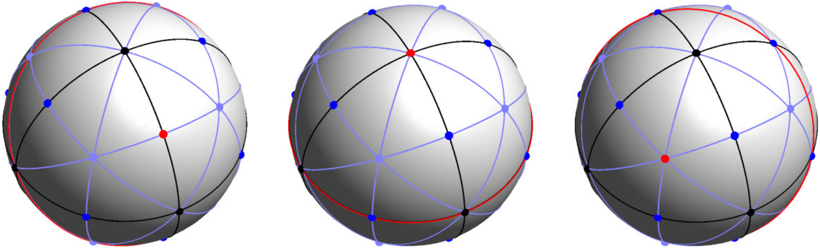

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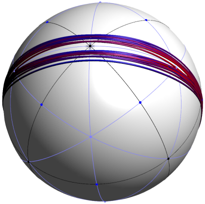
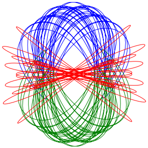
M-mode

- Least symmetric.
- Symmetry of rectangle.
- Moderately common.
- Like isosceles orbits.





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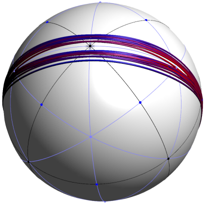

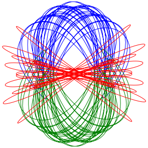

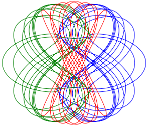

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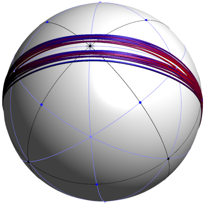

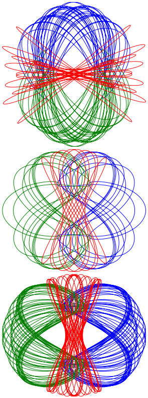

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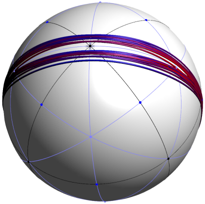

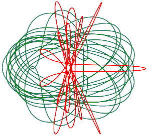

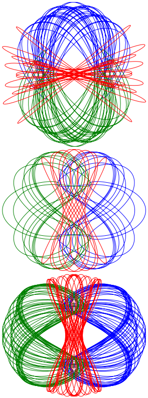

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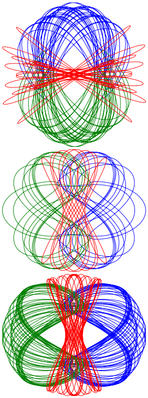

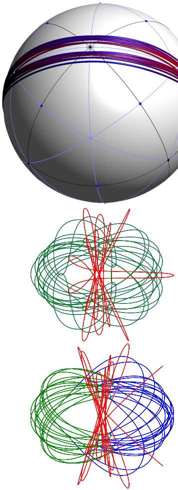

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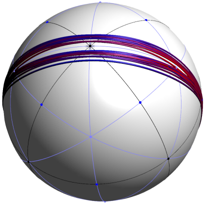
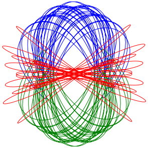
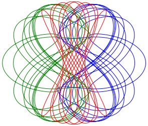
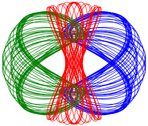
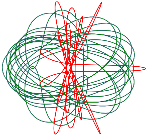
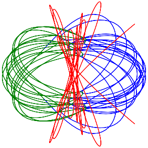


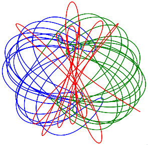






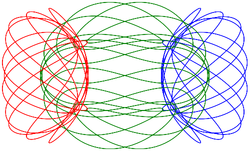


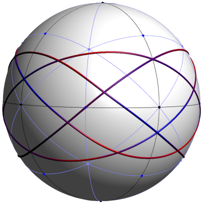




B-mode

- Moderately symmetric.
- Symmetry of square.
- Most common.
- Like collinear orbits.





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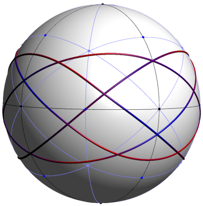

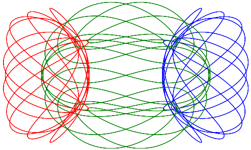

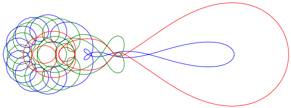

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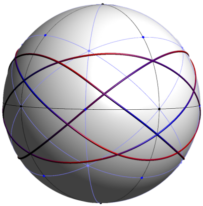

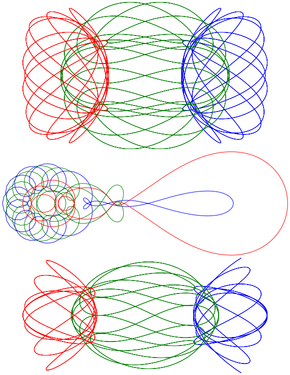

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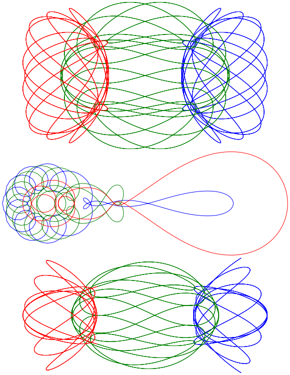

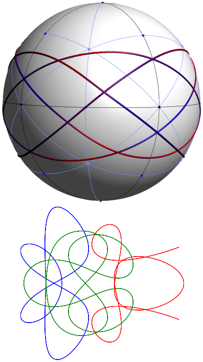

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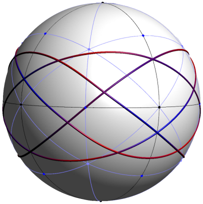

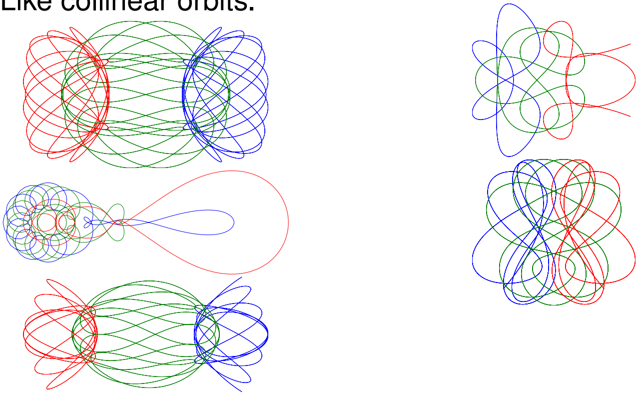

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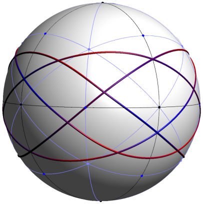
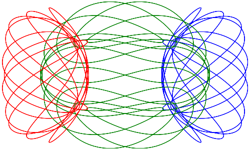
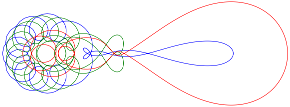
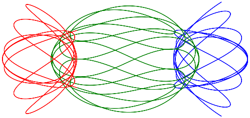
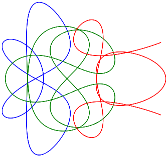
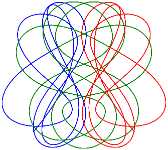
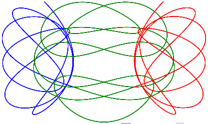
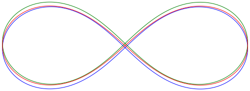
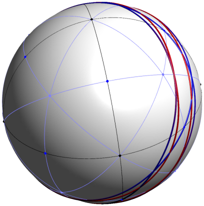








E-mode

- Most symmetric.
- Symmetry of hexagon.
- Very uncommon.
- Like simple choreographies.





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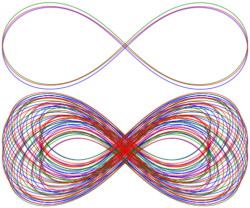

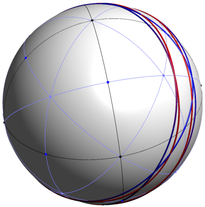

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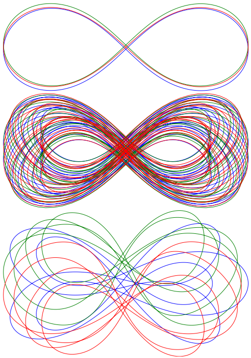

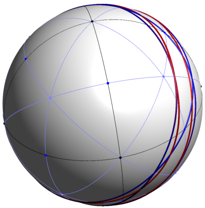

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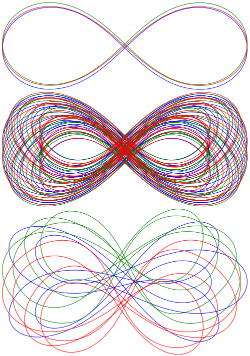

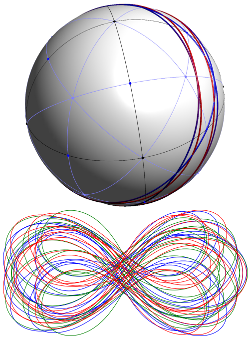

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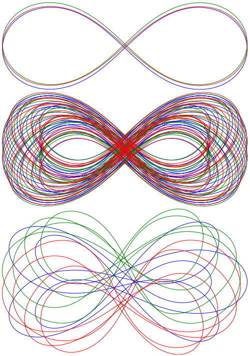

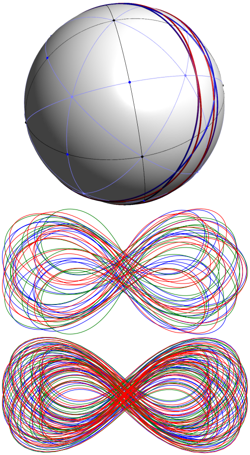

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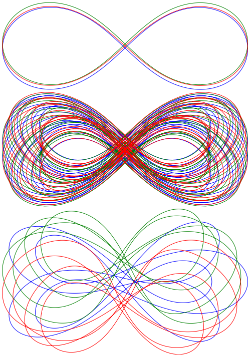

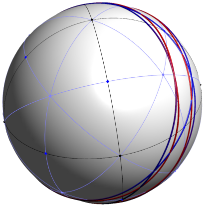

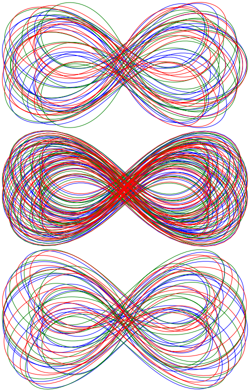

Conclusion

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